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Thermodynamics of the Three-dimensional Black Hole with a Coulomb-like Field

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Abstract: In this paper, we study the thermodynamical properties of the $(2+1)$ dimensional black hole with a Coulomb-like electric field and the differential form of the first law of thermodynamics is derived considering a virtual displacement of its event horizon. This approach shows that it is possible to give a thermodynamical interpretation to the field equations near the horizon. The $\Lambda = 0$ solution is studied and its interesting thermodynamical properties are commented.

Keywords: Thermodynamics; Black Hole; Non-linear Electrodynamics; Coulomb-like Electric Field
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As is well known, the electric field of the BTZ black hole is proportional to the inverse of r, hence its potential is logarithmic. If we are interested in a solution with a Coulomb-like electric field (proportional to the inverse of r^2), we need to consider non-linear electrodynamics. This kind of solution was reported by Cataldo et. al. [1], and describes charged-AdS space when considering a negative cosmological constant.

The thermodynamical properties of black holes are associated with the presence of the event horizon. In particular, Jacobson [2] and Padmanabhan [3] established that the first law in differential form,

$$dM = TdS + \Omega dJ + \Phi dQ, \tag{1}$$

can be obtained from the Einstein's field equations by using the idea of a virtual displacement of the horizon. The same idea was applied by Akbar [4] and Akbar and Siddiqui

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[5] to the BTZ black holes to show that the thermodynamical interpretation of the field equations holds for the static and non-static BTZ metrics in $(2+1)$ gravity to obtain the same results obtained in [6, 7, 8].

In this paper we investigate the thermodynamics of the three-dimensional black hole with a nonlinear electric field reported in [1], to show that the field equations include the first law of thermodynamics in differential form. We also consider the $\Lambda = 0$ black hole to show that it has interesting thermodynamical properties.

1. The Black Hole Solution

The metric reported by Cataldo et. al. [1] is a solution of the $(2+1)$ dimensional Einstein's field equations with a negative cosmological constant $\Lambda = -\frac{1}{l^2} < 0$,

$$G_{\mu\nu} - \frac{g_{\mu\nu}}{l^2} = \pi T_{\mu\nu}, \tag{2}$$

where we have used units such that $G = \frac{1}{8}$. As is well known, the electric field for a static circularly symmetric solution in three dimensions (charged BTZ solution [9]) is proportional to the inverse of r, i.e.

$$E \propto \frac{1}{r}, \tag{3}$$

and therefore, the potential is logarithmic,

$$A \propto \ln r. \tag{4}$$

To obtain a different electric field, Cataldo et. al. used a nonlinear electodynamics. In the non-linear theory, the action I does not depend only on the invariant $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, but it can be a generalization of it, for example

$$I \propto \int d^3x\sqrt{|g|}\left(F_{\mu\nu}F^{\mu\nu}\right)^p, \tag{5}$$

where p is some constant exponent. If the energy-momentum tensor is restricted to be traceless, the action becomes a function of $F^{3/4}$, and the static circularly symmetric solution obtained has the line element

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \tag{6}$$

where

$$f(r) = -M + \frac{r^2}{l^2} + \frac{Q^2}{6r}. \tag{7}$$

The electric field for this solution is

$$E(r) = \frac{Q}{r^2}, \tag{8}$$

which is the standard Coulomb field for a point charge. The metric depends on two parameters Q and M, that are identified as the electric charge and the mass, respectively.

1.1 Horizons

The horizons of this solution are defiened by the condition

$$f(r) = 0 \tag{9}$$

or

$$-M + \frac{r^2}{l^2} + \frac{Q^2}{6r} = 0, \tag{10}$$

that can be transformed into a third-order polynomial,

$$r^3 - \left(Ml^2\right) r + \frac{Q^2 l^2}{6} = 0. \tag{11}$$

This polynomial (11) can be written as

$$r^3 + pr + q = 0, \tag{12}$$

where

$$p = -Ml^2 = \frac{M}{\Lambda} \tag{13}$$

$$q = \frac{Q^2 l^2}{6} = -\frac{Q^2}{6\Lambda}. \tag{14}$$

To obtain the roots of this polynomial we need to establish a classification criteria based on the parameter

$$H = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \tag{15}$$

and using the quantity

$$R = \text{sign}(q)\sqrt{\frac{|p|}{3}} = \sqrt{\frac{Ml^2}{3}}. \tag{16}$$

With these definitions the roots are parameterized by the auxiliarly angle ϕ, that depends on the values of p and H, and we can establish three cases.

1.1.1 Case I. $p < 0$ and $H \leq 0$.

In this case we have a negative cosmological constant, $\Lambda < 0$ and

$$\left(\frac{-Ml^2}{3}\right)^3 + \left(\frac{Q^2 l^2}{12}\right)^2 \leq 0 \tag{17}$$

$$\frac{Q^4}{16} \leq \frac{M^3 l^2}{3}. \tag{18}$$

Therefore, the condition for this case is given, in terms of the cosmological constant, as

$$-\frac{16M^3}{3Q^4} \leq \Lambda < 0. \tag{19}$$

Under this condition, the auxiliary angle is defined by

$$\cos\phi = \frac{q}{2R^3} \tag{20}$$

or

$$\cos\phi = \frac{Q^2}{4Ml}\sqrt{\frac{3}{M}} \tag{21}$$

and the roots are all real,

$$r_1 = -2R\cos\left(\frac{\phi}{3}\right) \tag{22}$$

$$r_2 = -2R\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) \tag{23}$$

$$r_3 = -2R\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right). \tag{24}$$

1.1.2 Case II. $p < 0$ and $H > 0$.

This time we also have a negative cosmological constant, $\Lambda < 0$, but

$$\left(\frac{-Ml^2}{3}\right)^3 + \left(\frac{Q^2 l^2}{12}\right)^2 > 0, \tag{25}$$

which means

$$\frac{Q^4}{16} > \frac{M^3 l^2}{3}. \tag{26}$$

Then, the cosmological constant must be such that

$$\Lambda < -\frac{16M^3}{3Q^4}. \tag{27}$$

In this case, the auxiliary angle is defined by

$$\cosh\phi = \frac{q}{2R^3} \tag{28}$$

or

$$\cosh\phi = \frac{Q^2}{4Ml}\sqrt{\frac{3}{M}} \tag{29}$$

and the roots are

$$r_1 = -2R\cosh\frac{\phi}{3} \tag{30}$$

$$r_2 = R\cosh\frac{\phi}{3} + i\sqrt{3}R\sinh\frac{\phi}{3} \tag{31}$$

$$r_3 = r_2^* = R\cosh\frac{\phi}{3} - i\sqrt{3}R\sinh\frac{\phi}{3}. \tag{32}$$

1.1.3 Case III. $p > 0$ and $H > 0$.

Since $M > 0$, in this case the cosmological constant must be positive, $\Lambda > 0$, and

$$\left(\frac{M}{3\Lambda}\right)^3 + \left(\frac{Q^2}{12\Lambda}\right)^2 > 0 \tag{33}$$

or

$$\Lambda > 0 > -\frac{16M^3}{3Q^4}. \tag{34}$$

Since the cosmological constant in this case is positive, this condition is already fulfilled. The auxiliary angle is defined by

$$\sinh\phi = \frac{q}{2R^3} \tag{35}$$

or

$$\sinh\phi = \frac{Q^2}{4M}\sqrt{\frac{3\Lambda}{M}} \tag{36}$$

and the roots are

$$r_1 = -2R\sinh\frac{\phi}{3} \tag{37}$$

$$r_2 = R\sinh\frac{\phi}{3} + i\sqrt{3}R\cosh\frac{\phi}{3} \tag{38}$$

$$r_3 = r_2^* = R\sinh\frac{\phi}{3} - i\sqrt{3}R\cosh\frac{\phi}{3}. \tag{39}$$

For the black hole solution studied in this paper, the relevant cases are I and II, because the cosmological constant must be negative (i.e. we consider the charged-AdS space). In case I we have three real horizons, while in case II we have just one real horizon.

Note that in case I the limit condition $H = 0$ defines a extreme black hole with mass

$$M_{max} = \sqrt[3]{\frac{3}{16}\frac{Q^4}{l^2}} = \sqrt[3]{-\frac{3}{16}Q^4\Lambda} \tag{40}$$

and with the horizons

$$r_1 = -2R = -2\sqrt{\frac{Ml^2}{3}} \tag{41}$$

$$r_2 = r_3 = R = \sqrt{\frac{Ml^2}{3}}. \tag{42}$$

Note that r_1 is negative, so do not represent a physical horizon (indeed it is always negative). Therefore, the black holes of case I always have masses $M \leq M_{max}$ and only two physical horizons, r_2 and r_3, that, for the extremal black hole, coincide. The largest radius between r_2 and r_3 corresponds to the event horizon of the black hole r_H, while the other corresponds to the inner horizon.

1.2 Heat Capacity

The heat capacity of this black hole is defined by the relation

$$C_Q = \left(\frac{\partial M}{\partial T}\right)_Q, \tag{43}$$

thus, using (10) we obtain

$$C_Q = 4\pi r_H \left(\frac{12r_H^3 - Q^2l^2}{12r_H^3 + 2Q^2l^2}\right). \tag{44}$$

Therefore, C_Q is positive if

$$r_H^3 - \frac{Q^2l^2}{12} > 0 \tag{45}$$

or, using again equation (10), we conclude that the heat capacity is positive when the event horizon has a radius r_H that satisfies

$$r_H > \frac{Q^2}{4M}. \tag{46}$$

2. The First Law of Thermodynamics

In this section we will deduce the first law of thermodynamics for the three-dimensional black hole with Coulomb-like electric field using the field equations near the horizon. First, we will define the thermodynamical quantities in terms of the mass of the black hole given by equation (10). The surface gravity at the horizon is

$$\kappa = \frac{1}{2}\left.\frac{df}{dr}\right|_{r=r_H} = \frac{r_H}{l^2} - \frac{Q^2}{12r_H^2} \tag{47}$$

and the Hawking temperature can be expressed as

$$T = \frac{\kappa}{2\pi} = \frac{r_H}{2\pi l^2} - \frac{Q^2}{24\pi r_H^2}. \tag{48}$$

The Bekenstein-Hawking entropy is given by

$$S = \frac{2\pi r_H}{4G}, \tag{49}$$

where $2\pi r_H$ is the perimeter of the horizon. Since we are working in units such that $G = \frac{1}{8}$, the entropy becomes twice the perimeter,

$$S = 4\pi r_H. \tag{50}$$

The electrostatic potential at the horizon is defined in terms of the mass by

$$\Phi = \left.\frac{\partial M}{\partial Q}\right|_{r=r_H} = \frac{Q}{6r_H}. \tag{51}$$

Now, the Einstein tensor has non-zero components

$$\begin{aligned} G_t^t = G_r^r &= \frac{1}{2r}\frac{df}{dr} \\ G_\varphi^\varphi &= \frac{1}{r}\frac{d^2 f}{dr^2}. \end{aligned}$$

Therefore, the (r, r) component of the field equations is

$$G_r^r + \Lambda g_r^r = \pi T_r^r \tag{52}$$

and when this equation is evaluated at the horizon $r = r_H$, we have

$$\left.\frac{df}{dr}\right|_{r=r_H} - \frac{2r_H}{l^2} = 2\pi r_H T_r^r, \tag{53}$$

where we have used $\Lambda = -\frac{1}{l^2}$. The component T_r^r of the stress-energy tensor can be interpreted as the radial pressure ($T_r^r = -P$), then

$$\left.\frac{df}{dr}\right|_{r=r_H} - \frac{2r_H}{l^2} = -2\pi r_H P. \tag{54}$$

To give a thermodynamical interpretation of this equation, we consider a virtual displacement of the horizon, dr_H. Thus, multiplying on both sides of the equation by this factor,

$$\left.\frac{df}{dr}\right|_{r=r_H} dr_H - \frac{2r_H}{l^2} dr_H = -2\pi r_H P dr_H \tag{55}$$

$$\frac{1}{4\pi}\left.\frac{df}{dr}\right|_{r=r_H} d(4\pi r_H) - \frac{2r_H}{l^2} dr_H = -d\left(\pi r_H^2\right) P. \tag{56}$$

Using equation (10) as the definition of the mass on the horizon, we obtain the differential

$$dM = \frac{2r_H}{l^2} dr_H + \frac{\Phi}{2} dQ, \tag{57}$$

and the field equation becomes

$$\frac{1}{4\pi} \left. \frac{df}{dr} \right|_{r=r_H} d\left(4\pi r_H\right) - dM + \frac{\Phi}{2} dQ = -d\left(\pi r_H^2\right) P. \tag{58}$$

The definition of entropy and Hawking temperature, gives

$$TdS - dM + \frac{\Phi}{2} dQ = -PdA, \tag{59}$$

where A is the area enclosed by the horizon. Thus, the field equation takes the form

$$dM + \frac{\Phi}{2} dQ = TdS + \Phi dQ + PdA. \tag{60}$$

The extra term at the left hand side is just the electrostatic energy enclosed by the horizon, and therefore, we identify

$$M + \frac{\Phi}{2} Q = E \tag{61}$$

as the total energy inside the horizon, and the field equation takes the usual form of the first law of thermodynamics,

$$dE = TdS + \Phi dQ + PdA. \tag{62}$$

3. The $\Lambda = 0$ Black Hole

If we consider a zero cosmological constant, $\Lambda = 0$, the resulting black hole has interesting properties. The line element becomes

$$ds^2 = -\left(-M + \frac{Q^2}{6r}\right) dt^2 + \frac{dr^2}{\left(-M + \frac{Q^2}{6r}\right)} + r^2 d\varphi^2, \tag{63}$$

that shows how this spacetime is asymptotically flat. This charged black hole has just one horizon at

$$r_H = \frac{Q^2}{6M}. \tag{64}$$

The surface gravity at the horizon is given by

$$\kappa = -\frac{Q^2}{12 r_H^2} = -\frac{3M^2}{Q^2} \tag{65}$$

and the Hawking temperature is negative,

$$T = -\frac{Q^2}{24\pi r_H^2} = -\frac{3M^2}{2\pi Q^2}. \tag{66}$$

The possibility of a negative temperature has been associated with the existence of *exotic matter*, but as has been shown by Cataldo et. al. [1], the non-linear electrodynamics used as source in the field equation of this black hole satisfies the weak energy condition. However, a possibility of negative temperatures without exotic matter has been proposed in the de Sitter geometry [10], but due to the thermodynamical instability of this space, the negative temperature is prohibited.

However, one proposal that works fine is to consider, as recently done by Arraut, et. al. [11], the surface gravity as the absolute value

$$\kappa = \frac{1}{2}\left|\frac{df}{dr}\right|_{r=r_H} = \frac{Q^2}{12 r_H^2} = \frac{3M^2}{Q^2}. \tag{67}$$

The reason for this election is the equivalence principle because, in this case, the local inertial observer is moving while r increases, thus, a static observer has a negative scalar acceleration.

On the other hand, the heat capacity for the $\Lambda = 0$ black hole is given by

$$C_Q = -\frac{\pi Q^2}{3M} = -2\pi r_H, \tag{68}$$

i.e. minus the perimeter of the event horizon. Since the heat capacity of this black hole is always negative, $\frac{\partial T}{\partial M} < 0$, the behavior of T is the expected, i.e. as M increases, the temperature decreaces.

Conclusion

We have studied the thermodynamics of the $(2+1)$ dimensional black hole with a Coulomb-like electric field, obtained by the use of a non-linear electrodynamics. By considering a virtual displacement of the horizon, we have shown that the field equations have a thermodynamical interpretation since they can be rewritten as the differential form of the first law,

$$dE = TdS + \Phi dQ + PdA, \tag{69}$$

where E is the total energy inside the horizon (that corresponds to the mass of the black hole plus the electrostatic energy enclosed by the horizon). This fact shows how the thermodynamical properties are undoubtely related with the presence of horizons and maybe, they are just a consequence of the holographic properties of gravity. A further study on this area is in progress.

Finally, using a zero cosmological constant, $\Lambda = 0$, the solution becomes a black hole with a negative Hawking temperature and a negative heat capacity even though the non-linear electric field used as source satisfies the weak energy condition and does not behave as exotic matter. In order to obtain a positive temperature, we use the equivalence principle to consider the absolute value of the surface gravity as the relevant quantity.

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