

A Geometrical Model of Fermion Spin Correlations

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Abstract: Bell's Theorem places limits on correlations between local spin measurements of entangled particles whose properties are described by "hidden variables" established prior to measurement. Bell's derivation assumes that the density of states, or sampling rate, is independent of the orientation of the measuring device. However, points on a rotating sphere are sampled at different rates at different positions, making Bell's Theorem inapplicable. We model spin one-half fermions as having a spherical distribution of observables with azimuthal symmetry, and assume that a Stern-Gerlach device uniformly samples points on a spherical surface. Application of Bayes' Theorem yields the joint density of states for two device orientations. Numerical calculations based on this model yield the fermion spin correlations observed in Stern-Gerlach experiments.

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1. Introduction

In 1935, Einstein, Podolsky, and Rosen (EPR) argued that either quantum mechanics is an incomplete description of nature, or two non-commuting physical operators cannot both have "simultaneous reality". [1] Bell subsequently derived a theorem which constrains the relationship between three angular correlations of entangled particles whose state is completely described by "hidden variables". [2] Since Bell's constraint is actually violated in experiments, at least one assumption in Bell's proof is not valid for actual measurements. The assumption of locality has been frequently questioned, and it is now commonly believed that violation of Bell's inequalities "proves" that locality, the independence of events with space-like separation, is violated. The EPR paradox has been investigated extensively both experimentally and theoretically. A recent review was writ-

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ten by Reid et al. [3] Wikipedia also explains some of the issues and provides a list of references.[4] Though some claim to have solved the paradox, none of the explanations involves straightforward calculation of correlations between particles in definite states in ordinary spacetime.

In this paper we question a different assumption made by Bell. This is the assumption that parameters describing the particle are independent of the device orientation. This is not generally the case when sampling points on a sphere. If points on the surface of a rotating sphere are sampled by fixed measuring devices, the sampling rate depends on device position, increasing as the device position approaches $\pi/2$ radians from the rotation axis.

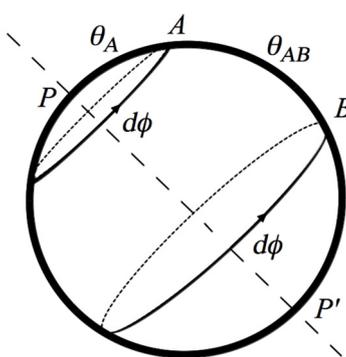


Fig. 1 Rotation about an axis $P - P'$ yields different sampling rates at different fixed positions A and B .

In Figure 1 for example, the arclength swept out by point A is $\sin \theta_A d\phi$, whereas the arclength swept out by point B is $\sin(\theta_A + \theta_{AB})d\phi$.

The issue here is not any causal link between the sphere and device orientations. Rather, it is geometrical relationship between angles, radii, and arclengths. Fixed angular intervals yield different sizes of differential solid angles depending on the location. Uniform angular increments will not yield spatially uniform sampling at different fixed orientations.

Bell's assumption appears to be motivated by the common interpretation of matter as "particles". A point-like particle might be described by a fixed set of parameters such as Bell used. Measurements of vector quantities would still depend on the relative orientation between the vector and the measuring device, but any measuring device would sample only the single point where the particle is located at a given time. Hence the unique density of states of the particle is also the density of states sampled by any measuring device.

However, matter may also be interpreted as consisting of waves, which are extended through space. There is a plethora of evidence for the wave nature of matter:

1. Lorentz invariance is a property of waves, and measurements exclusively involving waves would satisfy the laws of special relativity.[5]
2. Matter, like waves, exhibits interference phenomena.

3. Light can be converted into matter and anti-matter, and vice-versa.
4. Descriptions of both matter and light utilize a characteristic speed c .
5. Waves, like matter, are subject to uncertainty relations.
6. Waves, like fermions, have independent states separated by π rotation and may be described by Dirac bispinors.[6]
7. Quantum mechanical operators can be derived from a simple wave model. [7]
8. Gravity has no quantum mechanical interpretation, but since a gravitational field is equivalent to a variation in the speed of light [8], gravity may simply be interpreted as wave refraction.

Indeed, the "pilot wave" interpretation of quantum mechanics is very similar to a description of soliton waves. [9, 10] This interpretation of quantum mechanics is deterministic but its adherents still utilize non-locality for interpretation of EPR results.[11, 12] Starting from a zero-spin singlet state of entangled particles, the quantum potential acts non-locally to enforce the requirement that the total spin remain zero.

With a wave theory of matter, massive particles must be interpreted as soliton waves which extend through space. Any property of the wave, such as spin or polarization, may have different values at different points in space. In particular, an observable may have an angular distribution, so that measurements made at one orientation sample points of the wave differently than measurements made at a different orientation. For example, in Stern-Gerlach experiments with a primary magnetic field B_z increasing in the z -direction, a spherical wave surface may be regarded as being preferentially sampled at the highest point of the sphere along the z -axis.

Almost all tests of quantum mechanical non-locality involve angular measurements and violations of Bell's inequalities or related constraints[13-18]. An exception is Popper's experiment [19, 20] which tests the effect of spatial localization on one of two entangled particles. Realization of Popper's experiment has not yielded any evidence of non-locality.[21]

In quantum mechanics the correlations are computed from a product of complex amplitudes rather than a product of measured eigenvalues. This method of computing correlations has been applied to purely local measurements and found to yield the same results as quantum mechanics. [22] Hence the key assumption in Bell's derivation is not locality but the integral form of the correlation function.

One could make a simple argument that the correlation between normalized spin measurements should equal the correlation between the measurement vectors (the cosine of the angle between them). A topological argument is also possible. [23, 24] However, these approaches do not adequately address the relationships between physical processes, experimental data, and the theoretically computed correlations.

We will derive the correlations between binary fermion spin measurements from a simple model in which each measurement samples a single point from a spatially uniform distribution on a spherical surface. We will derive the spin correlations observed in EPR experiments, and show where Bell's Theorem breaks down for this model.

2. Bell's Derivation

First, let's review Bell's derivation. We consider the correlation between binary spin measurements made by two Stern-Gerlach devices oriented along directions \mathbf{a} and \mathbf{b} , respectively. The two-particle correlation $P(\mathbf{a}, \mathbf{b})$ is defined as:

$$P(\mathbf{a}, \mathbf{b}) \equiv \frac{1}{N} \sum_{i=1}^N A(\mathbf{a}, i) B(\mathbf{b}, i) \quad (1)$$

where $A = \pm 1$ and $B = \pm 1$ are the normalized results of measurements on one particle at orientation \mathbf{a} and another particle at orientation \mathbf{b} , i labels the individual measurements, and N is the total number of measurements. Bell assumed that the particles being measured could be described by a set of parameters λ , distributed with probability density $\rho(\lambda)$, independent of the device orientation. With these assumptions the correlation is written as:

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) \quad (2)$$

Bell compared correlations involving three different Stern-Gerlach devices measuring entangled particles for which $B(\mathbf{a}, \lambda) = -A(\mathbf{a}, \lambda)$:

$$P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) = - \int d\lambda \rho(\lambda) [A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) - A(\mathbf{a}, \lambda) A(\mathbf{c}, \lambda)] \quad (3)$$

$$= \int d\lambda \rho(\lambda) A(\mathbf{a}, \lambda) A(\mathbf{b}, \lambda) [1 - A(\mathbf{b}, \lambda) A(\mathbf{c}, \lambda)] \quad (4)$$

Since the measurement values are ± 1 , the following inequality holds:

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq \int d\lambda \rho(\lambda) [1 - A(\mathbf{b}, \lambda) A(\mathbf{c}, \lambda)] \quad (5)$$

Bell interprets this inequality as:

$$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| \leq 1 + P(\mathbf{b}, \mathbf{c}) \quad (6)$$

However, we will show that this interpretation is incorrect because a unique density of states $\rho(\lambda)$ cannot generally be used for all three correlations.

3. Model of Stern-Gerlach Measurement

We first consider measurements performed on a single soliton at different orientations of the Stern-Gerlach apparatus with a large field B_z and variations $\partial B_z / \partial z$ and $\partial B_y / \partial y$. The z axis is not fixed, but is defined locally to be the orientation of the Stern-Gerlach device (described by unit orientation vectors \mathbf{a} , \mathbf{b} , or \mathbf{c} which intersect a unit sphere at points A , B , or C , respectively). The x -axis is the direction of propagation. The orientation vectors \mathbf{a} and \mathbf{b} are in the $y - z$ plane, and \mathbf{a} is assumed to be fixed.

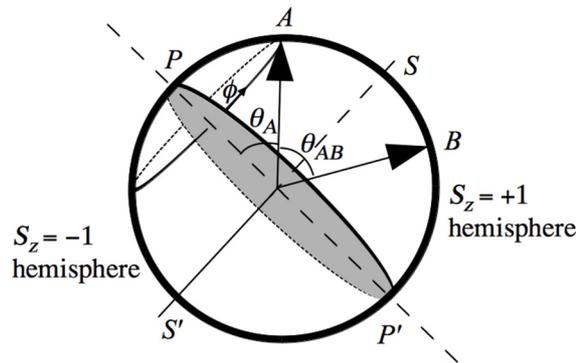


Fig. 2 Proposed Stern-Gerlach geometry, looking toward the x -axis. Points P , A , and B are in the $y - z$ plane.

We model the soliton as a unit sphere divided into two hemispheres by a great circle of arbitrary orientation, as shown in Figure 2. The points P and P' represent the intersection points of the dividing circle with the $y - z$ plane. We assume azimuthal symmetry with respect to rotations about an axis $S - S'$, or \mathbf{s} , through the center of the dividing circle and perpendicular to it. This is equivalent to fixing the angle about the \mathbf{s} axis. A spin measurement yields a value of $+1$ if the sampled point is in one hemisphere, and -1 if in the other hemisphere.

Let A and B refer to the measuring devices (or the points sampled by them). Assume device A samples a point on the sphere, with all points being equally likely. Due to azimuthal symmetry, the canonical rotation axes must be defined orthogonal to \mathbf{s} . Let θ_A be the angle in the $y - z$ plane between the points P and A , and let ϕ be the angle about \mathbf{p} between the great circle and the point A . The angle ϕ is assumed to change direction (preserve its sign) if $\theta_A > \pi$ so that the sign change is attributed to θ_A rather than ϕ . The angle in the $y - z$ plane between the point P and a second measuring device B is $\theta_B = \theta_A + \theta_{AB}$, where θ_{AB} is the angle between A and B orientations.

In keeping with Bell's notation, we denote the measured spin values as $A(\mathbf{a})$ and $B(\mathbf{b})$. Measurement by device A yields the spin $A(\mathbf{a}) = \text{Sign}[\sin \theta_A \sin \phi]$. The probability density of angle θ_A is the ratio between the differential solid angle and the total solid angle of the sphere:

$$dPr(\theta_A) = \frac{2\pi \sin \theta_A d\theta_A}{4\pi} \quad (7)$$

Including variation of ϕ yields a density of states proportional to the differential solid angle $d\Omega_A = |\sin \theta_A| d\theta_A d\phi$. Device B yields the spin $B(\mathbf{b}) = \text{Sign}[\sin \theta_B \sin \phi]$, with differential solid angle $d\Omega_B = |\sin \theta_B| d\theta_B d\phi$.

The density of states is the key to resolving the EPR paradox. Given a sphere with an identified axis \mathbf{p} , one is more likely to sample points nearly orthogonal to \mathbf{p} than nearly parallel to it. This is the origin of the $\sin \theta_A$ factor in the density of states.

Alternatively, one could argue that the point P should be equally likely to be anywhere

in the circle of θ_A , or that the point S should be uniformly distributed around the sphere. This would be the case in the absence of azimuthal symmetry, which reduces the number of states near $\theta_A = 0$ for device A (or near $\theta_B = 0$ for device B).

Hence the mean spins measured at A and B respectively, are:

$$\langle A(\mathbf{a}) \rangle = \frac{\int |\sin \theta_A| d\theta_A d\phi \text{Sign} [\sin \theta_A \sin \phi]}{\int |\sin \theta_A| d\theta_A d\phi} = 0 \quad (8)$$

$$\langle B(\mathbf{b}) \rangle = \frac{\int |\sin \theta_B| d\theta_B d\phi \text{Sign} [\sin (\theta_B) \sin \phi]}{\int |\sin \theta_B| d\theta_B d\phi} = 0 \quad (9)$$

Notice that for each device, the density of states is uniform with respect to the differential solid angle. However, for a given angular parameterization, the differential solid angle depends on the device orientation. Hence the density of states is a function of the orientation of the measuring device. This is inconsistent with Bell's derivation.

As with other angular parameterizations of the sphere, uniform sampling is only achieved at one device location. However, we can use Bayes' Theorem to compute the joint probability of sampling at two locations.

Let $d\text{Pr}(A)$ represent the probability density of sampling at angles (θ_A, ϕ) and $d\text{Pr}(B)$ represent the probability density of sampling at angles (θ_B, ϕ) . For a given configuration (given value of θ_{AB}), the point sampled by the second device is determined from the point sampled by the first device, so the conditional probabilities are $\text{Pr}(B|A) = \text{Pr}(A|B) = 1$. For equal differential solid angles, the differential probabilities for A and B are equal.

For computing correlations, we are interested in the joint probability density $d\text{Pr}(A \cap B)$. According to Bayes' Theorem:

$$d\text{Pr}(A \cap B) = \text{Pr}(B|A) d\text{Pr}(A) = \text{Pr}(A|B) d\text{Pr}(B) \quad (10)$$

Therefore we are free to use either $d\Omega_A$ or $d\Omega_B$ as the differential solid angle for integration. In other words, either θ_A or θ_B may be taken as an independent variable, with the other being a dependent variable.

Taking θ_A as the independent variable yields the correlation between measurements $A(\mathbf{a})$ and $B(\mathbf{b})$:

$$P(\mathbf{a}, \mathbf{b}) = \frac{\int |\sin \theta_A| d\theta_A d\phi \text{Sign} [\sin \theta_A \sin \phi] \text{Sign} [\sin (\theta_A + \theta_{AB}) \sin \phi]}{\int |\sin \theta_A| d\theta_A d\phi} \quad (11)$$

This simplifies to:

$$P(\mathbf{a}, \mathbf{b}) = \frac{\int |\sin \theta_A| d\theta_A \text{Sign} [\sin \theta_A \sin (\theta_A + \theta_{AB})]}{\int |\sin \theta_A| d\theta_A} \quad (12)$$

This expression was evaluated numerically for different values of θ_{AB} and compared with the quantum mechanical prediction of:

$$P(\mathbf{a}, \mathbf{b}) = \cos(\theta_{AB}) \quad (13)$$

Evaluating 80 equally spaced angles for each of θ_A and ϕ , and 40 angles for (θ_{AB}) , the resultant RMS error was 2.4×10^{-14} . [25] Hence we conclude that the above model adequately predicts the correlation between successive measurements on a single particle.

In correlations between two entangled particles, the two particles are created with opposite spin for all orientations (rotation of ϕ by π), thereby changing the sign of spin B . The two-particle correlation predicted by our model is therefore:

$$P(\mathbf{a}, \mathbf{b}) = -\cos(\theta_{AB}) \quad (14)$$

This is in agreement with quantum mechanical predictions and experimental observations of entangled particles.

4. The Flaw in Bell's Proof

Now we can see why Bell's theorem fails to predict the correct result for this model. Since different devices sample different parts of the spherical distribution, the density of states at the device, in terms of sampled angles, depends on its orientation. Given a set of differential angles ($d\theta_A$ and $d\phi$), the differential solid angles at different points are not equal.

For two devices we may take either as the "independent" measurement and the other as the "dependent" measurement. In other words, we may use either A or B as the soliton's local z -axis for defining angles. However, if we then compare a third device C , the expression for density of states at this device is not generally equal to that at either of the other two. The correlations $P(\mathbf{a}, \mathbf{b})$, $P(\mathbf{a}, \mathbf{c})$, and $P(\mathbf{b}, \mathbf{c})$ cannot all use the same expression for density of states. We cannot simply parameterize the density of states of the soliton as a unique distribution $\rho(\lambda)$, because the expression for density of states at the device depends on its orientation. That is why Bell's identification of Equations 5 and 6 is incorrect. Bell's analysis is evidently limited in validity to point-like particles for which each device is sampling the same physical entity.

The effect of azimuthal symmetry is also crucial to computing the correct correlations. Mathematically, azimuthal symmetry allows us to neglect rotations about the \mathbf{s} -axis, which would alter the density of states. Without azimuthal symmetry, Bell's inequality could still hold. One might naively expect that the vector \mathbf{s} should be uniformly distributed in space. Similarly, the fraction of all possible dividing circles between devices separated by angle θ must be equal to $\theta/2\pi$. However, azimuthal symmetry effectively reduces the weight of each circle parallel to the $y - z$ plane to the weight of a single point. This reduces the relative weight of sampled points near the dividing circle and increases the relative weight of sampled points far from the dividing circle. The net effect of azimuthal symmetry is thus an increase in the correlations.

5. Interpretation of Measurement

Since a deterministic model can explain fermion spin correlations, there is no need for the quantum mechanical interpretation of measurement which "collapses" the wave function into the measured eigenstate. Rather, the evidence supports the hypothesis of a definite soliton state.

Suppose we have entangled fermions with opposite spin. If the first soliton is selected for a particular result from device A , the correlation with this fermion's spin at device B is exactly negative of the correlation of the second fermion's spin at a device having the same orientation as B . Hence all of the measurements are consistent with the existence of pre-determined states.

The model is also consistent with the fact that only one component of spin is measurable at any instant. It is impossible to superpose multiple Stern-Gerlach devices to measure the precise orientation of the sphere because two superposed devices are equivalent to a single device measuring an intermediate angle. We can only measure spin at one point at a time (A in this case).

According to the above analysis, the soliton need not change its orientation as it passes through the Stern-Gerlach device. However, the experimental results require that information of spin S_y (and S_x) orthogonal to a device's orientation be destroyed if a particular state of S_z is selected. This loss of information on S_y (and S_x) could be attributed to rotation of the soliton about the measured axis. Net rotation occurs only if the wave propagation is disturbed by a measurement (e.g. blocking part of the wave). Any of the rotated states would have been equally likely *a priori*, so such rotation would not effect the correlations. Hence this model could be consistent with the loss of spin information due to measurement of other spin components.

We emphasize that the spherical shell model presented here is a simplification of a wave model of matter. It is not intended to be physically realistic, but only to provide a geometrical model for interpreting spin correlations. A complete wave model would be necessary to account for other phenomena such as interference.

The predictions of this paper may be at least approximately tested by a relatively simple experiment. Take a smooth ball, color one hemisphere white (+1) and one hemisphere black (-1). Make a holder to fit the ball with two small holes separated by an angle of 0.689 rad, which is the angle of maximum difference between cosine and linear correlations. For each trial, manipulate the ball to obtain a random orientation, then place it in the holder and look at the color through each of the two holes. The resulting correlation should be $P = 1 - 2(0.689)/\pi = 0.56$, consistent with linear correlation and Bell's theorem.

In order to include the effect of azimuthal symmetry, measure the angle of each sampled point from the dividing circle and compute the cosine of this angle (proportional to the circumference of the azimuthal circle through the sampled point). Then compute the weighted correlation using the inverse square of the average of the two cosines as a weighting factor. (If the two measurements were independent, accurate, and part of

an infinite sample set, then the appropriate weight would be the inverse of the product of the two cosines. The proposed weight factor reduces the influence of points near the poles. Lack of independence between the two position measurements is assumed to be of minimal importance since both points cannot be very near a pole.) The weighted correlation will certainly be higher than the non-weighted correlation, and should approach $\cos(0.689 \text{ rad}) = 0.77$. Vary the spacing between the holes to determine correlations for other angular separations.

Conclusions

The correlation between binary spin measurements is derived from a geometrical model of a spin one-half fermions as a spherical surface with azimuthal symmetry, separated into (+) and (−) spin states by a great circle. Bell's theorem is not valid for this model because for a given angular parameterization, the density of sampled states varies with the orientation of the measuring device. The model is consistent with an interpretation of elementary fermions as extended soliton waves rather than point-like particles.

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