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The Conformal Universe II: Conformal Symmetry, its Spontaneous Breakdown and Higgs Fields in Conformally Flat Spacetime

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Abstract: This is the second of a series of three papers on Conformal General Relativity. The conformal group is here introduced as the invariance group of the partial ordering of causal events in flat nD spacetime. Its general structure and field–theoretic representations are described, with particular emphasis on the remarkable properties of orthochronous inversions. Discrete symmetries and field representations in flat 4D spacetime are described in details. The spontaneous breakdown of global and local conformal symmetry is then discussed and the roles played by the ghost dilation field and physical scalar fields are evidenced. Hyperbolic coordinates of various types are introduced for the purpose of providing different but equivalent representations of physical systems, respectively grounded on the Riemann manifold and the Cartan manifold, mainly in view of further studies. Lastly, the Lagrangian theory of Higgs fields interacting with dilation field in a conformally flat 4D spacetime, as well as their motion equations and energy–momentum tensors, are described for both Riemann– and Cartan–manifold representations, in view of the detailed study of Higgs–field dynamics, which will be illustrated in the third paper.

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1. The conformal group and its representations

As explained in the last Section of Part I, the local conformal flatness of the 4D Riemann manifold peculiar to the semi–classical approximation survives gravitational quantum–

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field renormalization, thanks to the presence of massive gravitational ghosts, which are natural ingredients of local conformal invariance. Causality, although destroyed precisely by these ghosts on some sub–Planckian scale, nevertheless survives as a relationship among *observable events*, with the possible exception of the primordial symmetry–breaking event, provided that the mass of gravitational ghosts is sufficiently large. Therefore, wherever the Riemann manifold is the topological support of observable events, it is locally isomorphic to Minkowski spacetime.

The importance of this lies in the fact that the local conformal group is the widest invariance group of *partial ordering of causal events* which may be implemented on a differentiable manifold. Let us clarify this point for the general case of nD spacetime.

1.1 Conformal group as the invariance group of causality

Let us define an *nD Minkowski spacetime* $\mathcal{M}_n$ as the Cartesian product of an (n-1)–dimensional Euclidean space $\mathcal{R}^{n-1}$ by a real time axis T. The former is intended to represent the set of possible inertial observers at rest in $\mathcal{R}^{n-1}$ equipped with perfectly synchronized clocks, and the latter is intended to represent common time $x^0 \in T$ marked by the clocks. All observers are allowed to communicate with one another by signals of limited speed, the upper limit of which is conventionally assumed not to exceed 1 (the speed of light in $\mathcal{R}^{n-1}$). Let us indicate by $x^1, x^2, ..., x^{n-1}$ the coordinates of $\mathcal{R}^{n-1}$. Hence, regardless of any metrical considerations, we can write $\mathcal{M}_n = \mathcal{R}^{n-1} \times T$, which means that $\mathcal{M}_n$ is equivalent to the n–dimensional affine space. Clearly, points $x = \{x^0, x^1, x^2, ..., x^{n-1}\} \in \mathcal{M}_n$ also represent the set of all possible point–like events, observable in $\mathcal{R}^{n-1}$ at time x^0, partially ordered by the relation $x \leq y$ defined as follows:

x can influence y if and only if $y^0 - x^0 \leq \sqrt{(y^1 - x^1)^2 + \cdots + (y^{n-1} - x^{n-1})^2}$.

In contrast to causality in Newtonian spacetime, this partial ordering equips $\mathcal{R}^{n-1}$ with a natural topology, the basis of which may be formed by the open sets of points $y \in \mathcal{R}^{n-1}$, satisfying the inequalities

$$\sqrt{(y^1 - x^1)^2 + \cdots + (y^{n-1} - x^{n-1})^2} < \varepsilon$$

for arbitrarily small ε. The set of points $y \geq x$ with fixed x and variable y defines the *future cone* of x; the set of points $y \geq x$ with fixed y and variable x defines the *past cone* of y. The equation

$$(y - x)^2 \equiv (y^0 - x^0)^2 - (y^1 - x^1)^2 - \cdots - (y^{n-1} - x^{n-1})^2 = 0$$

defines the family of (double) light–cones of $\mathcal{M}_n$.

On this basis, the following theorem was proven by Zeeman in 1964 [1] (but before him, in a different way, by Alexandrov in 1953 [2] [3]):

Let Z be a one–to–one mapping of nD Minkowski spacetime $\mathcal{M}_n$ on to itself - no assumption being made about whether Z is linear or continuous. If Z preserves the partial ordering of events and $n > 2$, then Z maps light–cones onto light–cones and belongs to a group that is the direct product of the nD Poincaré group and the dilation group.

The same result was also obtained by other authors [4] [5] on the basis of weaker topological assumptions, and it is conceivable that it may be directly obtained by pure lattice–theoretic methods and suitable automorphism conditions, regardless of any embedding of causal events in an affine space.

To be specific, the invariance group of causality acts on x^μ, ($\mu = 0, 1, \ldots, n-1$), as follows

$$T(a) : x^\mu \to x^\mu + a^\mu \quad \text{(translations)}\,; \tag{1}$$

$$S(\alpha) : x^\mu \to e^\alpha x^\mu \quad \text{(dilations)}\,; \tag{2}$$

$$\Lambda(\omega) : x^\mu \to \Lambda(\omega)^\mu_\nu x^\nu \quad \text{(Lorentz rotations)}\,. \tag{3}$$

Here a^μ, α and the tensor $\omega \equiv \omega^{\rho\sigma} = -\omega^{\sigma\rho}$ are respectively the parameters of translations, dilation and Lorentz rotations. However, the invariance group of causality is somewhat larger [6] [7], since the partial ordering of events is also preserved, for instance, by the following map

$$I_0 : x^\mu \to -\frac{x^\mu}{x^2}\,,$$

where $x^2 = x^\mu x_\mu, x_\mu = \eta_{\mu\nu} x^\nu$, $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, \ldots, -1\}$, which we call the *orthochronous inversion* with respect to event $x = 0 \in \mathcal{M}_n$ [8]. The equalities $(I_0)^2 = 1$, $I_0 \Lambda(\omega) I_0 = \Lambda(\omega)$ and $I_0 S(\alpha) I_0 = S(-\alpha)$ are manifest. By translation, we obtain the orthochronous inversion with respect to any point $a \in \mathcal{M}_n$, which acts on x^μ as follows:

$$I_a : x^\mu \to -\frac{x^\mu - a^\mu}{(x-a)^2}\,.$$

Clearly, the causal ordering is also preserved by the transformations $E(b) = I_0 T(b) I_0$, which manifestly form an Abelian group and act on x^μ as follows

$$E(b) : \; x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2bx + b^2 x^2}\,, \tag{4}$$

where bx stands for $b^\mu x_\mu$. These are commonly known as *special conformal transformations*, but we will call them *elations*, since this is the name coined for them by Cartan in 1922 [9].

In conclusion, the topologically connected component of the complete invariance group of causal ordering in the nD Minkowski spacetime is the *n–dimensional conformal group* $\mathfrak{C}(1, n-1)$ formed of transformations (1)-(4), comprehensively depending on $n(n+3)/2+1$ real parameters.

From an invariantive point of view, $\mathfrak{C}(1, n-1)$ can be defined as the more general continuous group generated by infinitesimal transformations of the form $x^\mu \to x^\mu + \varepsilon u^\mu(x)$, where ε is an infinitesimal parameter and $u_\mu(x)$ are arbitrary real functions, satisfying the equation

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu \to \varepsilon \lambda(x)\, g_{\mu\nu} dx^\mu dx^\nu\,,$$

where $\lambda(x)$ is an arbitrary real function [10].

Indicating by P_μ, $M_{\mu\nu}$, D and K_μ the generators of $T(a), \Lambda(\omega), S(\alpha)$ and $E(b)$, respectively, we can easily find their actions on x^μ

$$P_\mu x^\nu = -i\delta^\nu_\mu\,, \quad M_{\mu\nu}x^\lambda = i(\delta^\lambda_\nu x^\mu - \delta^\lambda_\mu x^\nu)\,,$$
$$Dx^\mu = -ix^\mu\,, \quad K_\mu x^\nu = i(x^2\delta^\nu_\mu - 2x_\mu x^\nu)\,,$$

where δ^ν_μ is the Kronecker delta. Consequently, their actions on arbitrary differentiable functions f of x are

$$P_\mu f(x) = -i\partial_\mu f(x)\,; \quad M_{\mu\nu} f(x) = i(x_\mu\partial_\nu - x_\nu\partial_\mu) f(x)\,;$$
$$D\, f(x) = -ix^\mu\partial_\mu f(x)\,; \quad K_\mu f(x) = i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu)\, f(x)\,;$$

the Lie algebra of which satisfies the following commutation relations

$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0\,; \quad [P_\mu, K_\nu] = 2i(g_{\mu\nu}D + M_{\mu\nu})\,; \tag{5}$$
$$[D, P_\mu] = iP_\mu\,; \quad [D, K_\mu] = -iK_\mu\,; \quad [D, M_{\mu\nu}] = 0\,; \tag{6}$$
$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)\,; \quad [M_{\mu\nu}, K_\rho] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu)\,; \tag{7}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho})\,, \tag{8}$$

which form the prototype of the abstract Lie algebra of $\mathfrak{C}(1, n-1)$ [11] [12].

For the sake of clarity and completeness, we add to Eqs. (5)–(8) the discrete transforms

$$I_0 P_\mu I_0 = K_\mu\,; \;\; I_0 K_\mu I_0 = P_\mu\,; \;\; I_0 D I_0 = -D\,; \;\; I_0 M_{\mu\nu} I_0 = M_{\mu\nu}\,. \tag{9}$$

Note that I_0 and P_μ alone suffice to generate $\mathfrak{C}(1, n-1)$. In fact, using Eqs. (5)–(7) and the first of Eq.s (9), we can define all other group generators as follows:

$$K_\mu = I_0 P_\mu I_0\,, \quad D = \frac{i}{8} g^{\mu\nu}[K_\mu, P_\nu]\,, \quad M_{\mu\nu} = \frac{i}{2}[K_\nu, P_\mu] - g_{\mu\nu}D\,.$$

This tells many things about the partial ordering of causal events in $\mathcal{M}_n$. For instance, we may think of I_0 as representing the operator which performs the partial ordering of events as seen by a point–like agent located at $x = 0$, which receives signals from its own past and sends signals to its own future, of $T(a)$ as the operator which shifts the agent from $x = 0$ to $x = a$ in $\mathcal{M}_n$ and of I_a as a continuous set of involutions which impart a symmetric–space structure to the lattice of causal events.

Lastly note that, provided that n is even, we can include parity transformation P : $\{x^0, \vec{x}\} \to \{x^0, -\vec{x}\}$ as a second discrete element of the conformal group. Time–reversal must be instead excluded, since it does not preserve the causal ordering of events. This makes an important difference between General Relativity (GR) and Conformal General Relativity (CGR): *time reversal, so familiar to GR, is replaced by orthochronous inversion I_0 conventionally centered at an arbitrary point $x = 0$.*

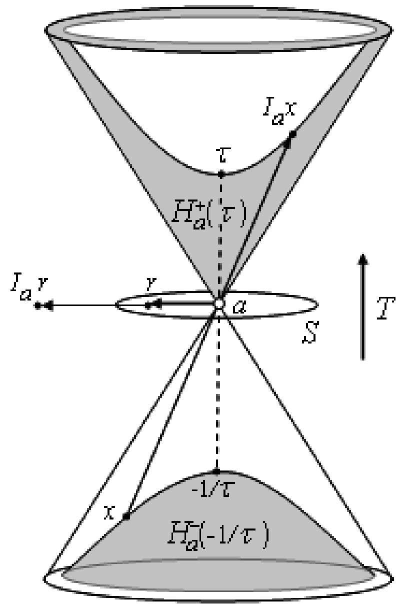

Fig. 1 The orthochronous inversion stemming from point a of the Minkowski spacetime.

1.2 Remarkable geometric properties of orthochronous inversions

Orthochronous inversion I_a, where a is any point of $\mathcal{M}_n$, has the following properties:

1) It leaves invariant the double cone centered at a, swapping the interiors of the future and past cones so as to preserve the time arrow and the collineation of all points on straight lines through a.

2) It partitions the events of the past and future cones stemming from a into a two–fold family of $(n-1)$-D hyperboloids parameterized by kinematic time

$$\tau = \pm\sqrt{(x^0 - a^0)^2 + \cdots + (x^{n-1} - a^{n-1})^2}\,,$$

as shown in Fig. 1.

3) It maps future region $H_a^+(\tau)$, extending from cone–vertex a to the $(n-1)$–D hyperboloid at τ, into region $H_a^-(-1/\tau)$ of the past cone defined by the $(n-1)$–D hyperboloid at $-\infty$ and the hyperboloid at $\tau' = -1/\tau$, and vice versa (Fig.1, gray areas). Similarly, it maps the set–theoretic complements of $H_a^+(\tau), H_a^-(-1/\tau)$ onto each other within the cones.

4) It performs polar inversion of points r internal to the space–like unit $(n-1)$–D sphere S centered at a and orthogonal to time axis T into points $r' = I_a r$ external to S, and vice versa.

5) Functions which are invariant under I_a only depend upon kinematic time τ. Thus, if they vanish near the vertex of the past cone, they also vanish at the infinite kinematic time of the future cone, and vice versa. This property has an important implication in that, if I_0 is invariant under the spontaneous breakdown of conformal symmetry, in an infinite time the universe reaches the same physical conditions in which it existed just a moment before the symmetry breaking event. In other words, the action–integral invariance under I_0 is compatible with assuming that *the time course of the universe may*

be described as a transition from a state of an unstable initial vacuum to one of a stable final vacuum.

1.3 Conformal transformations of local fields

When a differential operator g is applied to a differentiable function f of x, the function changes as $gf(x) = f(gx)$, which may be interpreted as the form taken by f in the reference frame of coordinates $x' = gx$. When a second differential operator g' acts on $f(gx)$, we obtain $g'f(gx) = f(gg'x)$, i.e., we have $g'gf(x) = f(gg'x)$, showing that g' and g act on the reference frame in reverse order.

The action of g on a local quantum field $\Psi(x)$ of dimension, or weight, w_Ψ, bearing a spin subscript ρ, has the general form $g\Psi_\rho(x) = \mathcal{F}_\rho^\sigma(g^{-1}, x)\Psi_\sigma(gx)$, where $\mathcal{F}(g^{-1}, x)$ is a matrix obeying the composition law $\mathcal{F}(g_2^{-1}, x)\mathcal{F}(g_1^{-1}, g_2x) = \mathcal{F}(g_2^{-1}g_1^{-1}, x)$. These equations are consistent with coordinate transformations, since the product of two transformations g_1, g_2 yields

$$g_2g_1\Psi_\rho(x) = \mathcal{F}_\rho^\sigma[(g_1g_2)^{-1}, x]\Psi_\sigma(g_1g_2x)\,,$$

with g_2, g_1 always appearing in reverse order on the right–hand member.

According to these rules, the action of $\mathfrak{C}(1, n-1)$ generators on an irreducible unitary representation $\Psi_\rho(x)$ of the Poincaré group, describing a field of dimension w_Ψ and spin subscript ρ, may be summarized as follows

$$[P_\mu, \Psi_\rho] = -i\partial_\mu\Psi_\rho\,; \tag{10}$$
$$[K_\mu, \Psi_\rho] = i[x^2\partial_\mu - 2x_\mu(x^\rho\partial_\rho - w_\Psi)]\Psi_\rho + ix^\nu(\Sigma_{\mu\nu})_\rho^\sigma\Psi_\sigma\,; \tag{11}$$
$$[D, \Psi_\rho] = -i(x^\mu\partial_\mu - w_\Psi)\Psi_\rho\,; \tag{12}$$
$$[M_{\mu\nu}, \Psi_\rho] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Psi_\rho - i(\Sigma_{\mu\nu})_\rho^\sigma\Psi_\sigma\,; \tag{13}$$

where $\Sigma_{\mu\nu}$ are the spin matrices, i.e., the generators of Lorenz rotations on the spin space. The corresponding set of finite conformal transformations are

$$\Psi_\rho(x) \xrightarrow{T(a)} \Psi_\rho(x+a)\,; \tag{14}$$
$$\Psi_\rho(x) \xrightarrow{E(b)} \mathcal{E}(-b, x)_\rho^\sigma\Psi_\sigma\Big(\frac{x - bx^2}{1 - 2bx + b^2x^2}\Big)\,; \tag{15}$$
$$\Psi_\rho(x) \xrightarrow{S(\alpha)} e^{-w_\Psi\alpha}\,\Psi_\rho(e^\alpha x)\,; \tag{16}$$
$$\Psi_\rho(x) \xrightarrow{\Lambda(\omega)} \mathcal{L}_\rho^\sigma(-\omega)\Psi_\sigma[\Lambda(\omega)x]\,; \tag{17}$$

where $\mathcal{E}(-b, x), \mathcal{L}(-\omega)$ are suitable matrices which perform the conformal transformations of spin components, respectively for elations and Lorentz rotations.

As regards the orthochronous inversion, we generally have

$$\Psi_\rho(x) \xrightarrow{I_0} \mathcal{I}_0(x)_\rho^\sigma\Psi_\sigma(-x/x^2)\,, \tag{18}$$

where matrix $\mathcal{I}_0(x)$ obeys the equation

$$\mathcal{I}_0(x)\mathcal{I}_0(-x/x^2) = 1\,. \tag{19}$$

For consistency with (14), (15) and Eqs. $E(b) = I_0(x)\, T(b)\, I_0(x)$, we also have

$$\mathcal{E}^{\sigma}_{\rho}(-b, x) = \mathcal{I}_0(x)\, \mathcal{I}_0(x - b)\,. \tag{20}$$

For the needs of a Langrangian theory, the adjoint representation of Ψ_α must also be defined. It is indicated by $\bar{\Psi} = \Psi^\dagger \mathcal{B}$, where $\mathcal{B}$ is a suitable matrix or complex number chosen to satisfy equation $\bar{\bar{\Psi}} = \Psi$, implying $\mathcal{B}\,\mathcal{B}^\dagger = 1$, and Hamiltonian self–adjointness condition. Therefore, under the action of a group element g, the adjoint representation is transformed as

$$\bar{\Psi}^{\rho}(x) \xrightarrow{\;g\;} \bar{\Psi}^{\sigma}(gx)\, \bar{\mathcal{F}}(g^{-1}, x)^{\rho}_{\sigma}\,,$$

where $\bar{\mathcal{F}}(g^{-1}, x) = \mathcal{B}^\dagger \mathcal{F}^\dagger(g^{-1}, x) \mathcal{B}$.

For a deeper insight into the theory of conformal group representations see Ref. [13].

1.4 Discrete symmetries of the conformal group in $\mathcal{M}_4$

In this and the next two subsections we focus on the transformation properties of a spinor field $\psi(x)$ since those of all other fields can be inferred by reducing direct products of spinor field representations.

As is well–known in standard field theory, the algebra of spinor representation contains the discrete group formed by the *parity operator* P, the *charge conjugation* C and the *time reversal* T. The latter commutes with P, and C, and the *elicity projectors* $P_\pm$, $P_+ + P_- = 1$, which are defined by equations $\psi_R = P_+\psi$ and $\psi_L = P_-\psi$, where R and L stand respectively for the right–handed and the left–handed spin components relative to linear momentum. However, as already pointed at the end of subsec. 1.1, passing from the Poincaré to the conformal group, we must exclude T, which violates causal ordering, and transfer the role of this operator to I_0.

Let us normalize Dirac matrices γ^μ in such a way that $\gamma^0 = (\gamma^0)^\dagger = (\gamma^0)^T$ and $\gamma^2 = (\gamma^2)^\dagger = (\gamma^2)^T$, where superscript T indicates matrix transposition, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then, the equalities $\psi_L = \frac{1}{2}(1 + \gamma^5)\,\psi$ and $\psi_R = \frac{1}{2}(1 - \gamma^5)\,\psi$ hold. As is well–known in basic Quantum Mechanics, in this representation P and C act on $\psi_{L,R}$ as follows:

$$\psi_{R,L}(x) \xrightarrow{\;P\;} \mathcal{P}\psi_{L,R}(Px);\;\; \psi_{R,L}(x) \xrightarrow{\;C\;} \mathcal{C}\psi^*_{L,R}(x);\;\; \psi_{R,L}(x) \xrightarrow{\;CP\;} \mathcal{CP}\psi^*_{R,L}(Px); \tag{21}$$

where $Px \equiv P\{x^0, \vec{x}\} = \{x^0, -\vec{x}\}$, $\mathcal{P} = \gamma^0$, $\mathcal{C} = i\gamma^2\gamma^0$, $\mathcal{CP} = i\gamma^2$ and $\psi^*_{R,L}$ are the complex conjugates of $\psi_{R,L}$. We can easily verify the equations

$$\mathcal{PC} = -\mathcal{CP}\,, \quad \mathcal{C}^{-1} = \mathcal{C}^\dagger = -\mathcal{C}\,, \quad \mathcal{C}\gamma^\mu = (\gamma^\mu)^T\mathcal{C}\,, \quad \mathcal{C}\gamma^5 = -\gamma^5\mathcal{C}\,, \quad \mathcal{P}\gamma^5 = -\gamma^5\mathcal{P}\,,$$

the last two of which show that both $\mathcal{P}$ and $\mathcal{C}$ interchange L with R (whilst $\mathcal{PC}$ leaves them unchanged). Note that, despite their anticommutativity, $\mathcal{P}$ and $\mathcal{C}$ act commutatively on fermion bilinears, of which all spinor observables are made of.

The general form of $\mathcal{I}_0(x)$ introduced in Eq.(18) is determined by requiring that I_0 commutes with P, C, $P_\pm$, as time–reversal T does, as the heuristic principle of perseverance of formal laws [14] suggests, and that it satisfies Eq.(19). In summary:

$$\gamma^0\mathcal{I}_0(Px) = \mathcal{I}_0(x)\,\gamma^0\,; \quad \gamma^2\gamma^0\mathcal{I}_0(x) = \mathcal{I}_0(x)\,\gamma^2\gamma^0\,;$$
$$\gamma^5\mathcal{I}_0(x) = \mathcal{I}_0(x)\,\gamma^5\,; \quad \mathcal{I}_0(x^\mu)\,\mathcal{I}_0(-x^\mu/x^2) = 1\,.$$

Of course, we also require that I_0 is not equivalent to 1 and CP. We can easily verify that all these conditions lead to the general formula

$$\mathcal{I}_0(x) = \pm(x^2)^\delta \frac{\not x}{|x|}\,\gamma^2\gamma^0 = \pm(x^2)^{\delta-1/2}\gamma^2\gamma^0\not x\,, \tag{22}$$

where: $|x| = \sqrt{x^2} = \sqrt{\not x^2}$; δ is a real number; $\not x = \eta_{\mu\nu}\gamma^\mu x^\nu$, with $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The sign being arbitrary, we take it equal to -1 for the sake of convenience. Note that $\not x$ is a pseudo–scalar, since $\mathcal{P}\not x\mathcal{P}^{-1} = \gamma^0\gamma_\mu\gamma^0(Px)^\mu = -\not x^\dagger$, .

To find δ, we impose the condition that the conformal invariant free–field Feynman propagator

$$\langle 0|T\{\psi_\alpha(x_1), \overline{\psi}_\beta(x_2)\}|0\rangle = \frac{i}{2\pi^2}\,\frac{\not x_1 - \not x_2}{[(x_1-x_2)^2 + i\epsilon]^2}\,,$$

where $T\{\cdots\}$ indicates time ordering, also be invariant under I_0, i.e.,

$$\langle 0|T\{I_0\psi_\alpha(x_1)I_0^{-1}, \overline{I_0\psi_\beta(x_2)I_0^{-1}}\}|0\rangle = \langle 0|T\{\psi_\alpha(x_1), \overline{\psi}_\beta(x_2)\}|0\rangle\,,$$

Using Eq.(22), we find the adjoint transformation

$$\bar\psi(x) = \psi^\dagger(x)\,\gamma^0 \xrightarrow{I_0} -(x^2)^{\delta-1/2}\,\psi^\dagger(-x/x^2)\,\not x^\dagger\gamma^0\gamma^2\gamma^0 = -(x^2)^{\delta-1/2}\,\overline{\psi}(-x/x^2)\,\not x\,\gamma^2\gamma^0\,,$$

where equalities $\gamma^2\gamma^0 = -\gamma^0\gamma^2$ and $\gamma^0\not x^\dagger\gamma^0 = \not x$ were used. Thus, the invariance condition takes the form

$$\left(x_1^2\,x_2^2\right)^{\delta-1/2}\frac{\gamma^0\gamma^2\not x_1(\not x_2/x_2^2 - \not x_1/x_1^2)\,\not x_2\,\gamma^0\gamma^2}{[(-x_1/x_1^2 + x_2/x_2^2)^2 + i\epsilon]^2} = \left(x_1^2\,x_2^2\right)^{\delta+3/2}\frac{\not x_1 - \not x_2}{[(x_1-x_2)^2 + i\epsilon]^2}\,,$$

which yields $\delta = -3/2$, i.e., just the dimension of ψ. Thus, Eq.(22) can be factorized as

$$\mathcal{I}_0(x) = \left(x^2\right)^{-3/2}\frac{i\not x}{|x|}(i\,\gamma^2\gamma^0) = (x^2)^D\mathcal{S}_0(x)\,\mathcal{C} = (x^2)^D\mathcal{C}\,\mathcal{S}_0(x)\,, \tag{23}$$

where D is the dilation generator, $\mathcal{C}$ the charge conjugation matrix and $\mathcal{S}_0(x) = i\,\not x/|x\,|$ a spin matrix satisfying the equation $\bar{\mathcal{S}}_0(x)\,\mathcal{S}_0(x) = \mathcal{S}_0(x)\,\bar{\mathcal{S}}_0(x) = 1$.

Note that $\mathcal{S}_0(x)$ acts as a self–adjoint reflection operator which transforms every object in $\mathcal{M}_4$ to its specular image with respect to the 3D space orthogonal to time–like 4–vector x^μ at $x = 0$. Indeed, we have

$$\mathcal{S}_0(x)\,\not y\,\bar{\mathcal{S}}(x) = \not y - 2\,\frac{(xy)}{x^2}\not x\,, \quad \mathcal{S}_0(x)\,\gamma^\mu\,\bar{\mathcal{S}}_0(x) = \gamma^\mu - 2\,\frac{x^\mu\not x}{x^2}\,. \tag{24}$$

1.5 Conformal transformations of tensors in $\mathcal{M}_4$

Since $\not{x}\not{y} = xy - ix^\mu y^\nu \sigma_{\mu\nu}$, where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, we obtain from Eqs. (20) and (23)

$$\mathcal{E}_\rho^\sigma(-b, x) = \frac{(1 + bx - ix^\mu b^\nu \sigma_{\mu\nu})_\rho^\sigma}{(1 + 2bx + b^2x^2)^2} \,. \tag{25}$$

Noting that $(x^\mu b^\nu \sigma_{\mu\nu})^2 = x^2b^2 - (bx)^2$, we define

$$\sigma = \frac{x^\mu b^\nu \sigma_{\mu\nu}}{\sqrt{x^2b^2 - (bx)^2}} \,,$$

so $\sigma^2 = 1$. We can then pose

$$1 + bx - ix^\mu b^\nu \sigma_{\mu\nu} = A\Big(\cos\frac{\theta}{2} - i\sigma \sin\frac{\theta}{2}\Big) = Ae^{-i\sigma\theta/2} \,,$$

with

$$A = \sqrt{1 + 2bx + b^2x^2} \quad \text{and} \quad \tan\frac{\theta}{2} = \frac{\sqrt{x^2b^2 - (bx)^2}}{1 + 2bx} \,. \tag{26}$$

In conclusion, we can write

$$\mathcal{E}(-b, x) = (1 + 2bx + b^2x^2)^{-3/2} e^{-i\sigma\theta/2} \,,$$

showing that $\mathcal{E}(-b, x)$ is the product of a local dilation and Lorentz rotation $e^{-i\sigma\theta/2}$ acting on the spinor space, both of which depend on x.

Since the transformation properties of all possible tensors can be derived by reducing direct products of spinor–field representations, we can infer the general form of inversion and elations spin–matrices for fields $\Phi(x)$ of any spin in $\mathcal{M}_4$ as

$$\mathcal{I}_0(x) = (x^2)^D \, \mathcal{S}_0(x) \, \mathcal{C} \,, \quad \mathcal{E}(-b, x) = (1 + 2bx + b^2x^2)^D \, \mathcal{R}_0(-b, x) \,, \tag{27}$$

where D is the dilation generator, $\mathcal{S}_0(x)$ is the spin–reflection matrix for $\Phi(x)$, $\mathcal{C}$ the charge conjugation matrix and $\mathcal{R}_0(-b, x) = e^{-i\Sigma\theta}$, with θ defined as in Eq.(26) and

$$\Sigma = \frac{x^\mu b^\nu \Sigma_{\mu\nu}}{\sqrt{b^2x^2 - (xb)^2}} \,,$$

as the spin–rotation matrix for $\Phi(x)$ at $x = 0$.

As an application of the results so far achieved, let us study the transformation properties of fermionic bilinear forms and corresponding boson fields under the action of the conformal group.

1.6 Orthochronous inversions of vierbeins, currents and gauge fields

In the last two subsections, the behavior of the fields under orthochronous inversion was studied in the particular case of flat or conformally flat spacetime, basing on the transformation properties of spinor fields. As soon as we try to transfer the very same concepts to Dirac Lagrangian densities and equations, we encounter immediately the problem of that Dirac's matrices γ^μ, as dealt with so far, have no objective meaning and must be replaced by expressions of the form $\gamma^\mu(x) = e^\mu_a(x)\,\gamma^a$, where $e^\mu_a(x)$ is the *vierbein* tensor and γ^a a standard representation of Dirac' matrices in 4D [15].

Since $e^\mu_a(x)$ is related to the metric tensor by the equation $e^\mu_a(x)\,e^{a\mu}(x) = g^{\mu\nu}(x)$, it has dimension -1 and therefore is transformed by I_0 as follows

$$e^\mu_a(x) \xrightarrow{I_0} \frac{1}{x^2}\left[e^\mu_a(-x/x^2) - \frac{x^\mu x_\nu}{x^2}\, e^\nu_a(-x/x^2)\right].$$

consistently with Eq.(24).

Let ψ^a be us a spinor field of dimension $-3/2$ satisfying canonical anti–commutation relations, where a is some family superscript, and consider the hermitian bilinear forms

$$J^{(a,b)\mu}(x) = \frac{1}{2}\, e^\mu_a(x)\,[\bar\psi^a(x), \gamma^a\,\psi^b(x)]\,, \quad J^{5(a,b)\mu}(x) = \frac{1}{2}\, e^\mu_a(x)[\bar\psi^a(x), \gamma^a\,\gamma^5\psi^b(x)]\,,$$
$$J^{a,b}(x) = \frac{1}{2}[\bar\psi^a(x), \psi^b(x)]\,, \quad J^{5a,b}(x) = \frac{1}{2}[\bar\psi^a(x), \gamma^5\psi^b(x)]\,. \tag{28}$$

These may be respectively envisaged as the currents and axial–vector currents of some local algebra, and scalar and pseudo–scalar densities of ψ^a; all of which being expected to be coupled with appropriate boson fields in some Langrangian density. We leave as an exercise for the reader to prove that I_0 act on them as follows:

$$J^{(a,b)\mu}_\mu(x) \xrightarrow{I_0} -\frac{1}{(x^2)^4}\left[J^{(a,b)\mu}(I_0x) - \frac{x^\mu x_\nu}{x^2}\, J^{(a,b)\nu}(I_0x)\right]; \tag{29}$$

$$J^{5(a,b)\mu}_\mu(x) \xrightarrow{I_0} -\frac{1}{(x^2)^4}\left[J^{5(a,b)\mu}(I_0x) - \frac{x^\mu x_\nu}{x^2}\, J^{5(a,b)\nu}(I_0x)\right]; \tag{30}$$

$$J^{(a,b)}(x) \xrightarrow{I_0} \frac{1}{(x^2)^3} J^{(a,b)}(I_0x)\,; \quad J^{5(a,b)}(x) \xrightarrow{I_0} \frac{1}{(x^2)^3} J^{5(a,b)}(I_0x)\,. \tag{31}$$

where $I_0x = -x/x^2$.

As explained at the beginning of Section 3 Part I, covariant vector fields $A_\mu(x)$ and covariant axial–vector fields $A^5_\mu(x)$ have dimension 0, while scalar fields $\varphi(x)$ and pseudoscalar fields $\pi(x)$ have dimension -1. Therefore, for consistency with field equations, they are transformed by I_0 as follows

$$A_\mu(x) \xrightarrow{I_0} -\left[A_\mu(I_0x) - \frac{x_\mu x^\nu}{x^2}\, A_\nu(I_0x)\right]; \tag{32}$$

$$A^5_\mu(x) \xrightarrow{I_0} -\left[A^5_\mu(I_0x) - \frac{x_\mu x^\nu}{x^2}\, A^5_\nu(I_0x)\right]; \tag{33}$$

$$\varphi(x) \xrightarrow{I_0} \frac{1}{x^2}\,\varphi(I_0x)\,; \quad \pi(x) \xrightarrow{I_0} \frac{1}{x^2}\,\pi(I_0x)\,. \tag{34}$$

Combining Eqs. (29)-(31) with Eqs. (32)-(34) and using the transformation laws $g_{\mu\nu}(x) \xrightarrow{I_0} (x^2)^2 g_{\mu\nu}(I_0x)$, $\sqrt{-g(x)} \xrightarrow{I_0} (x^2)^4\sqrt{-g(I_0x)}$, we obtain the transformation laws

$$\sqrt{-g(x)}\, J^{(a,b)}_\mu(x)\, A^\mu(x) \xrightarrow{I_0} \sqrt{-g(I_0x)}\, J^{(a,b)}_\mu(I_0x)\, A^\mu(I_0x);$$
$$\sqrt{-g(x)}\, J^{5(a,b)}_\mu(x)\, A^{5\mu}(x) \xrightarrow{I_0} \sqrt{-g(I_0x)}\, J^{5(a,b)}_\mu(I_0x)\, A^{(5)\mu}(I_0x)\,;$$
$$\sqrt{-g(x)}\, J^{(a,b)}(x)\,\varphi(x) \xrightarrow{I_0} \sqrt{-g(I_0x)}\, J^{(a,b)}(I_0x)\,\varphi(I_0x)\,;$$
$$\sqrt{-g(x)}\, J^{5(a,b)}(x)\,\pi(x) \xrightarrow{I_0} \sqrt{-g(I_0x)}\, J^{5(a,b)}(I_0x)\,\pi(I_0x)\,;$$

showing that all Lagrangian densities of interest are transformed by I_0 as

$$\sqrt{-g(x)}\, L(x) \xrightarrow{I_0} \sqrt{-g(I_0 x)}\, L(I_0 x)\,. \tag{35}$$

1.7 Action–integral invariance under orthochronous inversion I_0

On the 3D Riemann manifold, the action of I_0 in the past and future cones $H_0^+ \equiv H_0^+(+\infty)$ and $H_0^- \equiv H_0^-(0^-)$ possesses the self–mirroring linear properties described in subsec. 1.2, provided that the geometry is conformally flat. If the geometry is appreciably distorted by the gravitational field, self–mirroring properties are still found provided that the points of cones are parameterized by polar geodesic coordinates $\{\tau, \hat{x}\}$, as described in subsec. 4.3 of Part I, with $\tau > 0$ for H_0^+ and $\tau < 0$ for H_0^-. In this case, we can define I_0 as the operation which maps point $x = \Sigma(\tau) \cap \Gamma(\hat{x}) \in H_0^+$ to point $x' = \Sigma(-1/\tau) \cap \Gamma(-\hat{x}) \in H_0^-$, and vice versa.

These changes suggest that in order for the motion equations to reflect appropriately the conditions for the spontaneous breaking of conformal symmetry, the action integral A of the system must be restricted to the union of H_0^+ and H_0^-. Using Eq.(35), posing $x' = -x/x^2$ and immediately renaming x' as x, we establish immediately the invariance under I_0 of all action integrals of the form $A = A^- + A^+$, where

$$A^- = \int_{H_0^-} \sqrt{-g(x)}\, L(x)\, d^4x\,, \quad A^+ = \int_{H_0^+} \sqrt{-g(x)}\, L(x)\, d^4x\,, \tag{36}$$

since $H_0^\pm \xrightarrow{I_0} H_0^\mp$ and $A^\pm \xrightarrow{I_0} A^\mp$.

In conclusion

Conformal invariance and causality are deeply related in that the former ensures the maximal fulfillment of the latter. Conformal symmetry, both in its geometric and algebraic aspects, is dominated by the role played by the orthochronous inversion I_0 centered at a selected point 0 of Minkowski spacetime. In particular, symmetry under I_0 replaces time–reversal. Action–integral invariance under I_0 allows us to limit spacetime integration over the interiors of past and future cones. Importantly, the peculiar relationships established by I_0 between the internal regions of the cones lead us to predict that, if spacetime is empty just a moment before the occurrence of the symmetry–breaking event, so is it at infinite far future, and vice versa.

2. Spontaneous breakdown of conformal symmetry

Let us assume that the action of a system and the fundamental state of its fields, i.e., *vacuum state* $|\Omega\rangle$, are invariant under a finite or infinite group G of continuous transformations. The spontaneous breakdown of the symmetry associated with this invariance is characterized by four main facts: 1) loss of the group invariance of $|\Omega\rangle$; 2) residual invariance of $|\Omega\rangle$ under a subgroup $S \subset G$, called the *stability subgroup*; 3) formation of one or more boson fields called the *Nambu–Goldstone* (NG) *fields* with non–zero VEV and gapless energy spectrum - one for each group generator which does not preserve $|\Omega\rangle$; 4) contraction of the set–theoretic complement of $S \subset G$ into an invariant Abelian subgroup of $|\Omega\rangle$ transformations, called the *contraction subgroup*, which produces amplitude translations of NG–fields and parametric rearrangements of group representations [16]. If $|\Omega\rangle$ is invariant under spacetime translations, the energy spectrum bears zero–mass poles and NG–fields are fields of massless particles, which are called the *NG–bosons*.

If G is a finite group of continuous transformations, the symmetry is global and the symmetry breaking is equivalent to a global phase transition. If the system is not invariant under spacetime translations the energy spectrum, gapless though, is free of zero–mass poles, implying that NG–bosons do not exist.

If G is an infinite group of local continuous transformations, as is indeed the case with local gauge transformations and conformal diffeomorphisms, the stability group too is infinite and local, and, because of locality, NG–bosons appear. However, these can be sequestered by zero–mass vector bosons to yield massive vector bosons [17] [18] [19], or condense at possible topological singularities of S (point–like singularities, vortical lines, boundary asymmetries etc.) to produce extended objects. In the latter case, NG–field VEVs appear to depend upon manifold coordinates.

In Part I, the creation of the universe was imagined as a process generated by the spontaneous breakdown of a symmetry occurred at a point $x = 0$ of an empty spacetime, which primed a nucleating event followed by a more or less complicated evolution in the interior of the future cone stemming from 0. It is now clear that the symmetry with which we are concerned is expected to reflect the invariance of the total action integral with respect to the group of conformal diffeomorphisms. Since this is an infinite group of local continuous transformations, we also expect that: 1) the stability subgroup S be infinite, local and possibly provided with a topological singularity including the vertex of future cone; 2) NG–boson condensation takes place at this singularity, with the formation of an object extended in spacetime. This implies that NG–boson VEVs depend on hyperbolic spacetime coordinates.

The search for spontaneous breakdown with these characteristics is considerably simplified if we focus on the possible stability subgroups of the fundamental group G_C of conformal connections rather than of the full group of diffeomorphisms. This simplification is legitimated by the following considerations: since we presume that, before the nucleating event, the manifold was conformally flat, we can assume that the evolution of the system immediately after this event was isotropic and homogeneous. This means that the metric remained conformally flat during a certain kinematic–time interval, until gravitational forces began to enter into play, favouring the aggregation of matter. Since during this initial period the symmetry is virtually global, we can limit ourselves to searching for *subgroups of* G_C endowed with topological singularities.

The mechanism of the spontaneous breaking of the global conformal invariance was investigated by Fubini in 1976. We report here his main results.

2.1 Possible spontaneous breakdowns of global conformal symmetry

It is known that the 15–parameter Lie algebra of the conformal group in 4D spacetime $\mathfrak{C}(1,3)$ is isomorphic with that of the hyperbolic–rotation group $O(2,4)$ on the 6D linear space $\{x^0, x^1, x^2, x^3, x^4, x^5\}$ of metric $(x^0)^2 + (x^5)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2$. Assuming that the dilation invariance of the vacuum is broken spontaneously, the sole possible candidates for a stability subgroup turn out to be:

S_{C_0}: the *Poincaré group* generated by $M_{\mu\nu}$ and P_μ. With this choice, possible NG–bosons VEVs are invariant under translations and are therefore constant.

S_{C_+}: the *de Sitter group* generated by $M_{\mu\nu}$ and

$$R_\mu = \frac{1}{2}\left(P_\mu + K_\mu\right).$$

The 10–parameter Lie algebra of this group is isomorphic with that of $O(1,4)$, the group which leaves invariant the fundamental form $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2$ of de Sitter spacetime [20]. It commutes with orthochronous inversion I_0 and satisfies the commutation relations

$$[R_\mu, R_\nu] = i\, M_{\mu\nu}\,.$$

Since G_C is generated by I_0 and the subgroup of spacetime translations, clearly S_{C_+} can also be generated by I_0 and Lorentz rotation group Λ_0 centered at $x = 0$. The spontaneous breakdown of conformal symmetry thus destroys the invariance of the vacuum under coordinate translation and dilations and provides S_{C_+} with the topological singularity formed of the point $x = 0$ and the light cones stemming from it. Every NG–boson VEV $\sigma_+(x)$ must satisfy the equations

$$R_\mu \sigma_+(x) = 0\,, \quad M_{\mu\nu}\sigma_+(x) \equiv -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,\sigma_+(x) = 0\,,$$

the second of which implies that $\sigma_+(x)$ depends only on $x^2 = \tau^2$.

S_{C_-}: the *anti–de Sitter group* generated by $M_{\mu\nu}$ and

$$L_\mu = \frac{1}{2}\left(P_\mu - K_\mu\right).$$

The 10–parameter Lie algebra of this group is isomorphic with that of $O(2,3)$, the group which leaves invariant the fundamental form $(x^0)^2 + (x^4)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ of anti–de Sitter spacetime. It anti–commutes with I_0 and satisfy the commutation relations

$$[L_\mu, L_\nu] = -i\, M_{\mu\nu}\,.$$

In this case, any NG–boson VEV $\sigma_-(x)$ must satisfy the equation

$$L_\mu \sigma_-(x) = 0\,, \quad M_{\mu\nu}\sigma_-(x) \equiv -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,\sigma_-(x) = 0\,,$$

the second of which implies that $\sigma_-(x)$ too depends only on τ^2.

Comparing the results obtained for the de Sitter and anti–de Sitter groups, we note that L_μ and D are the generators of the set–theoretic complement of S_{C_+} and that R_μ and D are the generators of the set–theoretic complement of S_{C_-}. Now, from equations

$$[R_\mu, D] = i\, L_\mu\,, \quad [L_\mu, D] = i\, R_\mu\,,$$

we derive

$$[R_\mu, D]\, \sigma_-(\tau) = i\, L_\mu\, \sigma_-(\tau) = 0\,, \quad [L_\mu, D]\, \sigma_+(\tau) = i\, R_\mu\, \sigma_+(\tau) = 0\,,$$

showing that the set–theoretic complements of S_{C_+} and S_{C_-} act respectively on VEVs $\sigma_+(\tau)$ and $\sigma_-(\tau)$ as invariant Abelian subgroups of transformations, as was indeed expected. Since the explicit expressions of R_μ and L_μ acting on a scalar field $\sigma(x)$ of dimension -1 are respectively

$$R_\mu\sigma(x) \equiv -i\Big[\frac{1-x^2}{2}\,\partial_\mu + x_\mu(x^\nu\partial_\nu + 1)\Big]\sigma(x)\,,$$
$$L_\mu\sigma(x) \equiv -i\Big[\frac{1+x^2}{2}\,\partial_\mu - x_\mu(x^\nu\partial_\nu + 1)\Big]\sigma(x)\,,$$

as established by Eq.(11), we can easily verify that the above equations for NG–boson VEVs are equivalent to

$$\sigma_\pm(x) = \sigma_\pm(\tau)\,, \quad \partial^2\sigma_\pm(\tau) \pm c^2\, \sigma^3_\pm(\tau) = 0\,,$$

where

$$\partial^2 f(\tau) \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu f(\tau) = \left[\partial^2_\tau + \frac{3}{\tau}\partial_\tau\right] f(\tau)\,,$$

the general solutions of which are, respectively,

$$\sigma_+(\tau) = \frac{\sigma_0}{1+\tau^2}\,, \quad \sigma_-(\tau) = \frac{\sigma_0}{1-\tau^2}\,, \text{ where } \sigma_0 = \sqrt{\frac{8}{c}}\,.$$

However, these expressions are not very strictly determined, since by an arbitrary change of scale $\tau \to \tau/\tau_0$, $\tau_0 > 0$, they become

$$\sigma_+(\tau) = \frac{\sigma_0}{1+(\tau/\tau_0)^2}\,, \quad \sigma_-(\tau) = \frac{\sigma_0}{1-(\tau/\tau_0)^2}\,, \text{ where } \sigma_0 = \frac{1}{\tau_0}\sqrt{\frac{8}{c}}\,. \tag{37}$$

2.2 The NG scalar bosons of spontaneously broken conformal symmetry

The energy spectra of functions (37) described at the end of the previous subsection are manifestly gapless and free of zero–mass poles. It is thus evident that, on a suitable kinematic-time scale, $\sigma_+(\tau)$ and $\sigma_-(\tau)$ are the classic Lorentz–rotation–invariant solutions of the motion equations obtained from action integrals

$$A_\pm = \int\Big[\pm\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\sigma_\pm)(\partial_\nu\sigma_\pm) - \frac{c^2_\pm}{4}\,\sigma^4_\pm\Big]\, d^4x$$

(with positive potential energy), so that they can be interpreted respectively as a zero–mass physical scalar field and a zero–mass ghost scalar field. More precisely, since the VEV of a classic field coincides with the field itself, we can think of $\sigma_-(\tau)$ as the VEV $\langle\Omega|\sigma(x)|\Omega\rangle$ of a possible zero–mass scalar ghost field $\sigma(x)$, and of $\sigma_+(\tau)$ as the VEV $\langle\Omega|\varphi(x)|\Omega\rangle$ of a possible zero–mass physical scalar field $\varphi(x)$. It is therefore also natural to expect that the spontaneous breakdown of the full group of conformal diffeomorphisms is characterized by the generation of NG–fields of this kind. This means that we must be prepared to introduce into the conformal–invariant Lagrangian density on the Riemann or Cartan manifold one or more such scalar fields and interpret the classic fields as extended macroscopic objects formed of zero–mass NG–bosons condensed into stability subgroup singularities.

However, we cannot ignore the fact that the conformal invariance of the action integrals may be illusory, because of possible ultraviolet cut–off parameters brought into play by renormalization procedures. Since we are working in the semi–classical approximation we can ignore this problem, being confident that, in its quantum mechanical version, it may be solved by the conformal-invariant renormalization procedures suggested by Englert et al. (1976) [21].

Comparing these results with those discussed in Part I, we immediately realize that, for small τ, in the conformal flatness approximation and after a suitable change of scale, the NG–field $\sigma_-(\tau)$ determined by Fubini may be regarded as the "seed" of the ghost scalar field $\sigma(x)$ introduced in the *geometric* Lagrangian density described in Part I. Similarly, the NG–field of type $\sigma_+(x)$ may be regarded as the seeds of possible zero–mass scalar field $\boldsymbol{\varphi}$ of non–zero VEV belonging to the *matter* Lagrangian density.

Clearly, the former identification is perfectly consistent with assuming that $\partial_\mu\sigma(x)$ is the gauge field of the zero–curvature dilation connection. In fact, although $\sigma(x)$ has the properties of a pure gauge, the existence of paths extending from $\tau = 0$ to $\tau = +\infty$, which are imposed by the stability subgroup, works as a topological constraint which prevents $\sigma(x)$ from being eliminated by a simple gauge transformation.

A further property of spontaneous breaking comes from the continuity of matter energy–momentum tensor $\tilde{\Theta}^M_{\mu\nu}(x)$ on the Cartan manifold during the transition from $\tau = 0^-$ to $\tau = 0^+$, in particular, the continuity of its trace $\tilde{\Theta}^M(x)$.

Since the invariance of the action integral under I_0 imposes that the state of matter be the same at $\tau = 0^-$ and $\tau = 0^+$, we conclude that $\lim_{\tau\to+\infty}\tilde{\Theta}^M(x) = 0$, or, equivalently, to condition $\tilde{R}(x) \to 0$ for Cartan–manifold Ricci–scalar $\tilde{R}$. This is consistent with requiring that $\sigma(x)$ depends only on τ for $\tau \to 0^+$ (from above) and $\sigma(x) \to \sigma_0$ for $\tau \to +\infty$.

In conclusion

The main fact concerning the spontaneous breakdown of local conformal symmetry is the opening, in an empty Minkowski spacetime, of a future cone, in which the entire history of the universe remains confined. The earliest stage of this process is characterized by the spontaneous breakdown of global conformal symmetry, with the formation of field $\sigma(x)$, playing the role of the NG–field, which contains information regarding the expansion rate *of the conformal metric, and one or more zero–mass scalar fields* $\boldsymbol{\varphi}(x)$*, which become Higgs fields as soon as they start interacting with* $\sigma(x)$*. In this way, a huge transfer of energy from geometry to matter is made possible, giving rise to an enormously powerful inflation process, during which expanding spacetime is rapidly filled in with Higgs field and its decay products.*

An important aspect of this process is the invariance of total action $A = A^G + A^M$ *under the group of conformal diffeomorphisms with the inclusion of orthochronous inversion* I_0*. This is achieved by restricting the integration domain of the total Lagrangian density of* A *to the double cone* $H_0^- \cup H_0^+$*, stemming from symmetry breaking event* $x = 0$*, the boundaries of which form the topological singularities of the de Sitter and anti–de Sitter subgroups* S_{C_+} *and* S_{C_-}*. This is consistent with the fact that possible scalar field VEVs depend initially only on kinematic time* τ *relative to 0. However, this does not exclude that the dependence of* $\sigma(x)$ *and* $\boldsymbol{\varphi}(x)$ *on manifold coordinates may become more and more complicated as the system evolves, depending on the details of matter dynamics. In any case, these fields must recover their initial dependence on* τ *alone, with the energy–momentum tensor reaching zero, for* $\tau \to +\infty$*, because of the boundary conditions imposed by the* I_0 *symmetry.*

3. Relevant coordinate systems in conformal gravity

If the gravitational field is negligible, or if it can be represented in the linear approximation as a perturbation of flat metric tensor $\eta_{\mu\nu}$, then spacetime is conformally flat, which entails a drastic simplification of coordinate representations. In these conditions, the choice of coordinates is suggested by the symmetry properties of the stability subgroup. On the Riemann manifold, it is suggested by the future cone shape together with the associated family of synchronized–observer 3D subspaces. This leads naturally to *hyperbolic coordinates*. On the Cartan manifold, the scale factor of fundamental tensor must be also considered, and we are naturally led to introduce *conformal hyperbolic coordinates*.

3.1 Hyperbolic coordinates

After the spontaneous breaking of conformal symmetry, the partial ordering of causal events is more conveniently parameterized by *adimensional contravariant coordinates* x^μ, reflecting the action–integral invariance under orthochronous inversion I_0, that is, the hyperbolic coordinates centered at 0.

These are: *kinematic time* τ, *hyperbolic angle* ρ and *Euler angles* θ, ϕ, which are related to standard Minkowski coordinates by the equations

$$x^0 = \tau\, \cosh\rho\,; \quad x^1 = \tau\, \sinh\rho\, \sin\theta\, \cos\phi\,;$$
$$x^2 = \tau\, \sinh\rho\, \sin\theta\, \sin\phi\,; \quad x^3 = \tau\, \sinh\rho\, \cos\theta\,.$$

Posing $\hat{x} = \{\rho, \theta, \phi\}$, we can put $x^\mu = x^\mu(\tau, \hat{x})$ and write the 4–velocity along the x^μ direction as $u^\mu \equiv \partial_\tau x^\mu = x^\mu/\tau$. Then we have $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = (u^0)^2 - |\vec{u}|^2 = 1$.

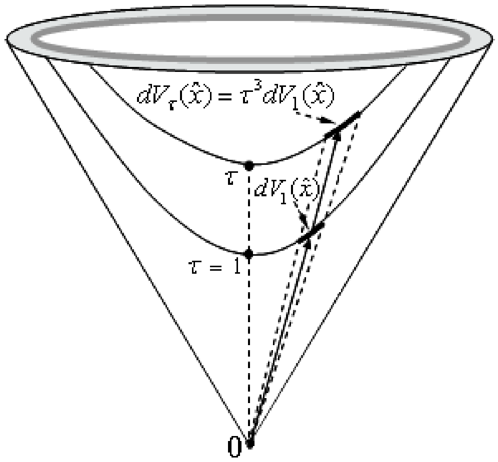

Fig. 2 Future cone $H^+(\tau)$. The position of a point in the cone is determined by τ and hyperbolic angles $\hat{x} = \{\rho, \theta, \phi\}$.

Fig.2 illustrates the future cone $H^+(\tau)$ together with the profiles of the unit hyperboloid, the hyperboloid at kinematic time τ, and the volume elements in hyperbolic coordinates

$$dV_1(\hat{x}) \equiv (\sinh\rho)^2 \sin\theta\, d\rho\, d\theta\, d\phi\,.$$

3D volume–elements on the hyperboloid at are related to them by the equation

$$dV_\tau(\hat{x}) \equiv \tau^3 dV_1(\hat{x})\,.$$

Accordingly, the 4D volume element of space–time at x writes as $d^4x \equiv dV_\tau(\hat{x})\, d\tau = \tau^3\, dV_1(\hat{x})\, d\tau$.

The squared line–element is easily found to be

$$ds^2 \equiv g_{\mu\nu}(\tau,\rho,\theta,\phi)\, dx^\mu dx^\nu = d\tau^2 - \tau^2\big[d\rho^2 + \big(\sinh\rho\big)^2 d\theta^2 + \big(\sinh\rho\,\sin\theta\big)^2 d\phi^2\big]\,,$$

and the squared gradient of a function $f(\tau,\hat{x})$

$$g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = (\partial_\tau)^2 - \frac{1}{\tau^2}|\vec{\nabla}_1 f|^2\,, \tag{38}$$

where

$$|\vec{\nabla}_1 f|^2 = (\partial_\rho f)^2 + \frac{(\partial_\theta f)^2}{(\sinh\rho)^2} + \frac{(\partial_\phi f)^2}{(\sinh\rho\,\sin\theta)^2}\,. \tag{39}$$

The Jacobian of the coordinate transformation is therefore

$$\sqrt{-g} = \sqrt{-g_{00}g_{11}g_{22}g_{33}} = \tau^3(\sinh\rho)^2\,\sin\theta\,,$$

and the covariant D'Alembert operator acts on a scalar function f as

$$D^2 f \equiv \frac{1}{\sqrt{-g}}\partial_\mu\Big(g^{\mu\nu}\sqrt{-g}\,\partial_\nu f\Big) = \partial_\tau^2 + \frac{3}{\tau}\,\partial_\tau f - \frac{1}{\tau^2}\,\Delta_1 f\,, \tag{40}$$

where

$$\Delta_1 f \equiv \frac{1}{(\sinh\rho)^2}\Big\{\partial_\rho\Big[(\sinh\rho)^2\,\partial_\rho f\Big] + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{(\sin\theta)^2}\partial_\phi^2 f\Big\}\,,$$

is the 3D Laplacian operator on the unit hyperboloid.

This operator has a complete set of orthonormalized eigenfunctions $\Phi_{\hat{k}}(\hat{x})$ labeled by the *hyperbolic momentum eigenvalues*

$$\hat{k} = \{k, l, m\}\,, \text{ where } 0 \leq k < +\infty\,;\; l = 0, 1, 2, \ldots\,;\; -l \leq m \leq l\,;$$

and a continuum spectrum of eigenvalues $-k^2 - 1$ [22]. More precisely, we have

$$\Phi_{\hat{k}}(\hat{x}) \equiv \Phi_{\{k,l,m\}}(\hat{x}) = \frac{1}{\sqrt{\sinh\rho}}\left|\frac{\Gamma(ik+l+1)}{\Gamma(ik)}\right| P^{-l-1/2}_{ik-1/2}(\cosh\rho)\, Y^m_l(\theta,\phi)\,;$$

$$\int_{V_{\hat{k}}} \Phi^*_{\hat{k}}(\hat{x}_1)\, \Phi_{\hat{k}}(\hat{x}_2)\, d^3\hat{k} \equiv \sum_{l,m} \int_0^{+\infty} \Phi^*_{\{k,l,m\}}(\hat{x}_1)\, \Phi_{\{k,l,m\}}(\hat{x}_2)\, dk = \delta^3_1(\hat{x}_2 - \hat{x}_1)\,;$$

$$\big(\Delta_1 + k^2 + 1\big)\, \Phi_{\hat{k}}(\hat{x}) = 0\,; \quad \int_{V_1} \Phi^*_{\hat{k}}(\hat{x})\, \Phi_{\hat{k}'}(\hat{x})\, dV_1(\hat{x}) = \delta^3_1(\hat{k} - \hat{k}')\,;$$

$$\int_{V_1} [\vec{\nabla}_1 \Phi^*_{\hat{k}}(\hat{x})] \cdot [\vec{\nabla}_1 \Phi_{\hat{k}'}(\hat{x})]\, dV_1(\hat{x}) = (k^2+1)\, \delta^3_1(\hat{k} - \hat{k}')\,; \quad \sum_{l,m} |\Phi_{\hat{k}}(\hat{x})|^2 = \frac{k^2}{2\pi^2}\,.$$

where $P^\mu_\lambda(\cosh\rho)$ are associated Legendre polynomials with subscripts and superscripts on the complex domain, $Y^m_l(\theta,\phi)$ are the familiar 3D spherical harmonics, $V_{\hat{k}}$ is the generalized integration volume of hyperbolic–momentum vectors $\hat{k}$, $dV_1(x) = (\sinh\rho)^2 \sin\theta\, d\rho\, d\theta\, d\phi$ and the Dirac deltas are defined as follows

$$\int_{V_1} \delta^3(\hat{x}_2 - \hat{x}_1)\, dV_1(\hat{x}) = 1\,, \quad \delta^3_1(\hat{k} - \hat{k}') = \delta_{l'l}\, \delta_{m'm}\, \delta(k - k')\,.$$

3.2 Conformal hyperbolic coordinates

The fundamental tensor of the Cartan manifold is related to hyperbolic coordinates by the square line element

$$d\tilde{s}^2 = e^{2\alpha(\tau,\hat{x})} d^2 s = e^{2\alpha(\tau,\hat{x})}\big\{d\tau^2 - \tau^2\big[d\rho^2 + \big(\sinh\rho\big)^2 d\theta^2 + \big(\sinh\rho\, \sin\theta\big)^2 d\phi^2\big]\big\} =$$
$$e^{2\alpha(\tau,\hat{x})} d\tau^2 - R^2_{RW}(\tau,\hat{x})\big[d\rho^2 + \big(\sinh\rho\big)^2 d\theta^2 + \big(\sinh\rho\, \sin\theta\big)^2 d\phi^2\big]\,.$$

Here, $R_{RW}(\tau,\hat{x}) = \tau\, e^{\alpha(\tau,\hat{x})}$ is the (inhomogeneous) Robertson–Walker radius of the universe in kinematic–time units, i.e., with speed of light $c = 1$ (hence $R_{RW} \to \tau$ in the infinite future). Defining $d\tilde{\tau} = e^{\alpha(\tau,\hat{x})} d\tau$ as the infinitesimal element of *proper time on the Cartan manifold* and interpreting $d\tilde{x} = \{d\tilde{\tau}, d\hat{x}\}$ as the differentials of a new coordinate system $x = \{\tilde{\tau}, \hat{x}\}$, called *conformal–hyperbolic coordinates*, the square line element takes the form

$$d\tilde{s}^2 = d\tilde{\tau}^2 - \tilde{R}^2_{RW}(\tilde{x})\big[d\rho^2 + \big(\sinh\rho\big)^2 d\theta^2 + \big(\sinh\rho\, \sin\theta\big)^2 d\phi^2\big]\,,$$

where $\tilde{R}_{RW}$ is R_{RW} as a function of $\tilde{x}$, the associated 3D volume–element of which is

$$\sqrt{-\tilde{g}(\tilde{x})}\, dV_1(\hat{x}) = \tilde{R}^3_{RW}(\tilde{x})\big(\sinh\rho\big)^2 \sin\theta\, dV_1(\hat{x})\,.$$

As regards the 4D volume element, we have

$$\sqrt{-\tilde{g}(\tilde{x})}\, d^4x = e^{4\alpha(\tilde{x})}\tau^3 dV_1(\hat{x}) = d\tilde{V}_{\tilde{\tau}}(\tilde{x})\, d\tilde{\tau}\,.$$

In this product, quantity $d\tilde{V}_{\tilde{\tau}}(\tilde{x}) = e^{3\alpha(\tilde{x})}\tau^3 dV_1(\hat{x}) = \tilde{R}_{RW}(\tilde{x})\, dV_1(\hat{x})$ represents the 3D volume–element of the space–like surface of the expanding universe, on which the set of comoving observers are synchronized at proper time $\tilde{\tau}$, mapped on the Riemann–manifold hyperboloid which contains the same set of comoving observers synchronized at kinematic time τ.

Replacing in Eqs. (38) and (40) kinematic time τ with proper time $\tilde{\tau}$, the associate differential operators act on scalar function f as follows

$$\begin{aligned}
&\tilde{g}^{\mu\nu}\big(\tilde{\partial}_\mu f\big)\big(\tilde{\partial}_\nu f\big) = \big(\partial_{\tilde{\tau}} f\big)^2 - \frac{1}{\tilde{R}^2_{RW}}\,|\vec{\nabla}_1 f|^2\,,\\
&\tilde{D}^2 f \equiv \frac{1}{\sqrt{-\tilde{g}}}\,\tilde{\partial}_\mu\Big(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\tilde{\partial}_\nu f\Big) = \partial^2_{\tilde{\tau}} f - \frac{1}{\tilde{R}^2_{RW}}\,\Delta_1 f + 3\,\tilde{g}^{\mu\nu}\big(\tilde{\partial}_\mu \ln \tilde{R}_{RW}\big)\big(\tilde{\partial}_\nu f\big) =\\
&\partial^2_{\tilde{\tau}} f - \frac{1}{\tilde{R}^2_{RW}}\,\Delta_1 f + 3\left[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\right]\partial_{\tilde{\tau}} f - \frac{3}{\tilde{R}^2_{RW}}\big(\vec{\nabla}_1 \tilde{R}_{RW}\big)\cdot\big(\vec{\nabla}_1 f\big)\,.
\end{aligned}$$

Here, $\tilde{\partial}_\mu$ and $\tilde{\alpha}$ are the components of the gradient operator and the exponent of the expansion factor in conformal hyperbolic coordinates respectively, the dot between round brackets stands for scalar product, and $\tau(\tilde{\tau})$ is the value of the kinematic time on the Riemann manifold corresponding to proper time $\tilde{\tau}$ on the Cartan manifold. The quantity

$$\partial_{\tilde{\tau}} \ln \tilde{R}_{RW}(\tilde{\tau}, \hat{x}) = \frac{\partial_{\tilde{\tau}}\tilde{R}_{RW}(\tilde{\tau}, \hat{x})}{\tilde{R}_{RW}(\tilde{\tau}, \hat{x})} = \tilde{H}_0(\tilde{\tau}, \hat{x})\,,$$

may be called the "local Hubble constant" in proper time units. It describes the local expansion rate of the universe on a large scale as a function of conformal hyperbolic coordinates. Here again, we must remark that this statement holds on the large scale as gravitational forces tend to destroy observer synchronization. The quantity coincides with the Hubble constant of standard cosmology. In fact, the local expansion rate is described by the 4–vector

$$\tilde{H}_\mu = \tilde{\partial}_\mu \ln \tilde{R}_{RW}(\tilde{\tau}, \hat{x})\,,$$

which comprises the Hubble constant and gradient $\vec{\nabla}_1 \ln \tilde{R}_{RW}$, accounting for local drifts of expansion possibly occurring if the expansion of the universe is not homogeneous.

For a homogeneous and isotropic universe, $\tilde{\alpha}$ depends only on $\tilde{\tau}$ and the covariant D'Alembert operator simplifies to

$$\tilde{D}^2 f = \partial^2_{\tilde{\tau}} f - \frac{1}{\tilde{R}^2_{RW}}\,\Delta_1 f + 3\left[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\right]\partial_{\tilde{\tau}} f$$

If the fundamental tensor is conformally flat and α depends only on τ, proper time is a pure function of kinematic time. In this case, passing from kinematic time τ_1 to kinematic time τ_2, the distance between any two comoving observers increases by the factor $R_{RW}(\tau_2)/R_{RW}(\tau_1)$, while physical unit scales are preserved. Note that, in pure hyperbolic coordinates, the distance increases by the factor τ_2/τ_1, while the unit scale changes by the factor $e^{\alpha(\tau_1)-\alpha(\tau_2)}$ (Fig.3).

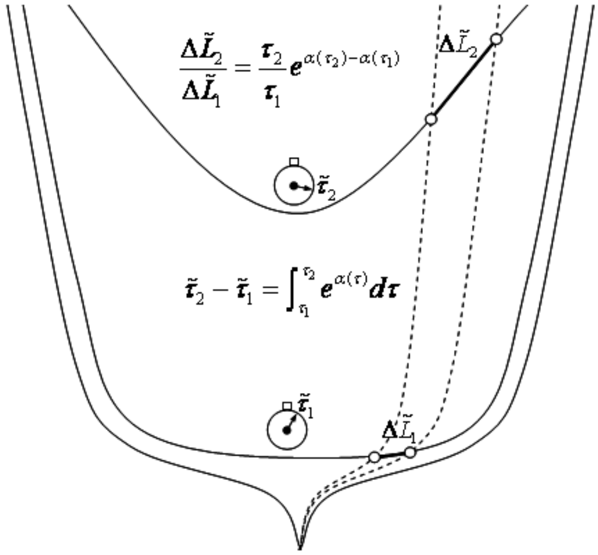

Fig. 3 Proper time and hyperbolic conformal coordinates on a conformally flat Cartan manifold.

In particular, passing from hyperbolic to conformal-hyperbolic coordinates, we must divide the energy density by $e^{4\alpha(\tau)}$ in order to preserve the unit of this quantity on the Cartan manifold. If the fundamental tensor is not conformally flat, the shape of the 3D–space of synchronized observers also depends on the gravitational field and ultimately on the time course of mass distributions. Therefore, neither hyperbolic coordinates nor conformal–hyperbolic coordinates can be exactly represented independently of the dynamic history of the system.

3.3 The proper–time eikonal

Determining conformal–hyperbolic coordinates, as described in the previous subsection, requires that we are able to express $\tilde{\tau}$ as a function of τ and $\hat{x}$. If scale–factor exponent α depends only on τ, we can integrate equation $d\tilde{\tau} = e^{\alpha(\tau)} d\tau$ and calculate the proper time as

$$\tilde{\tau} = \int_0^\tau e^{\alpha(\tau')} d\tau' \,,$$

but in the general case, if matter inhomogeneity cannot be ignored, we have

$$\frac{d\tilde{\tau}}{d\tau} = \frac{\partial\tilde{\tau}}{\partial\tau} + \frac{1}{\tau}\big(\vec{\nabla}_1\tilde{\tau}\big)\cdot\frac{d\hat{x}}{d\tau} \,,$$

clearly exhibiting an undesired dependence on path direction. Since the vanishing of dilation curvature requires that the determination of proper time is path independent, we are allowed to assume that the local direction of the path everywhere is that of minimum proper–time variation. We can therefore determine $\tilde{\tau}$ by solving the *eikonal equation*

$$\sqrt{\left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^2 - \frac{1}{\tau^2}|\vec{\nabla}_1\tilde{\tau}|^2} = e^{\alpha(\tau,\hat{x})} \,.$$

This leads us to regard $e^{\alpha(\tau,\hat{x})}$ as a sort of local refraction index and proper-time surfaces $\tilde{\tau}=\text{const}$ as equations describing a family of optical–geometric wave fronts propagating in the future cone. However, this picture can be complicated by the formation of caustics. If the geometry is conformally flat, these wave–fronts represent the loci of synchronized comoving observers. But in the presence of gravitational fields this simple picture does not generally hold.

3.4 Static hyperbolic coordinates

Although reflecting the geometry of the stability subgroup, the hyperbolic coordinates have a drawback in that Lagrangian densities and motion equations depend explicitly on kinematic time τ. This has two undesired consequences: the impossibility of separating the dependence of field amplitudes on space and time variables and the rising of frictional terms in the motion equations, which impart a non inertial character to reference frames. This point is of particular importance in sight of canonical quantization, since creation and annihilation operators are defined well only on the vacuum states of inertial systems. One should in fact consider that, in non inertial systems, quantum fluctuations of vacuum state materialize in part as thermal fluctuations (Fulling–Davis–Unruh effect) [23] [24] [25].

The problem is easily solved by introducing the adimensional *conformal time* parameter $t=\ln(\tau/\tau_0)$, with which the squared line element takes the form

$$d\hat{s}^2 = dt^2 - \left[d^2\rho + (\sinh\rho)^2 d\theta^2 + (\sinh\rho\sin\theta)^2 d\phi^2\right] = \frac{e^{-2t}}{\tau_0^2} ds^2\,.$$

We shall call $x^\mu=\{t,\rho,\theta,\phi\}$ the *static hyperbolic coordinates.* Hence, we have

$$\sqrt{-\hat{g}} = (\sinh\rho)^2\sin\theta\,;\quad d^4x = \sqrt{-\hat{g}}\,d\rho\,d\theta\,d\phi\,dt = dV_1(\hat{x})\,dt\,;$$
$$\hat{g}^{\mu\nu}\partial_\mu\partial_\nu = (\partial_t f)^2 - |\vec{\nabla}_1 f|^2\,;\quad \hat{\Box} f = \partial_t^2 f - \Delta_1 f\,;$$

where ∂_t is the partial derivative with respect to t. In Fig.4, the relation between hyperbolic and static hyperbolic coordinates is schematically illustrated.

Note that, in passing from kinematic to conformal time, we are forced to introduce an arbitrary kinematic–time constant τ_0, which leaves the problem of providing a physical meaning for it.

3.5 Conformal Hamiltonian in static hyperbolic coordinates

An important feature of action integrals in hyperbolic coordinates is that the Hamiltonian does not generate time translations but *kinematic–time dilations.* In fact, it coincides with the dilation operator D centered at the origin of the future cone, which is related to the Hilbert–Einstein energy momentum tensor $\Theta_{\mu\nu}(\tau,\hat{x})$ by the equation

$$D(\tau) = \int_{V_1}\tau^4 u^\mu u^\nu \Theta_{\mu\nu}(\tau,\hat{x})\,dV_1(\hat{x})\,,\ \text{where}\ \ u^\mu = \partial_\tau x^\mu = x^\mu/\tau\,, u^\mu u_\mu = 1\,.$$

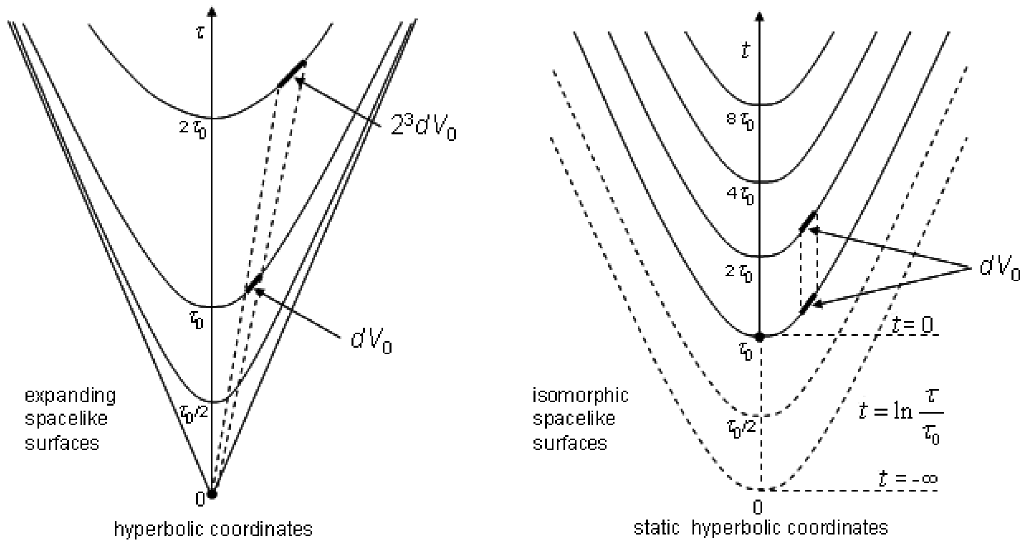

Fig. 4 In *hyperbolic coordinates*, the hyperboloids change shape in the course of kinematic time τ from $\tau = 0$ to $\tau = +\infty$, while volume elements expand proportionally to τ^3. By contrast, in *static hyperbolic coordinates*, all hyperboloids have the shape of the hyperboloid at $\tau = \tau_0$ and translate parallel to each other in the course of conformal time t from $t = -\infty$ to $t = +\infty$, while all volume elements maintain their original shape.

This can be easily understood by considering that the dilation current and dilation charge in general Minkowski coordinates $\{x^0, x^1, x^2, x^3\}$ are respectively defined as

$$J^D_\mu(x) = x^\nu \Theta_{\mu\nu}(x)\,, \quad D(x^0) = \int_{\Sigma(x^0)} J^D_i(x)\, d\Sigma^i(x^0)\,,\ i = 1, 2, 3\,,$$

where $d\Sigma^i(x^0)$ is the 3D volume element of the space–like surface at time x^0 [26]. In hyperbolic coordinates, the above expressions are obtained by the replacements

$$x^\nu \to \tau u^\nu\,; \quad J^D_i(x)\, d\Sigma^i(x^0) \to J^D_\mu u^\mu dV_\tau(\hat{x}) \equiv \tau^3 J^D_\mu u^\mu dV_1(\hat{x})\,.$$

In the context of quantum field theory, we have for a scalar field φ (dimension = -1)

$$i\big[D(\tau), \varphi(\tau, \hat{x})\big] = \partial_\tau\big[\tau\varphi(\tau, \hat{x})\big]\,; \tag{41}$$

$$i\big[D(\tau), \partial_\tau\varphi(\tau, \hat{x})\big] = \tau\partial_\tau\big[\partial_\tau\varphi(\tau, \hat{x})\big] + 2\,\partial_\tau\varphi(\tau, \hat{x}) = \partial^2_\tau\big[\tau\varphi(\tau, \hat{x})\big]\,. \tag{42}$$

We see that, in systems parameterized by hyperbolic coordinates, dilations act on a scalar field φ by kinematic–time translations of the adimensional product $\tau\varphi(\tau, \hat{x})$ [27], which implies a continuous field–amplitude rescaling in the course of kinematic time. In a similar way, they act on any field Φ of dimension w by kinematic–time translations of the adimensional product $\tau^{-w}\Phi(\tau, \hat{x})$.

Unfortunately, it is evident from the commutators of Eqs. (41) (42) that we cannot interpret $D(\tau)$ as the Hamiltonian for the adimensional fields defined above. However, if we express τ as a function of t, define the operator $\hat{H}(t)$, to be called the *conformal Hamiltonian*, by the equation $\hat{H}(t)\, dt = D(\tau)\, d\tau$ and pose $\hat{\Phi}(t, \hat{x}) = (\tau_0 e^t)^{-w}\Phi[\tau(t), \hat{x}]$, we find

$$i\big[\hat{H}(t), \hat{\Phi}(t, \hat{x})\big] = \partial_t\hat{\Phi}(t, \hat{x})\,; \quad i\big[\hat{H}(t), \partial_t\hat{\Phi}(t, \hat{x})\big] = \partial^2_t\hat{\Phi}(t, \hat{x})\,; \quad \text{etc},$$

showing that $\hat{H}$ is the correct generator of conformal time translation for any field theory on a conformally flat Riemann manifold represented in static hyperbolic coordinates.

The importance of this fact is the possibility of expanding all physical quantities depending on t in series of eigenfunctions of $\hat{H}(t)$; in particular, the possibility of decomposing field amplitudes expressed in static hyperbolic coordinates as linear combinations of creation and annihilation operators of the standard type, which is absolutely necessary for a correct description of quantum processes.

4. Two equivalent representations of action integrals

In the following, the representations of a physical system on the Riemann and Cartan manifolds are respectively called the *Riemann picture* and the *Cartan picture*.

The Riemann picture suits the need to describe the universe from the viewpoints of synchronized comoving observers living in the after–expansion epoch and endowed with kinematic–time clocks. Looking back to the past, these observers interpret the events which occurred during the universe expansion stage as subject to the action of the dilation field. In particular, all dimensional quantities, both geometric and physical, are imagined to undergo considerable changes of scale. Possible non–uniformities of this process are explained as coordinate distortions caused by the gravitational field. If the gravitational field is negligible, or if it can be approximated as a linear perturbation of the metric, the best description is provided by hyperbolic coordinates; otherwise, the metric tensor must be modified by including appropriate coefficients $a_{ij}(x)$, depending on matter distribution and field dynamics, as described in subsec. 4.3 of Part I.

In contrast, the Cartan manifold picture suits the need to describe the universe as an expanding system, in which sets of ideal synchronized comoving observers endowed with proper–time clocks are at rest on expanding 3D portions of spacetime. In this background, all bodies are seen to preserve their natural size and all quantities their measures. If the gravitational field is negligible, or if it can be represented as a linear perturbation, the best description is provided by conformal–hyperbolic coordinates; otherwise, the metric must be modified by including coefficients $a_{ij}(x)$, as specified above.

However, the two descriptions are perfectly equivalent and we can move from one to the other at any moment by a Weyl transformation.

Now let us be more precise about the existence of these equivalent descriptions. As clarified in subsec. 2.2. of Part I, the fundamental tensor of the Cartan manifold is related to the metric tensor of the Riemann manifold by the equation $\tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x)$. The addition of degree of freedom $\alpha(x)$ to the GR action integral allowed us to rewrite the Hilbert–Einstein Lagrangian density on a Riemann manifold as a Lagrangian density on the Cartan manifold. The latter was in turn equivalent to replacing the original Hilbert–Einstein Lagrangian density with an extended conformal–invariant Lagrangian density, which includes ghost scalar field $\sigma(x) = \sigma_0 e^{\alpha(x)}$, perhaps interacting with one or more physical scalar fields. In this way, if the matter Lagrangian density on the Riemann manifold is conformal–invariant, the explicit lack of conformal symmetry characteristic of

CGR may be reinterpreted as the effect of the spontaneous breakdown of local conformal symmetry of GR. This result condenses into the action–integral equivalence $\tilde{A} \sim A$

$$\tilde{A} = \int_C \sqrt{-\tilde{g}(x)}\tilde{L}_{\sigma_0}(x)d^4x \equiv \int_C e^{4\alpha(x)}\sqrt{-g(x)}\tilde{L}_{\sigma_0}(x)d^4x \sim A = \int_R \sqrt{-g(x)}\, L_{\sigma(x)}(x)d^4x,$$

where C and R stand respectively for the Cartan and Riemann manifolds, and $\tilde{L}_{\sigma_0}(x)$ and $L_{\sigma(x)}(x)$ are the total Lagrangian densities on the two manifolds. Factor $\sqrt{-g(x)}$ is evidenced, although its effective value is 1, since its dependence on $g_{\mu\nu}(x)$ cannot be ignored when variations $\delta A/\delta\, g^{\mu\nu}(x)$ are considered.

Note that, passing from Cartan to Riemann manifold, the time independence of physical units is lost. This is the price to be paid in order to achieve manifest conformal invariance.

Since orthochronous inversion can be included in the stability subgroup of the broken symmetry, the integration may be extended to the cone doublet $H_0^- \cup H_0^+$ stemming from the inversion centre, as explained in subsec. 1.7, of which only H_0^+ counts for obvious reasons as far as only the connected component of the conformal group is concerned. Hence, the equivalence takes the form

$$\tilde{A}^+ = \int_{\tilde{H}^+} \sqrt{-\tilde{g}(x)}\, \tilde{L}_{\sigma_0}(x)\, d^4x \sim A^+ = \int_{H^+} \sqrt{-g(x)}\, L_{\sigma(x)}(x)\, d^4x\,,$$

where $\tilde{H}^+$ and H^+ stand for the future cones in the Cartan and Riemann pictures, respectively. As already noted in Section 6 of Part I, $\tilde{L}_{\sigma_0}(x)$ and $L_{\sigma(x)}(x)$ must respectively incorporate the Weyl tensor terms $\tilde{C}^2(x)$ and $C^2(x)$ in order for the gravitational field to be renormalizable. If the universe expansion can be regarded as uniform and isotropic, the fundamental tensor of the Cartan manifold is conformally flat and the manifold can be simply parameterized by hyperbolic coordinates, in which case the equivalence simplifies to

$$\tilde{A}^+ = \int_0^{+\infty} d\tilde{\tau} \int_{\tilde{V}_1} \tilde{L}_{\sigma_0}(x)\, d\tilde{V}_{\tilde{\tau}}(\hat{x}) \sim A^+ = \int_0^{+\infty} d\tau \int_{V_1} L_{\sigma(x)}(x)\, dV_1(\hat{x})\,,$$

where $d\tilde{\tau} = e^{\alpha(\tau)}d\tau$ and $d\tilde{V}_{\tilde{\tau}}(\hat{x}) = \tilde{\tau}^3 e^{3\alpha(\tau)}\, dV_1(\hat{x})$. Since gravitational forces therefore are neglected, the fundamental tensor of the Cartan manifold turns out to depends only on the scale factor. Thus, the Riemann manifold becomes Minkowski spacetime, the Ricci scalar tensor vanishes, and therefore the Weyl tensor can be removed. The simplification is less drastic if the Weyl term is suppressed and the gravitational field is regarded as a linear perturbation of the metric tensor, so that the Ricci scalar tensor still appears in the Lagrangian densities. In this case, we speak of *approximate action integrals*. These can provide not only fairly good representations of the early stages of the universe, but also approximate representations of the mature universe.

4.1 Approximate Riemann picture and hyperbolic coordinates

The general form of the approximate action integral on a Riemann manifold spanned by hyperbolic coordinates is

$$A^{+} = \int_{0}^{+\infty} d\tau \int_{V_1} \tau^3 L[g_{\mu,\nu}, R, \sigma, \Phi]\, dV_1(\hat{x})$$

with the following identifications:
- $g_{\mu\nu}$, the Riemann metric tensor in the linear first–order approximation with $\sqrt{-g} = 1$;
- R, the approximated Ricci scalar tensor;
- τ, kinematic time;
- $\sigma(x) = \sigma_0 e^{\alpha(x)}$, the dilation field (dimension = -1) ($\sigma_0 = \sqrt{6} M_{rP}$);
- $\Phi(x)$, matter fields of various dimensions.

This picture is particularly useful from a mathematical standpoint, as the conformal invariance of the action is manifest. It is less useful for physical interpretations, as both geometric and physical quantities are subjected to local changes of scale. Dilation field σ appears explicitly as a ghost field of negative kinetic energy and positive potential energy, interacting with one or more physical scalar fields so as to favor positive energy transfers from geometry to matter.

4.2 Approximate Cartan picture and conformal–hyperbolic coordinates

The general form of the approximate action integral on the Cartan manifold spanned by hyperbolic coordinates $\tilde{x} = \tilde{\{x\}}$ has the general form

$$\tilde{A}^{+} = \int_{0}^{+\infty} d\tilde{\tau} \int_{\tilde{V}_\tau} \tilde{L}[\tilde{g}_{\mu,\nu}, \tilde{R}, \sigma_0, \Phi]\, d\tilde{V}_\tau(\hat{x})$$

with the following identifications:
- $\tilde{g}_{\mu\nu} = e^{2\alpha(x)} g_{\mu\nu}$, the fundamental tensor of the Cartan manifold with $g_{\mu\nu}$ in the linear first–order approximation and $\sqrt{-\tilde{g}} = e^{4\alpha(x)}$;
- $\tilde{R}$, the approximated conformal Ricci scalar;
- $\tilde{\tau}$, proper time;
- $\sigma_0 = \sigma_0 e^{\alpha(x)}$, the conformal–symmetry breaking parameter ($\sigma_0 = \sqrt{6} M_{rP}$);
- $\tilde{\Phi}(x) = e^{w_\Phi \alpha(x)} \Phi(x)$, matter fields of various dimensions.

These approximate action integrals are obtained from those on the Riemann manifold through the usual Weyl transformation. Hence, all possible dimensional constants appearing in the Lagrangian densities are functions of symmetry breaking parameter σ_0. In this picture, the dilation field disappears and the conformal symmetry appears to be explicitly broken.

5. Higgs field in conformally flat spacetime

In the next subsections we apply the concepts discussed so far to introduce the Lagrangian formalism on which the mechanism of matter generation is grounded, i.e., the interaction of the dilation field with one or more scalar fields, which thereby become Higgs fields. For the sake of simplicity, only the case of a single Higgs field is discussed. Its generalization to the case of Higgs field multiplets is straightforward.

5.1 Conformal Higgs field on the Riemann manifold

Here we focus on the simple case of a physical scalar field $\varphi(x)$ interacting with dilation field $\sigma(x)$ to produce a Higgs field. The interaction is described on the Riemann manifold by the approximate action integral $A^+ = A^{RM} + A^{RG}$, where

$$A^{RM} = \int_0^{+\infty} d\tau \int_{V_1} \frac{\tau^3}{2}\Big[g^{\mu\nu}\big(\partial_\mu\varphi\big)\big(\partial_\nu\varphi\big) - \frac{\lambda^2}{2}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2 + \frac{R}{6}\,\varphi^2\Big]\, dV_1\,, \tag{43}$$

$$A^{RG} = -\int_0^{+\infty} d\tau \int_{V_1} \frac{\tau^3}{2}\Big[g^{\mu\nu}\big(\partial_\mu\sigma\big)\big(\partial_\nu\sigma\big) + \frac{R}{6}\,\sigma^2\Big]\, dV_1\,. \tag{44}$$

Note that, at variance with the notation stated in subsec. 2.1 of Part I, the interaction between σ and φ is absorbed into A^{RM}. Here, $g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = (\partial_\tau f)^2 - \tau^{-2}|\vec{\nabla}_1 f|^2$, λ is the (adimensional) self–interaction constant of φ, $\sigma_0^2 = 6M_{rP}^2$, where M_{rP} is the reduced Planck–mass, and μ is a constant with the dimension of mass (in order that ratio μ/σ_0 be adimensional). The expression for A^{RM} is dictated by the condition that the self–interaction potential of $\tilde{\varphi}$ has the Mexican–hat form $\lambda^2\big(\tilde{\varphi}^2 - \mu^2/\lambda^2\big)^2$ typical of Higgs–field Lagrangian densities. Terms $R\varphi^2/6$ and $R\sigma^2/6$ ensure that A^{RM} and A^{RG} are manifestly conformal invariant, as for A^M and A^G in subsec. 3.1 of Part I.

The motion equations for φ and σ are therefore

$$D^2\varphi + \lambda^2\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\varphi - \frac{R}{6}\,\varphi = 0\,, \quad D^2\sigma + \frac{\mu^2}{\sigma_0^2}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\sigma - \frac{R}{6}\,\sigma = 0\,. \tag{45}$$

The corresponding energy-momentum tensors of matter and geometry are, respectively

$$\begin{aligned}\Theta^M_{\mu\nu} =\ & \big(\partial_\mu\varphi\big)\big(\partial_\nu\varphi\big) - \frac{g_{\mu\nu}}{2}\Big[g^{\rho\sigma}\big(\partial_\rho\varphi\big)\big(\partial_\sigma\varphi\big) - \frac{\lambda^2}{2}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2\Big] + \\ & \frac{1}{6}\big(g_{\mu\nu}D^2 - D_\mu D_\nu\big)\varphi^2 + \frac{\varphi^2}{6}\Big(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\Big)\,, \\ \Theta^G_{\mu\nu} =\ & -\big(\partial_\mu\sigma\big)\big(\partial_\nu\sigma\big) + \frac{g_{\mu\nu}}{2}\,g^{\rho\sigma}\big(\partial_\rho\sigma\big)\big(\partial_\sigma\sigma\big) - \frac{1}{6}\big(g_{\mu\nu}D^2 - D_\mu D_\nu\big)\sigma^2 - \\ & \frac{\sigma^2}{6}\Big(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\Big)\,.\end{aligned}$$

Contracting these tensors with $g^{\mu\nu}$ and using motion Eqs. (45), we find the corresponding traces

$$\Theta^M = \mu^2\frac{\sigma^2}{\sigma_0^2}\Big(\frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2} - \varphi^2\Big) = \mu^2\, e^{4\alpha}\Big(\frac{\mu^2}{\lambda^2} - e^{-2\alpha}\varphi^2\Big)\,, \quad \Theta^G = -\Theta^M.$$

The vanishing condition for total energy–momentum tensor $\Theta_{\mu\nu} = \Theta^M_{\mu\nu} + \Theta^G_{\mu\nu}$ provides the gravitational equation for the Higgs field interacting with the dilation field on the Riemann manifold. Considering the Higgs field as a fluid of energy density ρ_H, pressure p_H and 4–velocity $u^\mu = \partial_\tau x^\mu$, we obtain the following identifications

$$\Theta^M_{\mu\nu} = \big(\rho_H + p_H\big)\, u_\mu u_\nu - g_{\mu\nu} p_H \,; \quad \text{that is, } \rho_H = u^\mu u^\nu \Theta^M_{\mu\nu}\,,\ p_H = \frac{1}{3}\big(\rho_H - \Theta^M\big)\,,$$

which can be used to extract the expressions for ρ_H and p_H from $\Theta^M_{\mu\nu}$ as

$$\begin{aligned}\rho_H =& \frac{1}{2}\Big[\big(\partial_\tau\varphi\big)^2 - \frac{1}{\tau^2}|\vec{\nabla}_1\varphi|^2\Big] + \frac{\lambda^2}{4}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2 + \Big(\frac{1}{2\tau}\partial_\tau - \frac{1}{6\tau^2}\Delta_1\Big)\varphi^2 + \\ & \frac{\varphi^2}{6}\Big(u^\mu u^\nu R_{\mu\nu} - \frac{R}{2}\Big)\,,\\ p_H =& \frac{1}{6}\Big[\big(\partial_\tau\varphi\big)^2 - \frac{1}{\tau^2}|\vec{\nabla}_1\varphi|^2\Big] + \frac{\lambda^2}{12}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\Big(\varphi^2 + \frac{3\,\mu^2\sigma^2}{\lambda^2\sigma_0^2}\Big) + \\ & \Big(\frac{1}{6\tau}\partial_\tau - \frac{1}{18\tau^2}\Delta_1\Big)\varphi^2 + \frac{\varphi^2}{18}\Big(u^\mu u^\nu R_{\mu\nu} - \frac{R}{2}\Big)\,,\end{aligned}$$

where the formulae

$$\begin{aligned}& u^\mu\partial_\mu f = \partial_\tau f\,, \quad g^{\mu\nu}\big(\partial_\mu f\big)\big(\partial_\nu f\big) = \big(\partial_\tau\varphi\big)^2 - \frac{1}{\tau^2}|\vec{\nabla}_1\varphi|^2\,,\\ & \big(D^2 - u^\mu u^\nu D_\mu D_\nu\big) f = \frac{3}{\tau}\,\partial_\tau f - \frac{1}{\tau^2}\Delta_1 f\,,\end{aligned}$$

are used.

During the inflation stage, spacetime is conformally flat, the Ricci tensors vanish, and all particles are at rest with respect to the reference frame of synchronous comoving observers: hence, φ depends only on τ. In this case, we obtain the simplified total action integral

$$A^+ = A^{RM} + A^{RG} = \int_0^{+\infty} d\tau \int_{V_1} \frac{\tau^3}{2}\,\Big[\big(\partial_\tau\varphi\big)^2 - \frac{\lambda^2}{2}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2 - \big(\partial_\tau\sigma\big)^2\Big] dV_1\,, \quad (46)$$

leading to the simplified motion equations

$$\partial_\tau^2\varphi + \frac{3}{\tau}\,\partial_\tau\varphi + \lambda^2\,\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\varphi = 0;\ \ \partial_\tau^2\sigma + \frac{3}{\tau}\,\partial_\tau\sigma + \frac{\mu^2}{\sigma_0^2}\,\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\sigma = 0. \quad (47)$$

Correspondingly, the simplified total energy density and the pressure of the Higgs field simplify to

$$\rho_H = k_H + \frac{\lambda^2}{4}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2, \quad p_H = \frac{1}{3}\Big[k_H + \frac{\lambda^2}{4}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)\Big(\varphi^2 + \frac{3\,\mu^2\sigma^2}{\lambda^2\sigma_0^2}\Big)\Big]\,,$$

where

$$k_H = \frac{1}{2}\big(\partial_\tau\varphi\big)^2 + \frac{1}{\tau}\varphi\,\partial_\tau\varphi\,, \quad \frac{\lambda^2}{4}\,\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\frac{\sigma^2}{\sigma_0^2}\Big)^2,$$

are respectively kinetic energy density and potential energy density.

We see that ρ_H is always positive, while p_H may be both positive and negative. In particular, if k_H is negligible, p_H vanishes for $\Theta^M = 0$ and is negative for positive Θ^M, i.e., when $\varphi^2 < \mu^2\sigma^2/\lambda^2\sigma_0^2$. As noted in Part I Section 5, when p_H is negative, the creation of matter - in our case, Higgs bosons - takes place.

If both φ^2 and k_H are negligible, as is presumed to be the case at $\tau = 0$, we have $p_H \equiv -3\rho_H$, implying that the creation of matter starts immediately, or almost immediately, after the symmetry–breaking event. For $\tau \to +\infty$, we have $\sigma(x) \to \sigma_0$ and, consequently

$$\rho_H = k_H + \frac{\lambda^2}{4}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\Big)^2, \quad p_H = \frac{1}{3}\Big[k_H + \frac{\lambda^2}{4}\Big(\varphi^2 - \frac{\mu^2}{\lambda^2}\Big)\Big(\varphi^2 + 3\,\frac{\mu^2}{\lambda^2}\Big)\Big].$$

5.2 Conformal Higgs field on the Cartan manifold

The action integral of the dilation field interacting with a scalar field on the Cartan manifold can be obtained from the action integral on the Riemann manifold, described in the previous pages, by the following replacements

$$A^{RM} \to \tilde{A}^{CM}, \quad A^{RG} \to \tilde{A}^{CG}, \quad dV_1(\hat{x}) \to \tilde{R}_{RW}^{-3}(\tilde{x})d\tilde{V}_{\tilde{\tau}}(\hat{x}), \quad V_1 \to \tilde{V}_{\tilde{\tau}}, \quad e^{\alpha(x)}d\tau \to d\tilde{\tau},$$
$$\varphi(x) \to \tilde{\varphi}(x) = e^{-\alpha(x)}\varphi(x), \quad \sigma(x) \to \sigma_0, \quad e^{\alpha(x)}d\tau \to d\tilde{\tau}\,, \quad \partial_\mu \to \tilde{\partial}_\mu = e^{-\alpha(x)}\partial_\mu,$$

which yield the approximate action integrals, respectively of matter and geometry,

$$\tilde{A}^{CM} = \int_0^{+\infty} d\tilde{\tau} \int_{\tilde{V}_{\tilde{\tau}}} \frac{1}{2}\Big[\tilde{g}^{\mu\nu}\big(\tilde{\partial}_\mu\tilde{\varphi}\big)\big(\tilde{\partial}_\nu\tilde{\varphi}\big) - \frac{\lambda^2}{2}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)^2 + \frac{\tilde{R}}{6}\,\tilde{\varphi}^2\Big]\,V_{\tilde{\tau}}(\hat{x})\,,$$
$$\tilde{A}^{CG} = -\frac{\sigma_0^2}{12}\int_0^{+\infty} d\tilde{\tau}\int_{\tilde{V}_{\tilde{\tau}}} \tilde{R}\,d\tilde{V}_{\tilde{\tau}}(\hat{x})\,.$$

All quantities are marked by a tilde, to mean that they belong to the Cartan manifold spanned by conformal-hyperbolic coordinates. Note that the original conformal invariance of $\tilde{A}^{RM}$ and $\tilde{A}^{RG}$ has disappeared and the conformal symmetry appears explicitly broken instead in both $\tilde{A}^{CM}$ and $\tilde{A}^{CG}$.

For the sake of clarity, we expand here the kinetic term of the Lagrangian density for φ as follows

$$\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu\tilde{\varphi})(\tilde{\partial}_\nu\tilde{\varphi}) = (\partial_{\tilde{\tau}}\tilde{\varphi})^2 - \frac{1}{\tilde{R}_{RW}^2}|\vec{\nabla}_1\tilde{\varphi}|^2\,.$$

The motion equation for $\tilde{\varphi}$ is thus

$$\tilde{D}^2\tilde{\varphi} + \lambda^2\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)\tilde{\varphi} - \frac{\tilde{R}}{6}\,\tilde{\varphi} = 0\,, \tag{48}$$

where

$$\tilde{D}^2\tilde{\varphi} = \partial_{\tilde{\tau}}^2\tilde{\varphi} - \frac{1}{\tilde{R}_{RW}^2}\,\Delta_1\tilde{\varphi} + 3\,\Big[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\Big]\partial_{\tilde{\tau}}\tilde{\varphi} - \frac{3}{\tilde{R}_{RW}^2}\big(\vec{\nabla}_1\tilde{R}_{RW}\big)\cdot\big(\vec{\nabla}_1\tilde{\varphi}\big)\,. \tag{49}$$

Correspondingly, the energy–momentum tensors decompose into

$$\begin{aligned}\tilde{\Theta}^M_{\mu\nu} &= \frac{1}{6}\Big[4\big(\tilde{\partial}_\mu\tilde{\varphi}\big)\big(\tilde{\partial}_\nu\tilde{\varphi}\big) - \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\big(\tilde{\partial}_\rho\tilde{\varphi}\big)\big(\tilde{\partial}_\sigma\tilde{\varphi}\big)\Big] + \tilde{g}_{\mu\nu}\frac{\lambda^2}{4}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)^2 + \\ &\quad \frac{1}{3}\tilde{\varphi}\big(\tilde{g}_{\mu\nu}\tilde{D}^2 - \tilde{D}_\mu\tilde{D}_\nu\big)\,\tilde{\varphi} + \frac{\tilde{\varphi}^2}{6}\Big(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}\Big)\,, \\ \tilde{\Theta}^G_{\mu\nu} &= -\frac{\sigma_0^2}{6}\Big(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}\Big)\,,\end{aligned}$$

which are linked by the Cartan–Einstein gravitational equation $\tilde{\Theta}^M_{\mu\nu} + \tilde{\Theta}^G_{\mu\nu} = 0$, i.e.,

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \kappa\,\tilde{\Theta}^M_{\mu\nu}\,,\ \text{where}\,\kappa = \frac{6}{\sigma_0^2}\,.$$

By contraction with fundamental tensor $\tilde{g}^{\mu\nu}$ and using motion Eq.(48), we obtain the respective energy–momentum traces and the trace equation

$$\tilde{\Theta}^G = \frac{1}{\kappa}\tilde{R}\,,\quad \tilde{\Theta}^M = \mu^2\Big(\frac{\mu^2}{\lambda^2} - \tilde{\varphi}^2\Big) = -\tilde{\Theta}^G\,,\ \text{then}\ \tilde{R} = -\frac{6\,\mu^2}{\sigma_0^2}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)\,, \tag{50}$$

The expressions for energy density and pressure are obtained as in the previous Section, yielding

$$\begin{aligned}\tilde{\rho}_H &= \tilde{u}^\mu\tilde{u}^\nu\tilde{\Theta}^M_{\mu\nu} = \frac{1}{2}\Big[\big(\partial_{\tilde{\tau}}\tilde{\varphi}\big)^2 - \frac{1}{\tau^2(\tilde{x})}|\vec{\nabla}_1\tilde{\varphi}|^2\Big] + \frac{\lambda^2}{4}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)^2 + \\ &\quad \Big[\frac{1}{2\tau(\tilde{x})}\partial_{\tilde{\tau}} - \frac{1}{6\tau^2(\tilde{\tau})}\Delta_1\Big]\tilde{\varphi}^2 + \frac{\tilde{\varphi}^2}{6}\Big(\tilde{u}^\mu\tilde{u}^\nu\tilde{R}_{\mu\nu} - \frac{\tilde{R}}{2}\Big)\,, \\ \tilde{p}_H &= \frac{\tilde{\rho}_H - \tilde{\Theta}^M}{3} = \frac{1}{6}\Big[\big(\partial_{\tilde{\tau}}\tilde{\varphi}\big)^2 - \frac{1}{\tau^2(\tilde{x})}|\vec{\nabla}_1\tilde{\varphi}|^2\Big] + \frac{\lambda^2}{12}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)\Big(\tilde{\varphi}^2 + \frac{3\,\mu^2}{\lambda^2}\Big) + \\ &\quad \Big(\frac{1}{6\tau(\tilde{x})}\partial_{\tilde{\tau}} - \frac{1}{18\tau^2(\tilde{\tau})}\Delta_1\Big)\tilde{\varphi}^2 + \frac{\tilde{\varphi}^2}{18}\Big(\tilde{u}^\mu\tilde{u}^\nu\tilde{R}_{\mu\nu} - \frac{\tilde{R}}{2}\Big)\,.\end{aligned}$$

Here, $\tilde{u}^\mu = dx^\mu/d\tilde{\tau} = e^{-\alpha(x)}u^\mu$ and $\tau(\tilde{x})$ is the value of kinematic time τ as a function of conformal–hyperbolic coordinates $\tilde{x}$.

These action integrals simplify considerably if we assume that the geometry is conformally flat, which implies that $R = 0$ on the Riemann manifold, that the dilation field is constant σ_0, and that the Higgs field and kinematic time function $\tau(\tilde{x})$ depend only on proper time $\tilde{\tau}$. In this case, the simplified total action integral on the Cartan manifold takes the form

$$\tilde{A}^+ = \tilde{A}^{CM} + \tilde{A}^{CG} = \int_0^{+\infty} d\tilde{\tau}\int_{\tilde{V}_{\tilde{\tau}}}\frac{1}{2}\,\Big[\big(\tilde{\partial}_\tau\tilde{\varphi}\big)^2 - \frac{\lambda^2}{2}\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)^2 + \big(\tilde{\varphi}^2 - \sigma_0^2\big)\frac{\tilde{R}}{6}\Big]\,d\tilde{V}_{\tilde{\tau}}(\hat{x})\,.$$

Using Eq.(49) in the simplified form

$$\tilde{D}^2\tilde{\varphi} = \partial^2_{\tilde{\tau}}\tilde{\varphi} + 3\,\Big[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\Big]\partial_{\tilde{\tau}}\tilde{\varphi}\,,$$

and the last of Eq.(50), we obtain the motion equation for $\tilde{\varphi}(\tilde{\tau})$

$$\partial^2_{\tilde{\tau}}\tilde{\varphi} + 3\left[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\right]\partial_{\tilde{\tau}}\tilde{\varphi} + \Big(\lambda^2 - \frac{\mu^2}{\sigma_0^2}\Big)\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)\tilde{\varphi}\,.$$

This differs from the equation derived from the action integral with $\tilde{R} = 0$ by the replacement of the squared coupling constant λ^2 with $\lambda^2 - (\mu/\sigma_0)^2$ (as also found by Callan et al. in 1970). Since μ is expected to be of the order of magnitude of 130 GeV/c^2 and λ^2 of the order of magnitude of 10^{-2}, while $\sigma_0 \simeq 6 \times 10^{18}$ GeV/c^2, we see that correction of coupling constant λ due to the inclusion of the term $\tilde{\varphi}^2\tilde{R}/6$ in the action integral is absolutely negligible. We can therefore assume the simplified motion equation on the Cartan manifold to be

$$\partial^2_{\tilde{\tau}}\tilde{\varphi} + 3\left[\frac{1}{\tau(\tilde{\tau})} + \partial_{\tilde{\tau}}\tilde{\alpha}\right]\partial_{\tilde{\tau}}\tilde{\varphi} + \lambda^2\Big(\tilde{\varphi}^2 - \frac{\mu^2}{\lambda^2}\Big)\tilde{\varphi}\,. \tag{51}$$

For $\tilde{\tau} \to +\infty$, we have $\tilde{\varphi} \to \varphi$, $\tau(\tilde{\tau}) \to \tau$, $\partial_{\tilde{\tau}} \to \partial_\tau$, $\partial_{\tilde{\tau}}\tilde{\alpha} \to 0$, and all the equations which govern the dynamics of matter fields in the Cartan picture converge to corresponding equations of matter fields grounded on a Riemann manifold, in which the conformal symmetry appears explicitly broken.

In conclusion

At the end of the scale–expansion, all the properties associated with the underlying conformal symmetry dissolve and the universe continues to evolve following the laws of standard GR.

References

[1] Zeeman, E.C., *Causality implies the Lorentz Group, J. Math. Phys.* **5** (1964) 490.

[2] Alexandrov, A.D. and Ovchinnikova, V.V., *Notes on the foundations of the theory of relativity, Vestnik Leningrad Univ.* **11** (1953) 94.

[3] Alexandrov, A.D., *A contribution to chronogeometry, Can. J. of Math.* **19** (1967) 1119.

[4] Borchers, H.J. and Hegerfeldt, G.C., *The structure of space–time transformations, Commun. Math. Phys.* **28** (1972) 259.

[5] Göbel, R., *Zeeman Topologies on Space-Times of General Relativity Theory , Comm. Math. Phys.* 46 (1976) 289.

[6] Hawking, S.W., King, A.R. and McCarthy, P.J., *A new topology for curved space–time which incorporates the causal, differential, and conformal structures, J. Math. Phys.* **17** (1976) 174.

[7] Popovici, I. and Constantin Radulsecu, D., *Characterization of conformality in a Minkowski space, Ann. Inst. Henri Poincaré* **25** (1981) 131.

[8] Nobili, R., *Conformal covariant Wightman functions, Il Nuovo Cimento* bf 13 (1973) 129.

[9] Cartan, E., *Sur les espaces conformes généralisés et l'Universe optique, Comptes Rendus de l'Academie des Sciences*, **174** (1922) 734.

[10] Haag, R., *Local Quantum Physics, I.2.* Springer–Verlag, Berlin (1992).

[11] Mack, G. and Salam, A., *Finite–Component Field Representations of the Conformal Group, Annals of Physics*, **53** (1969) 174.

[12] Fubini, S., *A New Approach to Conformal Invariant Field Theories, Nuovo Cimento*, **34A** (1976) 521.

[13] Ciccariello, S. and Sartori, G. *Covariance Properties of Fields Under Inversion and Finite Conformal Transformations, Il Nuovo Cimento* **19** (1974) 470.

[14] Birkhoff, G. and von Neumann, J., *The Logic of Quantum Mechanics, Annals of Mathematics* **27** (1936) 823–843.

[15] Weinberg, S., *Gravitation and Cosmology, Chap.12.* John Wiley & Sons (1972)

[16] Umezawa, H., Matsumoto, H. and Tachiki, M., *Thermo Field Dymamics and Condensed States.* North–Holland Pub. Co., Amsterdam (1982)

[17] Englert, F. and Brout, R., *Broken Symmetry and the Mass of Gauge Vector Mesons, Phys. Rev. Lett.* **13** (1964) 321.

[18] Higgs P.W.,*Broken Symmetries and the Masses of Gauge Bosons, Phys. Rev. Lett.* **13** (1964) 508.

[19] Kibble, T. W. B., *Symmetry Breaking in Non–Abelian Gauge Theories, Phys. Rev.* **135** (1967) 1554.

[20] Moscella, U., *The de Sitter and anti–de Sitter Seightseeing Tour, Séminaire Poincaré.* **1** (2005) 1.

[21] Englert, F., Truffin, C. and Gastmans, R., *Conformal invariance in quantum gravity, Nuclear Physics B* **117** (1976) 407.

[22] Grib, A.A., Mamayev, S.G. and Mostepanenko, V.M., *Particle Creation from Vacuum in Homogeneous Isotropic Models of the Universe, General Relativity and Gravitation* **7** (1976) 535.

[23] Fulling S.A., *Nonuniqueness of canonical field quantization in Riemannian space–time, Phys. Rev. D* **7** (1973) 2850.

[24] Davies, P.C.W., *Quantum vacuum noise in physics and cosmology, CHAOS* **11** (1975) 539.

[25] Unruh,W.G., *Notes on black-hole evaporation, Phys. Rev. D* **14** (1986) 870.

[26] Callan, C.G., Coleman, S. and Jackiv, R., *A New Improved Energy–Momentum Tensor, Annals of Physics*, **59** (1970) 42.

[27] Brout, R., Englert, F. and Gunzig, E., *The Creation of the Universe as a Quantum Phenomenon, Annals od Physics*, bf 115 (1978) 78.