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Covariant Analytic Mechanics with Differential Forms and Its Application to Gravity

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Abstract: We discuss fundamentals of the covariant analytic mechanics with differential forms. We apply it to typical field theories, such as a scalar field, the electromagnetic field, and a non-abelian gauge field, as well as the Newtonian mechanics of a harmonic oscillator. A significant feature of the covariant analytic mechanics is that the canonical equations, in addition to the Euler-Lagrange equation, are not only manifestly Lorentz covariant but also gauge covariant. In the latter half of the paper, we apply the covariant analytic mechanics to Einstein's general theory of relativity, and show that the gravitational field can be successfully treated within the framework of it. We obtain the canonical equations of gravity with manifest diffeomorphism covariance.

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1. Introduction

In his original paper, Nakamura [1] proposed an idea of the covariant analytic mechanics which treats space and time on an equal footing. Although it is physically equivalent to the traditional one, the covariant analytic mechanics has advantages that the canonical equations are gauge covariant as well as manifestly Lorentz

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covariant. There is no need to introduce any tricks such as gauge fixing or the Dirac bracket even if gauge field theories are considered. Moreover, the covariant analytic mechanics fascinates us with its mathematical beauty. Its mathematical structure is simple and straightforward. The covariant analytic mechanics is interesting from both practical and theoretical reasons.

Unfortunately, the argument presented in [1] is composed of heuristic considerations or conjectures. There are some errors as well. In order to justify the idea of the covariant analytic mechanics, we need a logical argument. The purpose of the present paper is (1) to lay the theoretical foundation of the covariant analytic mechanics, (2) to correct errors contained in the original article, and (3) to apply it to typical field theories including gauge theories and gravity.

This paper is divided into two parts. In Part I, we discuss fundamentals of the covariant mechanics and apply it to the Newtonian mechanics of a harmonic oscillator, a scalar field, the electromagnetic field, and a non-abelian gauge field. Part II is devoted to Einstein's general theory of relativity. There, we show that the covariant analytic mechanics works successfully as well for the gravitational field.

Part I
Fundamentals

2. Covariant Analytic Mechanics

Let M be an n-dimensional C^∞-manifold and $\Omega^{p_1 p_2 \cdots p_k}(M) = \Omega^{p_1}(M) \times \cdots \times \Omega^{p_k}(M)$ where $\Omega^p(M)$ is the $\mathscr{F}(M)$-module of p-forms. For a map $\phi : \Omega^{p_1 \cdots p_k}(M) \to \Omega^r(M)$, we put $\beta = \phi(\alpha^1, \cdots, \alpha^k) \in \Omega^r(M)$ with $\alpha^i \in \Omega^{p_i}(M)$. If there exists $\omega_i \in \Omega^{r-p_i}(M)$ such that β behaves under variations $\delta\alpha^1, \cdots, \delta\alpha^k$ as $\delta\beta = \sum_i \delta\alpha^i \wedge \omega_i$, then we call ω_i the *left derivative* of β by α^i and denote $\omega_i = \partial\beta/\partial\alpha^i$. Similarly, we define the *right derivative* using $\delta\beta = \sum_i \omega_i \wedge \delta\alpha^i$. In the present paper, we use the left derivatives unless otherwise stated since both derivatives are essentially the same. We assume sum over repeated indices in what follows.

For a map $f : \Omega^{2l_1, \cdots, 2l_i}(M) \times \Omega^{2l_1+1, \cdots, 2l_i+1}(M) \times \Omega^{2m_1-1, \cdots, 2m_j-1}(M) \times \Omega^{2m_1, \cdots, 2m_j}$ (M) $\to \Omega^n(M)$, we define the *Lagrange form* $L = f(\varphi^1, \cdots, \varphi^i, d\varphi^1, \cdots, d\varphi^i, \psi^1, \cdots \psi^j, d\psi^1, \cdots d\psi^j) \in \Omega^n(M)$ where $\varphi^a \in \Omega^{2l_a}(M)$, $\psi^\alpha \in \Omega^{2m_\alpha - 1}(M)$ with $l_a, m_\alpha \in \boldsymbol{Z}$. Now, let D be an open connected subset in M having a smooth boundary, and its closure $\bar{D}$ be compact. Then, we identify the integral of L over $\bar{D}$ with an action and apply Hamilton's principle to it. Under variations $\delta\varphi^a$ and $\delta\psi^\alpha$, we obtain

$$\begin{aligned}\delta L &= \delta\varphi^a \wedge \frac{\partial L}{\partial\varphi^a} + \delta d\varphi^a \wedge \frac{\partial L}{\partial d\varphi^a} + \delta\psi^\alpha \wedge \frac{\partial L}{\partial\psi^\alpha} + \delta d\psi^\alpha \wedge \frac{\partial L}{\partial d\psi^\alpha}\\ &= \delta\varphi^a \wedge \left(\frac{\partial L}{\partial\varphi^a} - d\frac{\partial L}{\partial d\varphi^a}\right) + d\left(\delta\varphi^a \wedge \frac{\partial L}{\partial d\varphi^a}\right)\\ &\quad + \delta\psi^\alpha \wedge \left(\frac{\partial L}{\partial\psi^\alpha} + d\frac{\partial L}{\partial d\psi^\alpha}\right) + d\left(\delta\psi^\alpha \wedge \frac{\partial L}{\partial d\psi^\alpha}\right).\end{aligned}$$

Extension of du Bois Reymond's lemma to the present case is straightforward, and hence the Euler-Lagrange equations can be written as

$$\frac{\partial L}{\partial\varphi^a} - d\frac{\partial L}{\partial d\varphi^a} = 0, \qquad \frac{\partial L}{\partial\psi^\alpha} + d\frac{\partial L}{\partial d\psi^\alpha} = 0. \tag{1}$$

The first equation is for an even-form φ^a, and the second one is for an odd-form ψ^α. The + sign in the latter should be noted. Under the right derivatives, these equations are replaced by

$$\frac{\partial L}{\partial\varphi^a} + (-1)^n d\frac{\partial L}{\partial d\varphi^a} = 0, \qquad \frac{\partial L}{\partial\psi^\alpha} - (-1)^n d\frac{\partial L}{\partial d\psi^\alpha} = 0, \tag{2}$$

where $n = \dim M$.

We now define the *conjugate momentum form* p_a and π_α as

$$p_a = \frac{\partial L}{\partial d\varphi^a}, \qquad \pi_\alpha = \frac{\partial L}{\partial d\psi^\alpha}. \tag{3}$$

In contrast to the usual analytic mechanics, the number of components of the momentum $p_a \in \Omega^{n-2l_a-1}(M)$ does *not* coincide with that of the canonical variable $\varphi^a \in \Omega^{2l_a}(M)$. In fact, the former has $n!/[(n-2l_a-1)!(2l_a+1)!]$ components and the latter $n!/[(n-2l_a)!(2l_a)!]$. The same can be said about $\pi_\alpha \in \Omega^{n-2m_\alpha}(M)$ and $\psi_\alpha \in \Omega^{2m_\alpha-1}$. Although this disagreement sounds problematic at first, it does not cause any trouble in the following.

Let us assume that the system is non-singular. This means that (3) can be solved with respect to $d\varphi^a$ and $d\psi^\alpha$ and they are represented uniquely with φ^a, p_a, ψ^α, and π_α. In contrast to the traditional analytic mechanics, this assumption is satisfied even by gauge theories (see section 3). Then, define the *Hamilton form* $H \in \Omega^n(M)$ as follows.

$$H = d\varphi^a \wedge p_a + d\psi^\alpha \wedge \pi_\alpha - L \tag{4}$$

Under the right derivatives, it must be replaced by $H = p_a \wedge d\varphi^a + \pi_\alpha \wedge d\psi^\alpha - L$. Note that the Hamilton form H defined here cannot be interpreted as energy density. This is because the covariant analytic mechanics treats space and time on an equal footing so that H has nothing to do with the traditional Hamiltonian density $\mathscr{H}$. Under variations $\delta\varphi^a$, δp_a, $\delta\psi^\alpha$, and $\delta\pi_\alpha$, we have

$$\begin{aligned}\delta H = (-1)^{n-1}\delta p_a \wedge d\varphi^a + \delta d\varphi^a \wedge p_a + \delta\pi_\alpha \wedge d\psi^\alpha + \delta d\psi^\alpha \wedge \pi_\alpha \\ - \delta\varphi^a \wedge \frac{\partial L}{\partial \varphi^a} - \delta d\varphi^a \wedge \frac{\partial L}{\partial d\varphi^a} - \delta\psi^\alpha \wedge \frac{\partial L}{\partial \psi^\alpha} - \delta d\psi^\alpha \wedge \frac{\partial L}{\partial d\psi^\alpha}.\end{aligned}$$

Substituting (3), we obtain

$$\delta H = (-1)^{n-1}\delta p_a \wedge d\varphi^a + \delta\pi_\alpha \wedge d\psi^\alpha - \delta\varphi^a \wedge \frac{\partial L}{\partial \varphi^a} - \delta\psi^\alpha \wedge \frac{\partial L}{\partial \psi^\alpha}.$$

This result together with (1) furnishes Hamilton's canonical equations

$$d\varphi^a = -(-1)^n \frac{\partial H}{\partial p_a}, \qquad dp_a = -\frac{\partial H}{\partial \varphi^a}, \qquad d\psi^\alpha = \frac{\partial H}{\partial \pi_\alpha}, \qquad d\pi_\alpha = \frac{\partial H}{\partial \psi^\alpha}. \tag{5}$$

Here φ^a and p_a are an even-form and its conjugate momentum; ψ^α and π_α are an odd-form and its conjugate momentum. Under the right derivatives, these equations are replaced by

$$d\varphi^a = -(-1)^n \frac{\partial H}{\partial p_a}, \quad dp_a = (-1)^n \frac{\partial H}{\partial \varphi^a}, \quad d\psi^\alpha = \frac{\partial H}{\partial \pi_\alpha}, \quad d\pi_\alpha = -(-1)^n \frac{\partial H}{\partial \psi^\alpha}. \tag{6}$$

3. Examples

3.1 Harmonic Oscillator

The covariant analytic mechanics is a field theory in essence. Nevertheless, it can be applied to the Newtonian mechanics as well since it can be regarded as a 1-dimensional field theory. As an example, let us consider a harmonic oscillator. Its Lagrange form is

$$L = \frac{1}{2}(dq \wedge *dq - \omega^2 q \wedge *q) \tag{7}$$

where $q \in \Omega^0(\boldsymbol{R})$ represents the position of the oscillator, and ω is a positive constant. The "metric" on $\boldsymbol{R}$ is taken to be $(+)$. In the present case, q and $*dq$ are 0-forms so that the $\wedge$ sign in the above is superficial; L is a 1-form.

Under a variation δq, we have $\delta L = \delta dq \wedge *dq - \omega^2 \delta q \wedge *q$. Hence, we obtain $\partial L/\partial q = -\omega^2 * q$ and $\partial L/\partial dq = *dq$. Since q is a 0-form, the Euler-Lagrange equation is $\partial L/\partial q - d(\partial L/\partial dq) = 0$, namely

$$d*dq + \omega^2 * q = 0. \tag{8}$$

Using the codifferential $\delta = *d*$ and the Laplacian $\triangle = \delta d + d\delta$, we can rewrite (8) as $\triangle q + \omega^2 q = 0$, which is equivalent to $\ddot{q} + \omega^2 q = 0$ in the traditional notation.

The conjugate momentum form is defined as $p = \partial L/\partial dq = *dq \in \Omega^0(\boldsymbol{R})$. From this equation, we have $dq = *p$. Then, the Hamilton form $H = dq \wedge p - L$ becomes

$$H = \frac{1}{2}(p \wedge *p + \omega^2\, q \wedge *q).$$

Under variations δq and δp, we have $\delta H = \delta p \wedge *p + \omega^2\, \delta q \wedge *q$, which means $\partial H/\partial p = *p$ and $\partial H/\partial q = \omega^2 * q$. The canonical equations for the 0-form q are $dq = \partial H/\partial p$ and $dp = -\partial H/\partial q$, namely

$$dq = *p, \qquad dp = -\omega^2 * q.$$

From the first equation, we have $p = *dq$. Substituting this into the second, we obtain $d*dq + \omega^2 * q = 0$ which coincides with the Euler-Lagrange equation (8).

3.2 Scalar Field

As a second example, let us consider a real scalar field in the 4-dimensional Minkowski space with the metric $(+---)$. Its Lagrange form is

$$L = \frac{1}{2}(d\phi \wedge *d\phi - \mu^2 \phi \wedge *\phi) \tag{9}$$

where $\phi \in \Omega^0(\boldsymbol{R}^4)$ represents a scalar field, and μ is a positive constant. Note that the Lagrange form (9) of a scalar field and that of a harmonic oscillator (7) are exactly the same in shape, although the former is a 4-form and the latter is a 1-form. Thus the calculation below becomes similar except for signs. Under a variation $\delta\phi$, we have $\delta L = \delta d\phi \wedge *d\phi - \mu^2 \delta\phi \wedge *\phi$. Hence, we obtain $\partial L/\partial \phi = -\mu^2 * \phi$ and $\partial L/\partial d\phi = *d\phi$. The Euler-Lagrange equation for the 0-form ϕ is $\partial L/\partial \phi - d(\partial L/\partial d\phi) = 0$, namely

$$d*d\phi + \mu^2 * \phi = 0, \tag{10}$$

Using the codifferential $\delta = *d*$ and the Laplacian $\triangle = \delta d + d\delta$, we can rewrite (10) as $\triangle\,\phi - \mu^2\phi = 0$ which is equivalent to $\partial^\mu\partial_\mu\phi + \mu^2\phi = 0$ in the traditional notation.

The conjugate momentum form is defined as $\pi = \partial L/\partial d\phi = *d\phi \in \Omega^3(\boldsymbol{R}^4)$. From this equation, we have $d\phi = -*\pi$. Then, the Hamilton form $H = d\phi \wedge \pi - L$ becomes

$$H = \frac{1}{2}(\pi \wedge *\pi + \mu^2\phi \wedge *\phi).$$

Under variations $\delta\phi$ and $\delta\pi$, we have $\delta H = \delta\pi \wedge *\pi + \mu^2\delta\phi \wedge *\phi$, which means $\partial H/\partial\pi = *\pi$ and $\partial H/\partial\phi = \mu^2*\phi$. The canonical equations for the 0-form ϕ are $d\phi = -\partial H/\partial\pi$ and $d\pi = -\partial H/\partial\phi$, namely

$$d\phi = -*\pi, \qquad d\pi = -\mu^2*\phi.$$

From the first equation, we have $\pi = *d\phi$. Substituting this into the second, we obtain $d*d\phi + \mu^2*\phi = 0$ which coincides with the Euler-Lagrange equation (10).

3.3 Electromagnetic Field

As a third example, let us consider an electromagnetic field in the 4-dimensional Minkowski space with the metric $(+---)$. Its Lagrange form is

$$L = -\frac{1}{2}\,F \wedge *F + J \wedge A$$

where $A \in \Omega^1(\boldsymbol{R}^4)$ represents an electromagnetic field, $F = dA \in \Omega^2(\boldsymbol{R}^4)$ is the Faraday form, and $J \in \Omega^3(\boldsymbol{R}^4)$ is a source. J is assumed to be independent of A. Under a variation δA of the electromagnetic field, we have $\delta L = -\delta dA \wedge *F$ $-\delta A \wedge J$. Hence, we obtain $\partial L/\partial A = -J$ and $\partial L/\partial dA = -*F$. The Euler-Lagrange equation for the 1-form A is $\partial L/\partial A + d(\partial L/\partial dA) = 0$, namely

$$d*F = -J. \tag{11}$$

This, together with the identity $dF = 0$, is nothing but the Maxwell equation.

The conjugate momentum form is defined as $\pi = \partial L/\partial dA = -*F \in \Omega^2(\boldsymbol{R}^4)$. From this equation, we have $F = *\pi$. Then, the Hamilton form $H = dA \wedge \pi - L$ becomes

$$H = \frac{1}{2}\,\pi \wedge *\pi - J \wedge A.$$

Under variations δA and $\delta\pi$, we have $\delta H = \delta\pi \wedge *\pi + \delta A \wedge J$, which means $\partial H/\partial\pi = *\pi$ and $\partial H/\partial A = J$. The canonical equations for the 1-form A are $dA = \partial H/\partial\pi$ and $d\pi = \partial H/\partial A$, namely

$$dA = *\pi, \qquad d\pi = J. \tag{12}$$

From the first equation, we have $\pi = -*F$. Substituting this into the second, we obtain $d*F = -J$ which coincides with the Euler-Lagrange equation (11). It should be noted that no gauge fixing is used in these calculations; both (11) and (12) are gauge covariant as well as manifestly Lorentz covariant.

3.4 Gauge Field

As a final example, we consider a non-abelian gauge field in the 4-dimensional Minkowski space with the metric $(+---)$. We assume that the gauge group G is a compact semi-simple Lie group as usual and denote by $\mathfrak{g}$ the Lie algebra of G. We expand the connection form A ($\mathfrak{g}$-valued 1-form) and the curvature form F ($\mathfrak{g}$-valued 2-form) on the G-principal fibre bundle $P \xrightarrow{\pi} \boldsymbol{R}^4$ as $A = \sum_a T^a \otimes A^a$ and $F^a = \sum_a T^a \otimes F^a$ with respect to the basis T^a $(1 \le a \le \dim G)$ of $\mathfrak{g}$. By construction, $A^a \in \Omega^1(\boldsymbol{R}^4)$ and $F^a \in \Omega^2(\boldsymbol{R}^4)$. Then, the Lagrange form of the gauge field is

$$L = -\frac{1}{2g^2}\, F^a \wedge *F^a + \frac{1}{g}\, J^a \wedge A^a. \tag{13}$$

Here $J^a \in \Omega^3(\boldsymbol{R}^4)$ represents a source of the gauge field, and g is a coupling constant. J^a is assumed to be independent of A^a. The second structure equation $F = dA + A \wedge A$ implies

$$F^a = dA^a + \frac{1}{2}\, f_{abc}\, A^b \wedge A^c, \tag{14}$$

where f_{abc} is the structure constant of G. The upper and lower indices are identified here since there is no need to distinguish them. Under a variation δA^a of the gauge field, (13) and (14) result in

$$\delta L = -\frac{1}{g^2}\, \delta F^a \wedge *F^a - \frac{1}{g}\, \delta A^a \wedge J^a, \qquad \delta F^a = \delta dA^a + f_{abc}\, \delta A^b \wedge A^c,$$

respectively. From these equations, we obtain

$$\delta L = -\frac{1}{g^2}\, \delta dA^a \wedge *F^a + \delta A^a \wedge \left(-\frac{1}{g^2}\, f_{abc}\, A^b \wedge *F^c - \frac{1}{g}\, J^a \right)$$

which means

$$\frac{\partial L}{\partial dA^a} = -\frac{1}{g^2} {*F^a}, \qquad \frac{\partial L}{\partial A^a} = -\frac{1}{g^2} f_{abc} A^b \wedge *F^c - \frac{1}{g} J^a.$$

The Euler-Lagrange equation for the 1-form A^a is $\partial L/\partial A^a + d(\partial L/\partial dA^a) = 0$, namely $d{*F^a} + f_{abc} A^b \wedge *F^c = -gJ^a$. With the covariant exterior differentiation d_A, it reads

$$d_A{*F^a} = -gJ^a. \tag{15}$$

This is nothing but the Yang-Mills equation.

The conjugate momentum form is defined as $\pi^a = \partial L/\partial dA^a = -g^{-2}{*F^a} \in \Omega^2(\boldsymbol{R}^4)$. From this equation, we have $F^a = g^2{*\pi^a}$, which is rewritten with (14) as

$$dA^a = -\frac{1}{2} f_{abc} A^b \wedge A^c + g^2{*\pi^a}$$

Then, the Hamilton form $H = dA^a \wedge \pi^a - L$ becomes

$$H = -\frac{1}{2} f_{abc}\, \pi^a \wedge A^b \wedge A^c + \frac{1}{2} g^2\, \pi^a \wedge *\pi^a - \frac{1}{g} J^a \wedge A^a.$$

Under variations δA^a and $\delta\pi^a$, we have

$$\delta H = \delta\pi^a \wedge \left(g^2{*\pi^a} - \frac{1}{2} f_{abc} A^b \wedge A^c \right) + \delta A^a \wedge \left(f_{abc}\, \pi^b \wedge A^c + \frac{1}{g} J^a \right)$$

which means

$$\frac{\partial H}{\partial \pi^a} = g^2{*\pi^a} - \frac{1}{2} f_{abc} A^b \wedge A^c, \qquad \frac{\partial H}{\partial A^a} = f_{abc}\, \pi^b \wedge A^c + \frac{1}{g} J^a.$$

The canonical equations for the 1-form A^a are $dA^a = \partial H/\partial\pi^a$ and $d\pi^a = \partial H/\partial A^a$, namely

$$F^a = g^2{*\pi^a}, \qquad d_A\pi^a = \frac{1}{g} J^a. \tag{16}$$

Here, we have used (14) and the covariant exterior differentiation d_A. From the first equation, we have $\pi^a = -g^{-2}\,{*F^a}$. Substituting this into the second, we obtain $d_A{*F^a} = -gJ^a$ which coincides with the Euler-Lagrange equation (15). Again, no gauge fixing is used in these calculations; both (15) and (16) are gauge covariant as well as manifestly Lorentz covariant.

4. Remarks

In the traditional analytic mechanics the Lagrangian density of the electromagnetic field is $\mathscr{L} = -(1/4)\, F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$, from which the Maxwell equation follows as an Euler-Lagrange equation. Nevertheless, the definition of the conjugate momentum $\pi_\nu = \partial\mathscr{L}/\partial\partial_0 A_\nu = -F^{0\nu}$ cause a problem, because it implies $\pi_0 = 0$. To circumvent the difficulty, one usually needs gauge fixing or Dirac's theory of constrained systems. Note that π_k $(k = 1, 2, 3)$ represents the electric field E^k, but π_0 does not represent a dynamical variable. This is why $\pi_0 = 0$ occurs in the traditional analytic mechanics.

In the covariant analytic mechanics, however, the electromagnetic field is represented as a 1-form and its conjugate momentum as a 2-form. In 4-dimensions, the former has 4 components and the latter 6 components; there are 10 canonical variables in all. Importantly, 6 components of the conjugate momentum are all dynamical (the electric and the magnetic fields) and there is no redundant degree of freedom. Meanwhile, the first canonical equation $dA = \partial H/\partial\pi$ has 6 components and the second equation $d\pi = \partial H/\partial A$ has 4 components; there are 10 canonical equations in all. This is why neither gauge fixing nor Dirac's theory is unnecessary in the covariant analytic mechanics. The same explanation applies to gauge theories.

Part II
Gravity

5. Notation

Let M be a 4-dimensional pseudo-Riemannian manifold with a metric g of which signature is $(+---)$, and let (θ^α) denote an orthonormal frame, a set of tensor-valued 1-forms, on an open subset $U \subset M$. Then, we have $g = \mathring{g}_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$ with $\mathring{g}_{\alpha\beta} = \mathrm{diag}(+---)$. Throughout Part II, we put $\eta = *1$, $\eta^\alpha = *\theta^\alpha$, $\eta^{\alpha\beta} = *(\theta^\alpha \wedge \theta^\beta)$, $\eta^{\alpha\beta\gamma} = *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma)$, and $\eta^{\alpha\beta\gamma\delta} = *(\theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta)$. Here $*$ denotes the Hodge operator. We assume the Levi-Cività connection on M and denote the connection 1-form on U as $\omega^\alpha{}_\beta$. All indices are raised and lowered with $\mathring{g}_{\alpha\beta}$ or its inverse $\mathring{g}^{\alpha\beta}$. Then, an identity $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$ holds. The curvature 2-form $\Omega^\alpha{}_\beta$ is given as $\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$. Expanding the curvature form as $\Omega^\mu{}_\nu = (1/2) R^\mu{}_{\nu\alpha\beta}\theta^\alpha \wedge \theta^\beta$, we define $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and $R = R^\mu{}_\mu$. The latter coincides with the scalar curvature.

6. Lagrange and Hamilton Forms of Gravity

We define the Lagrange 4-form of the gravitational field as

$$L = \mathscr{L}_G + \mathscr{L}_{\mathrm{mat}}, \tag{17}$$

where $\mathscr{L}_G$ and $\mathscr{L}_{\mathrm{mat}}$ are the Lagrange form of pure gravity and that of matter fields, respectively. We define the former as

$$\mathscr{L}_G = -\frac{1}{g'}\,N, \qquad N = \eta^{\rho}{}_{\alpha} \wedge \omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\rho}. \tag{18}$$

Here, $g' = 16\pi G/c^3$ represents the gravitational constant. It should be noted that there is a relation such as $*R = \eta_{\alpha\beta} \wedge \Omega^{\alpha\beta} = N + d(\eta_{\alpha\beta} \wedge \omega^{\alpha\beta})$. Under a variation $\delta\theta^{\alpha}$ of the frame field, N becomes

$$\delta N = \delta\theta^{\gamma} \wedge \{\eta_{\alpha\beta\gamma} \wedge \Omega^{\alpha\beta} + d(\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta})\} - \delta d\theta^{\gamma} \wedge (\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}).$$

Under the assumption of

$$\delta\mathscr{L}_{\mathrm{mat}} = -\delta\theta^{\gamma} \wedge *T_{\gamma}, \tag{19}$$

the variation of the Lagrange form (17) results in

$$\delta L = -\frac{1}{g'}\,\delta\theta^{\gamma} \wedge \left\{\eta_{\alpha\beta\gamma} \wedge \Omega^{\alpha\beta} + d(\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}) + g'{*T_{\gamma}}\right\} + \frac{1}{g'}\,\delta d\theta^{\gamma} \wedge \left(\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}\right).$$

Hence we obtain

$$\frac{\partial L}{\partial \theta^{\gamma}} = -\frac{1}{g'}\left\{\eta_{\alpha\beta\gamma} \wedge \Omega^{\alpha\beta} + d(\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}) + g'{*T_{\gamma}}\right\},$$
$$\frac{\partial L}{\partial d\theta^{\gamma}} = \frac{1}{g'}\,\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}.$$

As is discussed in Part I, the Euler-Lagrange equation for the 1-form θ^{γ} is $\partial L/\partial\theta^{\gamma} + d(\partial L/\partial d\theta^{\gamma}) = 0$ and in the present case it appears as

$$-\eta_{\alpha\beta\gamma} \wedge \Omega^{\alpha\beta} = g'{*T_{\gamma}}. \tag{20}$$

This leads to Einstein's equation of gravity $R^{\alpha}{}_{\beta} - (1/2)\delta^{\alpha}{}_{\beta}R = (8\pi G/c^3)T^{\alpha}{}_{\beta}$ if we expand tensor-valued 1-form T_{α} as $T_{\alpha} = T_{\alpha\beta}\theta^{\beta}$.

We now define the momentum 2-form π_{γ} and the Hamilton 4-form H as

$$\pi_{\gamma} = \frac{\partial L}{\partial d\theta^{\gamma}} = \frac{1}{g'}\,\eta_{\alpha\beta\gamma} \wedge \omega^{\alpha\beta}, \tag{21}$$

$$H = d\theta^{\gamma} \wedge \pi_{\gamma} - L = \mathscr{H}_G - \mathscr{L}_{\mathrm{mat}}, \tag{22}$$

where $\mathscr{H}_G$ is related to N, defined in (18), as

$$\mathscr{H}_G = -\frac{1}{g'}\, N. \tag{23}$$

$\mathscr{H}_G$ is the Hamilton form of pure gravity if we solve (21) with respect to $d\theta^\alpha$ to represent $\omega^\alpha{}_\beta$ in N with π_α and θ^α. However, this is not a trivial task. We consider the issue in the next section.

7. Solving Momentum Definition Conversely

Let us solve (21) with respect to $d\theta^\alpha$ to represent $\omega^\alpha{}_\beta$ with π_α and θ^α. To begin with, we expand the connection form as

$$\omega^{\mu\nu} = \Gamma^\mu{}_\gamma{}^\nu\, \theta^\gamma \tag{24}$$

with $\Gamma^\mu{}_\gamma{}^\nu \in \mathcal{F}(U)$, and substitute it together with $\eta_{\mu\nu\alpha} = \eta_{\mu\nu\alpha\beta}\, \theta^\beta$ into (21). Then, we obtain $\pi_\alpha = (1/2)\pi_{\alpha\beta\gamma}\, \theta^\beta \wedge \theta^\gamma$ with

$$\pi_{\alpha\beta\gamma} = \frac{1}{g'}\big(\eta_{\mu\nu\alpha\beta}\, \Gamma^\mu{}_\gamma{}^\nu - \eta_{\mu\nu\alpha\gamma}\, \Gamma^\mu{}_\beta{}^\nu\big). \tag{25}$$

Symmetries $\Gamma^\mu{}_\gamma{}^\nu = -\Gamma^\nu{}_\gamma{}^\mu$ and $\pi_{\alpha\beta\gamma} = -\pi_{\alpha\gamma\beta}$ should be noted. We can solve (25) with respect to $\Gamma^\rho{}_\beta{}^\sigma$ as $\Gamma^\rho{}_\beta{}^\sigma = -(g'/8)\, \eta^{\rho\sigma\gamma\alpha}(\pi_{\alpha\beta\gamma} + \pi_{\beta\alpha\gamma} + \pi_{\gamma\alpha\beta})$, which can be written as

$$\Gamma^\rho{}_\beta{}^\sigma = \frac{g'}{8}\, \eta^{\rho\sigma\gamma\alpha} *p_{\alpha\beta\gamma} \tag{26}$$

with

$$p_{\alpha\beta\gamma} = \pi_\alpha \wedge \eta_{\beta\gamma} + \pi_\beta \wedge \eta_{\alpha\gamma} + \pi_\gamma \wedge \eta_{\alpha\beta} \tag{27}$$

since

$$*p_{\alpha\beta\gamma} = -(\pi_{\alpha\beta\gamma} + \pi_{\beta\alpha\gamma} + \pi_{\gamma\alpha\beta}). \tag{28}$$

Substituting (26) into (24), we obtain

$$\omega^{\rho\sigma} = \frac{g'}{8}\, \eta^{\rho\sigma\gamma\alpha} *p_{\alpha\beta\gamma}\, \theta^\beta. \tag{29}$$

This is equivalent to $d\theta^\rho = (g'/8)\eta^{\rho\sigma\gamma\alpha} *p_{\alpha\beta\gamma}\, \theta_\sigma \wedge \theta^\beta$ owing to the first structure equation $d\theta^\rho = -\omega^{\rho\sigma} \wedge \theta_\sigma$. Combining above equations leads to

$$\pi_\alpha = -\frac{1}{2} *p_{\alpha\mu\nu}\, \theta^\mu \wedge \theta^\nu. \tag{30}$$

8. Variation of Hamilton Form

8.1 Hamilton Form of Gravity

A variation of the Hamilton form (22) reduces to that of N. Hence, we consider a variation of N in what follows. In order to achieve this goal, we start the consideration from

$$N = \frac{g'^2}{64}\, \eta^{\rho\sigma\gamma\alpha} \eta_\sigma{}^{\tau\nu\lambda} *p_{\alpha\beta\gamma} *p_{\lambda\mu\nu}\, \theta^\beta \wedge \theta^\mu \wedge \eta_{\tau\rho},$$

which is obtained by (18) and (29). With identities $\theta^\beta \wedge \theta^\mu \wedge \eta_{\tau\rho} = \left(\delta^\mu_\rho\, \delta^\beta_\tau - \delta^\mu_\tau\, \delta^\beta_\rho\right) \eta$ and $\eta^{\rho\sigma\gamma\alpha} \eta_\sigma{}^{\tau\nu\lambda} = \mathring{g}^{\tau\rho} \mathring{g}^{\nu\gamma} \mathring{g}^{\lambda\alpha} + \mathring{g}^{\tau\gamma} \mathring{g}^{\nu\alpha} \mathring{g}^{\lambda\rho} + \mathring{g}^{\tau\alpha} \mathring{g}^{\nu\rho} \mathring{g}^{\lambda\gamma} - \mathring{g}^{\tau\gamma} \mathring{g}^{\nu\rho} \mathring{g}^{\lambda\alpha} - \mathring{g}^{\tau\alpha} \mathring{g}^{\nu\gamma} \mathring{g}^{\lambda\rho} - \mathring{g}^{\tau\rho} \mathring{g}^{\nu\alpha} \mathring{g}^{\lambda\gamma}$, it yields

$$N = n^{\alpha\beta\gamma\lambda\mu\nu} *p_{\alpha\beta\gamma} *p_{\lambda\mu\nu}\, \eta. \tag{31}$$

Here

$$\begin{aligned} n^{\alpha\beta\gamma\lambda\mu\nu} = \frac{g'^2}{64} \big(& \mathring{g}^{\beta\gamma} \mathring{g}^{\nu\alpha} \mathring{g}^{\lambda\mu} + \mathring{g}^{\beta\alpha} \mathring{g}^{\nu\mu} \mathring{g}^{\lambda\gamma} + \mathring{g}^{\mu\gamma} \mathring{g}^{\nu\beta} \mathring{g}^{\lambda\alpha} + \mathring{g}^{\mu\alpha} \mathring{g}^{\nu\gamma} \mathring{g}^{\lambda\beta} \\ & - \mathring{g}^{\beta\gamma} \mathring{g}^{\nu\mu} \mathring{g}^{\lambda\alpha} - \mathring{g}^{\beta\alpha} \mathring{g}^{\nu\gamma} \mathring{g}^{\lambda\mu} - \mathring{g}^{\mu\gamma} \mathring{g}^{\nu\alpha} \mathring{g}^{\lambda\beta} - \mathring{g}^{\mu\alpha} \mathring{g}^{\nu\beta} \mathring{g}^{\lambda\gamma} \big), \end{aligned} \tag{32}$$

of which symmetries are

$$n^{\alpha\beta\gamma\lambda\mu\nu} = -n^{\gamma\beta\alpha\lambda\mu\nu}, \qquad n^{\alpha\beta\gamma\lambda\mu\nu} = -n^{\alpha\beta\gamma\nu\mu\lambda}, \qquad n^{\alpha\beta\gamma\lambda\mu\nu} = n^{\lambda\mu\nu\alpha\beta\gamma}. \tag{33}$$

8.2 Preparation for Variation

When we calculate a variation of N, we meet $\delta{*p_{\alpha\beta\gamma}}$. We now turn our attention to it. For simplicity, we put $\tau_{\alpha\beta\gamma} = \pi_{\alpha\beta\gamma} + \pi_{\beta\alpha\gamma} + \pi_{\gamma\alpha\beta}$. Then, (28) appears as

$$*p_{\alpha\beta\gamma} = -\tau_{\alpha\beta\gamma}. \tag{34}$$

Using this equation, for an arbitrary 4-form ζ we have

$$(\delta{*p_{\alpha\beta\gamma}})\, \zeta = -\delta\tau_{\alpha\beta\gamma} \cdot \zeta = \delta\tau_{\alpha\beta\gamma} \cdot \zeta{*\eta} = \delta\tau_{\alpha\beta\gamma} \cdot \eta{*\zeta} = \{\delta(\tau_{\alpha\beta\gamma}\, \eta) - \tau_{\alpha\beta\gamma}\, \delta\eta\} *\zeta.$$

Here, we have used identities $*\eta = -1$ and $\zeta{*\eta} = \zeta \wedge *\eta = \eta \wedge *\zeta = \eta{*\zeta}$. Finally, noticing (34) again, which implies $p_{\alpha\beta\gamma} = \tau_{\alpha\beta\gamma}\eta$ as well, we have

$$(\delta{*p_{\alpha\beta\gamma}})\, \zeta = (\delta p_{\alpha\beta\gamma} + \delta\eta{*p_{\alpha\beta\gamma}}) *\zeta. \tag{35}$$

8.3 Variation of Hamilton Form

We now calculate a variation of N. Under variations $\delta\theta^\rho$ and $\delta\pi_\rho$, (31) becomes

$$\delta N = n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}} \big\{2(\delta{*p_{\alpha\beta\gamma}})\,\eta + \delta\eta{*p_{\alpha\beta\gamma}}\big\},$$

where we have used symmetry (33). Then, we apply (35) to the first term in the braces so that

$$\delta N = -n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}}\,(2\delta p_{\alpha\beta\gamma} + \delta\eta{*p_{\alpha\beta\gamma}})\,. \tag{36}$$

On the other hand, $\delta\eta_{\beta\gamma} = \delta\theta^\rho \wedge \eta_{\beta\gamma\rho}$ yields $\delta(\pi_\alpha \wedge \eta_{\beta\gamma}) = \delta\pi_\alpha \wedge \eta_{\beta\gamma} + \delta\theta^\rho \wedge \pi_\alpha \wedge \eta_{\beta\gamma\rho}$. Hence, (27) leads to

$$\begin{aligned}\delta p_{\alpha\beta\gamma} = \delta\pi_\rho \wedge \big(\delta^\rho_\alpha\,\eta_{\beta\gamma} + \delta^\rho_\beta\,\eta_{\alpha\gamma} + \delta^\rho_\gamma\,\eta_{\alpha\beta}\big)\\ + \delta\theta^\rho \wedge (\pi_\alpha \wedge \eta_{\beta\gamma\rho} + \pi_\beta \wedge \eta_{\alpha\gamma\rho} + \pi_\gamma \wedge \eta_{\alpha\beta\rho})\,.\end{aligned}$$

Substituting this and $\delta\eta = \delta\theta^\rho \wedge \eta_\rho$ into (36) and considering symmetry (33), we have

$$\begin{aligned}\delta N = \delta\pi_\rho \wedge \Big[-2n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}}\,\big(2\delta^\rho_\alpha\,\eta_{\beta\gamma} + \delta^\rho_\beta\,\eta_{\alpha\gamma}\big)\Big]\\ + \delta\theta^\rho \wedge \Big[-n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}}\,(4\pi_\alpha \wedge \eta_{\beta\gamma\rho} + 2\pi_\beta \wedge \eta_{\alpha\gamma\rho} + \eta_\rho{*p_{\alpha\beta\gamma}})\Big].\end{aligned}$$

This implies

$$\begin{aligned}\frac{\partial N}{\partial \pi_\rho} &= -2n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}}\,\big(2\delta^\rho_\alpha\,\eta_{\beta\gamma} + \delta^\rho_\beta\,\eta_{\alpha\gamma}\big)\,,\\ \frac{\partial N}{\partial \theta^\rho} &= -n^{\alpha\beta\gamma\lambda\mu\nu} {*p_{\lambda\mu\nu}}\,(4\pi_\alpha \wedge \eta_{\beta\gamma\rho} + 2\pi_\beta \wedge \eta_{\alpha\gamma\rho} + \eta_\rho{*p_{\alpha\beta\gamma}})\,.\end{aligned} \tag{37}$$

The first equation in the above can also be written as

$$\frac{\partial N}{\partial \pi_\rho} = -\frac{g'^2}{8}\,\eta_\sigma{}^{\rho\nu\lambda} {*p_{\lambda\mu\nu}}\,\theta^\mu \wedge \theta^\sigma, \tag{38}$$

since under (32) we obtain

$$\begin{aligned}n^{\alpha\beta\gamma\lambda\mu\nu}\big(2\delta^\rho_\alpha\,\eta_{\beta\gamma} + \delta^\rho_\beta\,\eta_{\alpha\gamma}\big) &= \frac{g'^2}{16}\,\big(\mathring{g}^{\nu\mu}\eta^{\rho\lambda} - \mathring{g}^{\lambda\mu}\eta^{\rho\nu} + \mathring{g}^{\mu\rho}\eta^{\lambda\nu}\big)\\ &= \frac{g'^2}{16}\,\mathring{g}^{\alpha\rho}\mathring{g}^{\beta\lambda}\mathring{g}^{\gamma\nu}\,\theta^\mu \wedge \eta_{\alpha\beta\gamma}\\ &= \frac{g'^2}{16}\,\eta_\sigma{}^{\rho\nu\lambda}\,\theta^\mu \wedge \theta^\sigma.\end{aligned}$$

9. Simplification of Equation (37)

In the Hamiltonian formulation in which π_γ is independent of θ^γ, we cannot simplify (37) any more. However, if we are allowed to use (21), equation (37) can be put into a more concise form. We now explain this.

9.1 Rewrite of (37)

Using (30) and symmetry $p_{\alpha\beta\gamma} = -p_{\gamma\beta\alpha}$, we can prove

$$\begin{aligned}
&4\pi_\alpha \wedge \eta_{\beta\gamma\rho} + 2\pi_\beta \wedge \eta_{\alpha\gamma\rho} \\
&\qquad = (*p_{\beta\rho\gamma} - *p_{\beta\gamma\rho})\eta_\alpha + 2(*p_{\alpha\rho\gamma} - *p_{\alpha\gamma\rho})\eta_\beta \\
&\qquad + (2*p_{\alpha\beta\rho} - *p_{\alpha\rho\beta} + *p_{\beta\alpha\rho})\eta_\gamma + (*p_{\alpha\gamma\beta} - 2*p_{\alpha\beta\gamma} - *p_{\beta\alpha\gamma})\eta_\rho
\end{aligned}$$

so that (37) becomes

$$\frac{\partial N}{\partial\theta^\rho} = n^{\alpha\beta\gamma\lambda\mu\nu}*p_{\lambda\mu\nu}\big\{2(*p_{\beta\gamma\rho} + *p_{\gamma\beta\rho})\eta_\alpha + 2(*p_{\alpha\gamma\rho} - *p_{\alpha\rho\gamma})\eta_\beta + *p_{\alpha\beta\gamma}\,\eta_\rho\big\}.$$

Substituting (32) into this equation, we obtain

$$\begin{aligned}
\frac{\partial N}{\partial\theta^\rho} &= \frac{g'^2}{8}\,\eta_\alpha(-*p^{\mu}{}_{\mu}{}^{\nu}\,*p_{\nu\rho}{}^{\alpha} - *p^{\mu\nu\alpha}\,*p_{\mu\rho\nu}) \\
&+ \frac{g'^2}{8}\,\eta_\alpha(-*p^{\mu\nu\alpha}\,*p_{\nu\mu\rho} - *p^{\mu}{}_{\mu}{}^{\nu}\,*p^{\alpha}{}_{\nu\rho} + *p^{\mu}{}_{\mu}{}^{\alpha}\,*p^{\nu}{}_{\nu\rho}) \\
&+ \frac{g'^2}{16}\,\eta_\rho(-*p^{\mu}{}_{\mu}{}^{\nu}\,*p^{\lambda}{}_{\lambda\nu} + *p^{\mu\nu\lambda}\,*p_{\nu\mu\lambda}).
\end{aligned} \tag{39}$$

Under (21), this equation is equivalent to

$$\frac{\partial N}{\partial\theta^\rho} = -\eta_{\alpha\beta\rho} \wedge \Omega^{\alpha\beta} - d(\eta_{\alpha\beta\rho} \wedge \omega^{\alpha\beta}). \tag{40}$$

We give the proof of this statement in the next section.

9.2 Proof of (40)

In terms of $\Omega^{\alpha\beta} = d\omega^{\alpha\beta} + \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma\beta}$, (40) appears as

$$\frac{\partial N}{\partial\theta^\rho} = -\omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma\beta} \wedge \eta_{\alpha\beta\rho} - \omega^{\alpha\beta} \wedge d\eta_{\alpha\beta\rho}.$$

Because of $D\eta_{\alpha\beta\rho} = 0$, where D denotes the covariant exterior differentiation, there is an identity $d\eta_{\alpha\beta\rho} = \omega^{\gamma}{}_{\alpha} \wedge \eta_{\gamma\beta\rho} + \omega^{\gamma}{}_{\beta} \wedge \eta_{\alpha\gamma\rho} + \omega^{\gamma}{}_{\rho} \wedge \eta_{\alpha\beta\gamma}$ so that the above equation leads to $\partial N/\partial\theta^{\rho} = A_{\rho} + B_{\rho}$ where

$$A_{\rho} = \omega^{\alpha\beta} \wedge \omega_{\beta}{}^{\gamma} \wedge \eta_{\alpha\gamma\rho}, \qquad B_{\rho} = \omega^{\alpha\beta} \wedge \omega_{\rho}{}^{\gamma} \wedge \eta_{\alpha\beta\gamma}. \tag{41}$$

Substituting (29), which is equivalent to substituting (21), into A_{ρ} in (41), we obtain $A_{\rho} = A^{(1)}_{\rho} + A^{(2)}_{\rho}$ with

$$\begin{aligned}
A^{(1)}_{\rho} &= \frac{g'^2}{32}\, \eta^{\alpha\beta\nu\lambda}\, \eta_{\beta}{}^{\gamma\upsilon\sigma} \big(\delta^{\mu}_{\gamma}\, \delta^{\tau}_{\rho} - \delta^{\mu}_{\rho}\, \delta^{\tau}_{\gamma}\big) {*p}_{\lambda\mu\nu}\, {*p}_{\sigma\tau\upsilon}\, \eta_{\alpha}, \\
A^{(2)}_{\rho} &= \frac{g'^2}{64}\, \eta^{\alpha\beta\nu\lambda}\, \eta_{\beta}{}^{\gamma\upsilon\sigma} \big(\delta^{\mu}_{\alpha}\, \delta^{\tau}_{\gamma} - \delta^{\mu}_{\gamma}\, \delta^{\tau}_{\alpha}\big) {*p}_{\lambda\mu\nu}\, {*p}_{\sigma\tau\upsilon}\, \eta_{\rho},
\end{aligned}$$

and these equations can be rewritten as

$$\begin{aligned}
A^{(1)}_{\rho} &= \frac{g'^2}{8}\, \eta_{\alpha} \big({*p}^{\mu}{}_{\mu}{}^{\nu}\, {*p}_{\nu\rho}{}^{\alpha} + {*p}^{\mu\nu\alpha}\, {*p}_{\mu\rho\nu}\big), \\
A^{(2)}_{\rho} &= \frac{g'^2}{16}\, \eta_{\rho} \big({*p}^{\mu}{}_{\mu}{}^{\nu}\, {*p}^{\lambda}{}_{\lambda\nu} - {*p}^{\mu\nu\lambda}\, {*p}_{\nu\mu\lambda}\big).
\end{aligned}$$

Similarly, we can write B_{ρ} in (41) as $B_{\rho} = B^{(1)}_{\rho} + B^{(2)}_{\rho}$ with

$$\begin{aligned}
B^{(1)}_{\rho} &= \frac{g'^2}{32}\, \eta^{\alpha\beta\nu\lambda}\, \eta^{\xi\gamma\upsilon\sigma} g_{\xi\rho} \big(\delta^{\mu}_{\beta}\, \delta^{\tau}_{\gamma} - \delta^{\mu}_{\gamma}\, \delta^{\tau}_{\beta}\big) {*p}_{\lambda\mu\nu}\, {*p}_{\sigma\tau\upsilon}\, \eta_{\alpha}, \\
B^{(2)}_{\rho} &= \frac{g'^2}{64}\, \eta^{\alpha\beta\nu\lambda}\, \eta^{\xi\gamma\upsilon\sigma} g_{\xi\rho} \big(\delta^{\mu}_{\alpha}\, \delta^{\tau}_{\beta} - \delta^{\mu}_{\beta}\, \delta^{\tau}_{\alpha}\big) {*p}_{\lambda\mu\nu}\, {*p}_{\sigma\tau\upsilon}\, \eta_{\gamma},
\end{aligned}$$

and these equations can be rewritten as

$$\begin{aligned}
B^{(1)}_{\rho} = {} & \frac{g'^2}{8}\, \eta_{\alpha} \big(- {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}_{\nu\rho}{}^{\alpha} - {*p}^{\mu\nu\alpha} {*p}_{\mu\rho\nu}\big) \\
& + \frac{g'^2}{8}\, \eta_{\alpha} \big(- {*p}^{\mu\nu\alpha} {*p}_{\nu\mu\rho} + {*p}^{\mu}{}_{\mu}{}^{\alpha} {*p}^{\nu}{}_{\nu\rho} - {*p}^{\mu\alpha\nu} {*p}_{\mu\nu\rho} - {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}_{\nu}{}^{\alpha}{}_{\rho}\big) \\
& + \frac{g'^2}{8}\, \eta_{\rho} \big(- {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}^{\lambda}{}_{\lambda\nu} + {*p}^{\mu\nu\lambda} {*p}_{\nu\mu\lambda}\big), \\
B^{(2)}_{\rho} = {} & \frac{g'^2}{8}\, \eta_{\alpha} \big(- {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}_{\nu\rho}{}^{\alpha} - {*p}^{\mu\nu\alpha} {*p}_{\mu\rho\nu}\big) \\
& + \frac{g'^2}{8}\, \eta_{\alpha} \big(- {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}^{\alpha}{}_{\nu\rho} + {*p}^{\mu\alpha\nu} {*p}_{\mu\nu\rho} + {*p}^{\mu}{}_{\mu}{}^{\nu} {*p}_{\nu}{}^{\alpha}{}_{\rho}\big).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\frac{\partial N}{\partial\theta^\rho} &= A^{(1)}_\rho + A^{(2)}_\rho + B^{(1)}_\rho + B^{(2)}_\rho \\
&= \frac{g'^2}{8}\,\eta_\alpha\big(- *p^{\mu}{}_{\mu}{}^{\nu}\, *p_{\nu\rho}{}^{\alpha} - *p^{\mu\nu\alpha}\, *p_{\mu\rho\nu}\big) \\
&\quad + \frac{g'^2}{8}\,\eta_\alpha\big(- *p^{\mu\nu\alpha} *p_{\nu\mu\rho} - *p^{\mu}{}_{\mu}{}^{\nu} *p^{\alpha}{}_{\nu\rho} + *p^{\mu}{}_{\mu}{}^{\alpha} *p^{\nu}{}_{\nu\rho}\big) \\
&\quad + \frac{g'^2}{16}\,\eta_\rho\big(- *p^{\mu}{}_{\mu}{}^{\nu}\, *p^{\lambda}{}_{\lambda\nu} + *p^{\mu\nu\lambda}\, *p_{\nu\mu\lambda}\big),
\end{aligned}$$

which coincides with (39). This completes the proof.

10. Canonical Equations

As is discussed in Part I, the canonical equations for 1-form θ^ρ and its momentum π_ρ are

$$d\theta^\rho = \frac{\partial H}{\partial \pi_\rho}, \qquad d\pi_\rho = \frac{\partial H}{\partial \theta^\rho}, \tag{42}$$

where the Hamilton 4-form H is given in (22). Our final goal is to show that these are equivalent to the Euler-Lagrange equation (20).

To begin with, let us consider the first equation of (42). Since the Lagrange form of matter fields $\mathscr{L}_{\rm mat}$ is assumed to be independent of the momentum form of gravity π_ρ, the right-hand side of the first equation of (42) can be written as

$$\frac{\partial H}{\partial \pi_\rho} = \frac{\partial \mathscr{H}_G}{\partial \pi_\rho} = -\frac{1}{g'}\,\frac{\partial N}{\partial \pi_\rho} = \frac{g'}{8}\,\eta_\sigma{}^{\rho\nu\lambda} *p_{\lambda\mu\nu}\,\theta^\mu\wedge\theta^\sigma,$$

where we have used (23) and (38). As for the left-hand side, let us substitute the first structure equation $d\theta^\rho = -\omega^\rho{}_\sigma \wedge \theta^\sigma$. Then, the first equation of (42) becomes

$$-\,\omega^\rho{}_\sigma\wedge\theta^\sigma = \frac{g'}{8}\,\eta_\sigma{}^{\rho\nu\lambda} *p_{\lambda\mu\nu}\,\theta^\mu\wedge\theta^\sigma.$$

This leads to $\omega^{\rho\sigma} = (g'/8)\,\eta^{\rho\sigma\nu\lambda} *p_{\lambda\mu\nu}\,\theta^\mu$ which coincides with equation (29), and as is discussed in §7, it is equivalent to (21):

$$\pi_\rho = \frac{1}{g'}\,\eta_{\alpha\beta\rho}\wedge\omega^{\alpha\beta}. \tag{43}$$

Hence, we conclude that the first equation of (42) implies (43).

We now consider the second equation of (42). The right-hand side of the equation can be written as

$$\frac{\partial H}{\partial \theta^\rho} = \frac{\partial \mathscr{H}_G}{\partial \theta^\rho} - \frac{\partial \mathscr{L}_{\mathrm{mat}}}{\partial \theta^\rho} = -\frac{1}{g'}\frac{\partial N}{\partial \theta^\rho} + {*T_\rho}$$

where we have used equation (23) together with $\partial \mathscr{L}_{\mathrm{mat}}/\partial\theta^\rho = -{*T_\rho}$ which is obtained by (19). Here, let us use the first equation of (42), or equivalently (43). $\partial N/\partial\theta^\rho$ appeared above is originally (37), but we can use (40) as is explained in section 9. As for π_ρ in the left-hand side, we substitute (43). Then, after a small amount of manipulation, the second equation of (42) yields

$$-\eta_{\alpha\beta\rho} \wedge \Omega^{\alpha\beta} = g' {*T_\rho}.$$

This coincides with the Euler-Lagrange equation (20).

11. Summary

The covariant analytic mechanics proposed in Part I is an extension of the traditional analytic mechanics so as to treat space and time on an equal footing. Although it is physically equivalent to the traditional one, the covariant analytic mechanics has advantages that the canonical equations are gauge covariant as well as manifestly Lorentz covariant. We have shown that it works successfully for typical field theories, such as a scalar field, the electromagnetic field, and a non-abelian gauge field, as well as the Newtonian mechanics of a harmonic oscillator.

In Part II, we have shown that the theoretical framework of the covariant analytic mechanics successfully accommodates the general theory of relativity, too. In short, Einstein's renowned equation of gravity

$$R^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu R = \frac{8\pi G}{c^3} T^\mu{}_\nu$$

is mathematically equivalent to the covariant Euler-Lagrange equation

$$\frac{\partial L}{\partial \theta^\rho} + d\frac{\partial L}{\partial d\theta^\rho} = 0$$

or sets of the covariant canonical equations

$$d\theta^\rho = \frac{\partial H}{\partial \pi_\rho}, \qquad d\pi_\rho = \frac{\partial H}{\partial \theta^\rho}. \tag{44}$$

It is of importance that the covariant canonical equations (44) are obtained without fixing a gauge. In fact, they are manifestly diffeomorphism covariant.

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