Involute Curves Of Timelike Biharmonic Reeb Curves \((LCS)_3\) - Manifolds

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Abstract: In this paper, we study involute timelike biharmonic Reeb curves in \((LCS)_3\)-manifold. We characterize curvatures of timelike biharmonic Reeb curves in \((LCS)_3\)-manifold. We obtain parametric equation involute curves of the timelike biharmonic Reeb curves in \((LCS)_3\)-manifold.

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1. Introduction

A smooth map \(\phi : N \rightarrow M\) is said to be biharmonic if it is a critical point of the bienergy functional:

\[
E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,
\]

where \(\mathcal{T}(\phi) := \text{tr} \nabla^\phi d\phi\) is the tension field of \(\phi\).

The Euler–Lagrange equation of the bienergy is given by \(\mathcal{T}_2(\phi) = 0\). Here the section \(\mathcal{T}_2(\phi)\) is defined by

\[
\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr} R(\mathcal{T}(\phi), d\phi) d\phi,
\]

and called the bitension field of \(\phi\). Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study timelike biharmonic Reeb curves in \((LCS)_3\)-manifold. We characterize curvatures of timelike biharmonic Reeb curves in \((LCS)_3\)-manifold. Several

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interesting results on a \((LCS)_3\)-manifold are obtained in terms of timelike biharmonic Reeb curves. Finally, we obtain parametric equation of the timelike biharmonic Reeb curves in \((LCS)_3\)-manifold.

2. \((LCS)_3\)-Manifolds

**Definition 2.1.** In a Lorentzian manifold \((M, g)\) a vector field \(P\) defined by

\[ g(X, P) = A(X) \]

for any \(X \in \chi(M)\) is said to be a concircular vector field if

\[ (\nabla_X A)(Y) = \alpha \{ g(X, Y) + \omega(X) \alpha(Y) \}, \]

where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form [15].

Let \(M^3\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[ g(\xi, \xi) = -1. \quad (2) \]

Since \(\xi\) is a unit concircular vector field, it follows that there exists a non-zero 1-form \(\eta\) such that for

\[ g(X, \xi) = \eta(X), \quad (3) \]

the equation of the following form holds

\[ (\nabla_X \eta)(Y) = \alpha \{ g(X, Y) + \eta(X) \eta(Y) \} \quad (\alpha \neq 0) \]

for all vector fields \(X, Y\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies

\[ \nabla_X \alpha = (X \alpha) = d\alpha(X) = \rho \eta(X), \]

\(\rho\) being a certain scalar function given by \(\rho = -(\xi \alpha)\). If we put

\[ \phi X = \frac{1}{\alpha} \nabla_X \xi, \]

then from (2.3) and (2.5) we have

\[ \phi X = X + \eta(X) \xi, \]

from which it follows that \(\phi\) is a symmetric \((1)\) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \(M^3\) together with the unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and \((1)\) tensor field \(\phi\) is said to be a Lorentzian
concircular structure manifold (briefly \((LCS)_3\)-manifold) [16]. In a \((LCS)_3\)-manifold, the following relations hold [15]:

\[
\begin{align*}
\eta(\xi) &= -1, \\
\phi \xi &= 0, \\
\eta(\phi X) &= 0, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X) \eta(Y), \\
\eta(R(X, Y) Z) &= (\alpha^2 - \rho) [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)], \\
S(X, \xi) &= (n - 1)(\alpha^2 - \rho) \eta(X), \\
R(X, Y) \xi &= (\alpha^2 - \rho) [\eta(Y) X - \eta(X) Y], \\
(\nabla_X \phi)(Y) &= \alpha [g(Y, X) \xi + 2\eta(X) \eta(Y) \xi + \eta(Y) X],
\end{align*}
\]

for all vector fields \(X, Y, Z\), where \(R, S\) denote respectively the curvature tensor and the Ricci tensor of the manifold.

3. Biharmonic Reeb Curves in the \((LCS)_3\)-Manifold

Let \(\gamma\) be a timelike curve on the \((LCS)_3\)-manifold parametrized by arc length. Let \(\{T, N, B\}\) be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along \(\gamma\) defined as follows:

- \(T\) is the unit vector field \(\gamma'\) tangent to \(\gamma\),
- \(N\) is the unit vector field in the direction of \(\nabla_T T\) (normal to \(\gamma\)), and
- \(B\) is chosen so that \(\{T, N, B\}\) is a positively oriented orthonormal basis.

Then, we have the following Frenet formulas:

\[
\begin{align*}
\nabla_T T &= \kappa N, \\
\nabla_T N &= \kappa T + \tau B, \\
\nabla_T B &= -\tau N,
\end{align*}
\]

where \(\kappa\) is the curvature of \(\gamma\) and \(\tau\) its torsion and

\[
\begin{align*}
g(T, T) &= -1, \quad g(N, N) = 1, \quad g(B, B) = 1, \\
g(T, N) = g(T, B) = g(N, B) = 0.
\end{align*}
\]

**Theorem 3.1.** [13] Let \((M, \phi, \xi, \eta, g)\) be a 3-dimensional \((LCS)_3\)-manifold and unit vector field \(X\) orthogonal to the Reeb vector field \(\xi\). Then,

\[
\begin{align*}
R(\xi, X) \xi &= (\alpha^2 - \rho) X, \\
R(X, \xi) X &= -(\alpha^2 - \rho) \xi.
\end{align*}
\]

**Theorem 3.2.** [13] \(\gamma\) is a timelike biharmonic Reeb curve which are either tangent or normal to the Reeb vector field in \((LCS)_3\)-manifold then

\[
\begin{align*}
\kappa &= \text{constant} \neq 0, \\
\kappa^2 - \tau^2 &= \alpha^2 - \rho, \\
\tau &= \text{constant}.
\end{align*}
\]
Proof. Using (12) and Frenet formulas (10), we have (16).

Corollary 3.3. [13] If \( \gamma \) is a timelike biharmonic Reeb curve which orthogonal to the Reeb vector field \( \xi \) in \((LCS)_3\)-manifold, then \( \gamma \) is a helix.

We consider the three-dimensional \((LCS)_3\)-manifold

\[
\mathbb{M} = \left\{ (x, y, z) \in \mathbb{R}^3 : x \neq \pm \sqrt{2}z^2, \ x \neq 0, \ z \neq 0 \right\},
\]

where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\mathbb{M} = \{ (x, y, z) \in \mathbb{R}^3 : x \neq \pm \sqrt{2}z^2, \ x \neq 0, \ z \neq 0 \}\).

Let \(\mathbb{M}\) be the Riemannian metric defined by

\[
g(e_1, e_1) = g(e_2, e_2) = 1, \ g(e_3, e_3) = -1,
\]

\[
g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.
\]

Let \(\eta\) be the 1-form defined by

\[
\eta(U) = g(U, e_3) \text{ for any } U \in \chi(\mathbb{M}).
\]

Let \(\phi\) be the (1) tensor field defined by

\[
\phi(e_1) = e_1, \ \phi(e_2) = e_2, \ \phi(e_3) = 0.
\]

Then using the linearity of \(\phi\) and \(g\) we have

\[
\eta(e_3) = -1,
\]

\[
\phi^2(U) = U - \eta(U)e_3,
\]

\[
g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W),
\]

for any \(U, W \in \chi(\mathbb{M})\).

Let \(\nabla\) be the Levi-Civita connection with respect to \(g\). Then, we have

\[
[e_1, e_2] = \frac{z}{x} e_2, \ [e_1, e_3] = -\frac{1}{z} e_1, \ [e_2, e_3] = -\frac{1}{z} e_2.
\]

Taking \(e_3 = \xi\) and using the Koszul’s formula, we obtain

\[
\nabla_{e_1} e_1 = -\frac{1}{z} e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\frac{1}{z} e_1,
\]

\[
\nabla_{e_2} e_1 = -\frac{z}{x} e_2, \quad \nabla_{e_2} e_2 = -e_1 - \frac{1}{z} e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{z} e_2,
\]

\[
\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\]

Moreover we put

\[
R_{ijk} = R(e_i, e_j)e_k.
\]
where the indices $i, j, k$ take the values 1, 2 and 3.

$$R_{122} = -[2(\frac{z}{x})^2 - \frac{1}{z^2}]e_1, \quad R_{133} = -\frac{2}{z^2}e_1, \quad R_{233} = -\frac{2}{z^2}e_2.$$  

**Theorem 3.4.** [13] Let $\gamma : I \rightarrow M$ be a unit speed timelike biharmonic Reeb curve which orthogonal to the Reeb vector field $\xi$ in $(LCS)_3$ manifold $M$. Then, the parametric equations of $\gamma$ are

$$x(s) = \frac{\sinh \varphi}{\varphi^2} \sqrt{\sinh^2 \varphi - 1 \cos(\varphi s + \sigma)}$$

$$-\frac{1}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1 \sin(\varphi s + \sigma)} + c_2,$$

$$y(s) = \frac{1}{12 \varphi^4} (\sinh^2 \varphi - 1) \frac{1}{(2 \varphi^3 s(3c_1^2 + 3 \sinh \varphi c_1 s + \sinh^2 \varphi s^2))}$$

$$-6 \varphi \sinh \varphi (\sinh \varphi s + c_1) \cos[2(\varphi s + \sigma)]$$

$$-3(\varphi^2 c_1^2 + 2 \varphi^2 \sinh \varphi c_1 s + \sinh^2 \varphi (-1 + \varphi^2 s^2)) \sin[2(\varphi s + \sigma)])$$

$$+ c_2 \sinh \varphi$$

$$-\frac{c_2}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1 \cos(\varphi s + \sigma)} + c_3,$$

$$z(s) = \sinh \varphi s + c_1,$$

where $\sigma, c_1, c_2, c_3$ are constants of integration.

Using Mathematica in above theorem, we have

4. **Involute Curves of Biharmonic Reeb Curves in the(LCS)$_3$-Manifold**

**Definition 4.1.** Let unit speed timelike curve $\gamma : I \rightarrow M$ and the curve $\Theta : I \rightarrow M$ be given. For $\forall s \in I$, then the curve $\Theta$ is called the involute of the curve $\gamma$, if the tangent
at the point $\gamma(s)$ to the curve $\gamma$ passes through the tangent at the point $\Theta(s)$ to the curve $\Theta$ and
\begin{equation}
g(T^*(s), T(s)) = 0.
\end{equation}

Let the Frenet-Serret frames of the curves $\gamma$ and $\zeta$ be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively.

**Theorem 4.2.** Let $\gamma : I \rightarrow \mathbb{M}$ be a unit speed timelike biharmonic Reeb curve which are either tangent or normal to the Reeb vector field in $(LCS)_3$- manifold $\mathbb{M}$, $\Theta$ its involute curve. Then, the parametric equations of $\Theta$ are
\begin{equation}
x(s) = \frac{\sinh \varphi}{\varphi^2} \sqrt{\sinh^2 \varphi - 1} \cos(\varphi s + \sigma) \\
+ \frac{1}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \sin(\varphi s + \sigma) \\
+ (C - s) (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \cos(\varphi s + \sigma) + c_2,
\end{equation}
\begin{equation}
y(s) = \frac{1}{12\varphi^4} (\sinh^2 \varphi - 1) (2\varphi^3 s(3c_1^2 + 3 \sinh \varphi c_1 s + \sinh^2 \varphi s^2)) \\
- 6\varphi \sinh \varphi (\sinh \varphi s + c_1) \cos[2(\varphi s + \sigma)] \\
- 3(\varphi^2 c_1^2 + 2\varphi^2 \sinh \varphi c_1 s + \sinh^2 \varphi(-1 + \varphi^2 s^2)) \sin[2(\varphi s + \sigma)]) \\
- \frac{c_2 \sinh \varphi}{\varphi^2} \sqrt{\sinh^2 \varphi - 1} \sin(\varphi s + \sigma) \\
- \frac{c_2}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \cos(\varphi s + \sigma) \\
+ (C - s) (\sinh \varphi s + c_1) \left( \frac{\sinh \varphi}{\varphi^2} \right) \sqrt{\sinh^2 \varphi - 1} \cos(\varphi s + \sigma) \\
+ \frac{1}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \sin(\varphi s + \sigma) \\
+ c_2 \sqrt{\sinh^2 \varphi - 1} \sin(\varphi s + \sigma) + c_3,
\end{equation}
\begin{equation}
z(s) = C \sinh \varphi + c_1,
\end{equation}

where $\sigma, c_1, c_2, c_3$ are constants of integration.

**Proof.** We assume that $\gamma : I \rightarrow \text{Heis}^3$ be a unit speed timelike biharmonic curve and $\Theta$ its involute curve on Heis$^3$. We find that the parametric equations of $\Theta$.

The involute curve of timelike biharmonic curve may be given as
\begin{equation}
\Theta(s) = \gamma(s) + u(s)T(s).
\end{equation}

From (4.3), then we have
\begin{equation}
\Theta'(s) = (1 + u'(s))T(s) + u(s)\kappa(s)N(s).
\end{equation}
Since the curve $\Theta$ is involute of the curve $\gamma$, $g\left(T^*(s), T(s)\right) = 0$. Then, we get

$$1 + u'(s) = 0 \text{ or } u(s) = C - s,$$

(22)

where $C$ is constant of integration.

Substituting (22) into (20), we get

$$\Theta(s) = \gamma(s) + (C - s) T(s).$$

(23)

On the other hand, (19) and (23), imply

$$T = \sqrt{\sinh^2 \varphi - 1} \cos (\varphi s + \sigma) e_1 + \sqrt{\sinh^2 \varphi - 1} \sin (\varphi s + \sigma) e_2 + \sinh \varphi e_3.$$

(24)

Therefore, from (23) and (24) we have

$$T = ((\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \cos (\varphi s + \sigma) \varphi^2

$$

$$+ \frac{1}{\varphi} (\sinh \varphi s + c_1) \sqrt{\sinh^2 \varphi - 1} \sin (\varphi s + \sigma) \varphi^2

$$

$$+ c_2) \sqrt{\sinh^2 \varphi - 1},$$

(25)

If we substitute (17) and (25) into (23), we have (19). This concludes the proof of Theorem.

We show that $\gamma$ and $\Theta$ in terms of Mathematica as follows:

![Graph of mathematical function](image)

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References


