Bi-parameter Semigroups of linear operators

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Abstract: Let $X$ be a Banach space. We define the concept of a bi-parameter semigroup on $X$ and its first and second generators. We also study bi-parameter semigroups on Banach algebras. A relation between uniformly continuous bi-parameter semigroups and $\sigma$-derivations is also established. It is proved that if $\{\alpha_{t,s}\}_{t,s \geq 0}$ is a uniformly continuous bi-parameter semigroup on a Banach algebra $X$, whose first and second generators are $d$ and $\sigma$, respectively, and if $d$ is also a $\sigma$-derivation then $d^n(ab) = (d + \sigma)^n(a) \ast (d + \sigma)^n(b)$ and $\alpha_{t,0}(ab) = \alpha_{t,1}(a) \ast \alpha_{t,1}(b)$ for all $a, b \in X$.

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1. introduction

Let $\mathcal{X}$ be a Banach space and let $\mathcal{L}(X')$ denote the Banach space of all bounded linear operators on $X$. A family $\{\alpha_t\}_{t \geq 0}$ in $\mathcal{B}(X)$ is called a uniformly (resp. strongly) continuous one-parameter semigroup on $X$, if

(i) $\alpha_0$ is the identity mapping $I$ on $X$;
(ii) $\alpha_{t+t'} = \alpha_t \alpha_{t'}$ for all $t, t' \in \mathbb{R}^+$;
(iii) $\lim_{t \downarrow 0} \alpha_t = I$ uniformly (resp. strongly) on $X$.

Namely, $\alpha$ is a representation of the semigroup $(\mathbb{R}^+, +)$ into $\mathcal{B}(X)$ which is continuous with respect to the uniform (resp. strong) operator topology on $\mathcal{B}(X)$. When $\{\alpha_t\}_{t \geq 0}$ is

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a one-parameter semigroup on \(\mathcal{X}\), the infinitesimal generator \(\delta\) of \(\alpha\) is defined by

\[
\delta(x) = \lim_{t \downarrow 0} \frac{1}{t} (\alpha_t(x) - x),
\]

whenever the limit exists and the domain \(D(\delta)\) of \(\delta\) is the set of all \(x \in \mathcal{X}\) for which this limit exists. If \(\{\alpha_t\}_{t \geq 0}\) is strongly continuous then \(D(\delta)\) is a dense linear subspace of \(\mathcal{X}\) and \(\delta\) is a closed linear operator on this domain and if the semigroup \(\{\alpha_t\}_{t \geq 0}\) is uniformly continuous, \(\delta\) is an everywhere defined bounded linear operator on \(\mathcal{X}\), see [12] for details. For example, let \(\mathcal{X}\) be the Banach space (algebra) of all bounded uniformly continuous functions on \(\mathbb{R}\) with the supremum norm. For each \(t \in \mathbb{R}^+\), consider the linear mapping on \(\mathcal{X}\) defined by \((\alpha_t(f))(h) = f(t + h), (f \in \mathcal{X})\). It is easy to see that the family \(\{\alpha_t\}_{t \in \mathbb{R}^+}\) is a one-parameter semigroup satisfying \(\|\alpha_t\| \leq 1\) and \((\delta(f))(h) = f'(h)\) if \(f \in D(\delta)\). Obviously \(D(\delta)\) is the linear subspace of \(\mathcal{X}\) consisting of those \(f\) in \(\mathcal{X}\) which are differentiable with \(f' \in \mathcal{X}\). This example shows that the infinitesimal generator of this one-parameter semigroup, can be obtained by taking derivative when it exists.

It is easy to see that if \(\delta\) is a bounded linear operator on a Banach space \(\mathcal{X}\), then \(\alpha_t = \exp(t\delta)\) \((t \geq 0)\) is a uniformly and hence strongly continuous one-parameter semigroup of operators on \(\mathcal{X}\). In fact every uniformly continuous one-parameter semigroup is necessarily of this form for some bounded linear operator \(\delta\) (see [12], Theorems I.2, I.3 and Corollary I.4).

If \(\{\alpha_t\}_{t \geq 0}\) is a uniformly continuous one-parameter semigroup of homomorphisms on a Banach algebra \(\mathcal{X}\), then its infinitesimal generator \(\delta\) satisfies the Leibniz’s rule \(\delta(xy) = \delta(x)y + x\delta(y)\) for all \(x, y \in \mathcal{X}\). Such a linear mapping is called a derivation. Also, if \(\delta\) is a bounded derivation on \(\mathcal{X}\) then \(\alpha_t = \exp(t\delta)\) \((t \geq 0)\) forms a uniformly continuous one-parameter semigroup of homomorphisms on \(\mathcal{X}\), see [12, Theorems 1.2, 1.3 and Corollary 1.4] and also [1, Proposition 18.7]. The theory of one-parameter semigroups on operator algebras and their infinitesimal generators have been largely motivated by models of quantum statistical mechanics. The reader is referred to [4, 5, 13] for more details.

Let \(\mathcal{X}\) be a Banach algebra and let \(\sigma\) be a linear mapping on \(\mathcal{X}\). A linear mapping \(d : \mathcal{X} \to \mathcal{X}\) is called a \(\sigma\)-derivation if it satisfies the generalized Leibniz rule \(d(xy) = d(x)\sigma(y) + \sigma(x)d(y)\) for all \(x, y \in \mathcal{X}\). For example, if \(\rho\) is a homomorphism and \(\sigma = \frac{\rho}{2}\) then \(\rho\) is a \(\sigma\)-derivation. Moreover, when \(\sigma\) is an automorphism we can consider \(\delta = d\sigma^{-1}\) and find out that \(\delta\) is an ordinary derivation. This shows that the theory of \(\sigma\)-derivations combines the two subjects of derivations and homomorphisms. \(\sigma\)-derivations are investigated by many physicists and mathematicians. Automatic continuity, innerness, approximately innerness and amenability are the most important subjects which are studied in the theory of derivations and \(\sigma\)-derivations, see [6, 7, 8, 9, 10, 11].

When \(\delta\) is a derivation on a Banach algebra \(\mathcal{X}\), using the parameter \(t\) we can consider \(\alpha_t = \exp(t\delta)\) and construct the one parameter semigroup \(\{\alpha_t\}_{t \geq 0}\) of homomorphisms on \(\mathcal{X}\). It seems that when we are dealing with a \(\sigma\)-derivation \(d\), we need to consider two parameters \(t\) and \(s\) corresponding to \(d\) and \(\sigma\), respectively. In what follows we define a uniformly (resp. strongly) bi-parameter semigroup of operators and its first and second
generators. We will show that each uniformly continuous bi-parameter semigroup of operators on a Banach space $X$ is of the form $\alpha_{t,s} = \exp(t(d + s\sigma))$ $(t, s \geq 0)$, where $d$ and $\sigma$ are bounded linear operators on $X$. We will also give a relation between uniformly continuous bi-parameter semigroups on Banach algebras and $\sigma$-derivations.

2. Bi-parameter Semigroups

We start with the definition of a bi-parameter semigroup.

**Definition 2.1.** Let $X$ be a Banach space. A family $\{\alpha_{t,s}\}_{t,s \geq 0}$ of bounded linear operators on $X$ is called a uniformly (resp. strongly) continuous bi-parameter semigroup if

(i) for each fixed $s \geq 0$, the family $\{\alpha_{t,s}\}_{t \geq 0}$ is a uniformly (resp. strongly) continuous one parameter semigroup with infinitesimal generator $\delta_s$;

(ii) for each $s \geq 0$, $D(\delta_s) = D(\delta_0)$;

(iii) for $s > 0$, the value

$$\frac{1}{s}(\lim_{t \downarrow 0} \frac{1}{t}(\alpha_{t,s}(x) - x) - \lim_{t \downarrow 0} \frac{1}{t}(\alpha_{t,0}(x) - x)) = \frac{1}{s}(\delta_s(x) - \delta_0(x))$$

is independent of $s$ for all $x \in D(\delta_0)$.

Take $d = \delta_0$ and $D = D(\delta_0)$. Note that for $x \in D$ and $s > 0$, $\sigma(x) := \frac{1}{s}(\delta_s(x) - \delta_0(x))$ is the average growth of $\delta_s$ in the interval $[0, s]$ at $x$, which by definition is independent of the choice of $s$. Obviously $\sigma$ is a linear mapping on $D$ and $\delta_s = d + s\sigma$. The operators $d$ and $\sigma$, defined on $D$, are said to be the first and second generators of the bi-parameter semigroup $\{\alpha_{t,s}\}_{t,s \geq 0}$, respectively. The ordered pair $(d, \sigma)$ is simply called the generator of $\{\alpha_{t,s}\}_{t,s \geq 0}$.

If $d, \sigma$ are bounded linear operators on $X$ then as in the case of one-parameter semigroups [12], we examine $\alpha_{t,s} = \exp(t(d + s\sigma)) = \exp(t\delta_s)$ and get the following result.

**Proposition 2.2.** If $\{\alpha_{t,s}\}_{t,s \geq 0}$ is a uniformly continuous bi-parameter semigroup, then its first and second generators are bounded. Conversely, if $d$ and $\sigma$ are two bounded linear operators on a Banach space $X$ then $\alpha_{t,s} = \exp(t(d + s\sigma))$ is a uniformly continuous bi-parameter semigroup whose generator is $(d, \sigma)$.

It is clear that the first and second generators of a uniformly continuous bi-parameter semigroup are unique. Also, if $d$ and $\sigma$ are bounded linear operators then $\alpha_{t,s} = \exp(t(d + s\sigma))$ is a uniformly continuous bi-parameter semigroup with generator $(d, \sigma)$.

Is this semigroup unique? The answer is affirmative as we see below.

**Proposition 2.3.** Let $\{\alpha_{t,s}\}_{t,s \geq 0}$ and $\{\beta_{t,s}\}_{t,s \geq 0}$ be two uniformly continuous bi-parameter semigroups with the same generator $(d, \sigma)$. Then $\alpha_{t,s} = \beta_{t,s}$, for every $t, s \geq 0$. 
Proof 2.4. Fix \( s \geq 0 \), then \( \{\alpha_{t,s}\}_{t \geq 0} \) and \( \{\beta_{t,s}\}_{t \geq 0} \) are one parameter semigroups with infinitesimal generator \( \delta_s \). So \( \alpha_{t,s} = \beta_{t,s} \) for all \( t \geq 0 \). Since \( s \) is arbitrary we have the result.

Corollary 2.5. Uniformly continuous bi-parameter semigroups are of the form \( \exp(t(d + s\sigma)) \) for bounded linear operators \( d \) and \( \sigma \).

3. \( \sigma \)-Derivations and Bi-parameter Semigroups

Let \( d, \sigma \) be linear operators on a linear space \( X \). We construct a family of linear mappings \( \{Q_{n,k}\} \) \( (n \in \mathbb{N}, \ 0 \leq k \leq 2^n - 1) \), called the binary family corresponding to \( (d, \sigma) \), as follows.

Write the positive integer \( k \) in base 2 with exactly \( n \) digits, and put the operator \( d \) in place of 1’s and \( \sigma \) in place of 0’s. For example, \( 7 = (111)_2 \), \( 11 = (01011)_2 \), \( Q_{3,7} = ddd = d^3 \) and \( Q_{5,11} = \sigma d \sigma d \sigma = \sigma d \sigma d^2 \) (cf. [9]).

The following lemma is stated and proved in [9, Lemma ...]. We give the proof, for the sake of convenience.

Lemma 3.1. Let \( n \in \mathbb{N} \) and let \( k \in \{0, ..., 2^n - 1\} \). Then

(i) \( dQ_{n,k} = Q_{n+1,2^n+k} \);
(ii) \( \sigma Q_{n,k} = Q_{n+1,k} \).

Proof 3.2. Suppose that \( k = (c_n \ldots c_2 c_1)_2 \) where \( c_j \in \{0,1\} \) for \( j = 1, ..., n \), be the representation of \( k \) in the base 2 with \( n \) digits. Then

(i) \( dQ_{n,k} = Q_{n+1,1c_n \ldots c_2 c_1}_2 = Q_{n+1,k+2^n} \);
(ii) \( \sigma Q_{n,k} = Q_{n+1,0c_n \ldots c_2 c_1}_2 = Q_{n+1,k} \).

Lemma 3.3. If \( n \in \mathbb{N} \) and \( k \in \{0, ..., 2^n - 1\} \). Then

\[ (d + \sigma)^n = \sum_{k=0}^{2^n-1} Q_{n,k}. \]

Proof 3.4. We prove the assertion by induction on \( n \). For \( n = 1 \) the result is clear.
Now suppose that it is true for $n$. By Lemma 3.1, we obtain

\[
(d + \sigma)^{n+1} = (d + \sigma)(d + \sigma)^n \\
= (d + \sigma)\left( \sum_{k=0}^{2^n-1} Q_{n,k} \right) \\
= \sum_{k=0}^{2^n-1} dQ_{n,k} + \sum_{k=0}^{2^n-1} \sigma Q_{n,k} \\
= \sum_{k=0}^{2^n-1} Q_{n+1,2^n+k} + \sum_{k=0}^{2^n-1} Q_{n+1,k} \\
= \sum_{k=0}^{2^n+1-1} Q_{n+1,k} + \sum_{k=0}^{2^n-1} Q_{n+1,k} \\
= \sum_{k=0}^{2^n+1-1} Q_{n+1,k}.
\]

**Definition 3.5.** Let $X$ be a Banach space and let $\{\alpha_{t,s}\}_{t,s \geq 0}$ be a uniformly continuous bi-parameter semigroup of bounded linear operators on $X$ with generator $(d, \sigma)$, that is $\alpha_{t,s} = \exp(t(d + s\sigma))$. Take $\delta_s = d + s\sigma$ $(s \geq 0)$. Take

\[
Y = \{ \sum_{n=0}^{\infty} r_n t^n \delta^n : r_n \in \mathbb{C}, t, s \geq 0, \text{ and the series is convergent in norm of } L(X) \},
\]

\[
H = \{ T(a) : T \in Y \text{ and } a \in X \}.
\]

Let $n, m$ be nonnegative integers and $r, w \in \mathbb{C}$. We define a mapping $\star : H \times H \to X$ as follows

\[
rt^n(d + s\sigma)^n(a) \star wt^m(d + s\sigma)^m(b) = \begin{cases} 
0 & n \neq m \text{ or } r \neq w \\
rt^n s^n \sum_{k=0}^{2^n-1} Q_{n,k}(a)Q_{n,2^n-1-k}(b) & n = m, r = w
\end{cases}
\]

and for $r_i, w_i \in \mathbb{C}$

\[
\sum_{i=0}^{\infty} r_i t^i(d + s\sigma)^i(a) \star \sum_{i=0}^{\infty} w_i t^i(d + s\sigma)^i(b) = \sum_{i=0}^{\infty} \left( r_i t^i(d + s\sigma)^i(a) \star w_i t^i(d + s\sigma)^i(b) \right)
\]

whenever the limit exists; otherwise we define

\[
\sum_{i=1}^{\infty} r_i t^i(d + s\sigma)^i(a) \star \sum_{i=1}^{\infty} w_i t^i(d + s\sigma)^i(b) = 0.
\]
In particular,

\[ \alpha_{t,s}(a) \star \alpha_{t,s}(b) = \sum_{n=0}^{\infty} \left( \frac{t^n(d + s\sigma)^n}{n!} \right)(a) \star \left( \frac{t^n(d + s\sigma)^n}{n!} \right)(b). \]  

(1)

Since \( d \) and \( \sigma \) are bounded operators, the series in (1) converges.

**Lemma 3.6.** Let \( \{\alpha_{t,s}\}_{t,s \geq 0} \) be a uniformly continuous bi-parameter semigroup with generator \( (d, \sigma) \). Then

\[ \alpha_{t,1}(a) \star \alpha_{t,1}(b) - ab = (\alpha_{t,1}(a) - a) \star (\alpha_{t,1}(b) - b). \]  

(2)

**Proof 3.7.** By definition of \( \star \), we have

\[ \alpha_{t,1}(a) \star \alpha_{t,1}(b) - ab = \sum_{n=0}^{\infty} \frac{t^n(d + \sigma)^n}{n!}(a) \star \frac{t^n(d + \sigma)^n}{n!}(b) - ab \]

\[ = ab + t(d + \sigma)(a) \star t(d + \sigma)(b) + \frac{t(d + \sigma)^2}{2!}(a) \star \frac{t(d + \sigma)^2}{2!}(b) + \cdots - ab \]

\[ = t(d + \sigma)(a) \star t(d + \sigma)(b) + \frac{t(d + \sigma)^2}{2!}(a) \star \frac{t(d + \sigma)^2}{2!}(b) + \cdots. \]

On the other hand

\[ (\alpha_{t,1}(a) - a) \star (\alpha_{t,1}(b) - b) = \sum_{n=1}^{\infty} \frac{t^n(d + \sigma)^n}{n!}(a) \star \sum_{n=1}^{\infty} \frac{t^n(d + \sigma)^n}{n!}(b) \]

\[ = t(d + \sigma)(a) \star t(d + \sigma)(b) + \frac{t(d + \sigma)^2}{2!}(a) \star \frac{t(d + \sigma)^2}{2!}(b) + \cdots. \]

Thus we have the equality in (2).

**Lemma 3.8.** Let \( \{\alpha_{t,s}\}_{t,s \geq 0} \) be a uniformly continuous bi-parameter semigroup with generator \( (d, \sigma) \). If \( \sigma = I \), the identity mapping, then

\[ \alpha_{t,1}(a) \star \alpha_{t,1}(b) = \alpha_{t,0}(a) \cdot \alpha_{t,0}(b). \]
Proof 3..9. We have

\[ \alpha_{t,1}(a) \star \alpha_{t,1}(b) = \exp^{t(d+I)}(a) \star \exp^{t(d+I)}(b) \]

\[ = \left( \sum_{n=1}^{\infty} \frac{t^n(d + I)^n(a)}{n!} \right) \star \left( \sum_{n=1}^{\infty} \frac{t^n(d + I)^n(b)}{n!} \right) \]

\[ = \sum_{n=1}^{\infty} \frac{t^n(d + I)^n(a)}{n!} \star \left( \sum_{n=1}^{\infty} \frac{t^n(d + I)^n(b)}{n!} \right) \]

\[ = \sum_{n=1}^{\infty} \frac{\left( \sum_{k=0}^{n} \binom{n}{k} d^k(a) \right)}{n!} \star \left( \sum_{n=1}^{\infty} \frac{\left( \sum_{k=0}^{n} \binom{n}{k} d^k(b) \right)}{n!} \right) \]

\[ = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{t^n}{n!} \binom{n}{k} d^k(a) d^{n-k}(b) \]

\[ = \sum_{n=1}^{\infty} \sum_{k=0}^{n} t^n \frac{d^k(a)}{k!} \frac{d^{n-k}(b)}{(n-k)!} \]

\[ = \left( \sum_{n=1}^{\infty} \frac{t^n}{n!} d^n(a) \right) \cdot \left( \sum_{n=1}^{\infty} \frac{t^n}{n!} d^n(b) \right) \]

\[ = \alpha_{t,0}(a) \cdot \alpha_{t,0}(b). \]

Taking idea from the relation between uniformly continuous one parameter semigroups and derivations, we now are ready to state a relation between uniformly continuous bi-parameter semigroups and \( \sigma \)-derivations.

Theorem 3..10. Let \( \{\alpha_{t,s}\}_{t,s \geq 0} \) be a uniformly continuous bi-parameter semigroup with generator \( (d, \sigma) \). If \( d \) is also a \( \sigma \)-derivation then

(i) \( d^n(ab) = (d + \sigma)^n(a) \star (d + \sigma)^n(b) \);

(ii) \( \alpha_{t,0}(ab) = \alpha_{t,1}(a) \star \alpha_{t,1}(b) \).

In particular, if \( \sigma = I \) and \( d \) is a derivation then

\[ \alpha_{t,0}(ab) = \alpha_{t,0}(a) \cdot \alpha_{t,0}(b), \]

i.e., \( \alpha_{t,0} \) is a homomorphism.

Proof 3..11. We prove (i) by induction. For \( n = 1 \) the result is obvious. Now suppose
it is true for \( n \). From Definition 3.5 and Lemmas 3.1, 3.3 we have

\[
d^{n+1}(ab) = d\left(d^n(ab)\right)
= d((d + \sigma)^n(a) \star (d + \sigma)^n(b))
= d\left(\sum_{k=0}^{2^n-1} Q_{n,k}(a)Q_{n,2^n-1-k}(b)\right)
= \sum_{k=0}^{2^n-1} (dQ_{n,k}(a)\sigma Q_{n,2^n-1-k}(b) + \sigma Q_{n,k}(a)dQ_{n,2^n-1-k}(b))
= \sum_{k=0}^{2^n-1} (Q_{n+1,k+2^n}(a)Q_{n+1,2^n-1-k}(b) + Q_{n+1,k}(a)Q_{n+1,2^n-1-k+2^n}(b))
= \sum_{k=0}^{2^n-1} (Q_{n+1,k+2^n}(a)Q_{n+1,2^n+1-1-(k+2^n)}(b)) + \sum_{k=0}^{2^n-1} (Q_{n+1,k}(a)Q_{n+1,2^n+1-1-k+2^n}(b))
= \sum_{k=0}^{2^n+1-1} Q_{n+1,k}(a)Q_{n+1,2^n+1-1-k}(b)
= \left(\sum_{k=0}^{2^n+1-1} Q_{n+1,k}(a)\right) \star \left(\sum_{k=0}^{2^n+1-1} Q_{n+1,k}(b)\right)
= (d + \sigma)^{n+1}(a) \star (d + \sigma)^{n+1}(b).
\]

The assertion \((ii)\) follows by \((i)\) and the definition of \(\star\).

**Theorem 3.12.** Let \(\{\alpha_{t,s}\}_{t,s \geq 0}\) be a uniformly continuous bi-parameter semigroup with generator \((d, \sigma)\). If

\[
\alpha_{t,0}(ab) = \alpha_{t,1}(a) \star \alpha_{t,1}(b),
\]

then \(d\) is a \(\sigma\)-derivation. In particular, if \(\sigma = I\) then \(d\) is a derivation.

**Proof 3.13.** By assumption and the definition of \(\star\) we have

\[
d(ab) = \lim_{t \to 0} \frac{\alpha_{t,0}(ab) - ab}{t}
= \lim_{t \to 0} \frac{\alpha_{t,1}(a) \star \alpha_{t,1}(b) - ab}{t}
= \lim_{t \to 0} \frac{\left(\alpha_{t,1}(a) - a\right) \star \left(\alpha_{t,1}(b) - b\right)}{t}
= \lim_{t \to 0} \frac{\alpha_{t,1}(a) - a}{t} \star \lim_{t \to 0} \frac{\alpha_{t,1}(b) - b}{t}
= (d(a) + \sigma(a)) \star (d(b) + \sigma(b))
= d(a)\sigma(b) + \sigma(a)d(b).
\]
References


