Primordial Gravity's Breath

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Abstract: In a recent paper the Laser Interferometer Gravitational-wave Observatory (LIGO) Scientific Collaboration (LSC) obtained an upper limit on the stochastic gravitational-wave background (SGWB) of cosmological origin by using the data from a two-year science run of the LIGO. Such an upper limit rules out some models of early Universe evolution, like the ones with relatively large equation-of-state parameter and the cosmic (super) string models with relatively small string tension arising from some String Theory's models. This was also an upper limit for the SGWB which is proposed by the Pre-Big-Bang Theory. Another upper bound on the SGWB which is proposed by the Standard Inflationary Model is well known and often updated by using the Wilkinson Microwave Anisotropy Probe (WMAP) data. By using a conformal treatment, which represents a variation of early works, we release a formula that directly connects the average amplitude of the SGWB with the Inflaton field in the Standard Inflationary Scenario of General Relativity and an external Inflaton field. Then, by joining this formula with the equation for the characteristic amplitude h_c for the SGWB, the upper bounds on the SGWB from the WMAP and LSC data will be translated in lower bounds on the Inflaton field. The results show that the value of the Inflaton field that arises from the WMAP bound on the SGWB is totally consistent with the famous *slow roll* condition on Inflation. On the other hand, the value of the Inflaton field that arises from the LSC bound on the SGWB could be not consistent with such a condition. In any case, the analysis in this paper shows that the detection of the SGWB will permit a direct measure of the value of the Inflaton field by giving an extraordinary precious and precise information about the early Universe's dynamics. In other words, the detection of the SGWB will permit *to auscultate the primordial gravity's breath*

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1.   Introduction

The scientific community aims in a first direct detection of gravitational waves (GWs) in next years (for the current status of GWs interferometers see [1]) confirming the indirect, Nobel Prize Winner, proof of Hulse and Taylor [2].

Detectors for GWs will be important for a better knowledge of the Universe and either to confirm or to rule out, in an ultimate way, the physical consistency of General Relativity, eventually becoming an observable endorsement of Extended Theories of Gravity[3].

It is well known that an important potential source of gravitational radiation is the relic SGWB [4]. The potential existence of such a relic SGWB arises from general assumptions that mix principles of classical gravity with principles of quantum field theory [5-7]. As the zero-point quantum oscillations, which produce the relic SGWB, are generated by strong variations of the gravitational field in the early Universe, the potential detection of this relic SGWB is the only way to learn about the evolution of the primordial Universe, up to the bounds of the Planck epoch and the initial singularity [4, 7]. In fact, this kind of information is inaccessible to standard astrophysical observations [4, 7, 8]. The importance of this relic signal in cosmological scenarios has been discussed in an elegant way in [8].

The inflationary scenario for the early Universe [9, 10], which is tuned in a good way with the WMAP data on the Cosmic Microwave Background Radiation (CMBR) (in particular exponential Inflation and spectral index ≈ 1 [11]), amplified the zero-point quantum oscillations and generated the relic SGWB [6, 7].

A recent paper, which has been written by the LSC [4], has shown an upper limit on the SGWB by using the data from a two-year science run of LIGO. Such an upper limit rules out some models of early Universe evolution, like the ones with relatively large equation-of-state parameter and the cosmic (super) string models with relatively small string tension arising from some string theory models. It results also an upper limit for the SGWB which is proposed by the Pre-Big-Bang Theory (see [4] for details).

Another well known upper bound on the SGWB arises from the Standard Inflationary Model. Such an upper bound is often updated by using the WMAP data [4, 12].

In this paper a formula that directly connects the average amplitude of the SGWB with the Inflaton field will be obtained by using a variation of a conformal treatment analysed in [13] and [14]. By using such a formula and the equation for the characteristic amplitude h_c for the SGWB [15], the upper bounds on the SGWB from the WMAP and LSC data will be translated in lower bounds on the Inflaton field.

Our results show that the value of the Inflaton field that arises from the WMAP bound on the SGWB is totally consistent with the famous *slow roll* condition on Inflation [9, 10], while the value of the Inflaton field that arises from the LSC bound on the SGWB could not be consistent with this condition.

The analysis in this paper shows that the detection of the SGWB will permit, ultimately, a direct measure of the value of the Inflaton field by giving an extraordinary

precious and precise information on the early Universe's dynamics. In other words, the detection of the SGWB will permit *to auscultate the primordial gravity's breath.*

2.   The spectrum and the conformal treatment

Considering a relic SGWB, it can be characterized by a dimensionless spectrum [4, 7, 8]. The more recent values for the spectrum that arises from the WMAP data can be found in refs. [4, 12]. In such papers it is (for a sake of simplicity, in this paper natural units are used, i.e. $8\pi G = 1$, $c = 1$ and $\hbar = 1$)

$$\Omega_{gw}(f) \equiv \frac{1}{\rho_c} \frac{d\rho_{gw}}{d\ln f} \leq 10^{-13} \tag{1}$$

where

$$\rho_c \equiv 3H_0^2 \tag{2}$$

is the (actual) critical density energy, ρ_c of the Universe, H_0 the actual value of the Hubble expansion rate and $d\rho_{gw}$ the energy density of relic GWs in the frequency range f to $f + df$. This is the upper bound on the SGWB that observations put on the Standard Inflationary Model, i.e exponential Inflation and spectral index ≈ 1.

An higher bound results from the LIGO Scientific Community data in ref. [4]:

$$\Omega_{gw} \leq 6.9 * 10^{-6}. \tag{3}$$

This bound is at 95% confidence in the frequency band $41.5 - 169.25 Hz$, (see [4] for details).

In this case, the value is an upper limit for the SGWB which arises from the Pre-Big-Bang Theory [4, 16]. It also rules out some models of early Universe evolution, like the ones with relatively large equation of state parameter and the cosmic (super) string models with relatively small string tension arising from some string theory models (see [4]and references within).

We will consider a variation of a computation in [13]. In such a paper a conformal treatment has been applied to Scalar Tensor Gravity. A similar computation was also performed in [14] in the framework of $f(R)$ Theories of Gravity.

However, Scalar Tensor Gravity and $f(R)$ Theories are only particular cases where an external scalar field works like Inflaton, other cases could be, for example, the Higgs potential and a self-interacting scalar field.

In this work we discuss the Standard Model's case, in which the scalar field (Inflaton) arises from field theory [9].

In the Standard Scenario Inflation can solve many of the initial value, or 'fine-tuning', problems of the hot Big Bang model [9]. The fundamental assumption is that there is some mechanism to bring about the negative pressure state needed for quasi-exponential growth of the scale factor [9]. Inflaton is the name given to a relic scalar field φ, since its origin does not have to originate with a specified particle theory [9]. The original hope was that φ would help to determine the correct particle physics models but current model

building does not necessarily require specific particle phenomenology [9]. This is actually an advantage for the Standard Inflationary Scenario, as it retains its power to solve the initial value problems, yet it could arise from any arbitrary source (i.e., any arbitrary Inflation) [9]. In field theory, we consider a Lagrangian density, as opposed to the usual Lagrangian from classical mechanics [9]. In fact, in field theory scalar fields are taken to be continuous fields, whereas the Lagrangian in mechanics is usually based on discrete particle systems. The Lagrangian L is related to the Lagrangian density $\mathcal{L}$ by [9]

$$L = \int \mathcal{L} d^3 x. \tag{4}$$

The scalar field is represented by a continuous function $\varphi(x,t)$ which can be real or complex. Given a potential density of the field $V(\varphi)$, $\mathcal{L}$ reads [9]

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi). \tag{5}$$

Let us consider the standard Einstein-Hilbert action of General Relativity which is [17-19]

$$S = \int d^4 x \sqrt{-g}(R + \mathcal{L}_m), \tag{6}$$

where $\mathcal{L}_m$ is the Lagrangian of the matter.

One can define the "conformal scalar field" like a logarithm of the Inflaton field [13]

$$e^{2\Phi} \equiv \varphi. \tag{7}$$

By applying the conformal transformation [19]

$$\tilde{g}_{\alpha\beta} = e^{2\Phi} g_{\alpha\beta} \tag{8}$$

to the action (6) the conformal equivalent Hilbert-Einstein action [19]

$$A = \int \frac{1}{2k} d^4 x \sqrt{-\tilde{g}} [\tilde{R} + L_1(\Phi, \Phi_{;\alpha})], \tag{9}$$

is obtained. In this way, the analysis can be translated in a conformal frame. $L_1(\Phi, \Phi_{;\alpha})$ is the conformal scalar field contribution derived from

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + 2(\Phi_{;\alpha}\Phi_{;\beta} - g_{\alpha\beta}\Phi_{;\delta}\Phi^{;\delta} - \frac{1}{2}g_{\alpha\beta}\Phi^{;\delta}_{;\delta}) \tag{10}$$

and

$$\tilde{R} = e^{-2\Phi}(R - 6\Box\Phi - 6\Phi_{;\delta}\Phi^{;\delta}). \tag{11}$$

In the re-scaled action (9) the matter contributions have not been considered because our interaction with GWs concerns the linearized theory in vacuum.

Following [13], it is well known that the gravity-wave amplitude h_i^j (in the following we will consider the "plus" polarization h_+) is a conformal invariant and that the d'Alembert operator transforms as [13, 14]

$$\widetilde{\Box} = e^{-2\Phi}(\Box + 2\Phi^{;\alpha}\partial_{;\alpha}). \tag{12}$$

Thus, the background changes in the conformal frame while the tensor wave amplitude is fixed.

In order to study the cosmological stochastic background, the operator (12) has to be specified for a Friedman-Robertson-Walker metric [13, 14], obtaining

$$\ddot{h}_+ + (3H + 2\dot{\Phi})\dot{h}_+ + k^2 a^{-2} h_+ = 0, \tag{13}$$

being $\Box = \frac{\partial}{\partial t^2} + 3H\frac{\partial}{\partial t}$, $a(t)$ the scale factor and k the wave number.

Considering the conformal time $d\eta = dt/a$, Eq. (13) reads

$$\frac{d^2}{d\eta^2}h_+ + 2\frac{\gamma'}{\gamma}\frac{d}{d\eta}h_+ + k^2 h_+ = 0, \tag{14}$$

where $\gamma = ae^{\Phi}$. Inflation implies $\eta = \int(dt/a) = 1/(aH)$ and $\frac{\gamma'}{\gamma} = -\frac{1}{\eta}$ [13, 14].

Eq. (14) is formally equal to the equation of a damped harmonic oscillator $\mu(t)$

$$\ddot{\mu} + K\dot{\mu} + \omega_0^2\mu = 0, \tag{15}$$

where K is the damping constant and ω_0 the proper frequency of the harmonic oscillator, but in the case of Eq. (14) the *effective damping constant* $2\frac{\gamma'}{\gamma}$ depends on the conformal time. Hence, we are working with an *effective damped harmonic oscillator*. In any case, the solution of Eq. (14) has been found in [13, 14]

$$h_+(\eta) = k^{-3/2}\sqrt{2/k}[C_1(\sin k\eta - \cos k\eta) + C_2(\sin k\eta + \cos k\eta)]. \tag{16}$$

Inside the $1/H$ radius it is $k\eta \gg 1$. Furthermore, considering the absence of GWs in the initial vacuum state, only negative-frequency modes are present and then the adiabatic behavior is [13, 14]

$$h_+ = k^{1/2}\sqrt{2/\pi}\frac{1}{aH}C\exp(-ik\eta). \tag{17}$$

At the first horizon crossing ($aH = k$ at $t = 10^{-22}$ second after the Initial Singularity, [7]), the averaged amplitude $A_{h_+} = (k/2\pi)^{3/2}h_+$ of the perturbations is

$$A_{h_+} = \frac{1}{2\pi^2}C \tag{18}$$

when the scale a/k grows larger than the Hubble radius $1/H$, the growing mode of evolution is constant ("frozen", see [13, 14]). This situation corresponds to the limit $-k\eta \ll 1$ in equation (16).

The amplitude A_{h_+} of the wave is preserved until the second horizon crossing after which it can be observed, in principle, as an anisotropy perturbation of the CMBR [7, 8]. It can be shown that $\frac{\delta T}{T} \leq A_{h_+}$ is an upper limit to A_{h_+} since other effects can contribute to the background anisotropy [13, 14]. Then, it is clear that the only relevant quantity

is the initial amplitude C in equation (17) which is conserved until the re-enter. Such an amplitude directly depends on the fundamental mechanism generating perturbations that depends on the Inflaton scalar field which generates inflation.

Considering a single monocromatic GW, its zero-point amplitude is derived through the commutation relations [13, 14]

$$[h_+(t,x), \pi_{h_+}(t,y)] = i\delta^3(x-y) \tag{19}$$

calculated at a fixed time t.

As it is [13, 14]

$$\pi_{h_+} = e^{2\Phi}a^3\dot{h}_+, \tag{20}$$

equation (19) reads

$$[h_+(t,x), \dot{h}_+(y,y)] = i\frac{\delta^3(x-y)}{e^{2\Phi}a^3} \tag{21}$$

and the fields h_+ and $\dot{h}_+$ can be expanded in terms of creation and annihilation operators [13, 14]

$$h_+(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k[h_+(t)e^{-ikx} + h_+^*(t)e^{ikx}] \tag{22}$$

$$\dot{h}_+(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3k[\dot{h}_+(t)e^{-ikx} + \dot{h}_+^*(t)e^{ikx}]. \tag{23}$$

The commutation relations in conformal time are then [13, 14]

$$[h_+\frac{d}{d\eta}h_+^*, -h_+^*\frac{d}{d\eta}h_+] = i\frac{8\pi^3}{e^{2\Phi}a^3}. \tag{24}$$

Inserting (17) and (18), it is $C = \sqrt{2}\pi^2 He^{-\Phi}$ where H and Φ are calculated at the first horizon crossing and then

$$A_{h_+} = \frac{\sqrt{2}}{2}He^{-\Phi}, \tag{25}$$

which means that the amplitude of GWs produced during Inflation directly depends on the Inflaton field being $\Phi = \frac{1}{2}\ln\varphi$ [13]. Explicitly, it is

$$A_{h_+} = \frac{H}{\sqrt{2\varphi}}. \tag{26}$$

Thus, one immediately obtains

$$\varphi = \frac{H^2}{2A_{h_+}^2}. \tag{27}$$

that links directly the amplitude of relic GWs with the Inflaton scalar field φ which generates inflation. Then, we have re-obtained the important Eq. (27) in the general Standard Inflationary context of General Relativity plus an external scalar field which generates inflation. In this way, we have also completed the analyses of [13, 14], which concerned the particular cases of Scalar Tensor Gravity and $f(R)$ Theories.

3. Bounds from observations

The equation for the characteristic amplitude h_c is (see Equation 65 in [15])

$$h_c(f) \simeq 1.26 * 10^{-18} \left(\frac{1\text{Hz}}{f}\right) \sqrt{h_{100}^2 \Omega_{gw}(f)}, \qquad (28)$$

where $h_{100} \simeq 0.71$ is the best-fit value on the Hubble constant [11]. This equation gives a value of the amplitude of the relic SGWB in function of the spectrum in the frequency range of ground based detectors [15]. Such an amplitude is also the averaged strain applied on the detector's arms by the relic SGWB [15]. Such a range is given by the interval $10Hz \leq f \leq 10KHz$ [1].

Defining the average value of $h_c(f)$ like

$$A_{h_c} \equiv \frac{\int 1.26 * 10^{-18} \sqrt{h_{100}^2 \Omega_{gw}(f)} f^{-1} df}{\int df} \qquad (29)$$

one can assume that it is $A_{h_c} \simeq A_{h_+}$ [13].

In this way, it is also

$$\varphi \simeq \frac{H^2}{2A_{h_c}^2}. \qquad (30)$$

Now, by using Eq. (30), we can use the bounds (1) and (3) on the relic SGWB in order to obtain bounds on the Inflaton field φ. First of all, we emphasize that a redshift correction is needed because H in Eq. (30) is computed at the time of the first horizon crossing, while the value of A_{h_c} from the WMAP and LSC data is computed at the present time of the cosmological Era. The redshift correction on the spectrum is well known [7]:

$$\Omega_{gw}(f) = \Omega_{gw}^0(f)(1 + z_{eq})^{-1}, \qquad (31)$$

where $\Omega_{gw}^0(f)$ is the value of the spectrum at the first horizon crossing and $z_{eq} \simeq 3200$ [11] is the redshift of the Universe when the matter and radiation energy density were equal, see [7] for details.

Then, Eq. (30) becomes

$$\varphi \simeq \frac{H^2}{2A_{h_c}^2(1 + z_{eq})}. \qquad (32)$$

By considering the WMAP bound (1), the integrals in Eq. (29) have to be computed in the frequency range of ground based detectors which is the interval $10Hz \leq f \leq 10KHz$. One gets $A_{h_c}^2 \simeq 10^{-51}$.

By restoring ordinary units and recalling that $H \simeq 10^{22}Hz$ at the first horizon crossing [7], at the end, from Eq. (32), we get

$$\varphi \geq 10^2 grams. \qquad (33)$$

This result represents a lower bound for the value of the Inflaton field that arises from the WMAP data on the relic SGWB in the case of Standard Inflation [4, 12].

Now, let us consider the LSC bound (3). Such a bound is at 95% confidence in the frequency band $41.5 - 169.25 Hz$ [4], thus, in principle, we could not extend the integrals in Eq. (29) to the total interval $10Hz \leq f \leq 10KHz$. However, it is well known that for frequencies that are smaller than some hertz the spectrum which arises from the Pre-Big-Bang Theory rapidly falls, while at higher frequencies the spectrum is almost flat with a small decreasing [4, 16]. Thus, the integration of Eq. (29) in the interval $10Hz \leq f \leq 10KHz$ gives a solid upper bound for A_{h_c} in these models. One gets $A_{h_c}^2 \simeq 10^{-44}$. In this case, by restoring ordinary units and putting the value $H \simeq 10^{22}Hz$ in eq. (32) it is

$$\varphi \geq 10^{-5} grams. \tag{34}$$

This result represents a lower bound for the value of the Inflaton field that arises from the LSC data on the relic SGWB and it has to be applied to the case of the Pre-Big-Bang Theory [4, 16].

It is well known that the requirement for inflation, which is $p = -\rho$ [9, 10], can be approximately met if one requires $\dot{\varphi} << V(\varphi)$, where $V(\varphi)$ is the potential density of the field in Eq. (5). This leads to the famous *slow-roll approximation* (SRA), which provides a natural condition for Inflation to occur [9, 10]. The constraint on $\dot{\varphi}$ is assured by requiring $\ddot{\varphi}$ to be negligible. With such a requirement, the slow-roll parameters are defined (in natural units) by [9, 10]

$$\epsilon(\varphi) \equiv \tfrac{1}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2$$

$$\tag{35}$$

$$\eta(\varphi) \equiv \frac{V''(\varphi)}{V(\varphi)}.$$

Then, the SRA requirements are [9, 10]:

$$\epsilon \ll 1$$

$$\tag{36}$$

$$|\eta| \ll 1,$$

that are satisfied when it is [9, 10]

$$\varphi \gg M_{Planck}, \tag{37}$$

where the Planck mass, which is $M_{Planck} \simeq 2.177 * 10^{-5} grams$ in ordinary units and $M_{Planck} = 1$ in natural units has been introduced [9, 10].

Then, one sees immediately that the value of the Inflaton field of Eq. (33), that arises from the WMAP bound on the relic SGWB, is totally in agreement with the slow roll condition on Inflation. On the other hand, the value of the Inflaton field of Eq. (34), that arises from the LSC bound on the relic SGWB, is of the order of the Planck mass, thus, it could not be in agreement with the slow roll condition on Inflation.

The fact that the spectrum of the relic SGWB decreases with increasing Inflaton field is not surprising. In fact, even if the amplification of zero-point quantum oscillations increases the spatial dimensions of perturbations, it is well known that the curvature of the Universe is "redshifted" by Inflation, i.e. the inflationary scenario 'drives' the universe to a flat geometry [9, 10].

4. Conclusion remarks

By using a formula that directly connects the average amplitude of the relic SGWB with the Inflaton field and the equation for the characteristic amplitude h_c for the relic SGWB, in this paper the upper bounds on the relic SGWB from the WMAP and LSC data have been translated in lower bounds on the Inflaton field.

The results show that the value of the Inflaton field that arises from the WMAP bound on the relic SGWB is totally in agreement with the famous slow roll condition on Inflation [9, 10], while the value of the Inflaton field that arises from the LSC bound on the relic SGWB could not be in agreement with such a condition.

Finally, we further emphasize the importance of the formula (27). If the GWs interferometers will detect the relic SGWB in next years, such a formula will permit to directly compute the amount of Inflation in the early Universe. Hence, the analysis in this paper has shown that the detection of the SGWB will permit a direct measure of the value of the Inflaton field by giving an extraordinary precious and precise information on the early Universe's dynamics. In other words, the detection of the SGWB will permit *to auscultate the primordial gravity's breath.*

Acknowledgements

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