A Clarification on the Debate on "the Original Schwarzschild Solution"

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Abstract: Now that English translations of Schwarzschild's original paper exist, that paper has become accessible to more people. Historically, the so-called "standard Schwarzschild solution" was not the original Schwarzschild's work, but it is actually due to J. Droste and, independently, H. Weyl, while it has been ultimately enabled like correct solution by D. Hilbert. Based on this, there are authors who claim that the work of Hilbert was wrong and that Hilbert's mistake spawned black-holes and the community of theoretical physicists continues to elaborate on this falsehood, with a hostile shouting down of any and all voices challenging them. In this paper we re-analyse "the original Schwarzschild solution" and we show that it is totally equivalent to the solution enabled by Hilbert. Thus, the authors who claim that "the original Schwarzschild solution" implies the non existence of black holes give the wrong answer. We realize that the misunderstanding is due to an erroneous interpretation of the different coordinates. In fact, arches of circumference appear to follow the law $dl = rd\varphi$, if the origin of the coordinate system is a non-dimensional material point in the core of the black-hole, while they do not appear to follow such a law, but to be deformed by the presence of the mass of the central body M if the origin of the coordinate system is the surface of the Schwarzschild sphere.

1. Introduction

The concept of black-hole (BH) has been considered very fascinating by scientists even before the introduction of general relativity (see [1] for an historical review). A BH is a region of space from which nothing, not even light, can escape. It is the result of

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the deformation of spacetime caused by a very compact mass. Around a BH there is an undetectable surface which marks the point of no return. This surface is called an event horizon. It is called "black" because it absorbs all the light that hits it, reflecting nothing, just like a perfect black body in thermodynamics [2]. However, an unsolved problem concerning such objects is the presence of a space-time singularity in their core. Such a problem was present starting by the firsts historical papers concerning BHs [3, 4, 5]. It is a common opinion that this problem could be solved when a correct quantum gravity theory will be, finally, obtained, see [6] for recent developments.

On the other hand, fundamental issues which dominate the question about the existence or non-existence of BH horizons and singularities and some ways to avoid the development of BH singularities within the classical theory, which does not require the need for a quantum gravity theory, have been discussed by various authors in the literature, see references from [7] to [16]. In fact, by considering the exotic nature of BHs, it may be natural to question if such bizarre objects could exist in nature or to suggest that they are merely pathological solutions to Einstein's equations. Einstein himself thought that BHs would not form, because he held that the angular momentum of collapsing particles would stabilize their motion at some radius [17].

Recently, the debate became very hot as English translations of Schwarzschild's original work now exist and that work has become accessible to more people [18, 19]. Historically, the so-called "*Schwarzschild solution*" was not the original Schwarzschild's work, but it is actually due to J. Droste [20] and, independently, H. Weyl [21], while it has been ultimately enabled like correct solution by D. Hilbert [22]. Let us further clarify this point by adding some historical notes. In 1915, A. Einstein developed his theory of general relativity [23]. A few months later, K. Schwarzschild gave the solution for the gravitational field of a point mass and a spherical mass [3]. A few months after Schwarzschild, J. Droste, a student of H. Lorentz, independently gave an apparently different solution for the point mass and wrote more extensively about its properties [20]. In such a work Droste also claimed that his solution was physically equivalent to the one by Schwarzschild. In the same year, 1917, H. Weyl re-obtained the same solution by Droste [21]. This solution had a peculiar behaviour at what is now called the Schwarzschild radius, where it became singular, meaning that some of the terms in the Einstein equations became infinite. The nature of this surface was not quite understood at the time, but Hilbert [22] claimed that the form by Droste and Weyl was preferable to that in [3] and ever since then the phrase "*Schwarzschild solution*" has been taken to mean the line-element which was found in [20, 21] rather than the original solution in [3]. In 1924, A. Eddington showed that the singularity disappeared after a change of coordinates (Eddington coordinates [24]), although it took until 1933 for G. Lemaître to realize, in a series of lectures together with Einstein, that this meant the singularity at the Schwarzschild radius was an unphysical coordinate singularity [25].

In 1931, S. Chandrasekhar calculated that a non-rotating body of electron-degenerate matter above 1.44 solar masses (the Chandrasekhar limit) would collapse [5]. His arguments were opposed by many of his contemporaries like Eddington, Lev Landau and the

same Einstein. In fact, a white dwarf slightly more massive than the Chandrasekhar limit will collapse into a neutron star which is itself stable because of the Pauli exclusion principle [1]. But in 1939, J. R. Oppenheimer and G. M. Volkoff predicted that neutron stars above approximately 1.5 - 3 solar masses (the famous Oppenheimer–Volkoff limit) would collapse into BHs for the reasons presented by Chandrasekhar, and concluded that no law of physics was likely to intervene and stop at least some stars from collapsing to BHs [26]. Oppenheimer and Volkoff interpreted the singularity at the boundary of the Schwarzschild radius as indicating that this was the boundary of a bubble in which time stopped. This is a valid point of view for external observers, but not for free-falling observers. Because of this property, the collapsed stars were called "*frozen stars*" [27] because an outside observer would see the surface of the star frozen in time at the instant where its collapse takes it inside the Schwarzschild radius. This is a known property of modern BHs, but it must be emphasized that the light from the surface of the frozen star becomes redshifted very fast, turning the BH black very quickly. Originally, many physicists did not accept the idea of time standing still at the Schwarzschild radius, and there was little interest in the subject for lots of time. But in 1958, D. Finkelstein, by re-analysing Eddington coordinates, identified the Schwarzschild surface $r = 2M$ (in *natural units*, i.e. $G = 1$, $c = 1$ and $\hbar = 1$, i.e where r is the radius of the surface and M is the mass of the BH) as an *event horizon*, "*a perfect unidirectional membrane: causal influences can cross it in only one direction*" [28]. This extended Oppenheimer's results in order to include the point of view of free-falling observers. Finkelstein's solution extended the Schwarzschild solution for the future of observers falling into the BH. Another complete extension was found by M. Kruskal in 1960 [29].

These results generated a new interest on general relativity, which, together with BHs, became mainstream subjects of research within the Scientific Community. This process was endorsed by the discovery of pulsars in 1968 [30] which resulted to be rapidly rotating neutron stars. Until that time, neutron stars, like BHs, were regarded as just theoretical curiosities; but the discovery of pulsars showed their physical relevance and spurred a further interest in all types of compact objects that might be formed by gravitational collapse.

In this period more general BH solutions were found. In 1963, R. Kerr found the exact solution for a rotating BH [31]. Two years later E. T. Newman and A. Janis found the asymmetric solution for a BH which is both rotating and electrically charged [32]. Through the works by W. Israel, B. Carter and D. C. Robinson the no-hair theorem emerged [1], stating that a stationary BH solution is completely described by the three parameters of the Kerr–Newman metric; mass, angular momentum, and electric charge [1].

For a long time, it was suspected that the strange features of the BH solutions were pathological artefacts from the symmetry conditions imposed, and that the singularities would not appear in generic situations. This view was held in particular by Belinsky, Khalatnikov, and Lifshitz, who tried to prove that no singularities appear in generic solutions [1]. However, in the late sixties R. Penrose and S. Hawking used global techniques

to prove that singularities are generic [1].

The term "*black hole*" was first publicly used by J. A. Wheeler during a lecture in 1967 [33] but the first appearing of the term, in 1964, is due by A. Ewing in a letter to the American Association for the Advancement of Science [34], verbatim: "*According to Einstein's general theory of relativity, as mass is added to a degenerate star a sudden collapse will take place and the intense gravitational field of the star will close in on itself. Such a star then forms a 'black hole' in the universe.*"

In any case, after Wheeler's use of the term, it was quickly adopted in general use.

Today, the majority of researchers in the field is persuaded that there is no obstacle to forming an event horizon. On the other hand, there are other researchers who demonstrated that various physical mechanisms can, in principle, remove both of event horizon and singularities during the gravitational collapse [7] - [16]. In particular, in [9] an exact solution of Einstein field equations which removes both of event horizon and singularities has been found by constructing the right-hand side of the field equations, i.e. the stress-energy tensor, through a non-linear electrodynamics Lagrangian which was previous used in super-strongly magnetized compact objects, such as pulsars, and particular neutron stars [35, 36].

On the other hand, there are researchers who invoke the non existence of BH by claiming that the Schwarzschild's original work [3] gives a solution which is physically different from the one derived by Droste [20] and Weyl [21]. Let us see this issue in more detail. The new translations of Schwarzschild's original work can be found in ref. [18, 19]. These works commented on Schwarzschild's original paper [3]. In particular Abrams [18] claimed that the line-element (we use natural units in all this paper)

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}} \tag{1}$$

i.e. the famous and fundamental solution to the Einstein field equations in vacuum, gives rise to a space-time that is neither equivalent to Schwarzschild's original solution in [3]. Abrams also claimed that Hilbert [22] opined that the form of (1) by Droste and Weyl was preferable to that in [3] and ever since then the phrase "*Schwarzschild solution*" has been taken to mean the line-element (1) rather than the original solution in [3]. In a following work [37] Abrams further claimed that "*Black Holes are The Legacy of Hilbert's Error*" as Hilbert's derivation used a wrong variable. Thus, Hilbert's assertion that the form of (1) was preferable to the original one in [3] should be invalid. Based on this, there are authors who agree with Abrams by claiming that the work of Hilbert was wrong and Hilbert's mistake spawned the BHs and the community of theoretical physicists continues to elaborate on this falsehood, with a hostile shouting down of any and all voices challenging them, see for example references [38, 39, 40, 41].

In this paper we re-analyse *"the original Schwarzschild solution"* to Einstein field equations derived in [3]. Such a solution arises from an apparent different physical hypothesis which assumes arches of circumference to to *do not follow the law dl = rdφ*, but to be deformed by the presence of the mass of the central body M. This assumption enables the

origin of the coordinate system to be *not* a single point, but a spherical surface having radius equal to the gravitational radius, i.e. the surface of the Schwarzschild sphere. The solution works for the external geometry of a spherical static star and circumnavigates the Birkhoff theorem [4].

Then, the simplest case of gravitational collapse, i.e. the spherical radial collapse of a star with uniform density and zero pressure, will be analysed by turning attention to the interior of the collapsing object and the precise word line that its surface follows in the external geometry. The result of the analysis will show that the singularity within the totally collapsed spherical object remains. In fact, a coordinate transform that transfers the origin of the coordinate system, which is the surface of a sphere having radius equal to the gravitational radius, in a non-dimensional material point in the core of the BH re-obtains the solution (1). Thus, *"the original Schwarzschild solution"* [3] results physically equivalent to the solution (1) enabled like the correct on by Hilbert in [23], i.e. the solution that is universally known like the "Schwarzschild solution" [1]. This analysis ultimately shows that the authors who claim that the original Schwarzschild solution leaves no room for the science fiction of the BHs (see references[18, 19] and from [37] to [41]) give the wrong answer. The misunderstanding is due to an erroneous interpretation of the different coordinates. In fact, arches of circumference appear to follow the law $dl = rd\varphi$, if the origin of the coordinate system is a non-dimensional material point in the core of the BH, while they do not appear to follow such a law, but to be deformed by the presence of the mass of the central body M if the origin of the coordinate system is the surface of the Schwarzschild sphere. Thus, the only way to remove the singularity in the core of a BH within the classical theory of Einstein's general relativity is changing the hypotheses which govern the internal geometry of the collapsing star, following for example the ideas in references from [7] to [16].

2.   The *"original Schwarzschild solution"*

Following [42], the more general line-element which respects central symmetry is

$$ds^2 = h(r,t)dr^2 + k(r,t)(\sin^2\theta d\varphi^2 + d\theta^2) + l(r,t)dt^2 + a(r,t)drdt, \tag{2}$$

where

$$r \geq 0, \ \ 0 \leq \theta \leq \pi, \ \ 0 \leq \varphi \leq 2\pi. \tag{3}$$

We search a line-element solution in which the metric is spatially symmetric with respect to the origin of the coordinate system, i.e. that we find again the same solution when spatial coordinates are subjected to a orthogonal transformations and rotations and it is asymptotically flat at infinity [1]. In order to obtain the *"standard Schwarzschild solution"*, i.e. the line-element (1), to Einstein field equations in vacuum one uses transformations of the type [42]

$$r = f_1(r',t'), \ \ t = f_2(r',t'), \tag{4}$$

where f_1 and f_1 are arbitrary functions of the new coordinates r' and t'. At this point, if one wants the *"standard Schwarzschild solution"*, r and t have to be chosen in a way that $a(r, t) = 0$ and $k(r, t) = -r^2$ [42]. In particular, the second condition implies that the standard Schwarzschild radial coordinate is determined in a way which guarantees that the length of the circumference centred in the origin of the coordinate system is $2\pi r$ [42].

In our approach we will suppose again that $a(r, t) = 0$, but, differently from the standard analysis, we will assume that the length of the circumference centred in the origin of the coordinate system *is not* $2\pi r$. We release *an apparent different physical assumption*, i.e. that arches of circumference *are deformed* by the presence of the mass of the central body M. Note that this different physical hypothesis permits to circum-navigate the Birkhoff Theorem [4] which leads to the *"standard Schwarzschild solution"* [3]. In fact, the demonstration of the Birkhoff Theorem *starts* from a line element *in which $k(r, t) = -r^2$ has been chosen*, see the discussion in paragraph 32.2 of [1] and, in particular, look at Eq. (32.2) of such a paragraph.

Then, we proceed assuming $k = -mr^2$, where m is a generic function to be determined in order to obtain that the length of circumferences centred in the origin of the coordinate system are not $2\pi r$. In other words, m represents *a measure of the deviation from $2\pi r$* of circumferences centred in the origin of the coordinate system.

The line element (2) becomes

$$ds^2 = hdr^2 - mr^2(\sin^2\theta d\varphi^2 + d\theta^2) + ldt^2. \tag{5}$$

One puts

$$X \equiv \tfrac{1}{3}r^3$$

$$Y \equiv -\cos\theta \tag{6}$$

$$Z \equiv \varphi.$$

In the X, Y, Z coordinates the line-element (5) reads

$$ds^2 = ldt^2 + \frac{h}{r^4}dX^2 - mr^2[\frac{dY^2}{1-Y^2} + dZ^2(1-Y^2)]. \tag{7}$$

Let us consider three functions

$$A \equiv -\frac{h}{r^4}$$

$$B \equiv mr^2 \qquad\qquad (8)$$

$$C \equiv l$$

which satisfy the conditions

$$X \to \infty \; implies \; A \to \frac{1}{r^4} = \frac{1}{(3X)^{\frac{4}{3}}}, \; B \to r^2 = 3X^{\frac{2}{3}}, \; C \to 1$$

$$(9)$$

$$normalization \; condition \; AB^2C = 1.$$

The line-element (7) becomes

$$ds^2 = Cdt^2 - AdX^2 - B\frac{dY^2}{1-Y^2} - BdZ^2(1-Y^2). \qquad (10)$$

From the metric (10) one gets the Christoffell coefficients like (only the non zero elements will be written down)

$$\Gamma^t_{tX} = -\frac{1}{2C}\frac{\partial C}{\partial X} \qquad\qquad \Gamma^X_{XX} = -\frac{1}{2A}\frac{\partial A}{\partial X}$$

$$\Gamma^X_{YY} = \frac{1}{2A}\frac{\partial B}{\partial X}\frac{1}{1-Y^2} \quad \Gamma^X_{ZZ} = \frac{1}{2A}\frac{\partial B}{\partial X}(1-Y^2)$$

$$\Gamma^X_{tt} = -\frac{1}{2A}\frac{\partial C}{\partial X} \qquad\qquad \Gamma^Y_{YX} = -\frac{1}{2B}\frac{\partial B}{\partial X}$$

$$(11)$$

$$\Gamma^Y_{YY} = \frac{-Y}{1-Y^2} \qquad\qquad \Gamma^Y_{ZZ} = -Y(1-Y^2)$$

$$\Gamma^Z_{ZX} = -\frac{1}{2B}\frac{\partial B}{\partial X} \qquad\qquad \Gamma^Z_{ZX} = \frac{Y}{1-Y^2}.$$

By using the equation for the components of the Ricci tensor, the components of Einstein field equation in vacuum are [42]

$$R_{ik} = \frac{\partial\Gamma^l_{ik}}{dx^l} - \frac{\partial\Gamma^l_{il}}{dx^k} - \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} = 0. \qquad (12)$$

By inserting Eqs. (11) in Eqs. (12) one gets only three independent relations

$$\frac{\partial}{\partial X}\left(\frac{1}{A}\frac{\partial B}{\partial X}\right) - 2 - \frac{1}{AB}\left(\frac{\partial B}{\partial X}\right)^2 = 0 \tag{13}$$

$$\frac{\partial}{\partial X}\left(\frac{1}{A}\frac{\partial A}{\partial X}\right) - \frac{1}{2}\left(\frac{1}{A}\frac{\partial A}{\partial X}\right)^2 - \left(\frac{1}{B}\frac{\partial B}{\partial X}\right)^2 - \frac{1}{2}\left(\frac{1}{C}\frac{\partial C}{\partial X}\right)^2 = 0 \tag{14}$$

$$\frac{\partial}{\partial X}\left(\frac{1}{A}\frac{\partial C}{\partial X}\right) - \frac{1}{AC}\left(\frac{\partial C}{\partial X}\right)^2 = 0. \tag{15}$$

From the second of Eqs. (9) (normalization condition) one gets also

$$\frac{1}{A}\frac{\partial A}{\partial X} + \frac{2}{B}\frac{\partial B}{\partial X} + \frac{1}{C}\frac{\partial C}{\partial X} = 0. \tag{16}$$

Eq. (15) can be rewritten like

$$\frac{\partial}{\partial X}\left(\frac{1}{C}\frac{\partial C}{\partial X}\right) = \frac{1}{AC}\frac{\partial A}{\partial X}\frac{\partial C}{\partial X}, \tag{17}$$

which can be integrated, giving

$$\frac{1}{C}\frac{\partial C}{\partial X} = aA, \tag{18}$$

where a is an integration constant.

By adding Eq. (14) to Eq. (17) one gets

$$\frac{\partial}{\partial X}\left(\frac{1}{A}\frac{\partial A}{\partial X} + \frac{1}{C}\frac{\partial C}{\partial X}\right) = \left(\frac{1}{B}\frac{\partial B}{\partial X}\right)^2 + \frac{1}{2}\left(\frac{1}{A}\frac{\partial A}{\partial X} + \frac{1}{C}\frac{\partial C}{\partial X}\right)^2 \tag{19}$$

Considering Eq. (16) we obtain

$$2\frac{\partial}{\partial X}\left(\frac{1}{B}\frac{\partial B}{\partial X}\right) = -3\left(\frac{1}{B}\frac{\partial B}{\partial X}\right)^2, \tag{20}$$

which can be integrated, giving

$$\frac{1}{B}\frac{\partial B}{\partial X} = \frac{2}{3X+b}, \tag{21}$$

where b is an integration constant. A second integration gives

$$B = d(3X+b)^{\frac{2}{3}}. \tag{22}$$

where d is an integration constant. But the first of Eqs. (9) implies $d = 1$, thus

$$B = (3X+b)^{\frac{2}{3}}. \tag{23}$$

By using Eqs. (18) and (16) we obtain

$$\frac{\partial C}{\partial X} = aAC = \frac{a}{B^2} = a(3X+b)^{-\frac{4}{3}}.$$

By integrating and considering the first of Eqs. (9) one gets

$$C = 1 - a(3X + b)^{-\frac{1}{3}} \tag{24}$$

Then, from Eq. (16) one obtains

$$A = \frac{(3X + b)^{-\frac{4}{3}}}{1 - a(3X + b)^{-\frac{1}{3}}}. \tag{25}$$

By putting Eqs. (25) and (23) in Eq. (13) one immediately sees that this last equation is automatically satisfied.

We note that the function A results singular for values $a(3X + b)^{-\frac{1}{3}} = 1$. However, this is a mathematical singularity due to the particular coordinates t, X, Y, Z defined by the transformation (6). In fact, by assuming that such a singularity is located at $X = 0$ we get

$$b = a^3, \tag{26}$$

i.e. we find a relation between the two integration constants b and a.

At the end we obtain

$$A = (r^3 + a^3)^{-\frac{4}{3}}[1 - a(r^3 + a^3)^{-\frac{1}{3}}]^{-1}$$

$$B = (r^3 + a^3)^{\frac{2}{3}} \tag{27}$$

$$C = 1 - a(r^3 + a^3)^{-\frac{1}{3}}.$$

By inserting the functions (27) in Eq. (10) and using Eqs. (8) and (6) to return to the standard polar coordinates the line-element solution reads

$$ds^2 = \left[1 - \frac{a}{(r^3+a^3)^{\frac{1}{3}}}\right] dt^2 - (r^3 + a^3)^{\frac{2}{3}}(\sin^2\theta d\varphi^2 + d\theta^2) +$$

$$\tag{28}$$

$$- \frac{d(r^3+a^3)^{\frac{2}{3}}}{1 - \frac{a}{(r^3+a^3)^{\frac{1}{3}}}}.$$

Hence, we understand that the assumption to locate the mathematical singularity of the function A at $X = 0$ coincides with the physical condition that the length of the circumference centred in the origin of the coordinate system is $2\pi(r^3 + a^3)^{\frac{1}{3}}$, which is different from the value $2\pi r$. This is the apparent fundamental physical difference between this solution and the "*standard Schwarzschild solution*" (1), i.e. the one enabled by Hilbert in [22]. The value of the generic function m which permits that the length of circumferences centred in the origin of the coordinate system are not $2\pi r$ is

$$m = \frac{(r^3 + a^3)^{\frac{2}{3}}}{r^2}. \tag{29}$$

On the other hand, in order to determinate the value of the constant a, by following [42], one can use the weak field approximation which implies $g_{00} \cong 1 + 2\varphi$ at large distances, where $g_{00} = (1 - \frac{a}{(r^3+a^3)^{\frac{1}{3}}})$ in Eq. (28) and $\varphi \equiv \frac{-M}{r}$ is the Newtonian potential. Thus, for $a \ll r$, we immediately obtain: $a = 2M = r_g$, i.e. a results exactly the gravitational radius [1, 42].

Then, we can rewrite the solution (28) in an ultimate way like

$$ds^2 = \left[1 - \frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}} \right] dt^2 - (r^3 + r_g^3)^{\frac{2}{3}}(\sin^2\theta d\varphi^2 + d\theta^2)+$$

$$-\frac{d(r^3+r_g^3)^{\frac{2}{3}}}{1-\frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}}}.$$

(30)

Historically, the line-element (30) represents *"the original Schwarzschild solution"* to Einstein field equations as it has been derived for the first time by Karl Schwarzschild in [3] with a slight different analysis.

Some comments are needed. By looking Eq. (30) one understands that the origin of the coordinate that we have chosen by putting $r \geq 0$, $0 \leq \theta \leq \pi$ $0 \leq \varphi \leq 2\pi$ and with the additional assumption that the length of circumferences centred in the origin of the coordinate system are not Euclidean, *is not* a single point, but it is the surface of a sphere having radius r_g, i.e. the surface of the Schwarzschild sphere. By putting

$$\widehat{r} \equiv (r^3 + r_g^3)^{\frac{1}{3}},$$

(31)

Eq. (28) becomes

$$ds^2 = (1 - \frac{r_g}{\widehat{r}})dt^2 - \widehat{r}^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{d\widehat{r}^2}{1 - \frac{r_g}{\widehat{r}}}.$$

(32)

Eq. (32) looks *formally equal* to the *"standard Schwarzschild solution"* (1). But one could think that the transformation (31) is forbidden for the following motivation. It transfers the origin of the coordinate system, $r = 0$, $\theta = 0$, $\varphi = 0$, which is the surface of a sphere having radius r_g in the r, θ, φ coordinates, in a non-dimensional material point $\widehat{r} = 0$, $\theta = 0$, $\varphi = 0$ in the $\widehat{r}$, θ, φ coordinates. Such a non-dimensional material point corresponds to the point $r = -r_g$, $\theta = 0$, $\varphi = 0$ in the original r, θ, φ coordinates. Thus, the transformation (31) could not be a suitable coordinate transformation because it transfers a spherical surface, i.e. a bi-dimensional manifold, in a non-dimensional material point. We will see in the following that this interpretation *is not correct*.

On the other hand, we are searching a solution for the external geometry, thus we assumed $r \geq 0$ in Eq. (3) and from Eq. (31) it is *always* $\widehat{r} \geq r_g$ in Eq. (32) as it is $r \geq 0$ in Eq. (30). In this way, *there are not physical singularities* in Eq. (32). In fact, $r = 0$ in Eq. (30) implies $\widehat{r} = r_g$ in Eq. (32) which corresponds to the mathematical singularity at $X = 0$. This singularity is *not physical* but is due to the particular coordinates t, X, Y, Z defined by the transformation (6).

Again, we emphasize the apparent different assumption of our analysis. As it is carefully explained in [42], the *"standard Schwarzschild solution"* (1), arises from the hypothesis that the coordinates r and t of the two functions (4) are chosen in order to guarantee that the length of the circumference centred in the origin of the coordinate system is $2\pi r$. Indeed, in the above derivation of *"the original Schwarzschild solution"* (30), r and t are chosen in order to guarantee that the length of the circumference centred in the origin of the coordinate system is *not* $2\pi r$. In particular, choosing to put the mathematical singularity of the function A at $X = 0$ is equivalent to the physical condition that the length of the circumference centred in the origin of the coordinate system is $2\pi(r^3 + r_g^3)^{\frac{1}{3}}$. Then, one could think that by forcing the transformation (31) for $r \leq 0$, one returns to the standard Schwarzschild solution (1), but a bi-dimensional spherical surface, that is the surface of the Schwarzschild sphere, is forced to become a non-dimensional material point and we force a non-Euclidean geometry for circumferences to become Euclidean. In that case, such a mathematical forcing could be the cause of the singularity in the core of the black-hole. Thus, this singularity could be only mathematical and not physical. But in the following, by matching with the internal geometry, we will see that this interpretation is not correct and that the singularity in the core of the BH remains a physical singularity also in the case of the *"original Schwarzschild solution"* given by Eq. (30).

Notice that a large distances, i.e. where $r_g \ll r$, the solution (30) well approximates the standard Schwarzschild solution (1), thus, both of the weak field approximation and the analysis of astrophysical situations remain the same.

3. Matching with the internal geometry: singular gravitational collapse

In the following we adapt the classical analysis in [1] to the line-element (30). Let us consider a test particle moving in the external geometry (30). By following the magnitude of the 4-vector of energy-momentum is represented by the rest mass μ of the particle [1]

$$g_{ik}p^i p^k + \mu^2 = g^{ik}p_i p_k + \mu^2 = 0, \tag{33}$$

or

$$-\frac{E^2}{1 - \frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}}} + \frac{1}{1 - \frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}}} \frac{r^4}{(r^3+r_g^3)^{\frac{4}{3}}} \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{(r^3+r_g^3)^{\frac{2}{3}}} + \mu^2, \tag{34}$$

where $\lambda = \tau/\mu$, L and E represent the affine parameter being τ the proper time, the angular momentum and the energy of the particle [1].

Einstein equivalence principle [1] implies that test particles follow the same wordlines regardless of mass. Then, what is relevant for the motion of particles are the normalized quantities $\tilde{L} = L/\mu$ and $\tilde{E} = E/\mu$.

Thus, Eq. (34) can be rewritten as

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{(r^3+r_g^3)^{\frac{4}{3}}}{r^4}\left\{\tilde{E}^2 - \left(1 - \frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}}\right)\left(1 + \frac{\tilde{L}^2}{(r^3+r_g^3)^{\frac{2}{3}}}\right)\right\} =$$

(35)

$$= \frac{(r^3+r_g^3)^{\frac{4}{3}}}{r^4}\left(\tilde{E}^2 - \tilde{V}^2(r)\right),$$

where the "*effective potential*" is defined by

$$\tilde{V}(r) \equiv \sqrt{\left(1 - \frac{r_g}{(r^3 + r_g^3)^{\frac{1}{3}}}\right)\left(1 + \frac{\tilde{L}^2}{(r^3 + r_g^3)^{\frac{2}{3}}}\right)}.$$

(36)

From Eq. (35) the proper time can be explicitly written down

$$\tau = \int d\tau = \int \left[\frac{dr}{\left(\tilde{E}^2 - \tilde{V}^2(r)\right)}\frac{r^2}{(r^3 + r_g^3)^{\frac{2}{3}}}\right].$$

(37)

In the following we discuss the collapse of a star with uniform density and zero pressure. Because no pressure gradients are present to deflect their motion, the particles on the surface of any ball of dust must move along radial geodesic in the external geometry of Eq. (30). The angular momentum vanishes and the integral (37) reduces to

$$\tau = \int d\tau = \int \left[\frac{dr}{\sqrt{\left(\frac{r_g}{(r^3+r_g^3)^{\frac{1}{3}}} - \frac{r_g}{(R^3+r_g^3)^{\frac{1}{3}}}\right)}}\frac{r^2}{(r^3 + r_g^3)^{\frac{2}{3}}}\right],$$

(38)

where $R \equiv \frac{r_g}{1-\tilde{E}^2}$ is the "*apastron*", i.e. the radius at which the particle has zero velocity [1].

Eq. (38) can be integrated in parametric form:

$$r = \frac{1}{2}\left[(R^3 + r_g^3)(1 + \cos\eta)^3 - r_g^3\right]^{\frac{1}{3}}$$

(39)

and

$$\tau = \frac{(R^3 + r_g^3)^{\frac{1}{3}}}{2}\left(\frac{R^3 + r_g^3}{r_g}\right)^{\frac{1}{6}}(\eta + \sin\eta).$$

(40)

Eq. (40) is the proper time read by a clock on the surface of the collapsing star.

The collapse begins when the parameter η is zero ($r = R$, $\tau = 0$) and terminates, for the external geometry, at $r = 0$, $\eta = \frac{2r_g}{(R^3+r_g^3)^{\frac{1}{3}}-1}$.

Thus, the total proper time to fall from rest at $r = R$ into the surface of the sphere $r = 0$ is

$$\tau = \frac{(R^3 + r_g^3)^{\frac{1}{3}}}{2} \left(\frac{R^3 + r_g^3}{r_g} \right)^{\frac{1}{6}} \left[\arccos\left(\frac{2r_g}{(R^3 + r_g^3)^{\frac{1}{3}} - 1}\right) + \sin \arccos\left(\frac{2r_g}{(R^3 + r_g^3)^{\frac{1}{3}} - 1}\right) \right].$$
(41)

Let us focus the attention on the simplest ball of dust, an interior that is homogeneous and isotropic everywhere, except at the surface. This is exactly the case of an interior locally identical to a dust filled Friedmann closed cosmological model [1, 9]. In fact, the closed model is the only one of interest because it corresponds to a gas sphere whose dynamics begins at rest with a finite radius [1, 9]. The ordinary line-element is given by [1, 9]

$$ds^2 = -d\tau^2 + a(\tau)(+d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\varphi^2),$$
(42)

where $a(\tau)$ is the scale factor of the internal space-time. In the case of zero pressure the stress-energy tensor is

$$T = \rho u \otimes u,$$
(43)

where ρ is the density of the star and u the 4-vector velocity of the matter. Thus, the Einstein field equations give only one meaningful relation in terms of η [1, 9]

$$\left(\frac{da}{d\eta}\right)^2 + a^2 = \frac{\rho}{3}a^4,$$
(44)

which admits the familiar cycloidal solution [1, 9]

$$a = \frac{a_0}{2}(1 + \cos \eta),$$
(45)

and

$$\tau = \frac{a_0}{2}(\eta + \sin \eta).$$
(46)

where a_0 is a constant.

Homogeneity and isotropy are broken only at the star's surface which lies at a radius $\chi = \chi_0$ for all τ during the collapse [1, 9], as measured in terms of the co-moving hyperspherical polar angle χ. The match between the internal solution given by Eqs. (45) and (46) and the external solution given by Eqs. (39) and (40) is possible. As a verification of such a match let us examine the separate and independent predictions made by the internal and external solutions for the star's circumference [1]. From Eqs. (39) and (40) the external solution enables the relations:

$$C = 2\pi(r^3 + r_g^3)^{\frac{1}{3}} = 2\pi(R^3 + r_g^3)^{\frac{1}{3}}(1 + \cos \eta)$$
(47)

$$\tau = \frac{(R^3 + r_g^3)^{\frac{1}{3}}}{2} \left(\frac{R^3 + r_g^3}{r_g} \right)^{\frac{1}{6}} (\eta + \sin \eta).$$

From Eqs. (45) and (46) the internal solution enables the relations:

$$C = 2\pi(r^3 + r_g^3)^{\frac{1}{3}} = 2\pi \frac{a_0 \sin \chi_0}{2}(1 + \cos \eta)$$

(48)

$$\tau = \frac{a_0}{2}(\eta + \sin \eta).$$

Thus, the match works for all time during the collapse if and only if

$$R = \left(a_0^3 \sin^3 \chi_0 - r_g^3\right)^{\frac{1}{3}}$$

(49)

$$r_g = a_0 \sin^3 \chi_0.$$

By inserting the first of Eqs. (49) in Eq. (39) one gets

$$r = \frac{1}{2}\left\{[(a_0 \sin \chi_0)(1 + \cos \eta)]^3 - r_g^3\right\}^{\frac{1}{3}}.$$ (50)

Eq. (50) represents the run of the collapse for both the external and internal solutions for $0 \leq \eta \leq \frac{2r_g}{(R^3 + r_g^3)^{\frac{1}{3}} - 1}$. When $\eta = \frac{2r_g}{(R^3 + r_g^3)^{\frac{1}{3}} - 1}$ it is $r = 0$ and particles reach the Schwarzschild sphere which is the origin of the coordinate system. For $\eta > \frac{2r_g}{(R^3 + r_g^3)^{\frac{1}{3}} - 1}$ Eq. (50) represents only the trend of the internal solution and the r coordinate becomes negative (this is possible because the origin of the coordinate system is the surface of the Schwarzschild sphere). The r coordinate reaches a minimum $r = -r_g$ for $\eta = \pi$. Thus, we understand that at this point the collapse terminates and the star is totally collapsed in a singularity at $r = -r_g$. In other terms, *in the internal geometry all time-like radial geodesics of the collapsing star terminate after a lapse of finite proper time in the termination point $r = -r_g$ and it is impossible to extend the internal space-time manifold beyond that termination point.* Thus, the point $r = -r_g$ represents a singularity based on the rigorous definition by Schmidt [43].

Clearly, as all the particle of the collapsing star fall in the singularity at $r = -r_g$ values of $r > -r_g$ do not represent the internal geometry after the end of the collapse, but they will represent the external geometry. This implies that the external solution (30), i.e. *"the original Schwarzschild solution"* to Einstein field equations which has been derived for the first time by Karl Schwarzschild in [3] can be analytically continued for values of $-r_g < r \leq 0$ and it results physically equivalent to the solution (1) that is universally known like the *"Schwarzschild solution"*. In fact, now the transformation (31) can be enabled and the origin of the coordinate system, $r = 0$, $\theta = 0$, $\varphi = 0$, which is the surface of a sphere having radius r_g in the r, θ, φ coordinates, results transferred in a non-dimensional material point $\widehat{r} = 0$, $\theta = 0$, $\varphi = 0$ in the $\widehat{r}$, θ, φ coordinates. Such a non-dimensional material point corresponds to the point $r = -r_g$, $\theta = 0$, $\varphi = 0$ in the original r, θ, φ coordinates.

Then, the authors who claim that *"the original Schwarzschild solution"* leaves no room for the science fiction of the BHs, see [18, 19], [37] - [41], give the wrong answer.

We realize that the misunderstanding is due to an erroneous interpretation of the different coordinates. In fact, arches of circumference appear to be $2\pi r$ if the origin of the coordinate system is a non-dimensional material point in the core of the BH while they do not appear to be $2\pi r$, but deformed by the presence of the mass of the central body M if the origin of the coordinate system is the surface of the Schwarzschild sphere.

The only way to remove the singularity in the core of a BH within the classical theory of Einstein's general relativity is changing the hypotheses which govern the internal geometry of the collapsing star, following for example the ideas in references from [7] to [16].

4.   Conclusion remarks

In this paper we clarified a issue on the the debate on *"the original Schwarzschild solution"*. As English translations of Schwarzschild's original paper exist, that paper has become accessible to more people. A misunderstanding arises from the fact that, historically, the so-called "*standard Schwarzschild solution*" (1) was not the original Schwarzschild's work, but it is actually due to Droste [20] and Weyl [21]. The solution in refs. [20, 21] has been ultimately enabled like correct solution by Hilbert in [22]. Based on this, there are authors who claim that the work of Hilbert was wrong and that Hilbert's mistake spawned BHs and accuse the community of theoretical physicists to continue to elaborate on this falsehood, with a hostile shouting down of any and all voices challenging them [18, 19], [37] - [41].

With the goal to clarify the issue, we re-analysed *"the original Schwarzschild solution"* to Einstein field equations by showing that such a solution arises from an apparent different physical hypothesis which assumes arches of circumference to be *not* $2\pi r$, but deformed by the presence of the mass of the central body M. This assumption enables the origin of the coordinate system to be *not* a single point, but a spherical surface having radius equal to the gravitational radius, i.e. the surface of the Schwarzschild sphere. The solution works for the external geometry of a spherical static star and circumnavigates the Birkhoff theorem. After this, we discussed the simplest case of gravitational collapse, i.e. the spherical radial collapse of a star with uniform density and zero pressure, by turning attention to the interior of the collapsing object and the precise word line that its surface follows in the external geometry. The result is that the singularity within the totally collapsed spherical object remains. In fact, a coordinate transform that transfers the origin of the coordinate system, which is the surface of a sphere having radius equal to the gravitational radius, in a non-dimensional material point in the core of the black-hole, re-obtains the solution re-adapted by Hilbert. Thus, *"the original Schwarzschild solution"* [3] results physically equivalent to the solution enabled by Hilbert in [22], i.e. the solution that is universally known like "*the standard Schwarzschild solution*". We conclude that Hilbert was not wrong but they are definitively wrong the authors who claim that *"the original Schwarzschild solution"* implies the non existence of BHs [18, 19], [37] - [41]. The misunderstanding is due to an erroneous interpretation of the different coordinates. In

fact, arches of circumference appear to be $2\pi r$ if the origin of the coordinate system is a non-dimensional material point in the core of the black-hole, while they do not appear to be $2\pi r$, but deformed by the presence of the mass of the central body M if the origin of the coordinate system is the surface of the Schwarzschild sphere.

Therefore, the only way to remove the singularity in the core of a BH within the classical theory of Einstein's general relativity is changing the hypotheses which govern the internal geometry of the collapsing star, following for example the ideas in references from [7] to [16].

Acknowledgements

References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company (1973).

[2] P. C. W. Davies, Rep. Prog. Phys. 41, 1313–1355 (1978).

[3] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1916, 189-196 (1916).

[4] G. D. Birkhoff, *Relativity and Modern Physics*. Cambrigdge, MA: Harvard University Press. LCCN 23008297 (1923).

[5] S. Chandrasekhar, *The Highly Collapsed Configurations of a Stellar Mass*, Mont. Not. Roy. Astr. Soc. 91, 456–466 (1931).

[6] H. Nicolai, G. F. R. Ellis, A. Ashtekar and others, Gen. Rel. Grav. 41, 4, 673-1011, *Special Issue on quantum gravity* (April 2009).

[7] S. Robertson and D. Leiter, Mon. Roy. Astron. Soc. 50, 1391 (2004).

[8] R. Schild, D. Leiter and S. Robertson, Astron. J. 2, 420-432 (2006).

[9] C. Corda and H. J. Mosquera Cuesta, Mod. Phys. Lett A 25, 28, 2423-2429 (2010).

[10] A. Mitra, Phys. Rev. Lett. 81, 4774 (1998).

[11] A. Mitra, Phys. Rev. D, 74, 2, 024010 (2006).

[12] A. Mitra, Mon. Not. Roy. Astron. Soc. 369, 492 (2006).

[13] A. Mitra, Journ. Math. Phys. 50, 4, 042502-042503 (2009).

[14] J. P. S. Lemos, O.B. Zaslavskii, arXiv:1004.4651 (2010).

[15] A. G. Agnese and M. La Camera, Phys. Rev. D 31, 1280–1286 (1985).

[16] S. Robertson and D. Leiter, Astrophys. J. 596, L203-L206 (2003).

[17] A. Einstein, Ann. Math. 40, 4, 922–936 (1939).

[18] L. S. Abrams, Phys. Rev. D20, 2474 (1979), also in arXiv:gr-qc/0201044.

[19] S. Antoci and D.E. Liebscher, Gen. Rel. Grav. 35, 5, 945-950 (2003).

[20] J. Droste, Proc. K. Ned. Akad. Wet. 19, 197 (1917).

[21] H. Weyl, Ann. Phys. (Leipzig) 54, 117 (1917).

[22] D. Hilbert, Nachr. Ges. Wiss. G¨ttingen, Math. Phys. Kl., 53 (1917).

[23] A. Einstein, *"Zur allgemeinen Relativitätstheorie",* Sitzungsberichte der a Koniglich Preußischen Akademie der Wissenschaften 1915, 778-86 (1915).

[24] A. S. Eddington, Nature 113, 192 (1924).

[25] A. Einstein and G. Lemaître, TALKING OVER THE UNIVERSE AND ITS ORIGIN, Lectures in Pasadena, California, JWC-CX 1-13-33 [January 13, 1933] WC MET. (STAMPED JAN 14 1933).

[26] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. 55, 4, 374–381 (1939).

[27] R. Ruffini and J. A. Wheeler, Phys. Tod. 30–41 (1971).

[28] D. Finkelstein, Phys. Rev. 110, 965–967 (1958).

[29] M. Kruskal, Phys. Rev. 119, 1743–1745 (1960).

[30] A. Hewish, S. J. Bell and J. D. H Pilkington, Nature 217, 709–713 (1968).

[31] R. Kerr, Phys. Rev. Lett. 11, 237–238 (1963).

[32] E. T. Newman and A. Janis, Journ. Math. Phys. 6, 6, 915–917 (1965).

[33] S. Hawking, *A Brief History of Time,* Bantam Dell Publishing Group (1988).

[34] A. Ewing, Science News Letter of 18 (January 1964).

[35] H. J. Mosquera Cuesta and J. M. Salim, Mont. Not. Roy. Ast. Soc. 354, L55-L59 (2004).

[36] H. J. Mosquera Cuesta and J. M. Salim, Ap. J. 608, 925-929 (2004).

[37] L. S. Abrams, Can. J. Phys. 67, 919 (1989), also in arXiv:gr-qc/0102055.

[38] S. Antoci, in "Meteorological and Geophysical Fluid Dynamics (a book to commemorate the centenary of the birth of Hans Ertel)", W. Schröder Editor, Science Edition, Bremen, 2004 , also in arXiv:physics/0310104.

[39] S. J. Crothers, Prog. in Phys. Vol. 1 (2005).

[40] S. J. Crothers, Prog. in Phys. Vol. 2 (2005).

[41] A. Loinger and T. Marsico, Spacetime & Substance, No. 2 (17) (2003).

[42] L. Landau L and E. Lifshitz - *Classical Theory of Fields* (3rd ed.). London: Pergamon. ISBN 0-08-016019-0. Vol. 2 of the Course of Theoretical Physics (1971).

[43] B. G. Schmidt, Gen. Rel. Grav. 1, 269-280 (1971).