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Dimensionless Constants and Blackbody Radiation Laws

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Abstract: The fine structure constant $\alpha = \mathbf{e}^2/\hbar c \approx 1/137.036$ and the blackbody radiation constant $\alpha_R = \mathbf{e}^2(a_R/k_B^4)^{1/3} \approx 1/157.555$ are two dimensionless constants, derived respectively from a *discrete atomic spectra* and a *continuous radiation spectra* and linked by an infinite *prime* product. The blackbody radiation constant governs large density matter where oscillating charges emit or absorb photons that obey the Bose-Einstein statistics. The new derivations of Planck's law, the Stefan-Boltzmann law, and Wein's displacement law are based on the fine structure constant and a simple 3D interface model. The blackbody radiation constant provides a new method to measure the fine structure constant and links the fine structure constant to the Boltzmann constant.

Keywords: Blackbody Radiation; Planck's Law; Fine Structure Constant; Boltzmann Constant
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1. Introduction

Planck and Einstein each noted respectively in 1905 and 1909 that $\mathbf{e}^2/c \sim h$ have the same order and dimension.[1, 2] This was before Sommerfeld's introduction of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c$ in 1916.[3] Therefore, the search for a mathematical relationship between $\mathbf{e}^2/c \sim h$ began with blackbody radiation.[4] The Stefan-Boltzmann law states that the radiative flux density or irradiance is $J = \sigma T^4$ [$\mathrm{erg \cdot cm^{-2} \cdot s^{-1}}$] in CGS units. From the Planck law, the Stefan-Boltzmann constant $\sigma = 5.670400(40) \times 10^{-5}$ [$\mathrm{erg \cdot cm^{-2}K^{-4}s^{-1}}$] is

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$$\sigma = 2\pi \int_0^\infty \frac{x^3 dx}{e^x - 1} \cdot \frac{c k_B^4}{(hc)^3} = \frac{2\pi^5}{15} \frac{c k_B^4}{(hc)^3}$$
$$= 2\pi \Gamma(4) \zeta(4) \frac{c k_B^4}{(hc)^3} = \frac{4^2 \pi^5}{5!} \frac{c k_B^4}{(hc)^3} \tag{1}$$

The Stefan-Boltzmann law can be expressed as the volume energy density of a blackbody $\varepsilon_T = \boldsymbol{a}_R T^4$ [erg $\cdot$ cm^{-3}], where the radiation density constant $\boldsymbol{a}_R$ is linked to the Stefan-Boltzmann constant

$$\boldsymbol{a}_R = \frac{4\sigma}{c} = \frac{4^3 \pi^5}{5!} \frac{k_B^4}{(hc)^3} \tag{2}$$

In 1914, Lewis and Adams noticed that the dimension of the radiation density constant divided by the 4$^{\text{th}}$ power Boltzmann constant $\boldsymbol{a}_R / k_B^4$ is (energy $\times$ length)$^{-3}$, while $\mathbf{e}^2$ is (energy $\times$ length). However, they obtained an incorrect result equivalent to $\alpha^{-1} = \hbar c / \mathbf{e}^2 = 32\pi \left(\pi^5/5!\right)^{1/3} = 137.348$.[5] In 1915, Allen rewrote it as $\alpha = \mathbf{e}^2/\hbar c = (15/\pi^2)^{1/3}/(4\pi)^2$.[6]

In CGS units, $\mathbf{e}^2 = (4.80320427(12) \times 10^{-10})^2$ [erg$\cdot$cm], $\boldsymbol{a}_R = 7.56576738 \times 10^{-15}$ [erg $\cdot$ cm^{-3}K^{-4}], and $k_B^4 = (1.3806504(24) \times 10^{-16})^4$ [erg^{4}K^{-4}]. We get the experimental dimensionless constant[7]

$$\alpha_R = \mathbf{e}^2 \left(\frac{\boldsymbol{a}_R}{k_B^4}\right)^{1/3} = \frac{1}{\mathbf{157.5548787}}$$
$$= \mathbf{0.00634699482} \tag{3}$$

This is the dimensionless blackbody radiation constant α_R.[2]

2. Relationship to the Fine Structure Constant

The dimensionless blackbody radiation constant α_R is on the same order of the fine structure constant $\alpha = \mathbf{e}^2/\hbar c$, and equal to

$$\alpha_R = 0.8697668 \cdot \alpha$$
$$= \frac{2}{\pi}\left(\frac{\pi^5}{5!}\right)^{1/3} \alpha = \left(\frac{\Gamma(4)\zeta(4)}{\pi^2}\right)^{1/3} \alpha = \left(\frac{\pi^2}{15}\right)^{1/3} \alpha \tag{4}$$

Therefore, $\alpha_R \neq \alpha$, both α and α_R are experimental results incapable of producing the α math formula. Physically, the fine structure constant α is obtained from the atomic *discrete* spectra, while the blackbody radiation constant α_R is obtained from the thermal radiation of a 3D cavity in the *continuous* spectra. However, their relationship can be given by the Riemann zeta-function or by the modification of Euler's product formula

[2] Not to be confused with the Stefan-Boltzmann constant σ or hc/k (blackbody radiation constant)

$$\frac{\alpha_R^3}{\alpha^3} = \frac{\pi^2}{15} = \frac{\zeta(4)}{\zeta(2)} = \prod_{\mathrm{p}} \left(\frac{\mathrm{p}^2}{\mathrm{p}^2+1} \right) = \frac{2^2}{(2^2+1)} \frac{3^2}{(3^2+1)} \frac{5^2}{(5^2+1)} \frac{7^2}{(7^2+1)} \cdots \tag{5}$$

where the Euler product extends over all the *prime* numbers. In other words, the fine structure constant and the blackbody radiation constant can be linked by the prime numbers.

The pattern of Planck spectra is given by $f(x) = x^3/(e^x - 1)$ where the photon $h\nu$ is hidden in the argument $x = h\nu/k_B T$. The *photon* integral in (1) is equal to a dimensionless constant (**Fig. 1**)

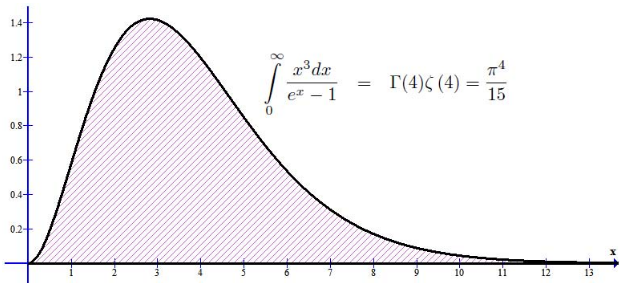

Figure 1 Photon integral is a dimensionless number $\Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 6.4939394$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \Gamma(4)\zeta(4) = \frac{\pi^4}{15} = 2 \cdot 3 \cdot \prod_{\mathrm{p}} \left(\frac{\mathrm{p}^4}{\mathrm{p}^4 - 1} \right) = 2 \cdot 3 \cdot \frac{2^4}{(2^4-1)} \frac{3^4}{(3^4-1)} \frac{5^4}{(5^4-1)} \frac{7^4}{(7^4-1)} \cdots \tag{6}$$

where the Euler product extends over all the *prime* numbers. The photon distribution integral (6) yields a zeta-function that is linked to the Euler prime products. (5) and (6) clearly show how the fine structure constant α for the *discrete* spectra in (4) is converted to the blackbody radiation constant α_R for the *continuous* spectra by multiplying a dimensionless constant (**Fig. 2**). (5) and (6) also indicate that this dimensionless constant can be expressed as an Euler infinite prime number product.

Figure 2 α and α_R from the *discrete and continuous* spectra.

3. Photon-Gas Model

From (5), the Stefan-Boltzmann law written as the volume energy density of a blackbody ε_T is related to the fine structure constant α and the oscillating charge $\mathbf{e}^2$ with different resonating frequencies in a cavity $\varepsilon_T = \boldsymbol{a}_R T^4 = \frac{4\sigma}{c} T^4$

$$\varepsilon_T = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 T^4 = \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{7}$$

and the radiative flux density is $J = \sigma T^4$

$$J = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4} k_B^4 T^4 = \frac{c}{4} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{8}$$

and the total brightness of a blackbody is $B = J/\pi$

$$B = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{c}{4\pi} k_B^4 T^4 = \frac{c}{4\pi} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{9}$$

and the inner wall pressure of the blackbody cavity is $P = \frac{4\sigma}{3c} T^4$

$$P = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{1}{3} k_B^4 T^4 = \frac{1}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{10}$$

According to the Bose-Einstein model of photon-gas,[8] the free energy in the thermodynamics is $F = -PV = -\frac{4\sigma}{3c} V T^4$

$$F = -\frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{V}{3} k_B^4 T^4 = -\frac{V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{11}$$

and the total radiation energy is $E = -3F = 3PV = \frac{4\sigma}{c} V T^4$

$$E = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 V k_B^4 T^4 = V \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{12}$$

where the photon gas $E = 3PV$ is the same as the extreme relativistic electron gas, and the entropy is $S = -\frac{\partial F}{\partial T} = \frac{16\sigma}{3c} V T^3$

$$S = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{4V}{3} k_B^4 T^3 = \frac{4V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{13}$$

and the specific heat of the radiation is $C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{16\sigma}{c} V T^3$

$$C_V = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 4V k_B^4 T^3 = 4V \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^3 \tag{14}$$

We have

$$N k_B T = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{V}{3} k_B^4 T^4 = \frac{V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{15}$$

From $PV = N k_B T$, the total number of photons in blackbody radiation is

$$N = \frac{\zeta(4)}{\zeta(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{V}{3} k_B^4 T^4 = \frac{V}{3} \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 T^4 \tag{16}$$

Landau assumed that the volume V in (11)∼(16) must be sufficiently large in order to change from *discrete* to *continuous* spectra.[7] Planck's law is violated at microscopic length scales. Experimentally, solids or dense-gas have the continuous spectra, and hot low-density gas emits the discrete atomic spectra. The photon $h\nu$ is hidden in $f(x) = x^3/(e^x - 1)$ where $x = h\nu/k_B T$, therefore, there is no $h\nu$ in (7)∼(16). In (7)∼(16), the charged oscillators $\alpha/\mathbf{e}^2 = 1/\hbar c$ or $\alpha_R/\mathbf{e}^2$ play a critical role in the electromagnetic coupling on a 3D surface (**Fig. 3**). Therefore, the traditional 3D box (or sphere) model is not necessarily composed of solid walls; the plasma gas photosphere layer of a star can have the same effect.

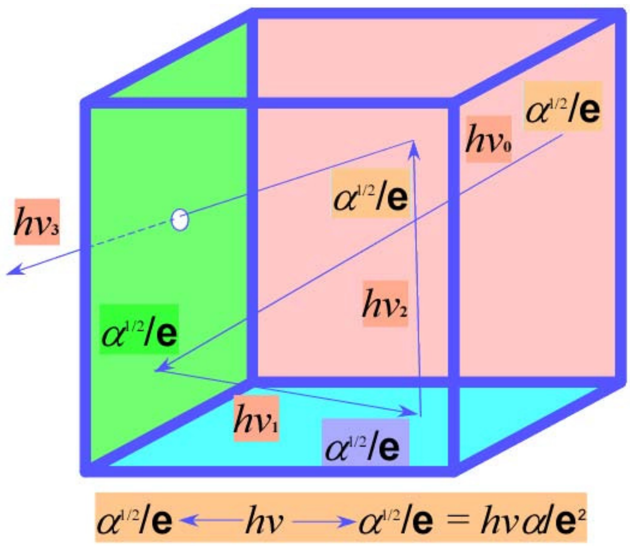

Figure 3 Blackbody radiation is related to α and $\mathbf{e}^2$ in a 3D cavity.

4. Planck's Law and the Stefan-Boltzmann law

Planck's original formula is a experimental fitting result. There are many derivations to explain the blackbody radiation law, including Planck in 1901, Einstein in 1917, Bose in 1924, Pauli in 1955.[9, 10, 11] We are not reinventing the blackbody radiation law, but instead pointing out that the surface charge is Planck's oscillator, and it is related to the fine structure constant. Planck's radiation law is derived as the result of the 3D interface interaction between photons and charged particles.

Using the 3D surface charge model in **Fig. 3** and the energy quanta $\varepsilon = h\nu$, Planck's law in terms of the spectral energy density in $[\mathrm{erg} \cdot \mathrm{cm}^{-3} \cdot \mathrm{sr}^{-1} \cdot \mathrm{Hz}^{-1}]$ can be rewritten as

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1} \tag{17}$$

$$= h \cdot \left(\frac{1}{\pi}\right)^2 \left(\alpha \frac{h\nu}{\mathbf{e}^2}\right)^3 \frac{1}{e^{h\nu/k_B T} - 1}$$
$$u(\varepsilon,\, T) = h \cdot \left(\frac{1}{\pi}\right)^2 \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{\varepsilon^3}{e^{\varepsilon/k_B T} - 1}$$

where $1/\pi$ is related to a solid angle in fractions of the sphere [$1sr = 1/4\pi$ fractional area]; and the interaction ratio of photon $h\nu$ and charge $\mathbf{e}^2$ is regulated by fine structure constant α, the cubic term involving the closed 3D cavity wall; Under the statistic thermodynamic equilibrium, Bose-Einstein distribution can be derived as

$$\frac{\sum_{n=0}^{\infty} n e^{-nh\nu/k_B T}}{\sum_{n=0}^{\infty} e^{-h\nu/k_B T}} = \frac{1}{e^{h\nu/k_B T} - 1} \tag{18}$$

The frequency of photons in $\varepsilon = h\nu$ is constantly shifted into a continuous spectra through photon-electron scattering, such as the Compton effect

$$\nu' = \frac{\nu}{1 + (h\nu/m_e c^2)(1 - \cos\theta)} \tag{19}$$

Fig. 3 shows that the scattering angle θ varies during each photon-electron interaction involving the fine structure constant. For $\varepsilon_T = \boldsymbol{a}_R T^4$, using $d\nu = (1/h)d\varepsilon$

$$\varepsilon_T = \int_0^\infty \frac{u(\varepsilon,\, T)d\varepsilon}{h} = \left(\frac{1}{\pi}\right)^2 \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 (k_B T)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{20}$$
$$= \frac{\Gamma(4)\zeta\,(4)}{\pi^2} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 (k_B T)^4 = \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 (k_B T)^4$$

This links the quantum theory to the classical theory of blackbody radiation with or without using the Planck constant

$$\boldsymbol{a}_R = \frac{\zeta\,(4)}{\zeta\,(2)} \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 = \left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{\zeta\,(4)}{\zeta\,(2)} \left(\frac{2\pi}{hc}\right)^3 k_B^4 \tag{21}$$
$$= \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 \cdot \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \frac{289}{290} \cdot \frac{361}{362} \cdots$$

Planck's law in terms of the spectral radiative intensity or the spectral radiance in [$\mathrm{erg} \cdot \mathrm{s}^{-1} \cdot \mathrm{cm}^{-2} \cdot \mathrm{sr}^{-1} \cdot \mathrm{Hz}^{-1}$] has a electromeganetic radiation with lightspeed c and $\sigma = c\boldsymbol{a}_R/4$, therefore, we multiply $c/4$ on $u(\nu,\, T)$ in (17)

$$I(\nu,\, T) = \frac{c}{4} \cdot u(\nu,\, T) = \frac{2\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} \tag{22}$$
$$= (hc) \cdot \left(\frac{1}{2\pi}\right)^2 \left(\alpha \frac{h\nu}{\mathbf{e}^2}\right)^3 \frac{1}{e^{h\nu/k_B T} - 1}$$
$$I(\varepsilon,\, T) = (hc) \cdot \left(\frac{1}{2\pi}\right)^2 \left(\frac{\alpha}{\mathbf{e}^2}\right)^3 \frac{\varepsilon^3}{e^{\varepsilon/k_B T} - 1}$$

For $J = \sigma T^4$,

$$J = \int_0^\infty \frac{I(\varepsilon, T)d\varepsilon}{h} = \left(\frac{c}{4}\right)\left(\frac{1}{\pi}\right)^2\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 (k_B T)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$
$$= \left(\frac{c}{4}\right)\frac{\Gamma(4)\zeta\,(4)}{\pi^2}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 (k_B T)^4 = \frac{c}{4}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 (k_B T)^4 \tag{23}$$

and the Stefan-Boltzmann constant is

$$\sigma = \frac{c}{4}\frac{\zeta\,(4)}{\zeta\,(2)}\left(\frac{\alpha}{\mathbf{e}^2}\right)^3 k_B^4 = \frac{c}{4}\left(\frac{\alpha_R}{\mathbf{e}^2}\right)^3 k_B^4 \tag{24}$$

The Planck constant h with the revolutionary concept of energy quanta is a bridge between classical physics and quantum physics. Einstein's proposal of the light quanta $h\nu$ in 1905 was based on the Planck constant. In QED, the photon is treated as a gauge boson, and the perturbation theory involves the finite power series in α. The discrete-continuous spectra is bridged by the Bose-Einstein distribution, and the *prime* sequences link the fine structure constant α to the blackbody radiation constant α_R.

5. Wien's Displacement Law

Wien's frequency displacement law is

$$\nu_{max} = b_\nu \cdot T \tag{25}$$

where $b_\nu = 5.878933(10) \times 10^{10}$ [Hz · K] in CODATA-2006. It has the numerical solution from

$$e^x(x-3) + 3 = 0 \tag{26}$$

where $x_\nu = h\nu/k_B T = 3 + W_0(-3e^{-3}) \approxeq 2.8214393721220788934$. Lambert W-function $W_0(x)$

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n \tag{27}$$

is a convergent series for $|x| < 1/e$, and

$$\nu_{max} = \frac{[3 + W_0(-3e^{-3})]k_B}{h} \cdot T \tag{28}$$

i.e.,

$$b_\nu = [3 + W_0(-3e^{-3})]\frac{\alpha}{2\pi}\frac{c}{e^2}\frac{k_B}{k_e} \tag{29}$$

Wien's wavelength displacement law is

$$\lambda_{max} = \frac{b_\lambda}{T} \tag{30}$$

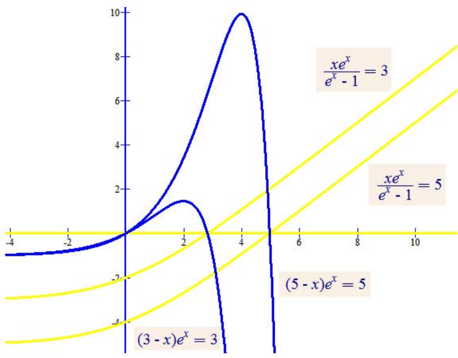

Figure 4 Numerical solution for $e^x(x - N) + N = 0$.

where $b_\lambda = 2.8977685(51) \times 10^{-3}$ [m · K] in CODATA-2006. It has a numerical solution from

$$e^x(x - 5) + 5 = 0 \tag{31}$$

where $x_\lambda = hc/\lambda_{max} k_B T = 5 + W_0(-5e^{-5}) \approxeq 4.96511423174427630 37$, and

$$b_\lambda = \frac{1}{5 + W_0(-5e^{-5})} \left(\frac{hc}{k_B}\right) \tag{32}$$

It is noticed that λ has a dimension of $V^{1/3} = k_B/\boldsymbol{a}_R = ck_B/\sigma = hc/k_B$. We can get another good approximate for b_λ in (32)

$$\begin{aligned} b_\lambda \approxeq \left(\frac{ck_B}{3\sigma}\right)^{1/3} &= \left(\frac{4}{3}\right)^{1/3} \frac{k_e e^2}{k_B \alpha_R} \\ = \left(\frac{20}{\pi^2}\right)^{1/3} \cdot \frac{\hbar c}{k_B} &= \left(\frac{5}{2\pi^5}\right)^{1/3} \frac{hc}{k_B} \end{aligned} \tag{33}$$

where $b_\lambda = 2.897720 \times 10^{-3}$ [m · K] with a uncertainty $u_r = 1.6 \times 10^{-5}$ to CODATA-2006. From $x_\lambda / x_\nu = 1.75978058603821178870$, we can get

$$\begin{aligned} b_\nu &\approxeq \frac{c \cdot x_\nu}{b_\lambda x_\lambda} = \frac{c \cdot [5 + W_0(-5e^{-5})]}{b_\lambda [3 + W_0(-3e^{-3})]} \\ &= 5.8790302 \times 10^{10} \text{ [Hz} \cdot \text{K]} \end{aligned} \tag{34}$$

From Wien's displacement law, the Boltzmann constant can be directly linked to the fine structure constant

$$\begin{aligned} k_B &= \frac{b_\nu h}{[3 + W_0(-3e^{-3})]} \\ &= \frac{b_\nu 2\pi k_e e^2}{[3 + W_0(-3e^{-3})]\alpha c} \\ &= \frac{2\pi k_e e^2}{b_\lambda [5 + W_0(-5e^{-5})]\alpha} \end{aligned} \tag{35}$$

The blackbody radiation constant is a new method to measure the fine structure constant. It links the fine structure constant to the Boltzmann constant.

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