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On a General Class of Solutions of a Nonholonomic Extension of Optical Pulse Equation

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Abstract: A Nonholonomic extension of an equation obeyed by short pulse in non-linear optics is obtained.A general class of solutions of such an equation is obtained with the help of Riemann-Hilbert technique.

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1. Introduction

Recent literature of integrable systems has been enriched by quite a few publications discussing a new class of systems which are nonholonomic in the sense that they are accompanied by differential constraints [1] those cannot be explicitly solved . This actually enlarges the integrable class itself. People have studied some interesting cases associated with AKNS [2] systems Kaup-Newell equation and so on . Here in this communication we have obtained a nonholonomic generalization of a very special type of equation which governs the propagation of a short optical pulse in nonlinear optics [3]. In the second part we have shown how a general class of solutions of such an equation can be generated through the Riemann Hilbert method [6].

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2. Formulation

To start with, we consider the Lax pair

$$\psi_x = L\psi \tag{1}$$

$$\psi_t = M\psi \tag{2}$$

$$L = \begin{pmatrix} -\phi_x & 1 \\ -\lambda & \phi_x \end{pmatrix} ; N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

whence consistency leads to

$$\phi_{xt} + A_x - C - \lambda B = 0 \tag{3}$$

$$B_x + 2A + 2\phi_x B = 0 \tag{4}$$

$$C_x + 2\lambda A - 2\phi_x C = 0 \tag{5}$$

Next we set

$$A = a_{-1}\lambda^{-1} + a_2 + a_1\lambda + a_0\lambda^2 \tag{6}$$

$$B = b_{-1}\lambda^{-1} + b_2 + b_1\lambda + b_0\lambda^2 \tag{7}$$

$$C = c_{-1}\lambda^{-1} + c_2 + c_1\lambda + c_0\lambda^2 \tag{8}$$

The unknown coefficients are determined in a recursive manner and are given as

$$\begin{aligned} a_0 &= b_0 = 0 \\ c_0 &= \eta_2 \\ a_1 &= \eta_2 \phi_x \\ b_1 &= -\eta_2 \end{aligned} \tag{9}$$

$$\begin{aligned} c_1 &= \frac{1}{2}\eta_2\left(\phi_{xx} + \phi_x^2\right) \\ a_2 &= -\frac{1}{4}\eta_2\left(\phi_{xxx} - 2\phi_x^3\right) \\ b_2 &= \frac{1}{2}\eta_2\left(\phi_{xx} - \phi_x^2\right) \\ c_{-1} &= \eta_1 \exp\left(2\phi\right) \end{aligned} \tag{10}$$

where η_1 and η_2 are constants .
Along with ;

$$a_{-1x} = \eta_1 \exp\left(2\phi\right) \tag{11}$$

$$b_{-1xx} + 2\phi_{xx}b_{-1} + 2\phi_x b_{-1x} + 2\eta_1 \exp\left(2\phi\right) = 0 \tag{12}$$

$$c_{2xx} - 2\phi_{xx}c_2 - 2\phi_x c_{2x} + 2\eta_1 \exp\left(2\phi\right) = 0 \tag{13}$$

and the nonlinear equation ;

$$\phi_{xt} + a_{2x} - c_2 - b_{-1} = 0 \tag{14}$$

It is interesting to note that c_2 , b_{-1} are to be determined only as solutions of eq(8) and eq(9) , which really leads to nonlocal nonholonomic constraints . To proceed further we consider the special case $\eta_1 = 0$, which leads to $a_{-1} =$ constant ; along with $c_2 = \eta_3 \exp(2\phi)$ and $b_{-1} = \eta_4 \exp(-2\phi)$ so that the nonlinear equations turns out to be ;

$$\phi_{xt} = \phi_{xxxx} - 6\phi_x^2 \phi_{xx} + 2\eta \sinh(2\phi) . \tag{15}$$

by proper choice of η_i. Equation (11) is nothing but the nonlinear equation obeyed by a short optical pulse propagating in an optical fibre . So, if we assume $\eta_1 \neq 0$, then eq(10) can be consider as a nonholonomic generalization of eq (11)[5] .

3. General Solutions

To analyze the solutions of the new equation [14] we take recourse to the Riemann-Hilbert technique [4]. The usual approach is to assume that the analytic wave function $\phi(\lambda)$, with the constraint

$$\phi_1(\lambda)\,\phi_2(\lambda) = G(\lambda) \tag{16}$$

where ϕ_1 , ϕ_2 are the boundary values of ϕ on the interior and exterior of a close Jordan curve Γ in the complex λ plane . In case of Riemann - Hilbert problem with simple poles we assume $G(\lambda) = 1$. Let ϕ_0 be the starting seed solution of eq [14] with ψ_0 , the corresponding Lax eigenfunction [5] . We can set

$$\psi = \chi\psi_0 \tag{17}$$

If the lax operator corresponding to ϕ_0 be denoted as L_0 , then one gets

$$\mathrm{L} = \chi_x \chi^{-1} + \chi L_0 \chi^{-1} \tag{18}$$

To formulate the Riemann-Hilbert problem we assume that χ contains simple pole in λ - plane ,

$$\chi = \left(1 + \frac{S}{\lambda - \lambda_1}\right) \tag{19}$$

along with

$$\chi^{-1} = \left(1 + \frac{R}{\lambda - \lambda_1}\right) \tag{20}$$

From the condition $\chi\chi^{-1} = 1$ one gets

$$\mathrm{S} = -\mathrm{R} = (\lambda_1 - \mu_1)\,P \tag{21}$$

where P is the projection operator ($P^2 = P$) . Also from Eq.[18] we get

$$\mathrm{L} = \mathrm{L}_0 - (\lambda_1 - \mu_1)[P, \sigma_-] \tag{22}$$

The Lax operator has some interesting symmetry property

$$\sigma_1 L^T \sigma_1 = L(-\phi, \lambda) \tag{23}$$

$$\sigma_2 L^T \sigma_2 = -L\left(\phi, \lambda\right) \tag{24}$$

with σ_1 σ_2 being Pauli matrices and $\sigma_- = \sigma_1 - i\sigma_2$. An easy seed solution is $\phi = 0$, whence we require the Lax eigenfunction as solution of ;

$$\psi_x = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \psi \tag{25}$$

$$\text{and} \quad \psi_t = \begin{pmatrix} \eta_1 x \lambda^{-1} & -\eta_1 x^2 \lambda^{-1} - \eta_2 \lambda \\ \eta_1 \lambda^{-1} - \eta_1 x^2 + \eta_2 \lambda^2 & -\eta_1 x \lambda^{-1} \end{pmatrix} \psi \tag{26}$$

writing out in component form we get one partial differential equation

$$\psi_{2t} + \left(-\eta x^2 + \eta\lambda^{-1} - \lambda^2\right) \frac{\psi_{2x}}{\lambda} = -\eta x \lambda^{-1} \psi_2 \tag{27}$$

Now it is easy to show that an equation of the form ;

$$f(x)\omega_x + g(y)\omega_y = [\delta_1(x) + \delta_2(y)]\omega \tag{28}$$

has a solution of the form

$$\omega = exp\left[\int \frac{\delta_1\left(x\right)}{f\left(x\right)} dx + \int \frac{\delta_2\left(y\right)}{g\left(y\right)} dy\right] \Theta\left(\int \frac{1}{f\left(x\right)} dx - \int \frac{1}{g\left(y\right)} dy\right) \tag{29}$$

where Θ is an arbitrary function of its argument. Finally using the explicit of function δ_1 and δ_2 from Eqs.[26], we arrive at

$$\psi_1 = \frac{\mu(\theta)}{\sqrt{\lambda - \eta\lambda^{-1}x^2}} \tag{30}$$

$$\psi_2 = \frac{\eta\lambda^{-1} x \mu(\theta)}{\sqrt{\lambda - \eta\lambda^{-1}x^2}} + \frac{\mu_x(\theta)}{\sqrt{\lambda - \eta\lambda^{-1}x^2}} \tag{31}$$

where $\theta = t + \frac{1}{2a\eta\lambda^{-1}} \ln\left(\frac{x+a}{x-a}\right)$, along with $a^2 = \frac{\lambda^3}{\eta}$. The Projection operator is now constructed as,

$$\mathrm{P} = \frac{\mid p >< q \mid}{< p \mid q >} \tag{32}$$

where $\mid p >$ and $\mid q >$ are given by;

$$\mid p >= \begin{pmatrix} a_1\psi_1(\lambda_1) \\ a_2\psi_2(\lambda_1) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{33}$$

$$\mid q >= \begin{pmatrix} b_1\psi_1(\mu_1) \\ b_2\psi_2(\mu_1) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{34}$$

where Eq.(22) leads to

$$\phi_x = (\lambda_1 - \mu_1)\frac{u_1 v_2}{u_1 v_1 + u_2 v_2} \tag{35}$$

The form of the solution is clearly different from the usual soliton like profile sustained by original optical pulse equation . The absence of any wavefront $(x - vt)$ suggests that it is a nonpropagating solution .

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