A Universal Nonlinear Control Law for the
Synchronization of Arbitrary 4-D
Continuous-Time Quadratic Systems

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Abstract: In this letter we show the existence of a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 4-D continuous-time quadratic systems.

Keywords: Synchronization; Universal Nonlinear Control Law; 4-D Continuous-Time Quadratic Systems

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1. Introduction

There are several methods of chaos synchronization. For example in [9], a method is introduced to synchronize two identical chaotic systems with different initial conditions. An adaptive control approach is presented in [10], a backstepping design is given in [13], an active control method is presented in [5-11,12], and a nonlinear control scheme is given in [4,6,8]. In fact, there are many applications of chaos synchronization in physical, chemical, and ecological systems, and in secure communications [1,2,3,7,9,10]. For 4-D continuous-time quadratic systems, some of these methods were applied to Lorenz-Stenflo systems, Qi systems, and other hyperchaotic quadratic systems as shown in [14,15,16,17,18,19,20]. In this letter, we apply nonlinear control theory to synchronize

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two arbitrary, 4-D, continuous-time, quadratic systems. This control law is a universal synchronization approach since it does not need any conditions on the considered systems.

2. Synchronization Using A Universal Nonlinear Control Law

In this section, we consider two arbitrary, 4-D, continuous-time, quadratic systems. The one with variables \(x_1, y_1, z_1,\) and \(u_1\) will be controlled to be the new system given by

\[
\begin{align*}
\begin{cases}
x'_1 = \mu_1 + \beta_{11}x_1 + \beta_{12}y_1 + \beta_{13}z_1 + \beta_{14}u_1 + f_1(x_1, y_1, z_1, u_1) \\
y'_1 = \mu_2 + \beta_{21}x_1 + \beta_{22}y_1 + \beta_{23}z_1 + \beta_{24}u_1 + f_2(x_1, y_1, z_1, u_1) \\
z'_1 = \mu_3 + \beta_{31}x_1 + \beta_{32}y_1 + \beta_{33}z_1 + \beta_{34}u_1 + f_3(x_1, y_1, z_1, u_1) \\
u'_1 = \mu_4 + \beta_{41}x_1 + \beta_{42}y_1 + \beta_{43}z_1 + \beta_{44}u_1 + f_4(x_1, y_1, z_1, u_1)
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= a_4x_1^2 + a_5y_1^2 + a_6z_1^2 + a_7u_1^2 + a_8x_1y_1 + a_9x_1z_1 + p_1 \\
f_2 &= b_4x_1^2 + b_5y_1^2 + b_6z_1^2 + b_7u_1^2 + b_8x_1y_1 + b_9x_1z_1 + p_2 \\
f_3 &= c_4x_1^2 + c_5y_1^2 + c_6z_1^2 + c_7u_1^2 + c_8x_1y_1 + c_9x_1z_1 + p_3 \\
f_4 &= d_4x_1^2 + d_5y_1^2 + d_6z_1^2 + d_7u_1^2 + d_8x_1y_1 + d_9x_1z_1 + p_4 \\
p_1 &= a_{10}y_1z_1 + a_{11}x_1u_1 + a_{12}z_1u_1 + a_{13}y_1u_1 \\
p_2 &= b_{10}y_1z_1 + b_{11}x_1u_1 + b_{12}z_1u_1 + b_{13}y_1u_1 \\
p_3 &= c_{10}y_1z_1 + c_{11}x_1u_1 + c_{12}z_1u_1 + c_{13}y_1u_1 \\
p_4 &= d_{10}y_1z_1 + d_{11}x_1u_1 + d_{12}z_1u_1 + d_{13}y_1u_1
\end{align*}
\]

and the one with variables \(x_2, y_2, z_2,\) and \(u_2\) as the response system

\[
\begin{align*}
\begin{cases}
x''_2 = \delta_1 + \rho_{11}x_2 + \rho_{12}y_2 + \rho_{13}z_2 + \rho_{14}u_2 + g_1(x_2, y_2, z_2, u_2) + v_1(t) \\
y''_1 = \delta_2 + \rho_{21}x_2 + \rho_{22}y_2 + \rho_{23}z_2 + \rho_{24}u_2 + g_2(x_2, y_2, z_2, u_2) + v_2(t) \\
z''_1 = \delta_3 + \rho_{31}x_2 + \rho_{32}y_2 + \rho_{33}z_2 + \rho_{34}u_2 + g_3(x_2, y_2, z_2, u_2) + v_3(t) \\
u''_1 = \delta_4 + \rho_{41}x_2 + \rho_{42}y_2 + \rho_{43}z_2 + \rho_{44}u_2 + g_4(x_2, y_2, z_2, u_2) + v_4(t)
\end{cases}
\end{align*}
\]
where

\[
\begin{align*}
    g_1 &= h_4 x_2^2 + h_5 y_2^2 + h_6 z_2^2 + h_7 u_2^2 + h_8 x_2 y_2 + h_9 x_2 z_2 + p_5 \\
    g_2 &= m_4 x_2^2 + m_5 y_2^2 + m_6 z_2^2 + m_7 u_2^2 + m_8 x_2 y_2 + m_9 x_2 z_2 + p_6 \\
    g_3 &= r_4 x_2^2 + r_5 y_2^2 + r_6 z_2^2 + r_7 u_2^2 + r_8 x_2 y_2 + r_9 x_2 z_2 + p_7 \\
    g_4 &= s_4 x_2^2 + s_5 y_2^2 + s_6 z_2^2 + s_7 u_2^2 + s_8 x_2 y_2 + s_9 x_2 z_2 + p_8 \\
    p_5 &= h_{10} y_2 z_2 + h_{11} x_2 u_2 + h_{12} z_2 u_2 + h_{13} y_2 u_2 \\
    p_6 &= m_{10} y_2 z_2 + m_{11} x_2 u_2 + m_{12} z_2 u_2 + m_{13} y_2 u_2 \\
    p_7 &= r_{10} y_2 z_2 + r_{11} x_2 u_2 + r_{12} z_2 u_2 + r_{13} y_2 u_2 \\
    p_8 &= s_{10} y_2 z_2 + s_{11} x_2 u_2 + s_{12} z_2 u_2 + s_{13} y_2 u_2
\end{align*}
\]

(4)

Here \((\mu_i, \delta_i)_{1 \leq i \leq 4} \in \mathbb{R}^8\) and \((\beta_{ij}, \rho_{ij})_{1 \leq i \leq 4, 1 \leq j \leq 4} \in \mathbb{R}^{16}\) and \((a_i, b_i, c_i, d_i, h_i, m_i, r_i, s_i)_{4 \leq i \leq 13} \in \mathbb{R}^{80}\) are bifurcation parameters, and \(v_1(t), v_2(t), v_3(t),\) and \(v_4(t)\) are the unknown nonlinear controller such that two systems (1)-(2) and (3)-(4) can be synchronized.

First, let us define the following quantities depending on the above two systems:

\[
\begin{align*}
    \eta_1 &= h_7 u_2^2 - a_9 u_1 x_2 + h_{13} u_1 y_2 - a_{12} u_1 z_2 + \rho_{14} u_1 - a_7 u_2^2 + h_9 u_2 x_1 \\
    \eta_2 &= -a_{13} u_2 y_1 + h_{12} u_2 z_1 - \beta_{14} u_2 + h_4 x_1^2 + h_8 x_1 y_2 + h_9 x_1 z_2 + \rho_{11} x_1 \\
    \eta_3 &= -a_4 x_2^2 - a_8 x_2 y_1 - a_9 x_2 z_1 - \beta_{11} x_2 + h_5 y_1^2 + h_{10} y_1 z_2 + \rho_{12} y_1 - a_5 y_2^2 \\
    \eta_4 &= -a_{10} y_2 z_1 - \beta_{12} y_2 + h_6 z_1^2 + \rho_{13} z_1 - a_6 z_2^2 - \beta_{13} z_2 - \mu_1 + \delta_1 \\
    \eta_5 &= m_7 u_1^2 - b_9 u_1 x_2 + m_{13} u_1 y_2 - b_{12} u_1 z_2 + \rho_{21} u_1 - b_7 u_2^2 + m_9 u_2 x_1 \\
    \eta_6 &= -b_{13} u_2 y_1 + m_{12} u_2 z_1 - \beta_{14} u_2 + m_4 x_1^2 + m_8 x_1 y_2 + m_9 x_1 z_2 + \rho_{21} z_1 \\
    \eta_7 &= -b_4 x_2^2 - b_8 x_2 y_1 - b_9 x_2 z_1 - \beta_{21} x_2 + m_5 y_1^2 + m_{10} y_1 z_2 + \rho_{22} y_1 - b_5 y_2^2 \\
    \eta_8 &= -b_{10} y_2 z_1 - \beta_{22} y_2 + m_6 z_1^2 + \rho_{23} z_1 - b_6 z_2^2 - \beta_{23} z_2 - \mu_2 + \delta_2 \\
    \eta_9 &= r_7 u_1^2 - c_9 u_1 x_2 + r_{13} u_1 y_2 - c_{12} u_1 z_2 + \rho_{34} u_1 - c_7 u_2^2 + r_9 u_2 x_1 \\
    \eta_{10} &= -c_{13} u_2 y_1 + r_{12} u_2 z_1 - \beta_{34} u_2 + r_4 x_1^2 + r_8 x_1 y_2 + r_9 x_1 z_2 + \rho_{11} x_1 \\
    \eta_{11} &= -c_4 x_2^2 - c_8 x_2 y_1 - c_9 x_2 z_1 - \beta_{31} x_2 + r_5 y_1^2 + r_{10} y_1 z_2 + \rho_{32} y_1 - c_5 y_2^2 \\
    \eta_{12} &= -c_{10} y_2 z_1 - \beta_{32} y_2 + r_6 z_1^2 + \rho_{33} z_1 - c_6 z_2^2 - \beta_{33} z_2 - \mu_3 + \delta_3 \\
    \eta_{13} &= s_7 u_1^2 - d_9 u_1 x_2 + s_{13} u_1 y_2 - d_{12} u_1 z_2 + \rho_{44} u_1 - d_7 u_2^2 + s_9 u_2 x_1 \\
    \eta_{14} &= -d_{13} u_2 y_1 + s_{12} u_2 z_1 - \beta_{44} u_2 + s_4 x_1^2 + s_8 x_1 y_2 + s_9 x_1 z_2 + \rho_{41} x_1 \\
    \eta_{15} &= -d_4 x_2^2 - d_8 x_2 y_1 - d_9 x_2 z_1 - \beta_{41} x_2 + s_5 y_1^2 + s_{10} y_1 z_2 + \rho_{42} y_1 - d_5 y_2^2 \\
    \eta_{16} &= -d_{10} y_2 z_1 - \beta_{42} y_2 + s_6 z_1^2 + \rho_{43} z_1 - d_6 z_2^2 - \beta_{43} z_2 - \mu_4 + \delta_4
\end{align*}
\]

(5)

(6)
\[
\begin{align*}
\xi_1 &= \beta_{11} + \rho_{11} + (a_4 + h_4) (x_1 + x_2) + a_9 u_1 + a_8 y_1 + a_9 z_1 + h_9 u_2 + h_8 y_2 + h_9 z_2 \\
\xi_2 &= \beta_{12} + \rho_{12} + (a_5 + h_5) (y_1 + y_2) + a_{10} z_1 + z_2 h_{10} \\
\xi_3 &= \beta_{13} + \rho_{13} + (a_6 + h_6) (z_1 + z_2) + u_1 a_{12} + u_2 h_{12} \\
\xi_4 &= \beta_{14} + \rho_{14} + (a_7 + h_7) (u_1 + u_2) + y_1 a_{13} + y_2 h_{13} \\
\xi_5 &= \eta_1 + \eta_2 + \eta_3 + \eta_4
\end{align*}
\]
\[
\begin{align*}
\xi_6 &= \beta_{21} + \rho_{21} + (b_4 + m_4) (x_1 + x_2) + b_9 u_1 + b_8 y_1 + b_9 z_1 + m_9 u_2 + m_8 y_2 + m_9 z_2 \\
\xi_7 &= \beta_{22} + \rho_{22} + (b_5 + m_5) (y_1 + y_2) + b_{10} z_1 + z_2 m_{10} \\
\xi_8 &= \beta_{23} + \rho_{23} + (b_6 + m_6) (z_1 + z_2) + u_1 b_{12} + u_2 m_{12} \\
\xi_9 &= \beta_{24} + \rho_{24} + (b_7 + m_7) (u_1 + u_2) + y_1 b_{13} + y_2 m_{13} \\
\xi_{10} &= \eta_5 + \eta_6 + \eta_7 + \eta_8
\end{align*}
\]
\[
\begin{align*}
\xi_{11} &= \beta_{31} + \rho_{31} + (c_4 + r_4) (x_1 + x_2) + c_9 u_1 + c_8 y_1 + c_9 z_1 + r_9 u_2 + r_8 y_2 + r_9 z_2 \\
\xi_{12} &= \beta_{32} + \rho_{32} + (c_5 + r_5) (y_1 + y_2) + c_{10} z_1 + z_2 r_{10} \\
\xi_{13} &= \beta_{33} + \rho_{33} + (c_6 + r_6) (z_1 + z_2) + u_1 c_{12} + u_2 r_{12} \\
\xi_{14} &= \beta_{34} + \rho_{34} + (c_7 + r_7) (u_1 + u_2) + y_1 c_{13} + y_2 r_{13} \\
\xi_{15} &= \eta_9 + \eta_{10} + \eta_{11} + \eta_{12}
\end{align*}
\]
\[
\begin{align*}
\xi_{16} &= \beta_{41} + \rho_{41} + (d_4 + s_4) (x_1 + x_2) + d_9 u_1 + d_8 y_1 + d_9 z_1 + s_9 u_2 + s_8 y_2 + s_9 z_2 \\
\xi_{17} &= \beta_{42} + \rho_{42} + (d_5 + s_5) (y_1 + y_2) + d_{10} z_1 + z_2 s_{10} \\
\xi_{18} &= \beta_{43} + \rho_{43} + (d_6 + s_6) (z_1 + z_2) + u_1 d_{12} + u_2 s_{12} \\
\xi_{19} &= \beta_{44} + \rho_{44} + (d_7 + s_7) (u_1 + u_2) + y_1 d_{13} + y_2 s_{13} \\
\xi_{20} &= \eta_{13} + \eta_{14} + \eta_{15} + \eta_{16}
\end{align*}
\]

Now let the error states be \( e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1, \) and \( e_4 = u_2 - u_1. \) Then the error system is given by
\[
\begin{align*}
e_1' &= \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 + \xi_5 + v_1 (t) \\
e_2' &= \xi_6 e_1 + \xi_7 e_2 + \xi_8 e_3 + \xi_9 e_4 + \xi_{10} + v_2 (t) \\
e_3' &= \xi_{11} e_1 + \xi_{12} e_2 + \xi_{13} e_3 + \xi_{14} e_4 + \xi_{15} + v_3 (t) \\
e_4' &= \xi_{16} e_1 + \xi_{17} e_2 + \xi_{18} e_3 + \xi_{19} e_4 + \xi_{20} + v_4 (t)
\end{align*}
\]
In this letter, we propose the following universal control law for the system (3)-(4):

\[
\begin{align*}
v_1(t) &= -(1 + \xi_1) e_1 - \xi_5 \\
v_2(t) &= -(\xi_2 + \xi_6) e_1 - (1 + \xi_7) e_2 - \xi_{10} \\
v_3(t) &= -(\xi_3 + \xi_{11}) e_1 - (\xi_8 + \xi_{12}) e_2 - (1 + \xi_{13}) e_3 - \xi_{15} \\
v_4(t) &= -(\xi_4 + \xi_{16}) e_1 - (\xi_9 + \xi_{17}) e_2 - (\xi_{14} + \xi_{18}) e_3 - (1 + \xi_{19}) e_4 - \xi_{20}
\end{align*}
\]

(12)

Then the two 4-D, continuous-time, quadratic systems (1)-(2) and (3)-(4) approach synchronization for any initial condition, since the error system (11) becomes

\[
\begin{align*}
e_1' &= -e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 \\
e_2' &= -\xi_2 e_1 - e_2 + \xi_8 e_3 + \xi_9 e_4 \\
e_3' &= -\xi_3 e_1 - \xi_8 e_2 - e_3 + \xi_{14} e_4 \\
e_4' &= -\xi_4 e_1 - \xi_9 e_2 - \xi_{14} e_3 - e_4
\end{align*}
\]

(13)

and if we consider the Lyapunov function \( V = e_1^2 + e_2^2 + e_3^2 + e_4^2 \), then it is easy to verify the asymptotic stability of the error system (13) by Lyapunov stability theory since we have \( \frac{dV}{dt} = -e_1^2 - e_2^2 - e_3^2 - e_4^2 < 0 \) for all \( (\mu_i, \delta_i)_{1 \leq i \leq 4} \in \mathbb{R}^8 \) and \( (\beta_{ij}, \rho_{ij})_{1 \leq i, j \leq 4} \in \mathbb{R}^{16} \) and \( (a_i, b_i, c_i, d_i, h_i, m_i, r_i, s_i)_{4 \leq i \leq 13} \in \mathbb{R}^{80} \) and for all initial conditions. If the two systems (1)-(2) and (3)-(4) are chaotic, then the control law (12) guaranties also their synchronization for any initial condition. An elementary example of this situation can be found in [14].

Also, we notice that any 4-D, continuous-time, quadratic, chaotic system can be stabilized (resp. controlled) to a stable 4-D, continuous-time, quadratic system that converges to an equilibrium point (resp. to a 4-D, continuous-time, quadratic system that converges to a periodic solution). Furthermore, any 4-D, continuous-time, quadratic system can be chaotified to a chaotic 4-D, continuous-time, quadratic system.

**Conclusion**

We have presented a universal nonlinear control law (without any conditions) for the synchronization of arbitrary 4-D, continuous-time, quadratic systems. This control law (12) can be considered either as a stabilization, or as a control, or as a chaotification approach for a general 4-D, continuous-time, quadratic system.

**References**


