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Synchronization of Different Chaotic Fractional-Order Systems via Approached Auxiliary System the Modified Chua Oscillator and the Modified Van der Pol-Duffing Oscillator

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Abstract: In this paper we propose the study of synchronization between two different chaotic fractional-order systems via approached auxiliary system, we choose the modified Chua oscillators as a master system and the modified Van der Pol-Duffing oscillator (MVDPD) as a slave system, this method is also detected for both well known systems Chen and Lu. Routh-Hurwitz criterion is used for the study of stability of error system between the master-slave systems. Numerical results show the effectiveness of the theoretical analysis.

Keywords: Chaos; Chaotic Fractional-Order System, Auxiliary System; Synchronization; Routch –Hurwitz Criterion
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1. Introduction

In recent years considerable interest is assigned to the applications of fractional derivatives (of fractional order) in several areas. It has been found in the interdisciplinary fields, many systems can be described by Fractional differential equations. Recently, synchronization of fractional order chaotic systems is starting to attract increasing attention due to its potential applications in secure communications and process control [2].

During this decade, several types of synchronization (complete synchronization or identical generalized phase, projective) [12] have been studied and many methods have been proposed, but all these types and methods are encompassed in two modes of synchronization. The first method is based on a mutual coupling between two or more chaotic

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systems. The second is called master or slave-way coupling: Its principle is to choose a generator of chaos called "transmitter". The latter is described by equations recurring and characterized by its state variables constituting the state vector. Some components of this vector are transmitted to a second system, called "receiver". This paper will look at the second mode, and we chose the systems of Van der Pol-Duffing modified (MVDPD) and the system of Chua modified and we try to study the synchronization of both systems then those of Chen and LU using the criterion of Routh-Hurwitz generalized to fractional order to study the stabilization of error system.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order but there are several definitions of fractional derivatives. In this paper, we use the Caputo-type fractional derivative defined in [10] by:

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^(n)(\tau) d\tau = j^{n-q}(\frac{d^n}{dt^n} f(t))$$

Where $n = [q]$the value of is q rounded up to the nearest integer, $\Gamma$is the gamma function.

For the numerical solutions of the systems that we study, we use the Adams-Bash forth-Moulton predictor-corrector scheme [4].

2. Systems Description

The first system in which we are interested is the Van der pol-Duffing oscillator (MVDPD) which is an improved model of an autonomous chaotic system by King and Gaito in 1992 [14], in this paper we will be interested in playing fractional version of the system.

While the second is derived from the famous Chua's circuit [15-17] in fact, it is now obvious that both oscillators can be represented by the simplified and generic electrical circuit of Fig.1, The difference in their qualitative dynamical behaviour lies only on the physical realization of the nonlinear resistance (N), and of course on the parameters selection.

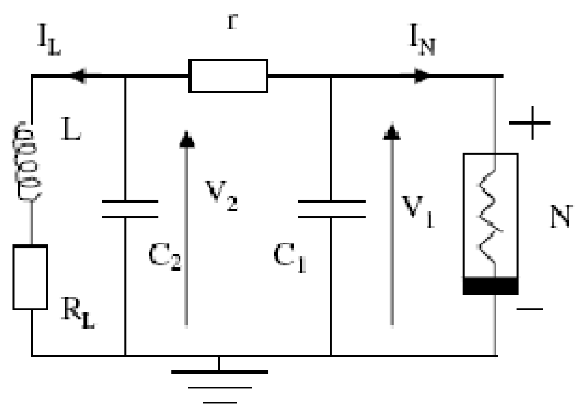


Fig. 1 The electrical model of the MVDPD oscillator.

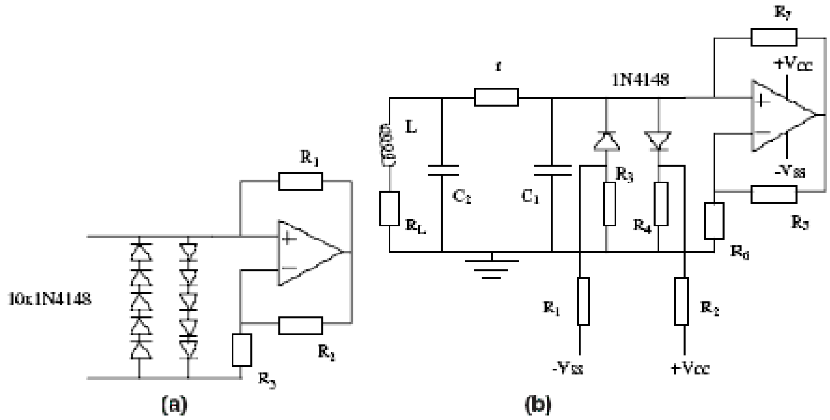


Fig. 2 (a) The physical realization of the nonlinear resistance and (b) the normal Chua oscillator.

2.1 The Fractional –Order (MVDPD) System

The fractional –order (MVDPD) system is given as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = m(y - x^3 + \alpha x + \mu) \\ \frac{d^q y}{dt^q} = x - y - z \\ \frac{d^q z}{dt^q} = \beta y - \gamma z \end{cases} \quad (1)$$

- That since the parameter μ is in general cancelled with the offset current of the op-amp; we further cancel it from our equations.
- Where q is the fractional order satisfying $q \in (0, 1]$, the parameters β, m, γ and α are all positive real, where $q = 1$, system **(1)** is the original integer-order (MVDPD) system exhibits chaotic behaviours with the parameter values $\beta = 200, m = 100, \gamma = 0.2$ *and* $\alpha = 0.1$
- To evaluate equilibrium points, let

$$\frac{d^q x}{dt^q} = 0, \frac{d^q y}{dt^q} = 0 \text{ and } \frac{d^q z}{dt^q} = 0$$

Then $E_0(0,0,0)$,$E_+(x_+,y_+,z_+)$, $E_-(x_-,y_-,z_-)$
Are the equilibrium points of system (1)
Where:

$$x_+ = \frac{1}{\beta}\frac{\beta+\gamma}{\beta^3+\gamma^3+3\beta\gamma^2+3\beta^2\gamma}\sqrt{\alpha\beta^6+\beta^5\gamma+4\alpha\beta^5\gamma+\beta^2\gamma^4+3\beta^3\gamma^3+3\beta^4\gamma^2+\alpha\beta^2\gamma^4+4\alpha\beta^3\gamma^3+6\alpha\beta^4\gamma^2}$$

$$y_+ = \frac{1}{\beta}\frac{\gamma}{\beta^3+\gamma^3+3\beta\gamma^2+3\beta^2\gamma}\sqrt{\alpha\beta^6+\beta^5\gamma+4\alpha\beta^5\gamma+\beta^2\gamma^4+3\beta^3\gamma^3+3\beta^4\gamma^2+\alpha\beta^2\gamma^4+4\alpha\beta^3\gamma^3+6\alpha\beta^4\gamma^2}$$

$$z_+ = \frac{1}{\beta^3+\gamma^3+3\beta\gamma^2+3\beta^2\gamma}\sqrt{\alpha\beta^6+\beta^5\gamma+4\alpha\beta^5\gamma+\beta^2\gamma^4+3\beta^3\gamma^3+3\beta^4\gamma^2+\alpha\beta^2\gamma^4+4\alpha\beta^3\gamma^3+6\alpha\beta^4\gamma^2}$$

Using the parameter values:
$m = 100$, β=200 , γ=0.2 and α=0.1 The system (1) has tree point's equilibrium

$E_0(0,0,0), E_+(0.3178, 3, 1749 \times 10^{-0.4}, 0.31749)$

and

$E_-(-0.3178, -3, 1749 \times 10^{-0.4}, -0.31749)$

And their eigenvalues are given as:

$E_0(0,0,0)...\lambda_1 = 13.436, \lambda_{2,3} =$ -2.318 $\pm$ 12.046i

$\mathrm{E}_+(0.3178, 3.1749 \times 10^{-0.4}, 0.31749)...\lambda_1 = -23.515,\ \lambda_{2,3} =\ 1.0081 \pm 13.075\mathrm{i}$ But

$\mathrm{E}_-(-0.3178, -3.1749 \times 10^{-0.4}, -0.31749)...\lambda_1 = -23.515,\ \lambda_{2,3} =\ 1.0081 \pm 13.075\mathrm{i}$
for the stability analysis we have this theorem introduced in [21].

Theorem 1

The following autonomous system:

$$D^q x = Ax.\ \mathrm{x(0)=x_0}$$

$$with\ 0 \prec q \prec 1,\ \mathrm{x} \in \mathrm{R}^n \text{ and } \mathrm{A} \in \mathrm{R}^n \times \mathrm{R}^n$$

Is asymptotically stable if and only if $|\arg(\lambda)| \succ q\pi/2$

Is satisfied for all eigenvalues $(\lambda)$of matrix A.Also, this system is stable if and only if $|\arg(\lambda)| \geq q\pi/2$is satisfied for all eigenvalues of matrix A and those critical eigenvalues which satisfy $|\arg(\lambda)| = q\pi/2$

Have geometric multiplicity one.

Fig.3 shows stable and unstable regions according to the above theorem.

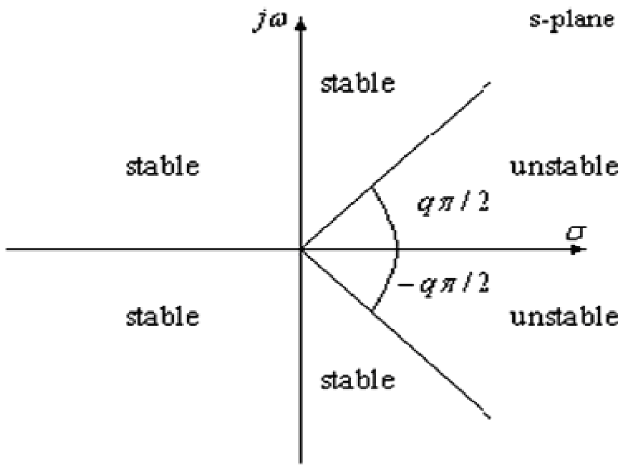


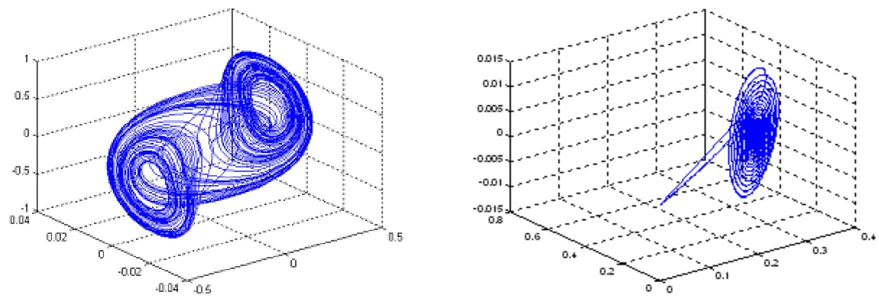
Fig. 3 Stability region of the fractional-order system.

- The equilibrium point $E_0(0,0,0)$is a saddle point

of index **1**[3], however, the equilibrium points E_+ and $\mathrm{E}_-$are saddle points of index **2** [3].

Thus, by the theorem 1 the necessary condition for the fractional-order (MAVPD) system (1) to remain chaotic is

$$q \succ \frac{2}{\pi}\arctan(\frac{|Im(\lambda_{2,3}|}{Re\lambda_{2,3}})$$



(a) The system is chaotic for $q = 0.98$. (b) Its stable for $q = 0.94$

Fig. 4

Consequently, the lowest fractional order q for which the fractional-order (MAVPD) system (1) demonstrates chaos using the above mentioned parameters is given by the inequality $q \succ 0.95146$.

Fig 4 shows the system (1) is chaotic for $\alpha = 0.98$ and is stable for $q = 0.94$

2.2 The Fractional-Order Chua System

Our second model is derived from the well-known and famous Chua's circuit[17] where the nonlinear element(commonly called Chua's diode) is implemented using two diodes only, in addition to an op-amp and some resistors (see Fig.3(b) for an example), Several other practical implementations of a Chua's diode characterized by a five-segments piecewise-linear current-voltage characteristic have been proposed [16].An implementation of Chua's circuit with a cubic nonlinearity was first described by Zhong[17].The advantage of the cubic nonlinearity are that it requires no absolute-valued functions, it is smooth and thus more suitable for mathematical calculations. Physically, the original Chua's circuit does not involve any resistor in series with the inductor. This resistor appears in the Chua oscillator of Ref.[18]. Coming back to the model of Fig.2 (b), we now rename the circuit elements by introducing an index "c" on them, and taking the current-voltage characteristics of the nonlinear resistance in the form:

$$i(V) = \nu_c + a_2 V + b_2 V^3.$$

Therefore, it is obvious that the Chua's oscillator will also be described by a set of first-order coupled differential equations similar to those of (MAVPD) system.

One can therefore conclude that the MAVPD and the modified Chua oscillators are in fact the same model, with their specific qualitative dynamical behaviour depending

Only on the selection of their parameters, since an inductor can always be physically constructed with a lower (and negligible) series resistance, and because we need to stay closer to the original Chua circuit oscillator can thus be described by the following equations:

$$\begin{cases} \dot{x} = \eta(y - x^3 + \mu x), \\ \dot{y} = x - y - z, \\ \dot{z} = \rho y, \end{cases} \tag{2}$$

Where the parameters η, μ and ρ are all positive real

In order to obtain the so-called double scroll attractor which is specific to the family of Chua's circuits, we use the same selection of parameters as in [18], that is $\eta = 10, \rho = 16$ and $\mu = 0.143$.

2.3 Fractional Version of Modified Chua's System

The fractional –order Chua system is given as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = \eta(y - x^3 + \mu x) \\ \frac{d^q y}{dt^q} = x - y - z \\ \frac{d^q z}{dt^q} = \rho y \end{cases} \tag{3}$$

Where q is the fractional order satisfying $q \in (0, 1]$, the parameters $\rho, \eta,$ and μ are all positive real, where $q = 1$, system **(3)** is the original integer-order Chua system exhibits chaotic behaviors with the parameter values $\rho = 16, \eta = 10,$ and μ= 0.143

The system (2) has tree point's equilibrium given by:

$$E_0(0,0,0),\ \mathrm{E}_+(\sqrt{\mu}, 0,\ \sqrt{\mu})\ \text{and} E_-(-\sqrt{\mu},\ 0,\ \text{-}\sqrt{\mu})$$

Using the parameter values:

$\rho = 16$, η=10 , and μ=0.143 the points equilibrium are given by:

$E_0(0,0,0)$, E_+(0.37815, 0, 0.37815) and E_-(-0.37815, 0,-0.37815) And their eigenvalues are given as:

$$E_0(0,0,0)...\lambda_1 = 14.889, \lambda_{2,3} = -0.79474 \pm 3.8386i$$

$$\mathrm{E}_+(0.37815, 3.0, 0.37815)...\lambda_1 = -4.2846,\ \ \lambda_{2,3} =\ 0.21236 \pm 3.2611\mathrm{i}$$

$$\mathrm{E}_-(-0.37815, 3.0, -0.37815)...\ \lambda_1 = -4.2846,\ \ \lambda_{2,3} =\ 0.21236 \pm 3.2611\mathrm{i}$$

- The equilibrium point $E_0(0,0,0)$ is a saddle point

of index **1**,however, the equilibrium points E_+ and E_- are saddle points of index **2**

Thus, by the theorem 1 the necessary condition for the fractional-order (Chua) system (3) to remain chaotic is

$$q \succ \frac{2}{\pi}\arctan(\frac{|Im(\lambda_{2,3}|}{Re\lambda_{2,3}})$$

Consequently, the lowest fractional order q for which the fractional-order (Chua) system (3) demonstrates chaos using the above mentioned parameters is given by the inequality $q \succ 0.9591$.

Fig 5 shows the system (3) is chaotic for $q = 0.98$ and is stable for $q = 0.94$

3. Synchronization of two Different Chaotic Fractional Systems with Auxiliary System

In this study we will study the synchronization between two different fractional systems with auxiliary system

3.1 The Method of Auxiliary System Approach

The principle of this method is based on the fact that even if the sending system $x(t)$conducted two identical receptor systems $y(t)$ and $z(t)$ starting with different initial conditions in the attraction basin, then the Stability analysis of synchronization in the space $X \oplus Y$, which may generally have a very complicated shape $y(t) = \phi(x(t))$, may be replaced by the stability analysis is quite simple $z(t) = y(t)$in the space $Z \oplus Y$

For this purpose we assume the following driver, slave and auxiliary system

$$x^q = F(x(t)) \tag{4}$$

$$y^q = G(y(t), g, x(t)) \tag{5}$$

And auxiliary system

$$z^q = G(z(t), g, x(t)) \tag{6}$$

Which is identical to the receiving system(5), Clearly, when the system receiver (5) and his auxiliary (6) have same transmission signal $x(t)$, then the fields vector in phase space of the receiver and auxiliary system are identical and the systems can grow on identical attractors.

It is easy to show that the linear stability of the manifold $z(t) = y(t)$ is equivalent to the linear stability of manifold of synchronized movements in $X \oplus Y$ which is determined by $\phi(.)$.the linear zed equations that govern the evolution of the quantities

$\zeta_y(t) = y(t) - \phi(x(t))$and $\zeta_z(t) = z(t) - \phi(x(t))$

Are

$$\zeta_y^q(t) = DG(\phi(x(t), g, x(t)) \times \zeta_y(t)$$

$$\zeta_z^q(t) = DG(\phi(x(t), g, x(t)) \times \zeta_z(t)$$

With $DG(w, h_u(t)) = \frac{\partial G(w,h_u(t)}{\partial w}$

Since the linear zed equations for $\zeta_y(t)$ et $\zeta_z(t)$ are identical, linear zed equations for $\zeta_z(t) - \zeta_y(t) = z(t) - y(t)$ have the same Tacobian matrix $DG(., g, x(t))$ in the previous equation.

So, if the collector synchronized movements in $X \oplus Y \oplus Z$is linearly stable for$z(t)-y(t)$, then it is linearly stable for $\zeta_y(t) = y(t) - \phi(x(t))$ and vice versa

The study of synchronization goes back to the study of stability in the vicinity of the origin of a new system that gives it the name of "system error". The latter represents the disturbance that may exist between the transmitting and receiving system.

To study the stability of the system error we will use the criterion of **Routh-Hurwitz** generalized to fractional order [1]

3.2 Some Stability Conditions

Let (x_e, y_e, z_e)be an equilibrium solution of the following three dimensional fractional-order systems:

$$\begin{cases} \frac{d^q x(t)}{dt^q} = f(x,y,z) \\ \frac{d^q y(t)}{dt^q} = g(x,y,z) \\ \frac{d^q z(t)}{dt^q} = h(x,y,z) \end{cases} \tag{7}$$

Where $q \in (0,1]$.the eigenvalues equation of the equilibrium point (x_e, y_e, z_e) is given by the following polynomial:$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$

And its discriminant D (P) is given as:

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2$$

(1) If $D(P) \succ 0$, then the necessary and sufficient condition for the equilibrium point (x_e, y_e, z_e), to be locally asymptotically stable, is :
$a_1 \succ 0, a_2 \succ 0, a_1a_2 - a_3 \succ 0$.

(2) If $D(P) \prec 0, a_1 \succ 0, a_2 \succ 0, a_3 \succ 0$. then (x_e, y_e, z_e) is locally asymptotically stable for,$q \prec 2/3$, However, if $D(P) \prec 0, a_1 \prec 0, a_2 \prec 0$, q $\succ$ 2/3, then all roots of equation satisfy the condition $|\arg(\lambda)| \prec q\pi/2$.

(3) if $D(P) \prec 0, a_1 \succ 0, a_2 \succ 0$,$a_1a_2 - a_3 = 0$. then (x_e, y_e, z_e) is locally asymptotically stable for all $q \in (0,1)$.

(4) the necessary condition for the equilibrium point (x_e, y_e, z_e) To be locally asymptotically stable, is $a_3 \succ 0$

Synchronization of Modified Chua's System and MAVPD System

Let us take in this paragraph the two preceding studies, the first is the fractional modified Chua system, the second is the fractional MAVPD system and we will detect their synchronization with an auxiliary system.

For this we assume the modified Chua system as transmitter (master):

$$\begin{cases} \frac{d^q x_1}{dt^q} = \eta(y_1 - x_1^3 + \mu x_1) \\ \frac{d^q y_1}{dt^q} = x_1 - y_1 - z_1 \\ \frac{d^q z_1}{dt^q} = \rho y_1 \end{cases} \tag{8}$$

And the MAVPD system as receiving (slave).

$$\begin{cases} \frac{d^q x_2}{dt^q} = m(y_2 - x_2^3 + \alpha x_2) - k(x_2 - x_1) \\ \frac{d^q y_2}{dt^q} = x_2 - y_2 - z_2 \qquad \qquad \text{.............(9)} \\ \frac{d^q z_2}{dt^q} = \beta y_2 - \gamma z_2 \end{cases} \tag{9}$$

The master system is coupled with the slave system only by the $x(t)$ scalar

We choose the auxiliary system that is identical to the slave system (9) (witch different initial conditions).

$$\begin{cases} \frac{d^q x_3}{dt^q} = m(y_3 - x_3 + \alpha x_3) - k(x_3 - x_1) \\ \frac{d^q y_3}{dt^q} = x_3 - y_3 - z_3 \\ \frac{d^q z_3}{dt^q} = \beta y_3 - \gamma z_3 \end{cases} \tag{10}$$

Subtraction of the two systems (9) and (10), gives us the following error system:

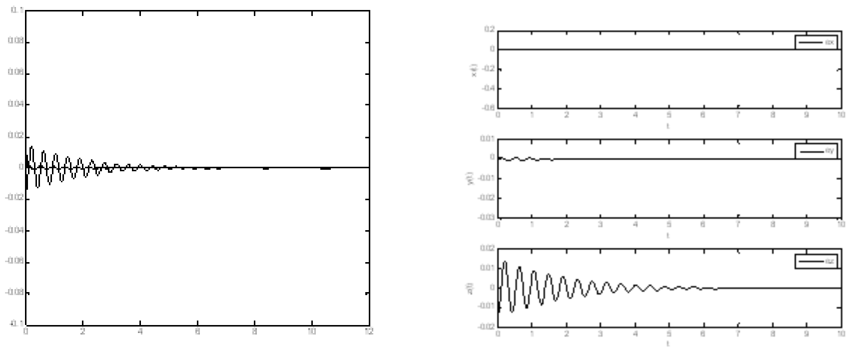
$$\begin{cases} \frac{d^q e_1}{dt^q} = (m\alpha - k)e_1 + me_2 - m(x_3^3 - x_2^3) \\ \frac{d^q e_2}{dt^q} = e_1 - e_2 - e_3 \\ \frac{d^q e_3}{dt^q} = \beta e_2 - \gamma e_3 \end{cases} \tag{11}$$

Where $e_1 = x_3 - x_2, e_2 = y_3 - y_2, e_3 = z_3 - z_2$

The system (11) can be written as following matrix:

$$\begin{pmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} + \phi(x, y, z) \tag{12}$$

Where $A \begin{pmatrix} m\alpha - k \text{ m } 0 \\ \text{1 -1 -1} \\ 0\ \beta\ \gamma \end{pmatrix}$



(a) (b)

Fig. 5 Graphs of the time variation of the synchronization errors $e_1 = x_3 - x_2, e_2 = y_3 - y_2, e_3 = z_3 - z_2$ (a) and (b) the error system converge to zero if $k = 90.685$

$\phi(x, y, z) = \begin{pmatrix} -m(x_3^3 - x_2^3) \\ 0 \\ 0 \end{pmatrix}$ is a nonlinear function satisfies the Lipschitz condition , so locality to zero it converges to zero

To study the stability of system (12) we use the conditions of criterion **Routh-Hurwitz** generalized to fractional order [1]

The characteristic polynomial of matrix A is given by:

$$\begin{aligned} P(x) &= x^3 + (k + \gamma - m\alpha + 1)x^2 + (k - m + \beta - m\alpha + \gamma(k - m\alpha + 1)x \\ &+ \beta k + \gamma k - \beta\beta m + \gamma k - \gamma m - \alpha\gamma m = 0 \end{aligned} \tag{13}$$

Proposition

If$k \leq \frac{m(\gamma+\alpha\beta+\alpha\gamma)}{\beta+\gamma}$, system (9) can not synchronize system (8) with auxiliary system (10)

Proof:

By applying the stability conditions to esq. (13) By condition **4** the necessary condition for the equilibrium point$(x_e, y_e, z_e)$to be locally asymptotically stable, is $a_3 \succ 0$, consequently if $k \leq \frac{m(\gamma+\alpha\beta+\alpha\gamma)}{\beta+\gamma}$ system (9) can not synchronize system(8) with auxiliary system (10)

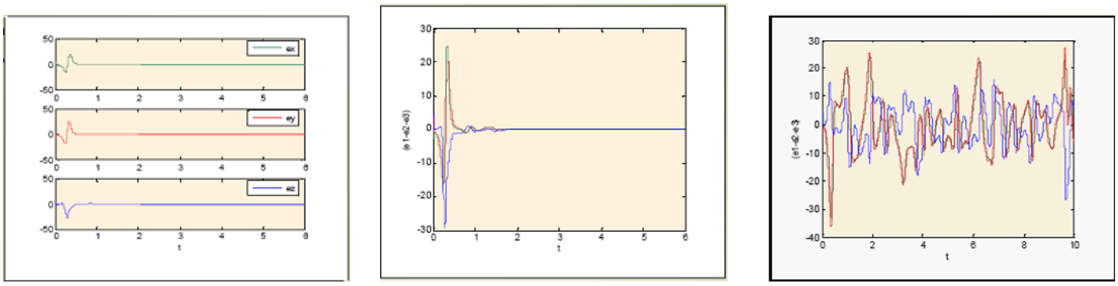
- Using the parameter values

$$m = 100 \ , \ \alpha{=}0.1 \ , \ \beta{=}200 \text{ and } \gamma{=}0.2$$

That the result of proposition the necessary condition is $k \leq 2$

In this case that the condition 4 is verified i.e.

$D(P) \prec 0, a_1 \succ 0, a_2 \succ 0, a_1 a_2 - a_3 = 0$ Must be $k = 11.448 or k = 90.685$ system (9) can synchronize system (8) with auxiliary system (10) for all $q \in (0, 1)$



(a) (b) (c)

Fig. 6 Graphs of the time variation of the synchronization errors $e_1 = x_3 - x_2, e_2 = y_3 - y_2, e_3 = z_3 - z_2$ (a) and (b) the error system converge to zero if $k = 22$ (c) the error system is instable if $k = 2$

Synchronization of Fractional Chen System and Fractional LU System

In this paragraph we replied this method of synchronization for two well known systems, the first is frictional Chen system and the second is fractional Lû system with auxiliary system

For this we assume the fractional Chen system as transmitter (master):

$$\begin{cases} \frac{d^q x_1}{dt^q} = a(y_1 - x_1), \\ \frac{d^q y_1}{dt^q} = (c - a)y_1 - x_1 z_1 + c y_1, \\ \frac{d^q z_1}{dt^q} = x_1 y_1 - b z_1, \end{cases} \tag{14}$$

Where $(a, b, c) = (35,\ 3,\ 28)$ Consequently, the lowest fractional order $q$for which the fractional-order Chen system (14) demonstrates chaos using the above mentioned parameters is given by the inequality.$q \succ 0.82$

And we assume the fractional Lû system as receiving (slave).

$$\begin{cases} \frac{d^q x_2}{dt^q} = \alpha(y_2 - x_2), \\ \frac{d^q y_2}{dt^q} = \delta y_2 - x_2 z_2 - k(y_2 - y_1), \\ \frac{d^q z_2}{dt^q} = x_2 y_2 - \beta z_2, \end{cases} \tag{15}$$

$(\alpha, \beta, \delta) = (36,\ 3,\ 20)$ and k is coupling parameter Consequently, the lowest fractional order q for which the fractional-order LU system (15) demonstrates chaos using the above mentioned parameters is given by the inequality.$q \succ\ 0.91605$

The master system is coupled with the slave system only by the $y(t)$ scalar

We choose the auxiliary system that is identical to the slave system (15) (with different initial conditions).

$$\begin{cases} \frac{d^q x_3}{dt^q} = \alpha(y_3 - x_3), \\ \frac{d^q y_3}{dt^q} = \delta y_3 - x_3 z_3 - k(y_3 - y_1), \\ \frac{d^q z_3}{dt^q} = x_3 y_3 - \beta z_3, \end{cases} \tag{16}$$

Subtraction of the two systems (16) and (15), gives us the following error system

$$\begin{cases} \frac{d^q e_1}{dt^q} = \alpha(e_2 - e_1) \\ \frac{d^q e_2}{dt^q} = \beta e_2 - k e_2 - x_3 e_1 - z_2 e_1 \\ \frac{d^q e_3}{dt^q} = x_3 e_2 - y_2 e_1 - \alpha e_3 \end{cases} \tag{17}$$

Where $e_1 = x_3 - x_2, e_2 = y_3 - y_2, e_3 = z_3 - z_2$

The system (17) can be written as following matrix:

$$\begin{pmatrix} \frac{d^q e_1}{dt^q} \\ \frac{d^q e_2}{dt^q} \\ \frac{d^q e_3}{dt^q} \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} + \phi(x, y, z) \tag{18}$$

Where $A \begin{pmatrix} -\alpha\ \alpha\ 0 \\ 0\ \delta\text{-k}\ 0 \\ 0\ 0\ \text{-}\beta \end{pmatrix}$

$\phi(x, y, z) = \begin{pmatrix} 0 \\ -z_2 e_1 - x_3 e_1 \\ x_3 e_2 - y_2 e_1 \end{pmatrix}$ is a nonlinear function satisfies the Lipschitz condition, so locality to zero it converges to zero

The characteristic polynomial of matrix A is given by

$$x^3 + (k+\alpha+\beta-\delta)x^2 + (-\alpha(-k+\delta)+\beta(k+\alpha-\delta))x - \alpha\beta(-k+\delta) \tag{19}$$

- By applying the stability conditions to esq.(19) with condition **4** the necessary condition for the equilibrium point (x_e, y_e, z_e) to be locally asymptotically stable, is $a_3 \succ 0$, consequently if $k \prec \delta$ system (14) can not synchronize system (15) with auxiliary system (16)
- . Using the parameter values$(\alpha, \beta, \delta) = (36,\ 3,\ 20)$

According to the previous result, the necessary condition for synchronization is achieved $k \geq 21$

Figure (6) show the synchronization is realized in case $k = 22$ but not realized in case $k = 2$

Conclusion

In this paper we have studied the synchronization between different chaotic fractional –order systems, with the auxiliary system, we have applied this method on both systems:

The modified Chua oscillator as transmitter system and Van der Pol-Duffing modified (MVDPD) as receiver system; we have also used it again on two well know systems Chen and LU.Using the criterion of routh Hurwitz to study the stability of the system error. Numerical results show the effectiveness of the theoretical analysis.

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