A Criterion for the Stability Analysis of Phase Synchronization in Coupled Chaotic System

Hadi Taghvafard*[1] and G. H. Erjaee†[1,2]

[1] *Mathematics Department, Shiraz University, Shiraz, Iran*
[2] *Mathematics Department, Qatar University, Doha, Qatar*

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Abstract: We report phase synchronization for the coupled diffusionless Lorenz system and for a new coupled chaotic system in four dimensional space. Stability is also examined by applying a measure to the linearlized evaluation difference matrix between coupled chaotic systems.

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1. Introduction

Recently, synchronization phenomena in coupled chaotic systems have received much attention [1 -17]. Pecora and Carroll have shown [1-4] that in coupled chaotic systems a complete synchronization occurs if the difference between the various states of synchronized systems converges to zero. They have also shown that synchronization stability depends upon the signs of the conditional Lyapunov exponents: i.e., if all of the Lyapunov exponents of the response system under the action of the driver are negative, then there is a complete and stable synchronization between the drive and response systems. Synchronization stability can also be verified using the Jacobian matrix in the linearized state difference between the drive and response chaotic systems [6]. Accordingly, despite the stability analysis in dynamical systems, if this Jacobian matrix is of full rank and all of its real parts of eigenvalues are negative, then the system will converge to zero, yielding complete synchronization. Phase synchronization is another type of synchronization phenomenon which occurs when the Jacobian matrix has some zero eigenvalues. In this

* Email:taghvafard@gmail.com
† Corresponding author, Email:erjaee@shirazu.ac.ir

case, the difference between various states of synchronized systems may not necessary converging to the zero, but will stay less than or equal to a constant.

The main goal of this paper is to discuss the stability analysis of phase synchronization in coupled chaotic systems which coupled by the nonlinear feedback function method [18]. Thus, a brief discussion of the nonlinear coupling feedback function method is presented in section 2, followed by the presentation of a criterion for the stability of phase synchronization in section 3. In section 4, we presented two numerical examples to corroborate our analytical assertion.

2. Phase Synchronization

We shall apply the nonlinear coupling feedback function method introduced by Ali and Fang [18] to coupled chaotic systems. Consider the differential equation $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$, where $\mathbf{F}(\mathbf{x}(t))$ is a vector-valued function and decomposed into linear, $\mathbf{A}(\mathbf{x}(t))$, and non-linear, $\mathbf{N}(\mathbf{x}(t))$, components, i.e.

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{A}(\mathbf{x}(t)) - \mathbf{N}(\mathbf{x}(t)). \tag{1}$$

Now consider two chaotic dynamical systems whose associated vector functions are decomposed as in (1) and coupled by using the non-linear parts of their vector functions as follows.

$$\dot{\mathbf{x_1}}(t) = \mathbf{A}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_1}(t)) + r[\mathbf{N}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_2}(t))], \tag{2}$$
$$\dot{\mathbf{x_2}}(t) = \mathbf{A}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_2}(t)) + r[\mathbf{N}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_1}(t))]. \tag{3}$$

Here, systems (2) and (3) serve as drive and response systems, respectively, and the parameter r measures the strength of their coupling. The stability of the synchronization can be studied by using the evolutional equation of the difference between systems (2) and (3). This equation is determined by the linear approximation

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A} + (2r-1)\frac{\partial \mathbf{N}}{\partial \mathbf{x}}\right]\mathbf{e}(t), \tag{4}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. It is well-known from linear stability theory in dynamical systems that if $r = 1/2$, then the stability type of the zero equilibrium in Equation (4) reflects the stability type of the synchronization between the two chaotic systems (2), (3) and depends upon the signs of the real parts of the eigenvalues of $\mathbf{A}$ [19]. However, in the case of phase synchronization, we are not able to use this criterion for stability because some of these eigenvalues have zero real parts. Instead, we will develop another criterion for stability of phase synchronization in the next section.

3. A Stability Criterion for Phase Synchronization

Suppose the two coupled chaotic systems in (2) and (3) have a phase synchronization. Define the matrix measures of a real square matrix $\mathbf{A} = (a_{ij})_{n\times n}$ by

$$\mu_*(\mathbf{A}) = \lim_{\epsilon\to 0} \frac{\|\mathbf{I} + \epsilon\mathbf{A}\|_* - 1}{\epsilon},$$

where $\mathbf{I}$ is a identity matrix. In this case, for the matrix norms

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|, \qquad \|\mathbf{A}\|_2 = [\lambda \max(\mathbf{A}^T\mathbf{A})]^{1/2},$$
$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|, \qquad \|\mathbf{A}\|_\omega = \max_j \sum_{i=1}^{n} \frac{\omega_i}{\omega_j}|a_{ij}|,$$

where $\omega_i > 0$, we have the matrix measures

$$\mu_1(\mathbf{A}) = \max_j \left\{ a_{jj} + \sum_{i=1,\ i\neq j}^{n} |a_{ij}| \right\}, \qquad \mu_2(\mathbf{A}) = \frac{1}{2}\lambda_{\max}(\mathbf{A}^T + \mathbf{A}),$$
$$\mu_\infty(\mathbf{A}) = \max_i \left\{ a_{ii} + \sum_{j=1,\ j\neq i}^{n} |a_{ij}| \right\}, \qquad \mu_\omega(\mathbf{A}) = \max_j \left\{ a_{jj} + \sum_{i=1,\ i\neq j}^{n} \frac{\omega_i}{\omega_j}|a_{ij}| \right\},$$

respectively.

Now suppose that in error system (4), with $r = 1/2$, matrix $\mathbf{A}$ doesn't have full rank. Then the global stability of this system for which $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t) = \mathbf{c}$, with a constant vector $\mathbf{c}$, can be determined by the following theorem.

Theorem 3..1. Let $\mu_*(\mathbf{A}) \le 0$ for some matrix measure μ_*. Then system (4) for $r = 1/2$ is globally asymptotically stable around a constant vector $\mathbf{c}$. Consequently, there is phase synchronization between systems (2) and (3) which is globally asymptotically stable.

Proof 3..2. Let $\mathbf{e}(t)$ be a solution of $\dot{\mathbf{e}}(t) = \mathbf{A}(\mathbf{e}(t) - \mathbf{c})$. Thus, one can obtain

$$\begin{aligned}
\frac{d|\mathbf{e}(t)-\mathbf{c}|}{dt} - \mu_*(\mathbf{A})|\mathbf{e}(t)-\mathbf{c}| &= \lim_{\epsilon\to 0^+} \frac{|\mathbf{e}(t+\epsilon)-\mathbf{c}| - |\mathbf{e}(t)-\mathbf{c}|}{\epsilon} - \lim_{\epsilon\to 0^+} \frac{\|\mathbf{I}+\epsilon\mathbf{A}\|_* - 1}{\epsilon}|\mathbf{e}(t)-\mathbf{c}| \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left[|\mathbf{e}(t+\epsilon)-\mathbf{c}| - \|\mathbf{I}+\epsilon\mathbf{A}\|_*|\mathbf{e}(t)-\mathbf{c}|\right] \\
&\le \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}\left|\mathbf{e}(t+\epsilon) - \mathbf{c} - [\mathbf{I}+\epsilon\mathbf{A}](\mathbf{e}(t)-\mathbf{c})\right| \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\epsilon}|\mathbf{e}(t+\epsilon) - \mathbf{e}(t) - \epsilon\mathbf{A}(\mathbf{e}(t)-\mathbf{c})| \\
&= |\dot{\mathbf{e}}(t) - \dot{\mathbf{e}}(t)| = 0.
\end{aligned}$$

Therefore, $\frac{d|\mathbf{e}(t)-\mathbf{c}|}{dt} \le \mu_*(\mathbf{A})|\mathbf{e}(t)-\mathbf{c}|$ which implies $|\mathbf{e}(t)-\mathbf{c}| \le e^{\mu_*(\mathbf{A})t}$. Now, if $\mu_*(\mathbf{A}) \le 0$ then system (4)is globally asymptotically stable around a constant vector $\mathbf{c}$. Note that the constant vector $\mathbf{c}$ depends upon the initial conditions.

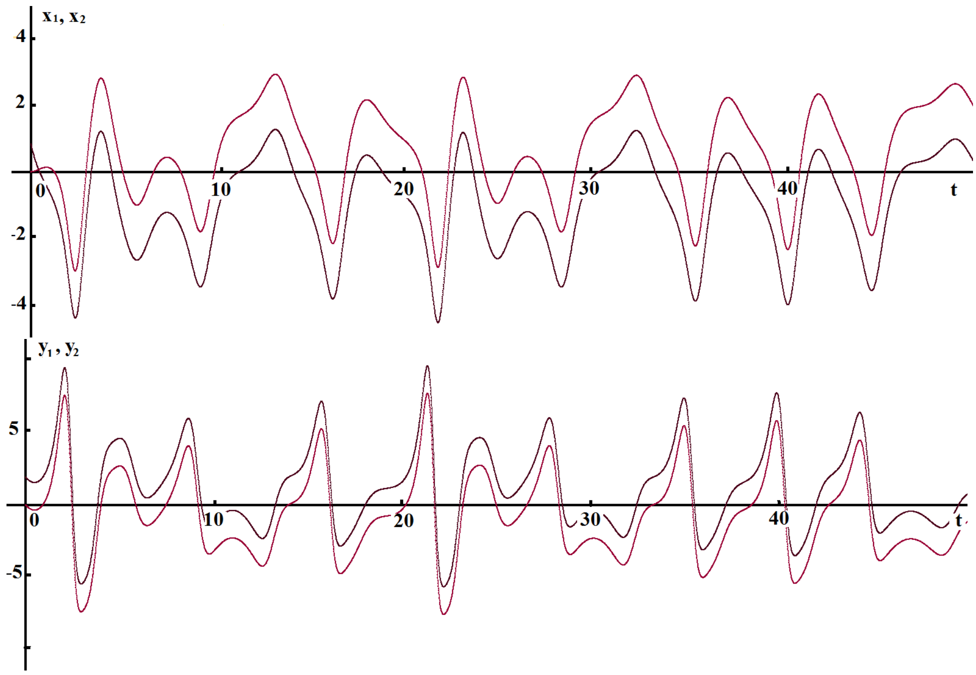

Fig. 1 Phase synchronization for the diffusionless Lorenz system

4. Numerical Results

We will first consider the diffusionless Lorenz system and then introduce a new four dimensional chaotic system which, we believe, is presented here for the first time. The diffusionless Lorenz system presented as follows [20].

$$\begin{cases} \dot{x_1} = -x - y, \\ \dot{y} = -xz, \\ \dot{z} = -xy + p. \end{cases} \tag{5}$$

It is well-known that this system is chaotic for $p \in (0, 5)$ [20]. Now we apply the nonlinear coupling feedback function method to this system to obtain

$$\begin{cases} \dot{x}_1 = -x_1 - y_1, \\ \dot{y}_1 = -x_1 z_1 + r(x_1 z_1 - x_2 z_2), \\ \dot{z}_1 = -x_1 y_1 + p + r(x_2 y_2 - x_1 y_1) \\ \dot{x}_2 = -x_2 - y_2, \\ \dot{y}_2 = -x_2 z_2 + r(x_2 z_2 - x_1 z_1), \\ \dot{z}_2 = -x_2 y_2 + p + r(x_1 y_1 - x_2 y_2). \end{cases} \tag{6}$$

As we can see in Fig.1, phase synchronization exists for system (6) with parameter values $r = 1/2$ and $p = 4$. As regards stability, cosider the matrix $\mathbf{A} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to

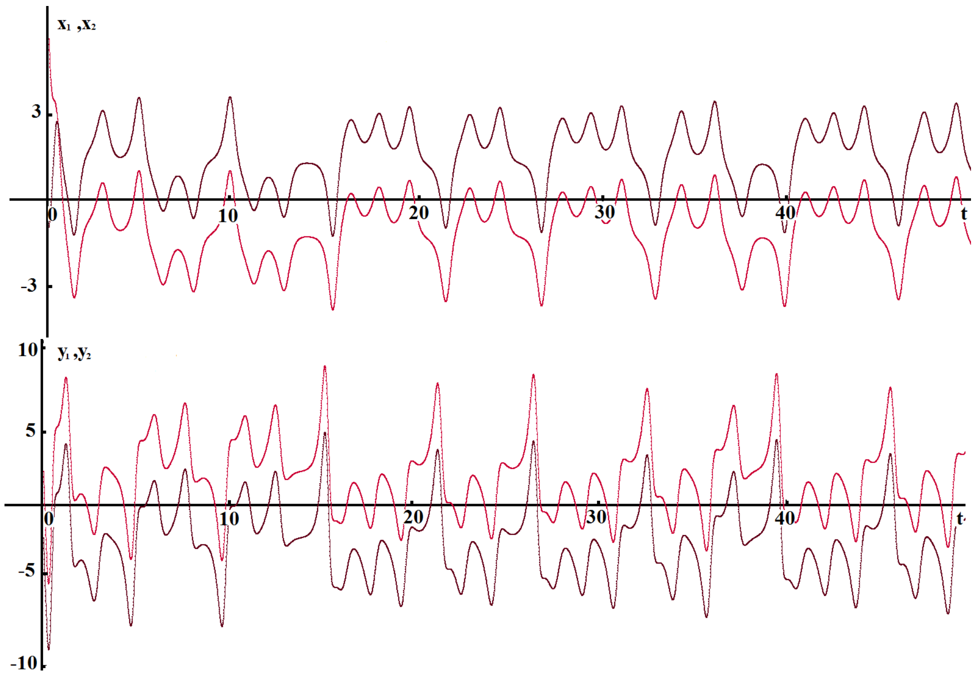

Fig. 2 Phase synchronization for the coupled system (8) in the text.

see that the eigenvalues of this matrix are -1 and zero with multiplicity 2. Since $\mu_\infty(\mathbf{A}) = 0$, the existence of phase synchronization in system (6) is globally asymptotically stable by Theorem 1.

As the second example, consider our new four dimensional system defined as follows.

$$\begin{cases} \dot{x} = -ax - by + w, \\ \dot{y} = -cy - axz, \\ \dot{z} = -z + axy + d. \\ \dot{w} = -fw - exz. \end{cases} \tag{7}$$

This system is chaotic for the parameter values $a = 3, b = 2, c = 0$ and $f = 1$. Applying the nonlinear coupling feedback function method to this system yields

$$\begin{cases} \dot{x}_1 = -ax_1 - by_1 + w_1, \\ \dot{y}_1 = -cy_1 - ax_1z_1 + ra(x_1z_1 - x_2z_2), \\ \dot{z}_1 = -z_1 + ax_1y_1 + d + ra(x_2y_2 - x_1y_1), \\ \dot{w}_1 = -fw_1 - ex_1z_1 + re(x_1z_1 - x_2z_2), \\ \dot{x}_2 = -ax_2 - by_2 + w_2, \\ \dot{y}_2 = -cy_2 - ax_2z_2 + ra(x_2z_2 - x_1z_1), \\ \dot{z}_2 = -z_2 + ax_2y_2 + d + ra(x_1y_1 - x_2y_2), \\ \dot{w}_2 = -fw_2 - ex_2z_2 + re(x_2z_2 - x_1z_1). \end{cases} \tag{8}$$

Figure 2 shows different states of phase synchronization in system (8). Here the eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} -3 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ are $-3, 0$ and -1 with multiplicity 2. Obviously, $\mu_\infty(\mathbf{A}) = 0$ is a matrix measure satisfying the condition of Theorem 1 and confirming the stability of phase synchronization.

5. Conclusions

As we have shown in this article, phase synchronization is an interesting case of synchronization that occurs in coupled chaotic systems. Such this phenomena occur if the real parts of some eigenvalues of the linearlized system found by the difference evolutional equation between coupled chaotic systems are zeros. In this case, stability can not be analyzed by the stability existence theorems used for dynamical systems. Nevertheless, as we shown here, if there is a non-positive matrix measure for the matrix **A** in equation (4), then stability can be determined by Theorem 1. Note, however, that zero matrix measure is sufficient for the stability of phase synchronization, and with the negative matrix measure we may demonstrate the stability of the synchronization that may exist in coupled chaotic systems. That is the consequence of the application of Theorem 1 to both phase synchronization and synchronization phenomena in coupled chaotic systems.

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