Bound State Solutions of the Klein Gordon Equation with the Hulthén Potential

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Abstract: An approximate solution of the Klein–Gordon equation for the Hulthén potential with equal scalar and vector potential is presented. Using the new improved approximation scheme to deal with the centrifugal term, we solve approximately the Klein–Gordon equation via the Nikiforov–Uvarov method for an arbitrary angular momentum quantum number. The corresponding eigen – energy and eigen functions are also obtained for the s-wave bound state.

Keywords: Klein Gordon Equation; Approximation Scheme; Hulthén Potential
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1. Introduction

In recent times the Nikiforov-Uvarov (NU) method has been used successfully in solving the Schrödinger, Dirac, Klein-Gordon, and Duffin-Kemmer-Petiau wave equations in the presence of some well known potential [1-5]. In relativistic mechanics, the solution of the Klein-Gordon and Dirac equation with some physical potential play a significant role in nuclear physics and other areas [6,7]. These relativistic equations contain two objects, the vector V(r) and scalar potential S(r).

The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$\left[-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+\left(S(r)+M\right)^2\right]\psi(r,\theta,\varphi)=0 \tag{1}$$

where M is the rest mass and for the case $S(r)=\pm V(r)$ has been studied recently [8,9].

However, the analytical solutions of the Klein-Gordon equations are possible only

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in the s-wave case with the angular momentum $\ell = 0$ for some exponential type potential models [10-11]. Conversely, when $\ell \neq 0$, one can only solve approximately the Klein-Gordon equations and the Dirac equation for some potential by using a suitable approximation scheme [6]. Generalized Hulthén potential which is reducible to the standard Hulthén potential, Woods-Saxon potential, exponential type screened potentials are studied [6,12]. The bound states relativistic solution of the standard Hulthén potential has also been presented [13,14].

With the conventional approximation scheme suggested by Greene and Aldrich [15] to deal with the centrifugal term, many authors have evaluated the bound state solutions of the Klein-Gordon and Dirac equations for different potentials [6,16].

The method Nikiforov-Uvarov (NU) [17] is based on solving the second order linear differential equations by reducing to a generalized equation of hypergeometric type. The NU-method is used to solve the Schrödinger, Klein-Gordon, Dirac and Duffin-Kemmer-Petiau equations with exponential-like potentials such as Woods-Saxon [18], Hulthén [19] and Pöschl – Teller [20].

Motivated by the success in obtaining approximately the bound state solution of the Klein-Gordon equation with Pöschl-Teller potential [21]. We attempt to solve approximately the arbitrary $l-$wave Klein-Gordon equation with Hulthén potential using the NU-method. The centrifugal term in the Klein-Gordon equation is deal with using a new improved approximation scheme [26].

2. Nikiforov-Uvarov Method

The NU method [17] is based on the solution of a generalized second order linear differential equation with special orthogonal functions [22]. The Schrödinger equation

$$\psi''(x) + [E - V(x)]\,\psi(x) = 0, \tag{2}$$

can be solved by this method. This can be done by transforming this equation of hypergeometric type with appropriate co-ordinate transformation, $s = s(x)$.

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{3}$$

In order to find the exact solution to equation (3), we set the wave function as

$$\psi(s) = \varphi(s)\chi(s), \tag{4}$$

and on substituting, equation (4) into equation (3) reduces equation (3) into hypergeometric type,

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \tag{5}$$

where the wave function $\psi(s)$ is defined as the logarithmic derivative [23]

$$\frac{\varphi^1(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)}, \tag{6}$$

where $\pi(s)$ is at most first-degree polynomials.

Likewise, the hypergeometric type function χ(s) in equation (5) for a fixed n is given by the Rodrigues relation

$$\chi_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma''(s)\rho(s)\right], \tag{7}$$

where B_n is the normalization constant and the weight function *ρ(s)* most satisfy the condition [17]

$$\frac{d}{ds} \left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s), \tag{8}$$

with

$$\tau(s) = \bar{\tau}(s) + 2\pi(s) \tag{9}$$

In order to accomplish the conditions imposed on the weight function ρ(s), it is necessary that the classical orthogonal polynomials τ(s) be equal to zero to some point of an interval (a, b) and its derivative at this interval at σ(s) > 0 will be negative, that is

$$\frac{d\tau(s)}{ds} < 0. \tag{10}$$

Therefore, the function π(s) and the parameter λ required for the NU-method are defined as follows:

$$\pi(s) = \frac{\sigma' - \bar{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \bar{\tau}}{2}\right)^2 - \bar{\sigma} + k\sigma}, \tag{11}$$

$$\lambda = k + \pi'(s) \tag{12}$$

The k-values in equation (11) are possible to evaluate if the expression under the square-root must be square of polynomials. This is possible, if and only if its discriminant is zero. With this, a new eigen-value equation becomes

$$\lambda = \lambda_n = -\frac{nd\tau}{ds} - \frac{n\,(n-1)}{2} \frac{d^2\sigma}{ds^2}, n = 0, 1, 2, ... \tag{13}$$

where τ(s) is as defined in equation (9) and on comparing equation (12) and equation (13), we obtain the energy eigen values.

3. Bound State Solutions of Klein-Gordon Equation

The three-dimensional Klein-Gordon equation with vector V(r) and scalar potential S(r) can be written as

$$\left[\nabla^2 + (V(r) - E)^2 - (S(r) + M)^2\right] \psi\,(\theta, r, \varphi) = 0, \tag{14}$$

where E is the relativistic energy; ∇^2 is the Laplace operator and where the velocity of light and Planck's constants have been set to unity. Writing the total spherical wave function as

$$\psi\,(r, \theta, \varphi) = \frac{1}{r} R(r) Y(\theta, \varphi), \tag{15}$$

separated equation (14) into variables and the resulting equations become [24]

$$\frac{d^2}{dr^2}R(r) + 2\left[MS(r) + EV(r) + S^2(r) - V^2(r) + \frac{\ell(\ell+1)}{r^2}\right] \tag{16}$$

$$R(r) = \left(E^2 - M^2\right)R(r) \tag{17}$$

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d}{d\theta}\Theta(\theta) + \left[\lambda - \frac{m^2}{\sin^2\theta} - (E+M)\right]\Theta(\theta) \tag{18}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0, \tag{19}$$

where $Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ and m^2 is the separate content, $\lambda = \ell(\ell+1)$.

The solution to equation (17) and (18) are well known, [25].

4. Solution of the Radial Equation via Nu-Method

The Hulthén potential is defined by

$$V(r) = \frac{-V_0 e^{-2\alpha r}}{(1 - e^{-2\alpha r})}, \tag{20}$$

where V_0 is the potential depth and α is an arbitrary constant. The radial equation of the Klein-Gordon equation for the special case V(r) = S(r) is

$$\frac{d^2R(r)}{dr^2} + \left[\frac{V_0 e^{-2\alpha r}}{(1-e^{-2\alpha r})}(M+E) + \frac{\ell(\ell+1)}{r^2}\right]R(\gamma r) = \left(E^2 - M^2\right)R(r) \tag{21}$$

The centrifugal term in equation (21) can be evaluated using the new improved approximation scheme [26]. However, with the centrifugal term present, equation (21) can not be solve analytically, that is for$\ell \neq 0$. In order to obtain the approximate analytical solutions of equation (21) for$\ell \neq 0$, we follow the new improved approximation scheme [26] to deal with the centrifugal term

$$\frac{1}{r^2} \approx 4\alpha^2\left[c_0 + \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right], \tag{22}$$

where $c_0 = \frac{1}{2}$ is an arbitrary dimensionless constant. In this study, we set c_0=0 which reduces the new improved approximation scheme to conventional approximation scheme suggested by Green and Aldrich [15].

Substituting equation (22) into equation (21) yields.

$$\frac{d^2R(r)}{dr^2} + \left[E^2 + \frac{\bar{V}_1 e^{-2\alpha r}}{(1-e^{-2\alpha r})} + \frac{4\alpha^2\ell(\ell+1)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}\right]R(r) = 0 \tag{23}$$

where $\bar{V}_1 = (E+M)V_0$ and $E^2 = (E^2 - M^2)$.

Introducing a new variable $s = e^{-2\alpha r}$, reduces equation (23) into hypergemetric type,

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dR}{ds} + \frac{1}{s^2(1-s)^2}\left[-\varepsilon^2 s^2 + \left(2\varepsilon^2 - \beta^2 + \gamma^2\right)s + \beta^2 - \varepsilon^2\right]R(s) = 0 \quad (24)$$

where the following dimensionless quantity has been used in obtaining equation (24);

$$\varepsilon^2 = -\frac{\bar{E}^2}{2\alpha^2}, \beta^2 = \frac{\bar{V}_1}{2\alpha^2}, \gamma^2 = 2\ell(\ell+1). \quad (25)$$

Now comparing equation (24) with equation (3), we get

$$\sigma(s) = s(1-s), \quad \bar{\tau}(s) = (1-s),$$

$$\bar{\sigma}(s) = -\varepsilon^2 s^2 + \left(2\varepsilon^2 - \beta^2 + \gamma^2\right)s + \beta^2 - \varepsilon^2. \quad (26)$$

Substituting equation (26) into equation (11) yields

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2}\sqrt{\left(4\varepsilon^2 - 4k + 1\right)s^2 + 4\left(\beta^2 - 2\varepsilon^2 - \gamma^2 + k\right)s + 4\left(\varepsilon^2 - \beta^2\right)}, \quad (27)$$

and we get two possible functions for each root k as

$$\pi(s) = -\frac{s}{2} \pm \frac{1}{2}\begin{cases} \left(2\sqrt{\varepsilon^2 - \beta^2} - i\sqrt{\gamma^2 + \frac{1}{4}}\right)s - 2\sqrt{\varepsilon^2 - \beta^2}, \\ for\, k = \beta^2 + \gamma^2 + \sqrt{\left(\varepsilon^2 - \beta^2\right)\left(-\gamma^2 - \frac{1}{4}\right)} \\ \\ \left(2\sqrt{\varepsilon^2 - \beta^2} + i\sqrt{\gamma^2 + \frac{1}{4}}\right)s - 2\sqrt{\varepsilon^2 - \beta^2}, \\ for\, k = \beta^2 + \gamma^2 - \sqrt{\left(\varepsilon^2 - \beta^2\right)\left(-\gamma^2 - \frac{1}{4}\right)} \end{cases} \quad (28)$$

From the four possible forms of the polynomials π(s), we select the one for which the $\tau'(s) < 0$, thus, using equation (9), we get

$$\tau(s) = 1 - 3s\left(\sqrt{\varepsilon^2 - \beta^2} + i\sqrt{\gamma^2 + \frac{1}{4}}\right)s - \sqrt{\varepsilon^2 - \beta^2}$$

$$\tau'(s) = -3 - \left(\sqrt{\varepsilon^2 - \beta^2} + i\sqrt{\gamma^2 + \frac{1}{4}}\right). \quad (29)$$

Therefore, the appropriate π(s) value is

$$\pi(s) = -\frac{s}{2} - \frac{1}{2}\left[\left(2\sqrt{\varepsilon^2 - \beta^2} + i\sqrt{\gamma^2 + \frac{1}{4}}\right)s - 2\sqrt{\varepsilon^2 - \beta^2}\right]. \quad (30)$$

The constant $\lambda = k + \pi'(s)$ is obtain as

$$\lambda = \beta^2 + \gamma^2 - i\sqrt{\left(\varepsilon^2 - \beta^2\right)\left(\gamma^2 + \frac{1}{4}\right)} + \frac{1}{2} - \sqrt{\varepsilon^2 - \beta^2} - i\sqrt{\gamma^2 + \frac{1}{4}}, \quad (31)$$

and from the definition of λ_n, we write

$$\lambda_n = 2n + 2n\sqrt{\varepsilon^2 - \beta^2} + in\sqrt{\gamma^2 + \frac{1}{4}} + n(n-1). \tag{32}$$

On comparing equations (31) and (32), we obtain the bound state eigen energy for the Klein-Gordon equation as

$$\bar{E} = -\frac{\bar{V}_1}{2\alpha^2} - 2\alpha^2 \left[\frac{\left(\frac{\bar{V}_1}{2\alpha^2}\right)}{\left(1 + 2n + i\sqrt{\gamma^2 + \frac{1}{4}}\right)} + \frac{2\ell(\ell+1)}{\left(1 + 2n + i\sqrt{\gamma^2 + \frac{1}{4}}\right)} + \frac{1}{2\left(1 + 2n + i\sqrt{\gamma^2 + \frac{1}{4}}\right)} - \frac{n + i\,(1+n)\sqrt{\gamma^2 + \frac{1}{4}}}{\left(1 + 2n + i\sqrt{\gamma^2 + \frac{1}{4}}\right)} \right]^2. \tag{33}$$

The corresponding wave function can now be calculated by first calculating the weight function in equation (8) as

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s), \tag{34}$$

yields

$$\rho(s) = (1-s)^{2\sqrt{\varepsilon^2 - \beta^2}+1}(1+s)^{i\sqrt{\gamma^2 + \frac{1}{4}}}. \tag{35}$$

Substituting equation (35) into the Rodrigues relation of equation (7), we obtain the eigen function $\chi_n(s)$ as

$$\begin{aligned}\chi_n(s) &= B_n(1+s)^{-i\left(\sqrt{\gamma^2+\frac{1}{4}}\right)}(1-s)^{-\left(2\sqrt{\varepsilon^2-\beta^2}+1\right)} \\ &\times \frac{d^n}{ds^n}\left[\left(1-s^2\right)^n (1-s)^{-\left(2\sqrt{\varepsilon^2-\beta^2}+1\right)} (1+s)^{-i\sqrt{\gamma^2+\frac{1}{4}}}\right] \\ &= B_n P_n^{(\mu,\upsilon)}(s),\end{aligned} \tag{36}$$

where B_n is the normalization constant and $\mu = i\sqrt{\gamma^2 + \frac{1}{4}}$ and $\upsilon = 2\sqrt{\varepsilon^2 - \beta^2} + 1$ and $\mathrm{P}_n(\mathrm{s})$ is the Jacobi polynomials.

The other part of the wave function in equation (6) is obtained by using $\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)}$ and solving the differential equation yields,

$$\varphi(s) = (1-s)^{2\sqrt{\varepsilon^2-\beta^2}+\frac{1}{2}}\,(1+s)^{i\sqrt{\gamma^2+\frac{1}{4}}} \tag{37}$$

Combining the Jacobi polynomials and equation (37), we obtain the wave function as

$$\psi_n(s) = A_n\,(1-s)^{2\upsilon}\,(1+s)^{\mu}\,P_n^{(\mu,\upsilon)}(s), \tag{38}$$

where A_n is a new normalization constant.

Conclusions

The solution of the radial Klein-Gordon equation for the Hulthén potentials with an arbitrary angular momentum $\ell \neq 0$ are obtained. Using the NU method, we obtain the approximate solution of the Klein-Gordon equation with equal scalar and vector potentials for s-wave bound state. By using a new improved approximation approach to deal with the centrifugal term, we obtain approximately the energy eigen value and the unnormalized radial eigen function in terms of the hypergeometric function for an arbitrary l-states.

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