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Algebraic Aspects for Two Solvable Potentials

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Abstract: We show that Lie algebras provide us with an useful method for studying real eigenvalues corresponding to eigenfunctions of Hamiltonian. We discuss the $SU(2)$ Lie algebra. We also discuss the eigenvalues for q-deformed Pöschl-Teller and Scarf potential via Nikiforov-Uvarov method.

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1. Introduction

The solution of the Schrödinger equation with physical potentials by using different techniques has an outgoing debate since the exact solution of the Schrödinger equation with any potential play an important role in quantum mechanics. Recently, there has been a growing interest in the study of Lie algebraic methods[1-5] which appear in different branches in physics and chemistry. For example, these methods provide a way to obtain the eigenfunctions of potentials in nuclear[6-7] and polyatomic molecules[8-9].

In this present paper, we study the Pöschl-Teller and Scarf potential in the framework of $SU(2)$ Lie algebra. To solve the differential equation, we use the Nikiforov-Uvarov method.

The arrangement of the present paper is as follows. A brief survey of Nikiforov-Uvarov method is given in Sec.2. In Sec.3, we have discussed $SU(2)$ Lie algebra. The q-deformed Pöschl-Teller interaction and the q-deformed Scarf interaction are discussed in Sec.4. Lastly, a closing discussion is given Sec.5.

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2. Nikiforov-Uvarov method

The conventional Nikiforov-Uvarov method[10], which received much interest, has been introduced for solving Schrödinger equation, Klein-Gordon and Dirac equations.

The differential equations whose solutions are the special functions of hypergeometric type can be solved by using the Nikiforov-Uvarov method which has been developed by Nikiforov and Uvarov[10]. In this method, the one dimensional Schrödinger equation is reduced to an equation by an appropriate coordinate transformation $x = x(s)$,

$$\frac{d^2\psi(s)}{ds^2} + \frac{\widetilde{\tau}(s)}{\sigma(s)}\frac{d\psi(s)}{ds} + \frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{1}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most of second degree, and $\widetilde{\tau}(s)$ is a polynomial, at most of first degree. In order to obtain a particular solution to Eq.(1), we set the following wave function as a multiple of two independent parts

$$\psi(s) = \phi(s)y(s) \tag{2}$$

According to Eq.(1) and Eq.(2) we have

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{3}$$

which demands that the following conditions be satisfied:

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{4}$$

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \qquad \tau'(s) < 0 \tag{5}$$

The condition $\tau'(s) < 0$ helps to generate energy eigenvalues and corresponding eigenfunctions. The condition $\tau'(s) > 0$ has widely discussed in[11] . The λ in (3) satisfies the following second-order differential equation

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \qquad n = 0, 1, 2, ... \tag{6}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree be negative. It is to be noted that λ or λ_n are obtained from a particular solution of the form $y(s) = y_n(s)$ which is a polynomial of degree n. The second part $y_n(s)$ of the wavefunction Eq.(2) is the hypergeometric-type function whose polynomial solutions are connected by Rodrigues relation[12-14]

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] \tag{7}$$

where C_n is normalization constant and the weight function $\rho(s)$ satisfies the relation as

$$\frac{d}{ds}[\sigma(s)\rho(s)] = \tau(s)\rho(s) \tag{8}$$

On the other hand, in order to find the eigenfunctions, $\phi_n(s)$ and $y_n(s)$ in Eqs.(4) and (7) and eigenvalues λ_n in Eq.(6), we need to calculate the functions:

$$\pi(s) = \left(\frac{\sigma' - \widetilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma} \tag{9}$$

$$k = \lambda - \pi'(s) \tag{10}$$

In principle, since $\pi(s)$ has to be a polynomial of degree at most one, the expression under the square root sign in Eq.(9) can be put into order to be the square of a polynomial of first degree[10], which is possible only if its discriminant is zero. Thus, the equation for k obtained from the solution of Eq.(9) can be further substituted in Eq.(10). In addition, the energy eigenvalues are obtained from Eqs.(6) and (10).

3. $SU(2)$ Lie Algebra

The generators J_x, J_y, J_z of the $SU(2)$ group characterized by the commutation relations

$$[J_x, J_y] = i\hbar J_z,\ [J_y, J_z] = i\hbar J_x,\ [J_z, J_x] = i\hbar J_y \tag{11}$$

The differential realization in spherical coordinate (r, θ, ϕ) of the $SU(2)$ generators are

$$J_z = -i\hbar\frac{\partial}{\partial\phi},\ J^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{12}$$

where $0 \le \phi < 2\pi$ and

$$\overrightarrow{J} = \overrightarrow{r} \times \overrightarrow{p} \tag{13}$$

We consider the Hamiltonian as $H = -J_z^2$ and the Casimir operator corresponding to the above generators is $C = J^2$. The Schrödinger equation is

$$C\psi = J^2\psi = j(j+1)\psi \tag{14}$$

Using Eqs.(11) and (14), we have

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \varepsilon\right]\psi = 0 \tag{15}$$

where

$$\varepsilon = \frac{j(j+1)}{\hbar^2} \tag{16}$$

To solve the Eq. (15), we separated $\psi(\theta, \phi)$ as

$$\psi(\theta, \phi) = \Theta(\theta)\Phi(\phi) \tag{17}$$

From Eq.(15) and Eq.(17), we have two second order differential equations

$$\frac{d^2\Theta(\theta)}{d\theta^2} + \cot\theta\frac{d\Theta(\theta)}{d\theta} + \left[\varepsilon - \frac{\mu^2}{\sin^2\theta}\right]\Theta(\theta) = 0 \tag{18}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \mu^2\Phi(\phi) = 0 \tag{19}$$

where μ is constant. The solution of the Eq.(19) is periodic and must satisfy the periodic boundary condition $\Phi(\phi + 2\pi) = \Phi(\phi)$, from which we have

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}\exp(i\mu\phi), \qquad \mu = 0, \pm 1, \pm 2, ... \tag{20}$$

After the substitution $s = \cos\theta$, the Eq.(18) becomes

$$\frac{d^2\Theta(\theta)}{ds^2} - \frac{2s}{1-s^2}\frac{d\Theta(\theta)}{ds} + \left[\frac{\varepsilon(1-s^2)-\mu^2}{(1-s^2)^2}\right]\Theta(\theta) = 0 \tag{21}$$

Now comparing Eq.(1) and Eq.(19), we have

$$\widetilde{\tau}(s) = -2s,\ \sigma(s) = 1 - s^2,\ \widetilde{\sigma}(s) = -\varepsilon s^2 + \varepsilon - \mu^2 \tag{22}$$

From Eq.(9) and Eq.(22), we have

$$\pi(s) = \pm\sqrt{(\varepsilon - k)s^2 - (\varepsilon - k) + \mu^2} \tag{23}$$

Due to Nikiforov-Uvarov method, the expression in the square root is taken as the square of a polynomial. Then, one gets the possible functions for each root k as

$$\pi(s) = \begin{cases} +\mu s & \text{if} \quad k = \varepsilon - \mu^2 \\ -\mu s & \text{if} \quad k = \varepsilon - \mu^2 \\ +\mu & \text{if} \quad k = \varepsilon \\ -\mu & \text{if} \quad k = \varepsilon \end{cases} \tag{24}$$

In order to obtain physical solution, $\tau(s)$ must satisfy $\tau'(s) < 0$, for which

$$\pi(s) = -\mu s \qquad \text{if} \qquad k = \varepsilon - \mu^2 \tag{25}$$

Hence from Eq.(5), We have

$$\tau(s) = -2(1 + \mu s), \qquad \tau'(s) = -2\mu \tag{26}$$

From Eqs.(6) and (10), the λ is given by

$$\begin{aligned} \lambda &= \lambda_n = 2n(1+\mu) + n(n-1) \\ \lambda &= \varepsilon - \mu(1+\mu) \end{aligned} \tag{27}$$

Eq.(27) and Eq.(16) gives

$$\begin{aligned} \varepsilon &= (n+\mu)^2 - \tfrac{1}{4} \\ j &= n + \mu \end{aligned} \tag{28}$$

According to Eqs.(4), (8), (22) and (26), the following expressions for $\phi(s)$ and $\rho(s)$ are obtained,

$$\phi(s) = (1 - s^2)^{\frac{\mu}{2}}, \qquad \rho(s) = (1 - s^2)^{\mu} \tag{29}$$

Using Eqs.(7), and (29), we have

$$y_n(s) = N_n P_n^{(\mu,\mu)}(s) \tag{30}$$

Using Eqs.(2), (29), and (30), we have

$$\Theta(\theta) = N_n (\sin\theta)^{\mu} P_n^{(\mu,\mu)}(\cos\theta) \tag{31}$$

where N_n is the normalization constant[15-16] satisfying.

$$N_n = \sqrt{\frac{(2j+1)(j-\mu)!}{2(j+\mu)!}} \tag{32}$$

Finally, from Eq.(17), Eq.(20) and Eq.(31), we have

$$\psi(\theta,\phi) = \sqrt{\frac{(2j+1)(j-\mu)!}{4\pi(j+\mu)!}} (\sin\theta)^{\mu} P_n^{(\mu,\mu)}(\cos\theta)\exp(i\mu\phi) \tag{33}$$

4. Pöschl-Teller and Scarf Potential

Set $s = \tanh_q z$ on Eq.(21), the equation becomes

$$\left[\frac{d^2}{dz^2} + \left(\Sigma + V_1 \mathrm{sech}_q^2 z\right)\right]\Theta(\theta) = 0 \tag{34}$$

where $\Sigma = -\mu^2$, $V_1 = q\,\varepsilon$ and the deformed hyperbolic function is defined as: $\sinh_q x = \frac{e^x - qe^{-x}}{2}$, $\cosh_q x = \frac{e^x + qe^{-x}}{2}$, $\tanh_q x = \frac{\sinh_q x}{\cosh_q x}$. The Eq.(34) is the Schrödinger equation for the Pöschl-Teller potential. The eigenvalue and the wavefunction of Eq.(34) are given in Ref.[17]. Again introducing $s = \coth_q z$ on Eq.(21), the equation becomes

$$\left[\frac{d^2}{dz^2} + \left(\Sigma - V_1 \mathrm{cosech}_q^2 z\right)\right]\Theta(\theta) = 0 \tag{35}$$

The Eq.(35) is the Schrödinger equation for the Scarf potential. The eigenvalue and the wavefunction of Eq.(35) are given in Ref.[18].

Conclusions

In this paper, we have derived the Schrödinger equation for Pöschl-Teller and Scarf potential by choosing an appropriate coordinate transformation. The Nikiforov-Uvarov method have been used to solve the second order differential equation. We have expressed the wave function in terms of Jacobi polynomial.

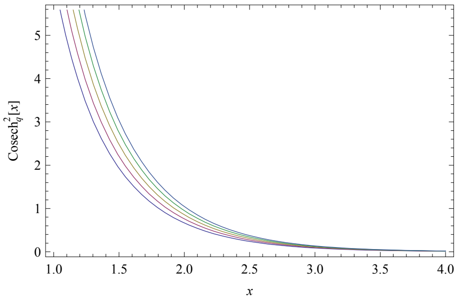

Fig. 1 A schematic representation of Pöschl-Teller potential for $q = 1$, and $\varepsilon = 35, 40, 45, 50, 55$.

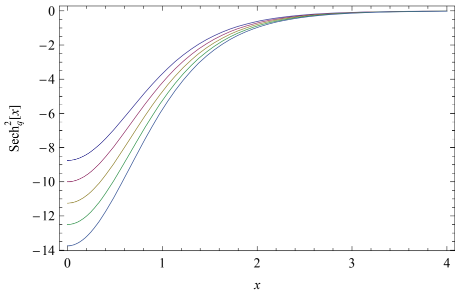

Fig. 2 A schematic representation of Scarf potential for $q = 1$, and $\varepsilon = 35, 40, 45, 50, 55$.

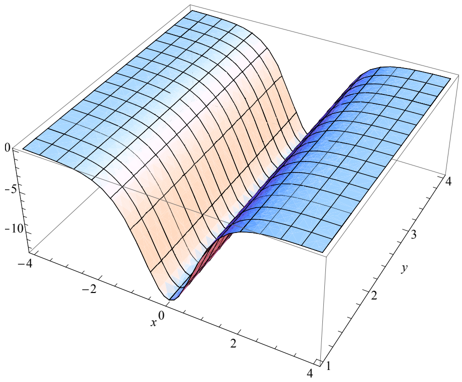

Fig. 3 A three dimensional representation of Pöschl-Teller potential for $q = 1$, and $\varepsilon = 50$.

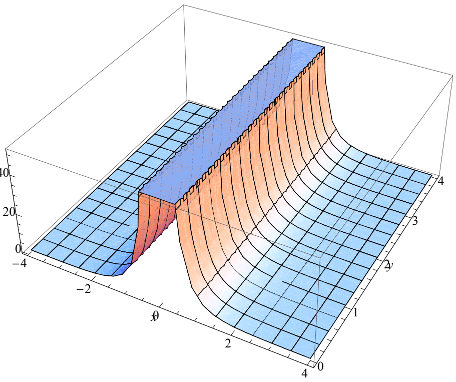

Fig. 4 A three dimensional representation of Scarf potential for $q = 1$, and $\varepsilon = 50$.

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