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A Study of the Dirac-Sidharth Equation

Raoelina Andriambololona*[1] and Christian Rakotonirina†[2]

[1] *Madagascar-Institut des Sciences et Techniques Nucléaires, Madagascar-INSTN*
[2] *Institute of High Energy Physics-Madagascar, iHEP-MAD*
Institut Supérieur de Technologie d'Antananarivo, IST-T, BP 8122, Madagascar

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Abstract: The Dirac-Sidharth equation has been constructed from the Sidharth Hamiltonian by quantification of the energy and momentum in Pauli algebra. We have solved this equation by using tensor product of matrices.

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1. Introduction

According to the special relativity of Einstein [1], we have the energy-momentum relation

$$E^2 = c^2 p^2 + m^2 c^4 \tag{1}$$

from which we can deduce the Klein-Gordon equation and the Dirac equation. This theory use the concept of continuous spacetime.
Quantized spacetime was introduced at the first time by Snyder [2, 3], which known as Snyder noncommutative geometry. That is because the commutation relations are modified and become [2, 3]

$$[x^\mu, x^\nu] = i\alpha \frac{\ell^2 c^2}{\hbar} \left(x^\mu p^\nu - x^\nu p^\mu\right), \tag{2}$$

$$[x^\mu, p_\nu] = i\hbar \left[\delta^\mu_\nu + i\alpha \frac{\ell^2 c^2}{\hbar^2} p^\mu p_\nu\right], \tag{3}$$

$$[p_\mu, p_\nu] = 0 \tag{4}$$

* raoelinasp@yahoo.fr
† rakotopierre@refer.mg, pierrekoto@yahoo.fr

where ℓ is any physical length scale. For axample, $\ell = \ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} cm$ the Planck length, the physical smallest possible length, where G is the gravitational constant. As consequence, the energy momentum relation gets modified and becomes (in natural units $c = \hbar = 1$)[4]

$$E^2 = p^2 + m^2 + \alpha l^2 p^4 \tag{5}$$

or (in SI units)[5, 6]

$$E^2 = c^2p^2 + m^2c^4 + \alpha \left(\frac{c}{\hbar}\right)^2 \ell^2 p^4 \tag{6}$$

where α a dimensionless constant.

$$\epsilon = \frac{\hbar c}{\sqrt{\alpha}\ell} \tag{7}$$

is the energy due to the Planck length scale $\ell = \ell_p$ [5, 6]. Then, [6]

$$E^2 = c^2p^2 + m^2c^4 + \frac{c^4p^4}{\epsilon^2} \tag{8}$$

The fundamental role of ϵ is explained in [5, 6, 7].
In fact, applying the Snyder-Sidharth Hamiltonian (5) Sidharth has deduced the Dirac-Sidharth equation [4, 8], i.e. the Dirac equation modified due to the non-commutative geometry of phase-space.
In this paper, in section 2, we will derive the Dirac-Sidharth equation, from the relation (6), by quantification of energy and momentum. In the Section.3, we will solve the Dirac-Sidharth equation by using tensor product of matrices.
We think that using differents mathematical tools in physics will make to appear differents hidden mathematical or physical properties.
First of all, let us give some properties of tensor product of matrices.

2. Tensor product of Matrices

Consider the $m \times n$-matrix $A = (A^i_j)$ and the $p \times r$-matrix $B = (B^i_j)$. The matrix defined by

$$A \otimes B = \begin{pmatrix} A^1_1 B & \dots & A^1_j B & \dots & A^1_n B \\ \vdots & & \vdots & & \vdots \\ A^i_1 B & \dots & A^i_j B & \dots & A^i_n B \\ \vdots & & \vdots & & \vdots \\ A^m_1 B & \dots & A^m_j B & \dots & A^m_n B \end{pmatrix}$$

obtained after performing the multiplications by scalar, $A^i_j B$, is called the tensor product of the matrix A by the matrix B. $A \otimes B$ is a $mp \times nr$-matrix.

$(B_1 \cdot A_1) \otimes (B_2 \cdot A_2) = (B_1 \otimes B_2) \cdot (A_1 \otimes A_2)$ for any matrices B_1, A_1, B_2, A_2 if the habitual matricial products $B_1 \cdot A_1$ and $B_2 \cdot A_2$ are defined.

For any matrices A and B, $A \otimes B = 0$ if, and only if $A = 0$ or $B = 0$.

3. A derivation of the Dirac-Sidharth Equation

For deriving the Dirac-Sidharth equation we use the method used by J.J. Sakurai [9] for deriving the Dirac equation.

The wave function of a spin-$\frac{1}{2}$ particle is two components. So, for quantifying the energy-momentum relation in order to have the modified Klein-Gordon equation [4, 8], or Klein-Gordon-Sidharth equation, of the spin-$\frac{1}{2}$ particle, the operators which take part in the quantification should be 2×2-matrices. So, let us take as quantification rules

$E \longrightarrow i\hbar\sigma^0 \frac{\partial}{\partial t} = i\hbar \frac{\partial}{\partial t}$

$\vec{p} \longrightarrow -i\hbar\sigma^1 \frac{\partial}{\partial x^1} - i\hbar\sigma^2 \frac{\partial}{\partial x^2} - i\hbar\sigma^3 \frac{\partial}{\partial x^3} = -i\hbar\vec{\sigma}\vec{\nabla} = \hat{p}_1\sigma^1 + \hat{p}_2\sigma^2 + \hat{p}_3\sigma^3$

where σ^1, σ^2, σ^3 are the Pauli matrices. Then we have, at first the Klein-Gordon-Sidharth equation

$$c^2\hbar^2 \left(\frac{\partial^2}{c^2 \partial t^2} - \Delta - m^2c^2 - \alpha \frac{\ell^2}{\hbar^2} \vec{\nabla}^4 \right) \phi = 0 \tag{9}$$

$$\begin{gathered} \left(i\hbar \frac{\partial}{\partial t} + ic\hbar\vec{\sigma}\vec{\nabla} \right) \frac{1}{mc^2} \left\{ \sum_{k=0}^{+\infty} (-1)^k \left[\frac{i\sqrt{\alpha}}{mc\hbar} \ell \left(-i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \right]^k \right\} \times \\ \left(i\hbar \frac{\partial}{\partial t} - ic\hbar\vec{\sigma}\vec{\nabla} \right) \phi = \left[mc^2 + i\sqrt{\alpha} \frac{c}{\hbar} \ell \left(-i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \right] \phi \end{gathered} \tag{10}$$

with application of the operator to two components wave function ϕ, which is solution of the Klein-Gordon-Sidharth equation. Let

$$\chi = \frac{1}{mc^2} \left\{ \sum_{k=0}^{+\infty} (-1)^k \left[\frac{i\sqrt{\alpha}}{mc\hbar} \ell \left(-i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \right]^k \right\} \times \left(i\hbar \frac{\partial}{\partial t} - ic\hbar\vec{\sigma}\vec{\nabla} \right) \phi \tag{11}$$

then, we have the following system of partial differential equations

$$\begin{cases} i\hbar \dfrac{\partial}{c\partial t}\chi + i\hbar\vec{\sigma}\vec{\nabla}\chi = mc\phi + i\sqrt{\alpha}\dfrac{\ell}{\hbar} \left(i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \phi \\ i\hbar \dfrac{\partial}{c\partial t}\phi - i\hbar\vec{\sigma}\vec{\nabla}\phi = mc\chi - i\sqrt{\alpha}\dfrac{\ell}{\hbar} \left(i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \chi \end{cases} \tag{12}$$

In additionning and in subtracting these equations, and in transforming the obtained equations under matrix form, we have the Dirac-Sidharth equation

$$i\hbar\gamma_D^\mu \partial_\mu \psi_D - mc\psi_D - i\sqrt{\alpha}\ell\hbar\gamma_D^5 \Delta\psi_D = 0 \tag{13}$$

in the Dirac (or "Standard") representation of the γ-matrices, where

$\gamma_D^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} = \sigma^3 \otimes \sigma^0$, $\gamma_D^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = i\sigma^2 \otimes \sigma^j$, $j = 1, 2, 3$,

$\gamma_D^5 = i\gamma_D^0\gamma_D^1\gamma_D^2\gamma_D^3 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} = \sigma^1 \otimes \sigma^0$, and $\psi_D = \begin{pmatrix} \chi + \phi \\ \chi - \phi \end{pmatrix}$,
$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.
We know that [10]

$$P\gamma^5 = -\gamma^5 P \tag{14}$$

It follows that the Dirac-Sidharth equation is not invariant under reflections [11]. The equation

$$i\hbar\gamma_W^\mu\partial_\mu\psi_W - mc\psi_W - i\sqrt{\alpha}\ell\hbar\gamma_W^5\Delta\psi_W = 0 \tag{15}$$

is the Dirac-Sidharth equation in the Weyl (or chiral) representation, where $\psi_W = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$.

So, χ is the left-handed two components spinor and ϕ the right-handed one. According to the equation (9), this method makes to appear that the right-handed two components spinor is solution of the Klein-Gordon-Sidharth equation.

4. Resolution of the Dirac-Sidharth equation

In this section, we search for solutions of the Dirac-Sidharth equation, in the form of plane waves in using tensor product of matrices. We had used this method, suggested by Raoelina Andriambololona, for solving the Dirac equation [12].
Let us look for a solution of the form

$$\psi_D = U(p)e^{\frac{i}{\hbar}(\vec{p}\vec{x} - Et)} \tag{16}$$

Let Ψ a four components spinor which is eigenstate both of $\hat{p}_j = -i\hbar\frac{\partial}{\partial x^j}$ and $\hat{E} = i\hbar\frac{\partial}{\partial t}$,
$\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$, and $\vec{n} = \frac{\vec{p}}{p} = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \end{pmatrix}$.
The Dirac-Sidharth equation becomes

$$\sigma^0 \otimes \sigma^0 U(p) - \frac{2}{\hbar}cp\sigma^1 \otimes \left(\frac{\hbar}{2}\vec{\sigma}\vec{n}\right)U(p) - mc^2\sigma^3 \otimes \sigma^0 U(p) + c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2 \otimes \sigma^0 U(p) = 0 \tag{17}$$

Let us take $U(p)$ of the form

$$U(p) = \varphi \otimes u \tag{18}$$

where u is the eigeinvector of the spin operator $\frac{\hbar}{2}\vec{\sigma}\vec{n}$. $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$ and u are two components.

Since $u \neq 0$, so

$$\left(\eta c p \sigma^1 + mc^2\sigma^3 - c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2\right)\varphi = E\varphi \tag{19}$$

with $\eta = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$

Solving this equation with respect to φ^1 and φ^2, we have

$$\Psi_+ = \sqrt{\frac{E+mc^2}{2E}}\begin{pmatrix} 1 \\ \frac{\eta c p - i\frac{c}{\hbar}\sqrt{\alpha}p^2\ell}{mc^2+E} \end{pmatrix} \otimes s e^{\frac{i}{\hbar}(\vec{p}\vec{x}-Et)} \tag{20}$$

the solution with positive energy, where $s = \frac{1}{\sqrt{2(1+n^3)}}\begin{pmatrix} -n^1 + in^2 \\ 1+n^3 \end{pmatrix}$ spin up,

$s = \frac{1}{\sqrt{2(1+n_3)}}\begin{pmatrix} 1+n^3 \\ n^1 + in^2 \end{pmatrix}$ spin down.

According to the equation (19), this method makes to appear the 2×2-matrix $h = \eta c p \sigma^1 - c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2 + mc^2\sigma^3$, or $h = \eta c p \sigma^1 - \frac{c^2p^2}{\epsilon}\sigma^2 + mc^2\sigma^3$ (if ℓ is the Planck length scale), whose eigeinvalues are the positive and the negative energies. h is like a vector in Pauli algebra. So, energy of the spin-$\frac{1}{2}$ particle can be associated to a vector in Pauli algebra, whose length or intensity is given by the dispersion relation.

$$h^2 = E^2 \tag{21}$$

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References

[1] Einstein, A. (1905). *Ann. der Phys.* **17** 1905, 891.

[2] Snyder, H.S. (1947). *Phys. Rev.* **72** 1947, 68-71.

[3] Snyder, H.S. (1947). *Phys. Rev.* **71** 1947, 38-41.

[4] Sidharth, B.G. (2004). *Int. J. Theor. Phys.* 43 (9), 2004, pp.1857-1861.

[5] Glinka, L.A. (2010). *Apeiron* 17 (4), 2010, pp. 223-242, E-print: arXiv:0902.4811v4 [hep-ph].

[6] Glinka, L.A. (2010). *Apeiron* 17 (4), 2010, pp. 243-271, E-print: arXiv:0905.3916v3 [hep-ph].

[7] Glinka, L.A. (2011), E-print: arXiv:1102.5002v2 [physics.gen-ph].

[8] Sidharth, B.G. (2005). *Int.J.Mod.Phys.E.* 14, (6), 2005, pp.923ff.

[9] Sakurai, J.J. (1967). *Advanced Quantum Mechanics*, Addison Wesley Publishing Company, 1967, pp.308-311.

[10] Bjorken J.D. and Drell, S.D. (1964). *Relativistic Quantum Mechanics*, Mc-Graw Hill, New York, 1964, p.39.

[11] Sidharth, B.G. (2009). E-print: arXiv: 0904.3639v2 [physics.gen-ph].

[12] Rakotonirina, C. (2003). *Produit Tensoriel de Matrices en Théorie de Dirac*, Thèse de Doctorat de Troisiéme Cycle, Université d'Antananarivo, Antananarivo, Madagascar, 2003.