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Position Vector Of Biharmonic Curves in the 3-Dimensional Locally ϕ-Quasiconformally Symmetric Sasakian Manifold

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Abstract: In this paper, we study biharmonic curves in locally ϕ-quasiconformally symmetric Sasakian manifold. Firstly, we give some characterizations for curvature and torsion of a biharmonic curve in in locally ϕ-quasiconformally symmetric Sasakian manifold. Moreover, we obtain the position vector of biharmonic curve in in locally ϕ-quasiconformally symmetric Sasakian manifold.

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1. Introduction

Let (N,h) and (M,g) be Riemannian manifolds. Denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention:

$$R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \Gamma(TN).$$

For a smooth map $\phi : N \longrightarrow M$, the Levi-Civita connection ∇ of (N,h) induces a conncetion ∇^ϕ on the pull-back bundle

$$\phi^* TM = \bigcup_{p \in N} T_{\phi(p)} M.$$

The section $\mathcal{T}(\phi) := \mathrm{tr}\nabla^\phi d\phi$ is called the tension field of ϕ. A map ϕ is said to be harmonic if its tension field vanishes identically.

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A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h.$$

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr} R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1}$$

and called the bitension field of ϕ. The operator Δ_ϕ is the rough Laplacian acting on $\Gamma(\phi^* TM)$ defined by

$$\Delta_\phi := -\sum_{i=1}^{n} \left(\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla^N_{e_i} e_i} \right),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In particular, if the target manifold M is the Euclidean space $\mathbb{E}^m$, the biharmonic equation of a map $\phi : N \to \mathbb{E}^m$ is

$$\Delta_h \Delta_h \phi = 0,$$

where Δ_h is the Laplace–Beltrami operator of (N, h).

Recently, there have been a growing interest in the theory of biharmonic maps which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

In this paper, we study biharmonic curves in locally ϕ-quasiconformally symmetric Sasakian manifold. Firstly, we give some characterizations for curvature and torsion of a biharmonic curve in in locally ϕ-quasiconformally symmetric Sasakian manifold. Moreover, we obtain the position vector of biharmonic helix in in locally ϕ-quasiconformally symmetric Sasakian manifold.

2. Locally ϕ-Quasiconformally Symmetric Sasakian Manifold

Let $(\mathbb{M}, g)$ be a 3-dimensional contact Riemannian manifold with contact form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. If ξ is a Killing vector field then $\mathbb{M}$ is called a $K-$contact Riemannian manifold [1]. If in such a manifold the relation

$$(\nabla_X \phi) Y = g(X, Y)\, \xi - \eta(Y)\, X \tag{2}$$

holds, where ∇ denotes the Levi-Civita connection of g, then $\mathbb{M}$ is called a *Sasakian manifold*.

Let R, Q, r denote the curvature tensor of type (1,3), Ricci operator and scalar curvature of $\mathbb{M}$, respectively. It is known that in a contact manifold $\mathbb{M}$ the Riemannian metric may be so chosen that the following relations hold

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \tag{3}$$
$$\phi^2 X = -X + \eta(X)\xi, \tag{4}$$
$$g(X,\xi) = \eta(X), \tag{5}$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{6}$$

for any vector fields X, Y. If $\mathbb{M}$ is a Sasakian manifold, then besides (3)-(6), the following relations hold

$$\nabla_X \xi = -\phi X, \quad (\nabla_X \eta) Y = g(X, \phi Y), \tag{7}$$
$$\Phi(X,Y) = (\nabla_X \eta) Y, \tag{8}$$
$$\Phi(X,Y) = -\Phi(Y,X), \tag{9}$$
$$\Phi(X,Y) = 0, \tag{10}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{11}$$
$$R(\xi,X)Y = (\nabla_X \phi) Y, \tag{12}$$
$$S(X,\xi) = (n-1)\eta(X). \tag{13}$$

Lemma 3.1. A 3-dimensional Sasakian manifold $\mathbb{M}$ is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant.

3. Biharmonic Curves in Locally ϕ-Quasiconformally Symmetric Sasakian Manifold $\mathbb{M}$

Let us consider biharmonicity of curves in 3-dimensional locally ϕ-quasiconformally symmetric Sasakian manifold. Let $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ be the Frenet frame field along γ . Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= -\tau \mathbf{N}, \end{aligned}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}} \mathbf{T}|$ is the geodesic curvature of γ and τ its geodesic torsion.

A helix is a curve with constant geodesic curvature and geodesic torsion. In particular, curves with constant nonzero geodesic curvature and zero geodesic torsion are called (Riemannian) circles. Note that geodesics are regarded as helices with zero geodesics curvature and torsion.

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}} \mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}} \mathbf{T}) \mathbf{T} = 0, \tag{14}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (14).

Theorem 3.1. *$\gamma : I \longrightarrow \mathbb{M}$ is a unit speed biharmonic curve in the locally ϕ-quasiconformally symmetric Sasakian manifold $\mathbb{M}$ if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 + \tau^2 &= 1, \\ \tau' &= 0. \end{aligned} \tag{15}$$

Proof. Using (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla^3_{\mathbf{T}}\mathbf{T} - \kappa R(\mathbf{T},\mathbf{N})\mathbf{T} \\ &= (-3\kappa\kappa')\mathbf{T} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{N} + (2\tau\kappa' + \kappa\tau')\mathbf{B} - \kappa R(\mathbf{T},\mathbf{N})\mathbf{T}. \end{aligned}$$

Using the Riemannian curvature in $\mathbb{M}$, we have

$$(3\kappa\kappa')\mathbf{T} + (\kappa'' - \kappa^3 - \kappa\tau^2 - \kappa)\mathbf{N} + (2\tau\kappa' + \kappa\tau')\mathbf{B} = 0 \tag{16}$$

By (16), we see that γ is a biharmonic curve if and only if

$$\begin{aligned} \kappa\kappa' &= 0, \\ \kappa'' - \kappa^3 - \kappa\tau^2 + \kappa &= 0, \\ 2\tau\kappa' + \kappa\tau' &= 0. \end{aligned} \tag{17}$$

These, together with (16), complete the proof of the theorem.

Lemma 3.2. *Let $\gamma : I \longrightarrow \mathbb{M}$ is a unit speed biharmonic curve in the locally ϕ-quasiconformally symmetric Sasakian manifold $\mathbb{M}$. Then, γ is a helix.*

Corollary 3.3.

$$\begin{aligned} \kappa &= \cos\rho, \\ \tau &= \sin\rho, \end{aligned} \tag{18}$$

where ρ is a constant angle.

Theorem 3.4. *Let $\gamma : I \longrightarrow \mathbb{M}$ is a unit speed biharmonic curve in the locally ϕ-quasiconformally symmetric Sasakian manifold $\mathbb{M}$. Then the position vector of γ is given by*

$$\begin{aligned} \gamma(s) = &\left(s.\sin^2\rho + c_1\cos\rho\sin s - c_2\cos\rho\cos s + c_3\right)\mathbf{T} \\ &+ \left(-\cos\rho + c_1\cos s + c_2\sin s\right)\mathbf{N} \\ &+ \left(s.\sin\rho\cos\rho - c_1\sin\rho\sin s + c_2\sin\rho\cos s + c_4\right)\mathbf{B}, \end{aligned} \tag{19}$$

where c_1, c_2, c_3, c_4 are constants of integration and ρ is a constant angle.

Proof. If $\gamma : I \longrightarrow \mathbb{M}$ is a biharmonic curve in the locally ϕ-quasiconformally symmetric Sasakian manifold M, then we can write its position vector as follows:

$$\gamma(s) = a(s)\mathbf{T} + b(s)\mathbf{N} + c(s)\mathbf{B} \tag{20}$$

for some differentiable functions a, b and c of $s \in I \subset \mathbb{R}$. These functions are called component functions (or simply components) of the position vector.

Differentiating (20) with respect to s and by using the corresponding Frenet equation, we find

$$\begin{aligned} a'(s) - b(s)\,\kappa &= 1, \\ b'(s) - a(s)\,\kappa - c(s)\,\tau &= 0, \\ c'(s) + b(s)\,\tau &= 0. \end{aligned} \tag{21}$$

From (21) we get the following differential equation:

$$b''(s) + \left(\kappa^2 + \tau^2\right) b(s) + \kappa = 0. \tag{22}$$

By using first equation of (15), we find

$$b''(s) + b(s) + \cos\rho = 0. \tag{23}$$

The solution of (23) is

$$b(s) = -\cos\rho + c_1 \cos s + c_2 \sin s, \tag{24}$$

where c_1 and c_2 are constants of integration.

From $a'(s) - b(s)\,\kappa = 1$ and using (24), we find the solution of this equation as follows:

$$a(s) = s.\sin^2\rho + c_1 \cos\rho \sin s - c_2 \cos\rho \cos s + c_3, \tag{25}$$

where c_3 is constant of integration.

By using (24), we find the solution of $c'(s) + b(s)\,\tau = 0$ as follows:

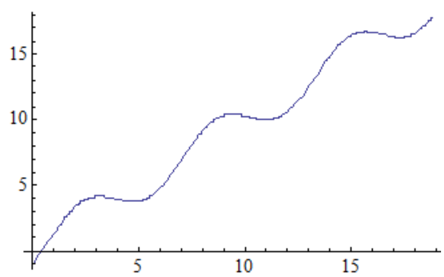
$$c(s) = s.\sin\rho\cos\rho - c_1 \sin\rho \sin s + c_2 \sin\rho \cos s + c_4, \tag{26}$$

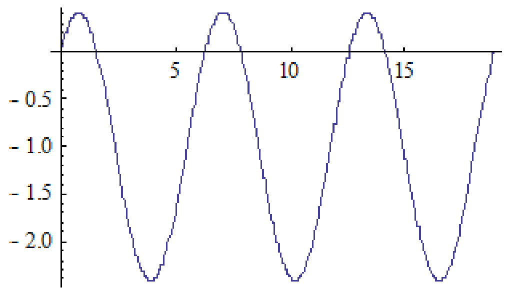
where c_4 is constant of integration.

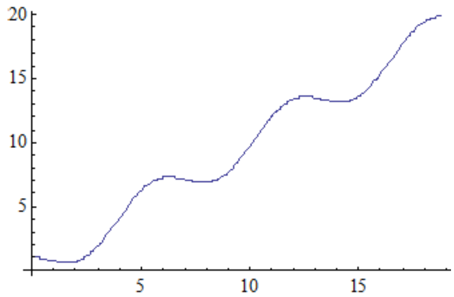
Substituting (24), (25) and (26) in (20) complete the proof of the theorem.

4. Applications

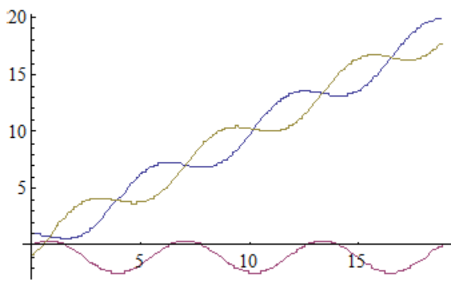
Using programme of Mathematica, we can draw the functions $a(s)$, $b(s)$, $c(s)$ from $\cos\rho = \frac{1}{\sqrt{2}}$, $c_1 = c_2 = c_3 = c_4 = 1$ are as follows, respectively:



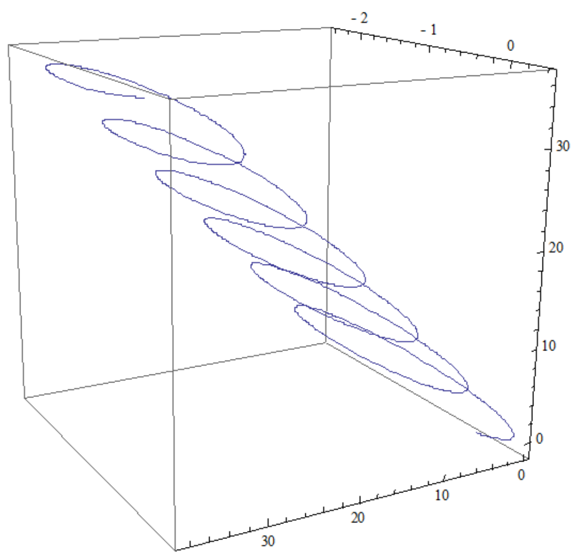




Similarly, we show the functions $a(s)$, $b(s)$, $c(s)$ together in one figure from $\cos\rho = \frac{1}{\sqrt{2}}$, $c_1 = c_2 = c_3 = c_4 = 1$ is as follows:



On the other hand, we get coordinate of γ as $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, we can draw γ from $\cos\rho = \frac{1}{\sqrt{2}}$, $c_1 = c_2 = c_3 = c_4 = 1$ is as follows:



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