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A Brief Historical Review of the Important Developments in Lanczos Potential Theory

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Abstract: In this paper we review some of the research that has emerged to form *Lanczos potential theory*. From Lanczos' pioneering work on quadratic Lagrangians, which ultimately led to the discovery of his famed tensor, through to the current developments in the area of exact solutions of the Weyl–Lanczos equations, we aim to exhibit what are generally considered to be the pivotal advances in the theory.

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1. Introduction

It is generally accepted that Lanczos' ground-breaking paper entitled 'The Splitting of the Riemann Tensor', [41] published in 1962 heralded *Lanczos potential theory*. However, the ideas and concepts that were to form the basis of the 1962 paper were born from two other very important but much overlooked papers: 'A Remarkable Property of the Riemann–Christoffel Tensor in Four Dimensions' [39] (published in 1938) and 'Lagrange Multipliers and Riemannian Space' [40] (published in 1949). The former introduces variational principles that are quadratic in the components of the Riemann tensor, and the latter introduces for the first time the Lanczos tensor albeit in a different guise.

As an introduction to the topic of Lanczos potential theory, the fundamental papers of Lanczos are first considered, demonstrating both the derivation of the theory and the fundamental field tensor, H_{abc}. In the following sections, both the 1949 [40] and 1962 [41] papers will be discussed, including the definition of Lanczos' Lagrangian, Lagrange

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multipliers and gauge conditions. This will form the foundation for later sections that discuss the application of these findings.

2. The Lanczosian Development of the Theory

Lanczos' investigations into Riemannian space-time stem back, for the purposes of this article, to his 1938, 1949 and 1962 papers, [39, 40, 41]. It is in the 1949 paper [40], however, that Lanczos' *spintensor* as he calls it first appears. This third-order tensor is derived from a quadratic action principle by means of a rather novel approach based on his investigations in the 1938 paper. It obviates the problems incurred when one attempts to discuss the field equations of the metric in terms of the classical field equations. This approach then effectively transforms the differential field equations to second order, thus keeping them comparable with those of physics.

Based on the work of Einstein's 1916 paper, [23], which establishes the connection between Riemannian curved space and physics, the challenge was set to find a new geometrical framework that would explain all physical phenomena — not just that of gravity. For example, by incorporating electricity into the fundamental field equations or by unifying all of the fundamental forces of physics. Some attempts worth noting that tried to generalise the field equations include the geometry of Weyl [76, 77], Eddington [18, 19], Schouten [61, 62] and Schroedinger [63, 64]. Others include the theories of Kaluza [38], Einstein [23, 24, 25, 26, 27, 28], Einstein and Mayer [29], Veblen [72, 73, 74], Veblen and Hoffmann [75], Hoffmann [36], Pauli [54, 55], Paul and Solomon [56, 57] and Bergmann [6, 7]. All of these theories tried but failed to build or define a space-time structure that unified all of the known fundamental fields.

In Lanczos' 1949 paper, the theme of examining the prospect of a theory that is not based purely on gravity is explored. Lanczos uses the assumption that the fundamental field equations should be derivable from an action principle and retains the classical framework of Riemannian geometry. Instead of using a linear action principle, Lanczos also postulated that the principle should be quadratic in the curvature components. Lanczos uses the quadratic action principle, but has to overcome one of the biggest criticisms of the basic field equations: the field equations for the metric tensor, g_{ik}, produce differential equations of fourth order rather than second order. As anything above second order does not have any physical meaning [22], this, of course, creates a problem when trying to relate the field equations to physics.

Lanczos utilised the variational principle [42] to investigate the basic structure of the field equations, and in turn drew comparisons with the Einsteinian theories of gravity [23, 24] and distant parallelism [26, 27]. The variational problem he employed assumes that the components of the Riemann curvature tensor, R_{ijkm}, are the basic metrical quantities rather than the metric tensor, g_{ik}. This restricts the Riemann tensor, R_{ijkm}, by the Bianchi identities, [see 65, p. 143] which causes the Bianchi identities to act as the auxiliary conditions of the variational problem. Consequently, a new fundamental field tensor acting as a Lagrange multiplier is introduced, which restricts the fundamental field equations to no more than second order. In the 1949 paper, the fundamental nature of the tensor is observed, but it is not until his 1962 paper [41], that applications are considered.

Lanczos relates his spintensor to a similar third order tensor, $\Lambda^i{}_{jk}$, that occurs in Einstein's theory of distant parallelism [27]. Both tensors are antisymmetric in j and k and are of third order. The main difference between the two quantities, however, is that Lanczos' tensor is a new fundamental tensor whereas Einstein's is a derived tensor from known quantities.

One important aspect of the 1949 paper is that it is firmly routed in four-dimensions. This is because it is based on a transformation of the Bianchi identities, which is only possible in four-dimensions. Although this is restrictive, it can be argued that it should not be considered a problem, as both Maxwell's action for electromagnetic fields and Dirac's equation of the electron strongly suggest a four-dimensional physical world.

Let us consider the 1949 paper in slightly more detail. We begin with Lanczos' original re-formulation of the Bianchi identities.

Defining the dual of the curvature tensor as

$$^*R^*_{ijkm} = R^{abuv}\epsilon_{abij}\epsilon_{uvkm}$$

[see 40, eq. II.1], where ϵ_{abuv} is the totally antisymmetric Levi-Civita tensor, allows the Bianchi identities for the dual to be written as a divergence

$$^*R^{*a}{}_{jkm;a} = 0 \tag{1}$$

[see 40, eq. II.2]. The above equation then allows the quantities R_{ijkm} and g_{ik} to be treated as separate metrical quantities rather than R_{ijkm} being expressed in terms of g_{ik} as long as the divergence condition (1) is satisfied. Hence, the divergence condition acts as an auxiliary condition of the variation. The consequence of the auxiliary condition yields a Lagrange multiplier, $H^i{}_{jk}$, which is antisymmetric in j and k, but where the symmetry properties of the first component are not known. The auxiliary condition of the variation for the Lagrange multiplier is therefore

$$-\frac{1}{2}\int H^i{}_{jk;m}\,{}^*R^*_{ijkm}d\tau$$

[see 40, eq. III.2], which is added to the action

$$\frac{1}{2}\int ({}^*R^*_{ijkm})^2 d\tau$$

[see 40, eq. III.3].

After applying the variation, Lanczos formed a view of the Riemann tensor that appears to be very similar in structure to that of the Riemann tensor expression involving Christoffel symbols

$$^*R^{*i}{}_{jkm} = H^i{}_{jk;m} - H_{mjk;}{}^i + H_j{}^i{}_{m;k} - H_k{}^i{}_{m;j} \tag{2}$$

[see 40, eq. III.5]. So in this case the tensor $H^i{}_{jk}$ appears to be analogous to the Christoffel symbol, Γ^i_{jk}, [see 65, p. 97], but with a few obvious differences. Γ^i_{jk} does not transform as a tensor whereas $H^i{}_{jk}$ transforms as a tensor of third order. Also, $\Gamma^i{}_{jk}$ is symmetric in j and k whereas $H^i{}_{jk}$ is anti-symmetric in j and k.

To derive Lanczos' field equations, (2) is substituted into the Bianchi identities, ${}^*R^{*i}{}_{jkm;a}$, and the tensor $P^i{}_{jk}$ is introduced

$$P^i{}_{jk} \equiv \Delta H^i{}_{jk} - H_{jk;}{}^i + H^i{}_{j;k} - H^i{}_{k;j} = 0$$

where rather than introducing the wave operator, $\Box$, Δ is the Laplacian operator, $\partial^2/\partial x_a^2$, as Lanczos works with an imaginary time coordinate, and

$$H_{jk} = H^a{}_{jk;a} \text{ and } H^i{}_j = H^i{}_j{}^a{}_{;a}.$$

Therefore, a system of 24 partial differential equations of second-order and 24 related field quantities has been generated.

In both the formalism of Einstein's field equations and his theory of distant parallelism, the importance of the divergence identities is seen. In the field equations, the vanishing divergence of the contracted curvature tensor, $R_{ik} - 1/2Rg_{ik}$, gives rise to the conservation laws of momentum and energy. In Lanczos' theory the two divergence identities are

$$P^a{}_{ik;a} \equiv 0 \text{ and } (P^k{}_{ia} - P^i{}_{ka})_{;a} \equiv 0$$

[see 40, eqns. V.1 and V.4], which as in Einstein's theory of distant parallelism reduces the number of independent field equations from 24 to 12.

Lanczos' 1949 paper does not claim to, and does not, solve any of the great unsolved problems of theoretical physics, but it does provide the basis of his 1962 paper, where the importance of his spintensor is realised. Although Lanczos did not explicitly mention it in his paper, the spintensor was shown by Takeno, [66], to generate the Weyl curvature tensor and it is a surprising fact that Lanczos was unaware of this when he wrote the 1962 paper.

2.1 The Splitting of the Riemann Tensor

Einstein examined in some detail the anti-self-dual part of the Riemann tensor, and it is from this analysis that the field equations were derived. As an analysis of the self-dual was never completed it was the aim of Lanczos' 1962 paper [41] to complete the investigation and carry out a similar analysis for the self-dual part of the Riemann tensor. His 1949 paper [40] had already defined the spintensor, H_{ijk}, and shown that by using it to integrate the field equations, the differential equations governing the field equations could be reduced from fourth to second order. In this paper he proves how the fundamental significance of the spintensor is implied by the relation to Dirac's equations [12, 13, 14, 15] of the electron.

One of the main questions facing Lanczos, was whether the Dirac equations could be derived from the Ricci tensor, R_{ik}. This is an important question that needed to be answered as it can hardly be denied that the electro-magnetic field does not have something to do with the four-dimensional nature of the physical universe. Both the double Maxwellian and Dirac equations have special significance in four-dimensions. As the Ricci tensor exists in all dimensions and does not show anything unusual in four-dimensions, Lanczos instead examined the Riemann tensor, R_{ijkm}, as analogies can be drawn from electromagnetic theory [47, 48] when considered in four-dimensions. In order to establish this analogy, it was necessary to find a component of the Riemann tensor that had the same symmetry properties of the Maxwell equations. As a result Lanczos stated that

'exactly those components of the Riemann tensor that are not embraced by the gravitational equations of Einstein, give us the clue toward a deeper understanding of the electromagnetic and wave-mechanical phenomena, with-in the framework of general relativity' [41, p. 382].

In fact only in four-dimensions is it possible to move between the Riemann tensor and its dual, analogous to the relationship between F_{ik} and ${}^*F^*{}_{ik}$. Hence,

$$ {}^*R^{*ijkm} = \frac{1}{4} R_{abuv} \epsilon^{abij} \epsilon^{uvkm} $$

[see 41, eq. 1.2]. The Riemann tensor can be split into two parts called the anti-self-dual and the self-dual:

$$ R_{ijkm} = \frac{1}{2}(A_{ijkm} + S_{ijkm}) $$

[see 41, eq. 1.5], where the anti-self-dual and the self-dual are defined respectively by

$$ A_{ijkm} = R_{ijkm} - {}^*R^*{}_{ijkm} $$

and

$$ S_{ijkm} = R_{ijkm} + {}^*R^*{}_{ijkm} $$

[see 41, eq. 1.4]. The Riemann tensor has 20 independent components, but they are not evenly split between the self-dual and the anti-self-dual 10+10 as may be assumed, but instead 9+11 where the anti-self-dual has 9 independent components and the self-dual 11. Contracting the Riemann tensor by the metric tensor, $g_{ik} = \delta_{ik}$, and applying the cyclic identity

$$ R_{ijkm} + R_{ikmj} + R_{imjk} = 0, $$

[see 41, eq. 1.13], we see that only two independent components exist which are self-dual and which contribute two components to the self-dual tensor, S_{ijkm}. In the anti-self-dual case these components cancel out. Both the anti-self-dual and the self-dual contain nine other components; this explains the unequal splitting of the Riemann tensor.

This method of splitting the Riemann tensor was first attributed to Rainich [58] and was later used by Einstein in his analysis of the anti-self-dual part of the Riemann tensor that resulted in the derivation of his field equations and hence the theory of general relativity. Although Einstein had great success when analysing the anti-self-dual part he soon abandoned his research of the Riemann tensor because his theory could not be combined with the electro-magnetic tensor. Lanczos therefore completed the analysis to yield a fuller understanding of the Riemann tensor without which we would lack important knowledge about Riemannian geometry.

Both Einstein and Lanczos used the variational principle to derive their field equations. As Einstein argued that the fundamental field equations of physics are inherently of first or second order, he chose a particularly simple Lagrangian for the variational derivation of his field equations. In the case of Einstein's Lagrangian, the resulting field equations are not higher than second order. On the other hand, Lanczos' field equations were reduced to first order. Of course it was hoped that the Lagrange multipliers possessed important physical significance; for example the 'potential energy' Lagrange multiplier of Hertz's forceless mechanics [35]. For this reason it has been debated whether Lanczos' spintensor, which forms one of the Lagrange multipliers, also has physical meaning. To strengthen the case that this is so, it can be argued that both the trace and trace-free part of the Riemann curvature tensor have physical meaning. Therefore, is it possible that upon splitting the Riemann tensor to derive Lanczos' spintensor, the latter also has a physical meaning

In constructing the action, Lanczos merely assumed that there existed a Lagrangian, L, that would generate the field equations. Using a number of pre-disposed conditions or auxiliary conditions, he then formulated his Lagrangian. Firstly, he considered L being constructed from the Riemann tensor and the metric tensor

$$A = \int L g^{\frac{1}{2}} dx_1 dx_2 dx_3 dx_4 \text{ with } L = L({}^*R^{*ijkm}, g_{ik})$$

[see 41, eqns. 2.2 and 2.3] where the important point to note in the construction of this Lagrangian is that the dual, ${}^*R^*{}_{ijkm}$, and the metric tensor, g_{ik}, are treated as independent quantities. Lanczos was able to introduce this property because of the presence of the auxiliary conditions. The first auxiliary condition to be introduced was that ${}^*R^{*ijkm}$ had to satisfy the Bianchi identities, ${}^*R^{*ijkm}{}_{;m} = 0$. The associated Lagrange multiplier for this first condition is Lanczos' spintensor, H_{ijk}. The second condition was that the Christoffel symbols, under variation, must satisfy $\Gamma^j_{ik} - \{^{\,j}_{ik}\} = 0$ where $\{^{\,j}_{ik}\} = \frac{1}{2} g^{jl}(g_{il,k} + g_{kl,i} - g_{ik,l})$. In this case the Lagrange multiplier is P^{ik}_j. Lastly, he introduced the contracted Riemann tensor, R_{ik}, where $R_{ik} \equiv \Gamma^k_{jk;i} - \Gamma^k_{ij;k} + \Gamma^k_{il}\Gamma^l_{jk} - \Gamma^k_{ij}\Gamma^l_{lk}$; hence $R_{ik} - \Gamma^k_{jk;i} + \Gamma^k_{ij;k} - \Gamma^k_{il}\Gamma^l_{jk} + \Gamma^k_{ij}\Gamma^l_{lk} = 0$. The corresponding Lagrange multiplier is ρ^{ik}. As all the auxiliary conditions are equal to zero, they can be added on to the initial Lagrangian without altering the properties of the original expression:

$$\begin{aligned} L' = {} & L({}^*R^{*ijkm}, g_{ik}) + H_{ijk}\,{}^*R^{*ijkm}{}_{;m} + P^{ik}_j(\Gamma^j_{ik} - \{^{\,j}_{ik}\}) \\ & + \rho^{ik}[R_{ik} + \Gamma^j_{kj;i} - \Gamma^j_{ik;j} + \Gamma^j_{il}\Gamma^l_{kj} - \Gamma^j_{ik}\Gamma^l_{lj}] \end{aligned}$$

[see 41, eq. 2.7]. Now that the Lagrangian has been established, varying with respect to appropriate geometrical quantities becomes a straightforward procedure.

A large proportion of the 1962 paper is dedicated to his spintensor, the Lagrange multiplier, H_{ijk}, but to gain a complete understanding of Lanczos' Lagrangian all the Lagrange multipliers need to be examined. The Lagrange multiplier, H_{ijk}, was referred to by Lanczos as the spintensor. It can be hypothesised that Lanczos gave it this name due to the connection between Dirac's equation of the electron [12, 13] and the splitting of the Riemann tensor. Examining H_{ijk} further it can be seen, due to the anti-symmetric nature displayed in the last two components, that it has 24 components whereas its conjugate contains 20. In order to restrict these components to match the desired 20, without any loss of generality, the following condition can be imposed,

$$^*H^{aj}{}_a = H_{uva}\epsilon^{uvaj} = 0$$

[see 41, eq. 2.9], or alternatively it can be written as the cyclic condition

$$H_{ijk} + H_{jki} + H_{kij} = 0 \tag{3}$$

[see 41, eq. 2.10].

Lanczos considered each multiplier in turn to examine its importance. The Lagrange multiplier, ρ^{ik}, was split into it's trace and trace-free parts, but the significance of the trace-free part was not identified. On splitting ρ^{ik} into it's trace and trace-free parts one yields

$$\rho^{ik} = Q^{ik} + qg^{ik} \qquad \text{where} \qquad 4q = \rho^{ik}g_{ik}$$
$$\text{and } Q^i_i = Q^{ik}g_{ik} = 0$$

[see 41, eqns. 2.11, 2.12 and 2.13], but this is as far as Lanczos' discussion of ρ^{ik} goes. The fundamental tensor is examined in some detail, so the only other Lagrange multiplier left to investigate is the canonical variable $P^{ik}{}_j$. As Lanczos proves that the variation of Γ^j_{ik} yields a relation for $P^{ik}{}_j$ that is not a new tensor, this multiplier ceases to be of further interest.

The variation of g_{ik} gives

$$\begin{aligned}\frac{\partial L}{\partial g_{ik}} + \frac{1}{2}Lg_{ik} = {} & Q_{ab}\,^*R^{*aibk} + Q^{ia}R^k_a + Q^{ka}R^i_a \\ & -\frac{1}{2}RQ^{ik} - qR^{ik} - \frac{1}{2}Rqg^{ik} \\ & -\frac{1}{2}(P^{iak} + P^{kai} - P^{ika})_{;a}\end{aligned}$$

[see 41, eq. 2.15]. In order to vary the Lagrangian by R^{ijkm}, we need to first specify the factor δ^*R^{ijkm}

$$B_{ijkm} = \frac{\partial L}{\partial\,^*R^{*ijkm}} - H_{ijk;m} - Q_{ik}g_{jm} + qg_{ik}g_{jm}$$

[see 41, eq. 2.16], applying the symmetry properties of ${}^*R^{*ijkm}$

$$\begin{aligned}\frac{1}{8}[&B_{ijkm} + B_{kmij} + B_{jimk} + B_{mkji}\\ -&B_{jikm} - B_{kmji} - B_{ijmk} - B_{mkij}\\ &\qquad -\frac{1}{3}\left(B_{abuv}\epsilon^{abuv}\right)\epsilon_{ijkm}]\end{aligned}$$

[see 41, eq. 2.17], allows the final variation to become

$$\frac{\partial L}{\partial\, {}^*R^{*ijkm}} = H_{ijk;m} + Q_{ik}g_{jm} - qg_{ik}g_{jm} \tag{4}$$

[see 41, eq. 2.19]. From this variation, Lanczos was able to introduce two gauge conditions where (4) remains invariant under the transformations

$$\begin{aligned}H_{ijk} &\to H'_{ijk} - \Phi_j g_{ik} + \Phi_i g_{jk}\\ Q_{ik} &\to Q'_{ik} + \Phi_{i;k} + \Phi_{k;i} - \frac{1}{2}\Phi^{\alpha}{}_{;\alpha}g_{ik}\\ q &\to q' - \frac{1}{2}\Phi^{\alpha}{}_{;\alpha}\end{aligned}$$

[see 41, eq. 2.20], where Φ_i is an arbitrary vector that can be used to normalise H_{ijk}. With the choice of Φ_i set to

$$\Phi_i = -\frac{1}{3}H_{ijk}g^{jk},$$

[see 41, eq. 2.21], we see the first gauge condition is

$$H_{ijk}g^{jk} = H_i{}^a{}_a = 0,$$

[see 41, eq. 2.23], which reduces the number of independent components from 20 to 16. In addition, introducing the divergence-free gauge condition

$$H_{ij}{}^k{}_{;k} = 0,$$

[see 41, eq. 3.5], does not reduce the number of components of H_{ijk}, but instead reduces the number of components to 10 by reducing the number of independent quantities of the Weyl-Lanczos equations (which will be introduced later) and hence equals the same number of components as the Weyl tensor. We therefore have a tensor, H_{ijk}, with the same number of components as the Weyl tensor, which will be subsequently shown to differentially generate the Weyl conformal tensor.

Of major concern was the fact that Lanczos' Lagrangian, $L = R_{ijkm}\,{}^*R^{ijkm}$, yields a function that vanishes identically [41, p. 384]. Because of this property, it is not possible to derive the field equations from this Lagrangian. Even though one cannot use the Lagrangian to derive the field equations, the study still highlights some interesting findings concerning an arbitrary Riemannian geometry of four dimensions. However, using a similar approach to that of Einstein [25] and Rainich [58] Lanczos solved the problem of the uneven (9+11) matching of components when splitting the Riemann tensor. By including the Ricci scalar and the metric tensor the new anti-self-dual and the self-dual become

$$U_{ijkm} = A_{ijkm} + \frac{1}{6}R(g_{ik}g_{jm} - g_{im}g_{jk})$$
$$V_{ijkm} = S_{ijkm} - \frac{1}{6}R(g_{ik}g_{jm} - g_{im}g_{jk})$$

[see 41, eq. 3.9], where

$$R_{ijkm} = \frac{1}{2}(U_{ijkm} + V_{ijkm})$$

[see 41, eq. 3.10], after some manipulation V_{ijkm} can be written as

$$V_{ijkm} = 2H_{ijk,m} - (H_{ik} + H_{ki})g_{jm} \tag{5}$$

[see 41, eq. 3.13]. Unlike the original form of the self-dual, S_{ijkm}, in this new formulation, V_{ijkm} is written in terms of a 'generating function', H_{ijk}. It can then be argued that Lanczos' spintensor is present in every Riemannian manifold of four dimensions (Lanczos 1949). Another observation is that H_{ijk} is also the gradient of an anti-symmetric tensor of second order

$$H_{ijk} = F_{ij;k}$$

[see 41, eq. 2.24], with

$$H_{ijk}g^{ik} = H^{a}{}_{ja} = 0 \text{ and } H_{ijk} + H_{jki} + H_{kij} = 0,$$

[see 41, eq. 2.23], which coincide with the double set of Maxwellian equations

$$F^{ia}{}_{;a} = 0 \text{ and } {}^{*}F^{ia}{}_{;a} = 0.$$

It was Einstein's discovery that posed the idea that all physical phenomena can be interpreted as geometrical properties of space-time. But how does one deal with the matter tensor and what is its structure? Lanczos argued that these questions could not be answered until a complete understanding of four-dimensional Riemannian geometry was obtained and that the spintensor, a third-order antisymmetric tensor with 16 independent components, would play a fundamental role.

Although understanding the implications of the field equations is a formidable task, Lanczos laid down three statements that he considered would encapsulate the general function of the field equations.

'Riemann's geometry remains untouched by any encroachments through additions or generalisations. . . '.

The presence of an unadulterated Riemann geometry of specifically four dimensions brings into existence a tensor of third order H_{ijk} of 16 components. . . '.

The new hierarchy Φ_i, g_{ik}, H_{ijk}, R_{ijkm}...'. [41, p. 389]

The introduction of the third order spintensor, H_{ijk}, bridges the gap between the second order metric tensor, g_{ik}, and the fourth order Riemann curvature tensor, R_{ijkm}. Lanczos places the importance of his spintensor between the metric tensor and the Riemann tensor. The tensor that he believed to be the most important was the vector potential, Φ_i. Lanczos considered these quantities to provide "... all the necessary and sufficient building blocks for a rational explanation of electricity and the quantum phenomena". The choice of naming his Lagrange multiplier H_{ijk} a 'spintensor' is perhaps explained by the fact that the Dirac equation describing an electron with spin without a mass term is analogous to the set of equations resulting from his method of splitting the Riemann tensor.

Lanczos' discovery of the spintensor is normally attributed to his 1949 and 1962 papers, but his considerations leading to his discovery can be traced back as far as his much earlier 1938 paper [39]. Again the 1938 paper centres around the earlier papers of Rainich [58] and Einstein [25] looking at the results relating to the self-dual and anti-self-dual, but it is here that he introduces the quadratic variational principle that becomes key to his findings in the 1962 paper. As mentioned earlier the spintensor is first introduced in the 1949 paper, but this is only discussed for weak fields; it is in the 1962 paper that the spintensor is proven to be universally valid. It is this result that proves that the spintensor holds for all four dimensional Riemannian manifolds when obtained from the variation of a Lagrangian.

3. Lanczos' Spintensor as a Potential

Lanczos' 1962 paper is generally considered as one of his most profound contributions to general relativity and Riemannian geometry. Yet it resided in virtual obscurity for some years. There are perhaps two explanations for this. Firstly, Lanczos makes no identification as to what is being generated by his spintensor. According to a conversation Lanczos had with Joseph Zund in 1967, Lanczos was apparently unaware that his spintensor generated the Weyl tensor, and furthermore, he appeared completely unaware of the Weyl tensor's existence. It is just conceivable that other relativists may also have been unaware of the significance of the spintensor's generating properties. Secondly, there has been, even to this day, controversy surrounding its conception by invoking the variational process. It has been argued that all manner of entities can be conceivably produced for a particular action, but unless there are sound geometrical arguments to supplement or support any variational inferences then one would be unwise to rely wholly upon results obtained in this way. The second of these objections will be rectified presently; however, it took two years before a relativist called Hyôitirô Takeno published a result that would ultimately raise the profile of Lanczos' spintensor and begin the flurry of research activity that it enjoys today.

In his 1964 paper 'On the Spintensor of Lanczos' Takeno [66] exhibited Lanczos' generating function together with its associated symmetries and gauge conditions in a systematic way. Although from a historical perspective, the significant result of this paper was Takeno's proof that the Weyl conformal tensor was being generated by Lanczos' spintensor, it also heralded the first attempt to generate explicit forms of the spintensor for spherically symmetric plane wave solutions to the Einstein field equations that incorporated electromagnetic fields. In addition, he gave a necessary and sufficient condition for a quantity $H'_{ijk} = H_{ijk} + A_{ijk}$ to also be a spintensor for a given space-time, where A_{ijk} satisfied the same symmetries, gauge conditions etc. as H_{ijk} and which Takeno termed an s-tensor. He continued to say that the s-tensor was a spintensor of a given space-time if and only if the space-time is conformally flat, which is equivalent to putting the LHS of (5) equal to zero.

Now, it is well known that the vector potential ϕ_i generates the Maxwell tensor F_{ij} differentially. If the LHS of (5) is considered to be the Weyl conformal tensor then in an analogous way, one may regard that Lanczos' spintensor acts as a potential for the Weyl tensor. Thence, the Lanczos potential, H_{ijk}, generates the Weyl tensor, C_{ijkl}, via the set of equations called the *Weyl–Lanczos equations*:

$$\begin{aligned} C_{abcd} = {} & H_{abc;d} + H_{cda;b} + H_{bad;c} + H_{dcb;a} \\ & + H^{e}{}_{ac;e} g_{bd} + H^{e}{}_{bd;e} g_{ac} \\ & - H^{e}{}_{ad;e} g_{bc} - H^{e}{}_{bc;e} g_{ad}. \end{aligned} \tag{6}$$

4. Lanczos Potential Theory and the 2-Spinor Formalism

If one considers the Weyl–Lanczos equations (6) for a moment it will quickly become apparent that solving them for any particular space-time poses quite a few problems. They are, of course, non-linear and the set comprises 16 individual differential equations with 16 independent components of H_{ijk}. Clearly, solving this system of equations as they stand would prove to be a formidable task unless highly symmetric space-times were considered, and even then this would be most arduous if undertaken analytically. However, the problem becomes more tractable if one employs the machinery of the two-component spinor formalism.

The task of re-writing the Lanczos potential, H_{ijk}, and the tensor form of the Weyl–Lanczos equations utilising 2-spinors was successfully accomplished in 1968 by Maher and Zund in their paper 'A spinor approach to the Lanczos spin tensor'[45]. In their paper the connecting quantities that allow transference between world-tensors and spinors are the *Infeld–van der Waerden symbols*, $\sigma^{\mathbf{a}}{}_{AB'}$. Collectively, this symbol represents, up to a factor, four 2×2 Hermitian matrices i.e. the three Pauli spin matrices and the unit matrix. However, it is now conventional to dispense with this symbol and merely define the spinor equivalent of a particular tensorial quantity. Thence, in spinor form, the Lanczos (tensor) potential can be written as

$$H_{abc} = \epsilon_{A'B'} H_{ABCC'} + \epsilon_{AB} \overline{H}_{A'B'C'C}, \tag{7}$$

where the bar denotes complex conjugation. So far neither the cyclic condition nor the gauge conditions have been imposed on (7); thus the only symmetry properties present are

$$H_{ABCC'} = H_{(AB)CC'}.$$

However, on applying these conditions, one finally arrives at the spinor equivalent of the Lanczos tensor:

$$H_{ABCC'} = H_{(ABC)C'}. \tag{8}$$

If one now replaces the indices of (8) with all of the various combinations of the dyad indices 0 and 1, one obtains the eight complex scalar quantities known as *Lanczos coefficients*. These eight quantities can be related to a spin basis o^A and ι^A as follows

$$\begin{aligned}
H_0 &= H_{0000'} = H_{ABCC'}o^A o^B o^C o^{C'}\\
H_1 &= H_{0010'} = H_{ABCC'}o^A o^B \iota^C o^{C'}\\
H_2 &= H_{0110'} = H_{ABCC'}o^A \iota^B \iota^C o^{C'}\\
H_3 &= H_{1110'} = H_{ABCC'}\iota^A \iota^B \iota^C o^{C'}\\
H_4 &= H_{0001'} = H_{ABCC'}o^A o^B o^C \iota^{C'}\\
H_5 &= H_{0011'} = H_{ABCC'}o^A o^B \iota^C \iota^{C'}\\
H_6 &= H_{0111'} = H_{ABCC'}o^A \iota^B \iota^C \iota^{C'}\\
H_7 &= H_{1111'} = H_{ABCC'}\iota^A \iota^B \iota^C \iota^{C'}.
\end{aligned} \tag{9}$$

Or equivalently a null tetrad basis, l, n, m and $\overline{m}$:

$$\begin{aligned}
H_0 &= H_{abc}l^a m^b l^c = H_{131}\\
H_1 &= H_{abc}l^a m^b \overline{m}^c = H_{134}\\
H_2 &= H_{abc}\overline{m}^a n^b l^c = H_{421}\\
H_3 &= H_{abc}\overline{m}^a n^b \overline{m}^c = H_{424}\\
H_4 &= H_{abc}l^a m^b m^c = H_{133}\\
H_5 &= H_{abc}l^a m^b n^c = H_{132}\\
H_6 &= H_{abc}\overline{m}^a n^b m^c = H_{423}\\
H_7 &= H_{abc}\overline{m}^a n^b n^c = H_{422}.
\end{aligned} \tag{10}$$

Now it is a relatively straightforward procedure of determining the spinor equivalent form of (6). We merely quote the result, but see, for example, O'Donnell (2003) [51] for a detailed derivation. Thence,

$$\begin{aligned} C_{abcd} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &= \nabla_{DD'}(H_{ABCC'}\epsilon_{A'B'} + \overline{H}_{A'B'C'C}\epsilon_{AB}) \\ &\quad -\nabla_{CC'}(H_{ABDD'}\epsilon_{A'B'} + \overline{H}_{A'B'D'D}\epsilon_{AB}) \\ &\quad +\nabla_{BB'}(H_{CDAA'}\epsilon_{C'D'} + \overline{H}_{C'D'A'A}\epsilon_{CD}) \\ &\quad -\nabla_{AA'}(H_{CDBB'}\epsilon_{C'D'} + \overline{H}_{C'D'B'B}\epsilon_{CD}) \\ &\quad -\nabla_{EE'}(H_A{}^E{}_{CC'}\epsilon_{A'}{}^{E'} + \overline{H}_{A'}{}^{E'}{}_{C'C}\epsilon_A{}^E)\epsilon_{BD}\epsilon_{B'D'} \\ &\quad -\nabla_{EE'}(H_B{}^E{}_{DD'}\epsilon_{B'}{}^{E'} + \overline{H}_{B'}{}^{E'}{}_{D'D}\epsilon_B{}^E)\epsilon_{AC}\epsilon_{A'C'} \\ &\quad +\nabla_{EE'}(H_A{}^E{}_{DD'}\epsilon_{A'}{}^{E'} + \overline{H}_{A'}{}^{E'}{}_{D'D}\epsilon_A{}^E)\epsilon_{BC}\epsilon_{B'C'} \\ &\quad +\nabla_{EE'}(H_B{}^E{}_{CC'}\epsilon_{B'}{}^{E'} + \overline{H}_{B'}{}^{E'}{}_{C'C}\epsilon_B{}^E)\epsilon_{AD}\epsilon_{A'D'}. \end{aligned}$$

Transvecting with $\epsilon^{A'B'}\epsilon^{C'D'}$ and simplifying yields

$$\begin{aligned} 2\Psi_{ABCD} &= \nabla_D{}^{E'}H_{ABCE'} + \nabla_C{}^{E'}H_{ABDE'} \\ &\quad +\nabla_B{}^{E'}H_{CDAE'} + \nabla_A{}^{E'}H_{CDBE'}. \end{aligned}$$

Further simplification can be achieved by employing the spinor differential gauge $\nabla^{CC'}H_{ABCC'} = 0$ resulting in the spinor form of the Weyl–Lanczos equations (6):

$$\Psi_{ABCD} = 2\nabla_D{}^{E'}H_{ABCE'}. \tag{11}$$

The benefits of this equation as it stands are extremely limited. Indeed, it is only when (11) is re-written in terms of spin coefficients that solutions are forthcoming. This is categorised by the now familiar form:

$$\begin{aligned} \Psi_0 &= -2[DH_4 - \delta H_0 \\ &\quad +(\overline{\epsilon} - 3\epsilon - \overline{\rho})H_4 + (\overline{\alpha} + 3\beta - \overline{\pi})H_0 + 3\kappa H_5 - 3\sigma H_1] \\ \Psi_1 &= -2[DH_5 - \delta H_1 \\ &\quad +(\overline{\epsilon} - \epsilon - \overline{\rho})H_5 + (\overline{\alpha} + \beta - \overline{\pi})H_1 + 2\kappa H_6 - 2\sigma H_2 - \pi H_4 + \mu H_0] \\ \Psi_1 &= -2[\overline{\delta}H_4 - \Delta H_0 \\ &\quad +(\overline{\beta} - 3\alpha - \overline{\tau})H_4 + (-\overline{\mu} + 3\gamma + \overline{\gamma})H_0 + 3\rho H_5 - 3\tau H_1] \\ \Psi_2 &= -2[DH_6 - \delta H_2 \\ &\quad +(\overline{\epsilon} + \epsilon - \overline{\rho})H_6 + (\overline{\alpha} - \beta - \overline{\pi})H_2 - 2\pi H_5 + 2\mu H_1 + \kappa H_7 - \sigma H_3] \\ \Psi_2 &= -2[\overline{\delta}H_5 - \Delta H_1 \\ &\quad +(\overline{\beta} - \alpha - \overline{\tau})H_5 + (-\overline{\mu} + \gamma + \overline{\gamma})H_1 + 2\rho H_6 - 2\tau H_2 - \lambda H_4 + \nu H_0] \\ \Psi_3 &= -2[DH_7 - \delta H_3 \\ &\quad +(\overline{\epsilon} + 3\epsilon - \overline{\rho})H_7 + (\overline{\alpha} - 3\beta - \overline{\pi})H_3 - 3\pi H_6 + 3\mu H_2] \\ \Psi_3 &= -2[\overline{\delta}H_6 - \Delta H_2 \\ &\quad +(\overline{\beta} + \alpha - \overline{\tau})H_6 + (-\overline{\mu} - \gamma + \overline{\gamma})H_2 - 2\lambda H_5 + 2\nu H_1 + \rho H_7 - \tau H_3] \\ \Psi_4 &= -2[\overline{\delta}H_7 - \Delta H_3 \\ &\quad +(\overline{\beta} + 3\alpha - \overline{\tau})H_7 + (-\overline{\mu} - 3\gamma + \overline{\gamma})H_3 - 3\lambda H_6 + 3\nu H_2]. \end{aligned}$$

Another seven years had to pass before anyone ventured to re-visit Lanczos potential theory. In fact it was Zund [78] again in his 1975 paper 'The theory of the Lanczos spinor' who took up the gauntlet. This paper took a more systematic approach in deriving the Weyl–Lanczos equation using the 2-spinor formalism. Moreover, it was the first attempt to obtain explicit forms of the Lanczos coefficients for a number of space-time metrics. Research in Lanczos potential theory was a trifle sparse throughout the remainder of the 1970's. However, again in 1975, Taub [67] considered Lanczos' 1962 paper, and being apparently oblivious to the earlier articles of Takeno and Maher & Zund, derived the 2-spinor form of both the Lanczos potential and the Weyl–Lanczos equations. It is interesting to note that he did not continue and obtain the spin coefficient form of the Weyl–Lanczos equations in that paper. Possibly, the solutions to the Weyl–Lanczos equations had no interest for him, which is strange considering his interest in exact solutions to Einstein's field equations. Also, Taub claims that Lanczos himself showed that the LHS of (5) was the Weyl conformal tensor, which was not quite the case. The most interesting aspect of Taub's paper is his identification of a differential operator exhibited in the neutrino equation, the vacuum Maxwell equations and the Bianchi identities in spinor form, which is analogous to the differential operator connecting the Lanczos spinor with the Weyl spinor in (11).

It could conceivably be asked whether it would be possible to construct for the Riemann tensor an analogous expression to (6). This area of endeavour has become formally known as the *Riemann–Lanczos* problem. The question was considered by Brinis Udeschini [68, 69, 70, 71] in a series of papers that spanned four years. Udeschini argued that the Riemann tensor can be generated by the derivatives of the Lanczos potential alone, i.e.

$$R_{abcd} = H_{abc;d} - H_{abd;c} + H_{cda;b} - H_{cdb;a}. \tag{12}$$

However, it was shown by Bampi & Caviglia [5] that the proposition (12) is false in six or less dimensions, and later by Massa & Pagani [46] for four dimensions. Nevertheless, the ideas encompassed in these papers furthered the scope of the theory by steering it in a different direction.

4.1 Proof of the Existence of the Lanczos Potential

As mentioned earlier there has been some controversy surrounding the conception of the Lanczos potential. This stems from the fact that the method employed was the variational process. Indeed, a proof of existence was part of the paper by Bampi & Caviglia that refuted proposition (12). It was based essentially upon reformulating equation (6) but without imposing the cyclic condition (3) and then employing Cartan's condition for involutiveness [9]. Reinhard Illge's 1988 paper 'On the potentials for several classes of spinor and tensor fields in curved spacetimes' [37] also gave a spinorial proof of the existence of the Lanczos potential, but the importance of this paper arises from the fact that it is the first to give a derivation of the wave equation for the Lanczos potential.

5. Solutions of the Weyl–Lanczos Equations

It must be considered that the ultimate goal in Lanczos potential theory is to determine unambiguously an algorithm that allows one to yield a complete set of Lanczos coefficients (9) and (10) for any given four dimensional pseudo-Riemannian space-time. Thus, for any given set of Weyl scalars solutions to (6) should be obtainable, in principle. Yet, research in this area although abundant has merely scratched the surface of this inscrutable dilemma. Indeed, after Takeno and Zund's attempts at solving these equations, the topic drew little attention until the late 1980s when Novello & Velloso in their paper 'The connection between general observers and Lanczos potential' [50] succeeded, albeit heuristically and with the aid of computational resources, in determining some algorithms for the Lanczos potential for three important space-times: Schwarzschild, Kasner and Gödel. The surprising aspect of their paper was the total omission of the 2-spinor formalism in any of the calculations, which would have afforded considerable reduction in computation. Their approach was essentially time-like and consisted of providing an *ad hoc* definition for a Lanczos potential in a given space-time. The various kinematical quantities were then manipulated to show that the given algorithm indeed satisfied the various kinematical assumptions together with the algebraic and differential Lanczos gauge conditions.

Unquestionably, Novello & Velloso's paper was responsible for sparking off a resurgence in Lanczos Potential theory. The space-times that were considered have been re-investigated regularly over the intervening years through a number of different approaches. Interestingly, at about the same time a team of investigators in Mexico had been working along similar lines only this time using the spinor formalism. A particularly important paper and one that heralded numerous others from the Mexican researchers was published in 1989 by de Parga, Chavoya & Bonilla [10] entitled 'Lanczos potential'. The paper gave in a concise manner the Weyl–Lanczos equations together with some possible form of the Lanczos potential for Petrov type N, III and O space-times. It was also the results of this paper that led to a great deal of speculation concerning the possibility that the Lanczos coefficients, (9) and (10), are merely spin coefficients in a different guise. A substantial amount of work followed from this paper with the primary aim of obtaining Lanczos coefficients for other space-times [11, 3, 32, 30, 31, 33, 43, 44]. Indeed, this particular line of investigation dominated the theory for some considerable time and led to a number of different approaches for generating Lanczos coefficients. In nearly all cases, the Lanczos coefficients obtained for various space-times could be directly formulated in terms of spin coefficients. However, it was shown by O'Donnell [53] that for the Schwarzschild metric a set of Lanczos coefficients could be obtained that could not be reduced merely to spin coefficients. Of course, the Lanczos coefficients for this metric had already been obtained a number of times using different methods, but this paper also illustrated, for the first time, the practicality of yielding different sets of Lanczos coefficients for the space-time — which although was expected from the theory was not previously realised.

It is well known that the majority of space-time metrics that exhibit interesting properties usually incorporate some number of non-null Killing vectors. Indeed, this was the ansatz that Dolan & Kim adopted, which would yield a completely different approach of determining Lanczos coefficients for various space-times including those mentioned earlier. Their paper 'Some solutions of the Lanczos vacuum wave equation' [16] utilised some of the results previously obtained in the Novello & Velloso paper [50]. Briefly, consider a non-null Killing vector ξ_a that satisfies the equation

$$\xi_{a;b} + \xi_{b;a} = 0.$$

Furthermore, let the Killing vector be hypersurface–orthogonal:

$$\xi_{[a;b}\xi_{c]} = 0.$$

Now define the unit tangent vector to the group trajectories: $u_a = \xi_a/\xi$ where $\xi^2 = \xi_a\xi^a$ remains invariant along the group trajectories. Because u_a is expansion-free, shear-free and hypersurface–orthogonal it can be shown that

$$u_{a;b} = a_a u_b$$

where a_a is an acceleration 4-vector of the group congruence:

$$a_a = u_{a;c}u^c.$$

So, for example, the Schwarzschild solution admits a time-like and a space-like hypersurface–orthogonal Killing vector. Employing Dolan & Kim's method the results in two analytical expressions for the Lanczos potential both in each case of satisfy Lanczos' gauge conditions. The Lanczos potential found for the time-like Killing vector was the same one exhibited in the Novello & Velloso paper [50]. See also Mora & Sanchez [49]. Theor Phys (2009))

Another method for generating Lanczos coefficients for various space-times was obtained by O'Donnell [52]. In the paper 'A method for finding Lanczos coefficients with the aid of symmetries' a set of algorithms involving the Lanczos potential and certain kinematical quantities for some arbitrary space-times, obtained using time-like vector fields, was considered. It was shown that such algorithms appear most naturally in the context of the spin coefficient formalism. Furthermore, explicit solutions were derived for the new spin coefficient algorithms. Based on the three algorithms presented in Novello & Velloso [50], subsequent formulations were obtained using the standard Newman–Penrose null tetrad formalism. The Lanczos coefficients obtained in this way for the three space-times in question conveyed some interesting properties for the corresponding Petrov types. The identification of these 'symmetries' could be utilised to help verify other possible explicit forms for the Lanczos coefficients for different space-times but having the same Petrov type as those obtained.

6. The Wave Equation for the Lanczos Potential

The tensor form of the wave equation for the Lanczos potential was first exhibited in [41]. Essentially, Lanczos employed (6) to expand the Schouten form of the Bianchi identities

$$C_{abc}{}^{d}{}_{;d} = J_{abc} \tag{13}$$

where $J_{abc} = R_{ca;b} - R_{cb;a} - \frac{1}{6}(g_{ca}R_{;b} - g_{cb}R_{;a})$, and then utilising the Ricci identity yields the result:

$$\begin{aligned}\Box H_{abc} = &\ J_{abc} - R_{apcq}H^{qp}{}_{b} + R_{bpcq}H^{qp}{}_{a} \\ &-R_{c}{}^{p}H_{abc} - R_{a}{}^{p}H_{cpb} + R_{b}{}^{p}H_{cpa} \\ &-\frac{1}{2}[(R_{bpqr}g_{ac} + R_{apqr}g_{bc})H^{qrp} + R_{abpq}H^{pq}{}_{c}].\end{aligned}$$

However, this equation does not reduce to $\Box H_{abc} = 0$ in vacuum. Marred by typographical errors, Lanczos' original form of the wave equation has undergone a number of re-derivations. Roberts [59] succeeded in producing a partially corrected version of the Lanczos wave equation:

$$\begin{aligned}\Box H_{abc} = &\ J_{abc} + [R_{bpqr}g_{ac} - R_{apqr}g_{bc}]H^{pqr} \\ &+R^{s}{}_{c}H_{abs} - R^{s}{}_{b}H_{csa} + R^{s}{}_{a}H_{csb} \\ &+R_{bqpc}H_{a}{}^{pq} - R_{aqpc}H_{b}{}^{pq} - R_{pqab}H_{c}{}^{pq}.\end{aligned}$$

Nevertheless, this too is not quite the correct version. In vacuum it does not reduce to $\Box H_{abc} = 0$, but produces an unnecessary constraint.

The first re-derivation of Lanczos' wave equation was due to Illge [37]. He used the 2-spinor formalism to show that if the Weyl spinor satisfied the spinor version of (13):

$$\nabla_{DC'}\Psi_{ABC}{}^{D} = J_{ABCC'}$$

then the Lanczos potential satisfies the wave equation

$$\begin{aligned}\Box H_{ABCC'} = &\ J_{ABCC'} + 3\nabla_{(A|C'|}\nabla^{EF'}H_{BC)EF'} \\ &+6\Phi_{C'G'C(A}H_{BC)}{}^{CG'} - \frac{1}{4}RH_{ABCC'}\end{aligned}$$

when both trace-free and divergence-free gauge conditions have been imposed. Note that we have adopted the notation: $\Box = \nabla_a\nabla^a = \nabla_{AB'}\nabla^{AB'}$. In vacuum this becomes:

$$\Box H_{ABCC'} = J_{ABCC'}.$$

The tensor version of Illge's wave equation is

$$\begin{aligned}\Box H_{abc} = &\ J_{abc} - 2R_{c}{}^{s}H_{abs} + R_{a}{}^{s}H_{bcs} + R_{b}{}^{s}H_{acs} \\ &+[H_{dbs}g_{ac} - H_{das}g_{bc}]R^{ds}.\end{aligned}$$

However, even Illge's version contained a typographical error, which was discovered by Dolan & Kim [17]. Their fully corrected version of the Lanczos wave equation is

$$\begin{aligned}\Box H_{abc} = & J_{abc} - 2R_c{}^s H_{abs} + R_a{}^s H_{bcs} + R_b{}^s H_{acs} \\ & +[H_{dbs}g_{ac} - H_{das}g_{bc}]R^{ds} + \frac{1}{2}RH_{abc}.\end{aligned}$$

In fact the only difference between this version and Illge's version is the factor $\frac{1}{2}RH_{abc}$. Illge's spinor version is, however, correct.

7. Physical Interpretation of the Lanczos Potential

This is an area of Lanczos potential theory that remains largely uninvestigated and is also arguably the most important factor of the theory that requires delineation. The fact that the Lanczos tensor acts as a potential in an analogous way to the vector potential in electromagnetic theory is not in itself proof that it is endowed with some physical presence. Nevertheless, it is a geometrical object of space-time and consequently there is good reason to presume that it can be interpreted physically.

It was Mark Roberts in his paper 'The physical interpretation of the Lanczos tensor' [60] who gave the first account of a possible physical interpretation of the Lanczos potential. In this paper, Roberts investigates a situation analogous to the Aharonov-Bohm effect [2].

Now, the Lorentz force accounts for the only physical presence of a Maxwell field on a point charge. It is a classical effect for regions with non-vanishing electric or magnetic fields. However, in the quantum realm one might have a vanishing electromagnetic field but non-vanishing vector potential, yet physical effects exist in this region: this is the Aharonov-Bohm effect. It was hoped that a similar kind of reasoning could be applied to the Lanczos potential, which might result in determining whether the potential has dynamical effects, analogous to the vector potential, thus endowing it with some kind of physical significance. It was concluded that in attempting to exhibit physical significance a covariant derivative corresponding to the electromagnetic covariant derivative (this is how the vector potential interacts with the Schrodinger equation) was required. A number of requirements for the existence of such a covariant derivative were postulated, but nothing conclusive emerged. Nevertheless, the Roberts paper stirs up some interesting conjectures as to a possible quantum mechanical description of the Lanczos potential.

The following year in 1997 Göran Bergqvist in his paper 'A Lanczos potential in Kerr Geometry' [8] also postulated a physical description for the Lanczos potential. The main conclusion was that a relationship existed between the Lanczos potential and the mass and momentum for the Kerr manifold. In this case the Lanczos potential defines a flat space-time connection. In their paper 'Lanczos potential for a rotating black hole' Gaftoi, López-Bonilla & Ovando [30] also postulated that, in appropriate coordinates, the Lanczos potential may act as an angular momentum density.

Appendix

Here we give accounts of some other research that might not directly fit into previous sections, but are still important contributions to the theory.

Avez [1] adopted the Weyl–Lanczos equations derived by Takeno and characterised a particular class of Einstein spaces. In fact he was interested in determining (globally) the effect of the Lanczos potential on space-time. His conclusion was that for a connected, not necessarily simply, space-time manifold then this space-time would necessarily incorporate a global coordinate system.

Atkins & Davis [4] adopted Lanczos' variational procedure, which yielded the Lanczos potential for non-Abelian gauge theories.

Hammon & Norris [34] were the first to give a precise explanation of the Lanczos gauge transformations from a proper gauge theoretic perspective and the corresponding gauge group was identified. They gave a fibre bundle characterisation of the connection structure which forms the basis of the Lanczos potential structure. This was, in fact, the generalised affine geometry of linear connections for a space-time manifold.

Edgar [21] obtained a new identity that made more transparent earlier results with respect to the tensor/spinor formulation of the Lanczos wave equation. He also made contributions to the identification of the Lanczos potential's existence and non-existence in dimensions other than four; see for example [20].

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