# A Boubaker Polynomials Expansion Scheme Solution to Random Love's Equation in the Case of a Rational Kernel

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**Abstract:** A polynomial expansion scheme is proposed as an analytical method for solving Love's integral equation in the case of a rational kernel. The tangible advantage of the used method, namely the Boubaker polynomials expansion scheme, is the proposition of a piecewise continuous infinitely derivable solution. Comparison with some results proposed in the related literature has been also carried out.


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## 1. Introduction

Love's integral equation (Fredholm equation of the second kind) [1-10] has shown a big interest in the several applied physics fields such as polymer structures, aerodynamics, fracture mechanics hydrodynamics and elasticity engineering. For literature related to the numerical solutions of singular integral equations of the deterministic type, a surveys of different analytical methods for the solution of random integral equations has been proposed by Bharucha-Reid [1] and Christensen *et al.* [2].

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## 2. Theory

Love's equations were early established by Love [6-8,11-14] for solving some magnetic and electrical fields problems. The actual study is concerned with calculation of the normalized field created conjointly by two similar plates of radius $R$, separated by a distance $k \times R$, where $k$ is a positive real parameter, and at equal or opposite potential, with zero potential at infinity, is the solution of the Love's [12-13] second kind integral equation:

$$\psi(x) = g(x) + \int_{-2}^{+2} K(x,t) \times \psi(t)dt \tag{1}$$

where $\psi$ is the normalized field to be determined , $g(x)$ is a given function and $K(x,t)$ is the rational kernel function with values in [-1;1] and defined by:

$$K(x,t) = \pm\frac{1}{\pi}\frac{k}{k^2 + (x-t)^2} \tag{2}$$

where the sign $\pm$ codes for equal or opposite potential cases.

## 3. Solution and Discussion

According to the Boubaker Polynomials Expansion Scheme *BPES* [15-26], the physically accepted weak solution is expanded as follows:

$$\psi(x) = \sum_{m=1}^{M_0} \beta_m . B_{4m}(x) \tag{3}$$

where $B_{4m}(x)$ are the 4m-order Boubaker polynomials [15-24], $\beta_m$ are unknown coefficients, and $M_0$is a given integer.

The relevant properties of the Boubaker polynomials are detailed in Appendix.

By introducing eq. (3) in eq. (1), one obtains:

$$\sum_{m=1}^{M_0} \beta_m . B_{4m}(x) = g(x) + \int_{-1}^{+1} \pm\frac{1}{\pi}\frac{k}{k^2 + (x-t)^2} \sum_{m=1}^{M_0} \beta_m . B_{4m}(t)dt \tag{4}$$

which is simplified to:

$$\sum_{m=1}^{M_0} \beta_m \left[1 - W_m(x)\right] \times B_{4m}(x) = g(x) \tag{5}$$

with:

$$W_m(x) = k \times \pm\frac{1}{\pi} \int_{-1}^{+1} \frac{B_{4m}(t)}{k^2 + (x-t)^2} dt \tag{6}$$

The integral form in Eq. (6) in calculated in reference to the $B_{4m}(x)$ expression [19-25]. It gives:

$$W_m(x) \ = 4R \times \pm\frac{1}{\pi} \sum_{p=0}^{2m} \left[\frac{(m-p)}{(4m-p)} C_{4m-p}^{p}\right] \times (-1)^p \times I_{2m-p+1}(x) \tag{7}$$

where :

$$I_q(x) \ = \int_{-1}^{+1} \frac{t^{2q-2}}{k^2+(x-t)^2} dt \tag{8}$$

$W_m(x)|_{m=1..M_0}$are calculated through calculation of $I_q(x)$ integrating the expression by part.
Once the expressions of $H_m(x)|_{m=1..M_0}$are determined, an appropriate uniform sampling $x_m|_{m=1..M_0}$on $x$ is carried out inside the interval [-1;1], so that the main equation eq. (5) becomes a matrix ($M_0$x$M_0$) system:

$$[P]\,[\beta] = [Q] \tag{9}$$

where: $[P]$ and $[Q]$are two arrays calculated using eq. (7-8) and $[\beta]$is the unknown array:

$$[\beta]^T = \begin{pmatrix} \beta_1 \ \beta_2 \ ... \ \beta_{M_0} \end{pmatrix} \tag{10}$$

Solution of the system represented in eq. (9) is achieved via iterative protocols and by stopping iterations when the error is inferior to $\theta_0$, a given value.

$$\| \ [P] \times [\beta]_{sol} - [Q] \ \| \leq \theta_0 \tag{11}$$

The solutions obtained in the particular case of $g$(x) is equal to unity and $\theta_0 = 10^{-4}$for different values of $k$, along with another result [27] , are gathered in Fig. 1. In order

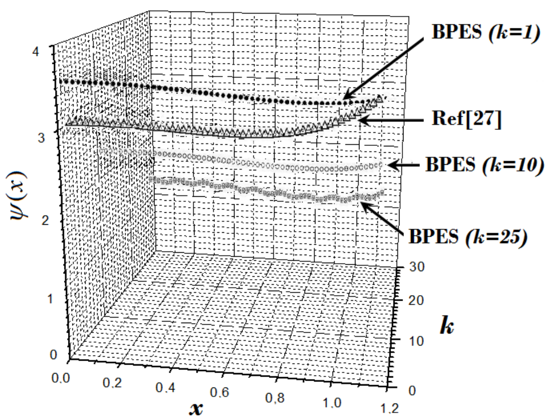


**Fig. 1** Solutions $\psi$(x) for $k = 1, 10$ and 25.

to compare accuracies, the mean quadratic error of the obtained solutions ($k$ ∈[1,27]) is plotted in Fig. 2. It can be noticed that the mean quadratic error decreases down to 6.5% for values of k superior to 23. This is a good proof of accuracy and rapid convergence.

## Conclusion and Perspectives

An analytical, piecewise continuous and differentiable solution to Love's equation has been presented. The Kernel function has been chosen as a rational function, case where

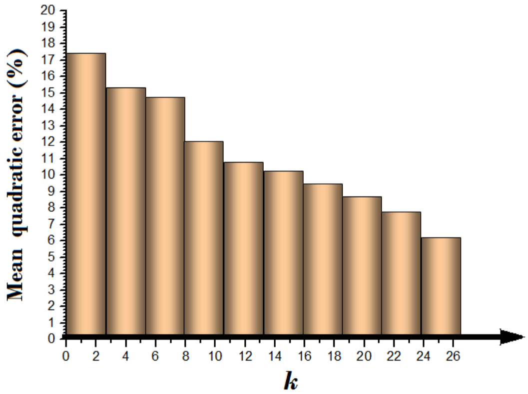


**Fig. 2** Mean quadratic error of the obtained solutions ($k \in [1,27]$)

exact analytical solutions are hard to derive. The used Boubaker Polynomials Expansion Scheme allowed yielding convergent and relatively accurate analytical solutions. This method can be used with more complicated kernels in order to give analytical solutions to several theoretical and applied physics problems.

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## Appendix: The Boubaker Polynomials

The Boubaker polynomials are integer-coefficient polynomial sequences which has been associated to several applied physics problems.
The first monomial definition of the Boubaker polynomials appeared in a physical study that yielded an analytical solution to heat equation inside a spray pyrolysis model. This monomial definition is traduced by:

$$B_n(X) = \sum_{p=0}^{\zeta(n)} \left[ \frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right] \cdot (-1)^p \cdot X^{n-2p}$$

where :

$$\zeta(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + ((-1)^n - 1)}{4}$$

(The symbol: $\lfloor \rfloor$ designates the floor function)

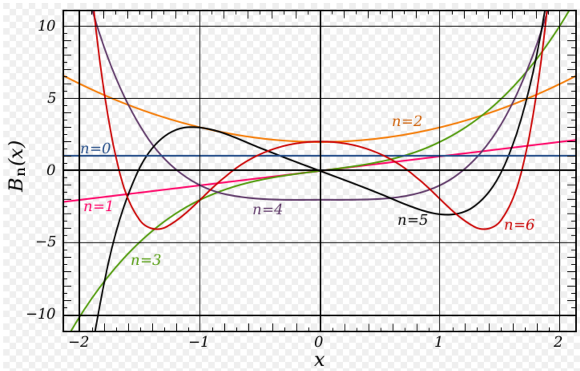


**Fig. 3** Plots of the first Boubaker polynomials

Their coefficients could be defined through a recursive formula:

$$\begin{cases} B_n(X) = \sum_{p=0}^{\zeta(n)} \left[ b_{n,j} X^{n-2j} \right], \\ b_{n,0} = 1, \\ b_{n,1} = -(n-4), \\ b_{n,j+1} = \frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \frac{(n-4j-4)}{(n-4j)} \times b_{n,j}, \\ b_{n,\zeta(n)} = \begin{cases} (-1)^{n/2} \times 2, & \text{if n even} \\ (-1)^{(n+1)/2} \times (n-2), & \text{if n odd} \end{cases} \end{cases}$$

The Boubaker polynomials have also a recursive relation:

$$\begin{cases} B_m(X) = X \times B_{m-1}(X) - B_{m-2}(X) \quad \text{for } m > 2, \\ B_2(X) = X^2 + 2; \\ B_1(X) = X; \\ B_0(X) = 1. \end{cases}$$

The ordinary generating function $OGF$ $f_B(X,t)$ of the Boubaker polynomials is:

$$f_B(X,t) = \frac{1+3t^2}{1+t(t-X)}$$

The characteristic differential equation of the Boubaker polynomials is:

$$\begin{cases} A_n y'' + B_n y' - C_n y = 0 \\ with : \begin{cases} A_n = (x^2-1)(3nx^2+n-2) \\ B_n = 3x(nx^2+3n-2) \\ C_n = -n(3n^2x^2+n^2-6n+8) \end{cases} \end{cases}$$

The Boubaker polynomials $B_n(x)$ are linked to the Fermat polynomials $F_n\left(x\right)$by the relation:

$$B_n(x) = \frac{1}{\left(\sqrt{2}\right)^n} F_n\left(\frac{2\sqrt{2}\times x}{3}\right) + \frac{1}{\left(\sqrt{2}\right)^{n-2}} F_{n-2}\left(\frac{2\sqrt{2}\times x}{3}\right); n = 0, 1, 2, ...$$

It has been demonstrated that each 4q-order Boubaker polynomial has got exactly $2q$-1 real positive roots, contained exclusively in the domain ]0;2[. The arithmetical properties of the minimal real positive root denoted $\alpha_n$gave the fundaments of the Boubaker Polynomials Expansion Scheme (BPES), which was used in different applied physics studies. According to the BPES definition, for a complex function f($x$) of a real argument x defined in the domain [-$a$; $a$], the $4n$-Boubaker polynomials expansion scheme (BPES) is performed by applying the expression:

$$f(x) = \frac{1}{2N_0} \sum_{q=1}^{N_0} \xi_q . B_{4q}(x\frac{\alpha_q}{a})$$

where $\alpha_q$ is $4q$-Boubaker polynomial minimal root,$N_0$ is a prefixed integer, and $\xi_q$ ($q$ = 1, ..., $N_0$) are complex coefficients.
According to this formulation, a weak solution to the equation:

$$\Im\left(f(x)\right) = Z_0$$

Where $\Im$ is a known linear operator, $Z_0$ is a given complex number, is obtained by calculating the set of complex coefficients $\xi_n$ ($n$ = 1, ..., $N_0$) which minimizes the real functional $\Lambda(x)$:

$$\Lambda(x) = \left| \Im\left(\frac{1}{2N_0} \sum_{q=1}^{N_0} \xi_q . B_{4q}(x\frac{\alpha_q}{a})\right) - Z_0 \right|$$

While solving a Dirichlet-Newmann boundary-type differential equation, the advantage of the BPES lies in embedding the exogenous boundary condition thanks to the following 4q-Boubaker polynomials properties:

- Values at boundaries, in the reduced real domain [0 ; $\alpha_q$]:

$$\begin{cases} \left.\sum\limits_{q=1}^{N} B_{4q}(x)\right|_{x=0} = 2N \neq 0; \\ \left.\sum\limits_{q=1}^{N} B_{4q}(x)\right|_{x=\alpha_q} = 0; \end{cases}$$

- first derivatives values at boundaries:

$$\begin{cases} \left.\sum\limits_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=0} = 0 \\ \left.\sum\limits_{q=1}^{N} \frac{dB_{4q}(x)}{dx}\right|_{x=\alpha_q} = \sum\limits_{q=1}^{N} H_q \end{cases}$$

where: $H_n = B'_{4n}(\alpha_n) = \left( \frac{4\alpha_n[2-\alpha_n^2]\times \sum\limits_{q=1}^{n} B_{4q}^2(\alpha_n)}{B^{(}_{4(n+1)}\alpha_n)} + 4\alpha_n^3 \right)$

- second derivatives values at boundaries:

$$\begin{cases} \sum\limits_{q=1}^{N} \frac{d^2 B_{4q}(x)}{dx^2}\bigg|_{x=0} = \frac{8}{3}\left(N(N^2-1)\right) \\ \sum\limits_{q=1}^{N} \frac{d^2 B_{4q}(x)}{dx^2}\bigg|_{x=\alpha_q} = \sum\limits_{q=1}^{N} G_q \end{cases}$$

where:

$$G_q = \frac{d^2 B_{4q}(x)}{dx^2}\bigg|_{x=\alpha_q} = \frac{3\alpha_q(4q\alpha_q^2+12q-2)H_q - 8q(24q^2\alpha_q^2+8q^2-3q+4)}{(\alpha_q^2-1)(12q\alpha_q^2+4q-2)}$$

Many arithmetical and differential properties of the Boubaker polynomials have been demonstrated, i. e.:

$$B_{4(q+1)} = (X^4 - 4X^2 + 2) \times B_{4(q)} - B_{4(q-1)} = B_4^*(X) \times B_{4(q)} - B_{4(q-1)}$$

$$B_{4q}^2(X) - B_{4(q-1)}(X) \times B_{4(q+1)}(X) = X^2(X^2-1)^2(3X^2+4) = B_8^*(X);\ \forall q > 1$$

and :

$$\sum_{q=1}^{n} B_{4q}^2(x) = \frac{1}{2x[2x^2-4]}\left[B'_{4(n+1)}(x)B_{4n}(x) - B'_{4n}(x)B_{4(n+1)}(x) - 4x^3\right]$$

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