Positive Energy Projectors and Spinors

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Abstract: In spectral geometry, physical concepts may become a source of geometric data. Here, we examine the vacuum given by a complex structure on phase space. The vacuum provides a soldering form for internal degrees of freedom providing them thus with spatial significance and eventually allowing them to be interpreted as spinors. To show more clearly the possibilities and limitations, the example of a discretized torus is discussed.
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1. Introduction

The vacuum of free quantum field theory is determined by a complex structure J on the one-particle complex Hilbert space $\mathcal{H}$, the classical phase space. It is shown here that such a structure can supply a represented algebra $\mathcal{A}$ with a soldering form that relates internal degrees of freedom, i.e., the eigenspaces of $\mathcal{A}$ in $\mathcal{H}$ with geometric structure. In this way, the standard soldering form of spin geometry can be recovered.

A case of particular interest is the torus since its spin structure was recently discussed not only in the classical but also in the noncommutative case [1] in the setting of Alain. Connes' axioms [2] for spectral geometry. Connes' axioms provide automatically for a spin structure and capture well much of the essentials of geometry. The here presented

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approach is not intended to achieve the same degree of completeness but rather to provide an alternative, physically motivated point of view on structures that may be eventually obtained otherwise.

The example illustrating the chosen approach in this work is the discretized torus $\mathbb{T}_{(n_1,n_2)}$. Its particularities are spelled out in Section 2 in order to fix the notation. Section 3 discusses invariant vacua given by invariant complex structures and determines their high-frequency behavior. The soldering form is obtained in Section 4. Corresponding facts on continuous tori are mentioned throughout for comparison. The significance of the presented approach is discussed in the Conclusion.

Given the physical motivation of the taken approach, it is interesting to compare the results with the situation of an ordinary spin structure, understood as the phase space (space of initial conditions) of a Dirac field on a corresponding 2 + 1-dimensional flat spacetime. Such a comparison justifies the interpretation of the high energy limit of the complex structure as the soldering form. This is worked out in the Appendix, after a short review of basic facts on spin structures over low-dimensional Minkowski space. It is also shown there for completeness that while the spin rotation matrix (which would also qualify for a soldering structure) is generally visible in the commutator of the physical coordinates x^i_P, this is not applicable in our case as the commutator vanishes in the high energy limit in two dimensions.

2. Preliminaries

The discretized torus $\mathbb{T}_{(n_1,n_2)}$ is the space of the group $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. The group G acts on $\mathbb{T}_{(n_1,n_2)}$ by translations and is the counterpart of the symmetry $U(1) \times U(1)$ of the ordinary (undiscretized) torus $\mathbb{T}^2$.

In particular, the action of $g = [g_1, g_2] \in G$ on point $x = [x_1, x_2] \in \mathbb{T}_{(n_1,n_2)}$ is given by:

$$g(x) = [g_1, g_2]([x_1, x_2]) = [g_1 + x_1 \mod n_1, g_2 + x_2 \mod n_2] \tag{1}$$

and in the generators $V_1 = (1, 0)$ and $V_2 = (0, 1)$ of G act by

$$V_1(x) = [1, 0]([x_1, x_2]) = [x_1 + 1 \mod n_1, x_2] \tag{2}$$

$$V_2(x) = [0, 1]([x_1, x_2]) = [x_1, x_2 + 1 \mod n_2] \tag{3}$$

The geometry of the discretized torus $\mathbb{T}_{(n_1,n_2)}$ consists of $n_1 n_2$ points and can be described via the Gel'fand transform by an $n_1 n_2$-dimensional commutative C^*-algebra $\mathcal{A} = C(\mathbb{T}_{(n_1,n_2)}, \mathbb{C})$ of complex functions on $\mathbb{T}_{(n_1,n_2)}$. The action of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ on $\mathbb{T}_{(n_1,n_2)}$ induces a corresponding action on the C^*-algebra $\mathcal{A}$:

$$g(a)(x) = a(g(x)) \qquad \text{for all } g \in G,\ a \in \mathcal{A} \text{ and } x \in \mathbb{T}_{(n_1,n_2)} \tag{4}$$

To simplify calculations, the same notation will be used for the algebra $\mathcal{A}$ and its representation on $\mathcal{H}$. In addition, it will be assumed that the action of the symmetry group G is unitarily implemented on $\mathcal{H}$ and the same notation will be used for the group and its unitary representation.

In order to fix $\mathcal{H}$ and at the same time to allow for intrinsic degrees of freedom to be later interpreted as spin, we choose $\mathcal{H} = L^2(\mathbb{T}_{(n_1,n_2)}) \otimes \mathbb{C}^2$, where $L^2(\mathbb{T}_{(n_1,n_2)})$ is the up to unitary equivalence unique representation space of the smallest faithful involutive representation of $\mathcal{A}$ with the obvious unitary implementation of the symmetry group G:

$$g(\phi)(x) = \phi(g(x)) \qquad \text{for all } g \in G, \phi \in L^2(\mathbb{T}_{(n_1,n_2)}) \text{ and } x \in \mathbb{T}_{(n_1,n_2)}, \tag{5}$$

to be extended trivially to $\mathcal{H}$:

$$g(\phi \otimes v)(x) = \phi(g(x)) \otimes v \tag{6}$$

for all $g \in G, \phi \in L^2(\mathbb{T}_{(n_1,n_2)}), v \in \mathbb{C}^2$ and $x \in \mathbb{T}_{(n_1,n_2)}$.

More generally, one may take $\mathcal{H} = L^2(\mathbb{T}_{(n_1,n_2)}, \mathbb{C}^2)$, i.e., the square integrable sections of a fibre bundle with base space $\mathbb{T}_{(n_1,n_2)}$ and fibre $\mathbb{C}^2$. The above choice corresponds to the fibre bundle being trivial. This is of course not a topological statement as the discretized torus carries the discrete topology but rather a statement on the action of the symmetry group G. Nontrivial actions could be obtained introducing a sign in (6):

$$g(\phi)(x) = (-1)^{s_1(\frac{g_1+x_1}{n_1})^\star + s_2(\frac{g_2+x_2}{n_2})^\star} \phi(g(x)), \tag{7}$$

where $s_1, s_2 \in \{0, 1\}$ determine the chosen action and $(\,\cdot\,)^\star$ denotes the integral part. The introduced signs do not change (4) and modify thus only the internal structure (allowing for the counterparts of the four inequivalent spin structures over $\mathbb{T}^2$), not changing the space geometry itself.

Also, denoting the vectors of the representation space by $L^2(\mathbb{T}_{(n_1,n_2)})$ as square-integrable functions is rather formal, as any function on a finite number of points equipped with the uniform discrete measure is square-integrable. Actually, it serves as a reminder of what is necessary in the undiscretized case, as a tool of comparison.

2.1 Continuous Tori $\mathbb{T}^2_\theta$

The noncommutative torus is the algebra generated by two unitaries U_1, U_2 subject to the relation

$$U_1 U_2 = \lambda U_2 U_1, \qquad \lambda = e^{i2\pi\theta}, \qquad \theta \in \mathbb{R}. \tag{8}$$

More precisely, algebra elements a are power series $a = \sum_{kl} a_{kl} U_1^k U_2^l$ with coefficients a_{kl} which vanish faster than any polynomial for $k, l \to \infty$. The commutative torus corresponds then to the choice $\theta = 0$.

The representation of the algebra on $\mathcal{H} = L^2(\mathbb{T}^2_\theta)$, with basis $|n_1, n_2\rangle$, $n_k \in \mathbb{Z}$, is given by

$$U_1 |n_1, n_2\rangle = \lambda^{n_2} |n_1 + 1, n_2\rangle, \tag{9}$$

$$U_2 |n_1, n_2\rangle = |n_1, n_2 + 1\rangle. \tag{10}$$

This representation possesses a cyclic separating vector $|0, 0\rangle$.

All of these tori, whether finite projective modules over the commutative torus (for θ rational) or with trivial center (for θ irrational) are continuous in the sense of allowing a continuous $U(1) \times U(1)$-symmetry.

3. Invariant Vacua

A complex structure on $\mathcal{H}$ is a linear map $J : \mathcal{H} \to \mathcal{H}$ satisfying:

$$J^2 = -1. \tag{11}$$

The complex structure's eigenspaces $\mathcal{H}_+$,$\mathcal{H}_-$ are the spaces of positive and negative frequencies. The corresponding eigenvalue projections are P_+, P_- and the following relationships hold:

$$P_\pm = \frac{1 \mp iJ}{2}. \tag{12}$$

A sensible restriction of the freedom in J is to require the fulfillment of the following conditions:

(1) **Invariance of the vacuum.** J is invariant under the action of the group G.
(2) **Charge conjugation.** There is an invariant anti-linear isomorphism between the eigenspaces of J.
(3) **Zeroth order condition.** J is a zeroth order pseudo-differential operator. It means that there is a finite limit to the symbol of the operator J in any direction in Fourier space at infinity.

Given the action (6) of the group G on the Hilbert space $\mathcal{H}$, J will be invariant under the action of G if

$$\langle \psi_1 \mid g^{-1}(J)\psi_2 \rangle - \langle \psi_1 \mid J\psi_2 \rangle = \langle g(\psi_1) \mid Jg(\psi_2) \rangle - \langle \psi_1 \mid J\psi_2 \rangle = 0 \tag{13}$$

for all $g \in G$ and $\psi_1, \psi_2 \in \mathcal{H}$. For this to hold it suffices to require that J is invariant under the action of its generators V_1, V_2 of G.

The above statements involving Fourier space assume the discrete (inverse) Fourier transform on the discretized circle, of which the torus is an easy 2-dimensional generalization. On $\mathbb{T}_{(n_1,n_2)}$, it is given by

$$f_{xy} = \frac{1}{\sqrt{n_1 n_2}} \sum_{p=0}^{n_1-1} \sum_{q=0}^{n_2-1} \tilde{f}_{pq} e^{\frac{2\pi i}{n_1} px} e^{\frac{2\pi i}{n_2} qy}, \qquad \tilde{f}_{pq} = \frac{1}{\sqrt{n_1 n_2}} \sum_{x=0}^{n_1-1} \sum_{y=0}^{n_2-1} f_{xy} e^{-\frac{2\pi i}{n_1} px} e^{-\frac{2\pi i}{n_2} qy}, \tag{14}$$

and may be compared with the (inverse) Fourier transform on the continuous torus:

$$f(\phi_1, \phi_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \tilde{f}_{mn} e^{im\phi_1} e^{in\phi_2}, \qquad \tilde{f}_{mn} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(\phi_1, \phi_2) e^{-im\phi_1} e^{-in\phi_2} d\phi_1 d\phi_2. \tag{15}$$

While for the ordinary circle, the high frequency behavior is given by the limit $n \to \infty$ of the Fourier index, for the discretized circles of the torus $\mathbb{T}_{(n_1,n_2)}$, the high frequency behavior is given by the Fourier indices closest to $\frac{n_i+1}{2}$. For n_i even, there are two such indices, $\frac{n_i}{2}$ and $\frac{n_i}{2} + 1$ and on those we will assume J to agree.

3.1 Discretized Torus $\mathbb{T}_{(n_1,n_2)}$

Condition (13) can be separated using its discrete Fourier transformed equivalent, since in the Fourier picture, (6) becomes:

$$\widetilde{g(\phi)}(p) = \frac{1}{\sqrt{n}}\sum_{x=0}^{n-1}(g\psi)(x)e^{-\frac{2\pi i}{n}px} \tag{16}$$

$$= \frac{1}{\sqrt{n}}\sum_{x=0}^{n-1}(\psi)(x+g \mod n)e^{-\frac{2\pi i}{n}px} \tag{17}$$

$$= \frac{1}{\sqrt{n}}\sum_{x=0}^{n-1}(\psi)(x)e^{-\frac{2\pi i}{n}p(x-g)} = e^{\frac{2\pi i}{n}pg}\tilde{\phi}(p). \tag{18}$$

We have then from (13):

$$e^{\frac{2\pi i}{n}(p-q)g}{J^q}_p - {J^q}_p = 0 \qquad \text{for any } p,q,g \in \mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}. \tag{19}$$

This is possible only if

$${J^q}_p = 0 \qquad \text{for all } p \neq q. \tag{20}$$

Thus, J is determined by a free choice of complex structures ${J^p}_p$ on complex 2-dimensional subspaces $e^{\frac{2\pi i}{n}px}\otimes\mathbb{C}^2$ of $\mathcal{H}$.

To completely characterize the freedom of choice, the compatible complex structures on $\mathbb{C}^2$ can be easily computed by generally solving condition (11). The solutions can be given as

$$\pm i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{21}$$

and

$$J = -in^k\sigma_k \tag{22}$$

with n^k a unit vector in 3-dimensional Euclidean space and σ_k the Pauli matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{23}$$

Clearly, (21) does not allow for a charge conjugation as the dimensions of the eigenspaces for the eigenvalues $\pm i$ are different and thus solution (21) has to be discarded.

3.2 Continuous Tori $\mathbb{T}^2_\theta$

Repeating the separation procedure (16)–(18) for a torus $\mathbb{T}^2_\theta$, we get

$$e^{i\phi(n_1-n_2)m}{J^{n_1}}_{n_2} - {J^{n_1}}_{n_2} = 0 \qquad \text{for any } n_1,n_2,m \in \mathbb{Z}\times\mathbb{Z} \tag{24}$$

and J on $\mathbb{T}^2_\theta$ is determined by a free choice of complex structures ${J^p}_p$ on complex 2-dimensional subspaces $e^{i\phi m}\otimes\mathbb{C}^2$, $m\in\mathbb{Z}$ of $\mathcal{H}$, as before.

4. The Soldering Form

We get the soldering form directly from the high-energy limit of the positive energy projection in direction $n^i = \frac{k^i}{|\vec{k}|}$ of momentum space, cf. Appendix A2,

$$\lim_{|\vec{k}|\to\infty} P_+ = \frac{1}{2}\left(1 - \vec{\not{n}}\gamma^0\right). \tag{25}$$

The corresponding vacuum complex structure $J = iP_+ - iP_-$ has as its limit then

$$\lim_{|\vec{k}|\to\infty} J = -i\vec{\not{n}}\gamma^0. \tag{26}$$

Its existence (in any direction) is assured by the zeroth order condition. In the discretized case, this seems to be an empty condition. That this component is finite is of course automatically assured by the discretization. This idea is however of conceptual value in giving geometric significance to the highest frequency component of J in Fourier space.

4.1 Continuous Tori $\mathbb{T}^2_\theta$

The Fourier space of the torus $\mathbb{T}^2_\theta$ is $\mathbb{Z}\times\mathbb{Z}$. While this lattice does not have a continuous rotational symmetry, we show that there still is an asymptotic $U(1)$-symmetry.

Consider the space of 1-dimensional rays $(kn_1, kn_2)_{k\in\mathbb{N}}$ in $\mathbb{Z}^2$. These uniquely determine 1-dimensional rays $(rn_1, rn_2)_{r\in\mathbb{R}}$ in $\mathbb{R}^2$. But 1-dimensional rays in the Euclidean geometry of $\mathbb{R}^2$ are parametrized by the unit vectors forming the unit circle S^1 with the symmetry group $U(1)$ and rays coming from rays in $\mathbb{Z}^2$ are dense in this circle.

This set-up allows for an asymptotic symmetry requirement of symmetry of Fourier coefficients by assigning to each ray from $\mathbb{Z}^2$ the limit of Fourier coefficients along that ray at infinity and requiring these limits to define by completion a continuous, $U(1)$-covariant function on S^1 which can be used to define a soldering form through (26).

4.2 Discretized Torus $\mathbb{T}_{(n_1,n_2)}$

In the discrete case, infinity in Fourier space is given by integer points on the boundary of the rectangle $I = [-(\frac{n_1+1}{2})^\star, +(\frac{n_1+1}{2})^\star] \times [-(\frac{n_2+1}{2})^\star, +(\frac{n_2+1}{2})^\star]$, cf. Figure 1.

A clear geometric meaning can be given to the high frequency component of the positive energy projection in the directions of x_1 and x_2, i.e., $P((\frac{n_1+1}{2})^\star, 0)$ and $P(0, (\frac{n_2+1}{2})^\star)$ or directly through the corresponding high energy limit of the complex structure $J((\frac{n_1+1}{2})^\star, 0)$ and $J(0, (\frac{n_2+1}{2})^\star)$, see (A.25).

The choices of these complex structures are free data within our framework and lead to in general noncommuting analogues of Clifford generators, which are according to (22):

$$iJ((\frac{n_1+1}{2})^\star, 0) = n_1^k\sigma_k \qquad iJ(0, (\frac{n_2+1}{2})^\star) = n_2^k\sigma_k. \tag{27}$$

Unless $n_1 = \pm n_2$, these operators span the vector space of Clifford generators and the usual spin structure is thus obtained, though with the following restriction.

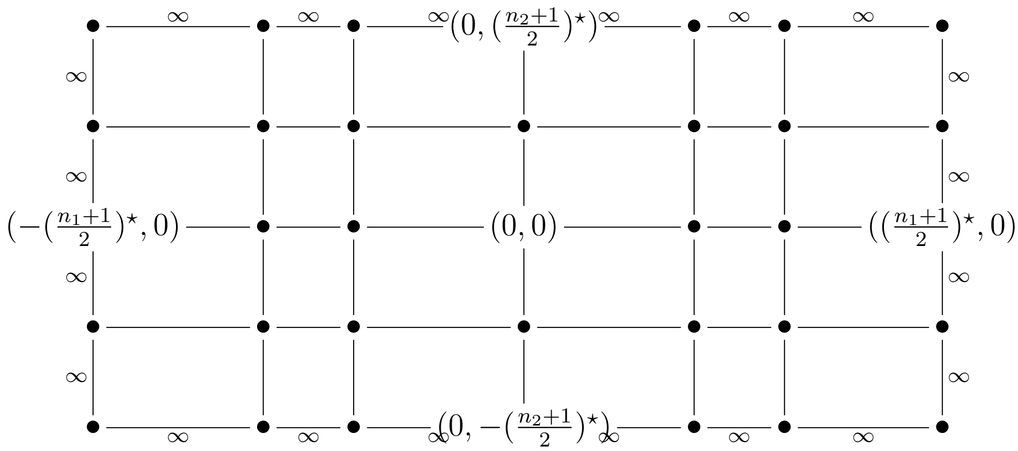

Fig. 1 Infinity in finite Fourier space of $\mathbb{T}_{(5,4)}$. The points on the boundary of the rectangle denoted by ∞ are what is to be considered infinity. Note, that with the particular values $n_1 = 5$, $n_2 = 4$, the infinite points of the upper side of the rectangle are identical with the ones on the lower while the points of the right and left boundary of the rectangle consist of distinct points. The points $((\frac{n_1+1}{2})^\star, 0)$ and $(0, (\frac{n_2+1}{2})^\star)$ are the infinities associated with the two discrete coordinate axes and are used as the high energy limit points for the corresponding coordinate directions.

It is tempting to extend these to some kind of algebra-valued form, as suggested by (A.24), but this idea cannot be applied without further modifications.

First, the linearity with respect to $\vec{n}$ in (A.24) is due to the Dirac operator being a first order differential operator while in a more general physical setting, a pseudodifferential operator is to be expected.

Second, we do not have at our disposal a rotational symmetry in a point of our space as the discretization of the torus destroyed that. This absence of structure is also a problem when dealing with noncommutative spaces, may however possibly be improved upon at least in this case using techniques similar to [3].

Thus the soldering form is not a form in the usual sense of spin manifolds. It does, however, provide a connection between spatial geometry (elementary lattice shifts) and internal degrees of freedom.

Conclusion

The above example is to be understood as a proof-of-concept: A complex structure given by the positive frequency projection motivated by quantum physics provides a link between spatial geometry and internal degrees of freedom and plays thus in this aspect the role of a soldering form.

This was already implicit in the remarkable work of M. H. L. Pryce [4] but not appreciated as a source of geometric information.

The discretization of the example did not change this robust fact while it destroyed the microlocal symmetry of the space in question. The resulting analogues of Clifford generators for elementary lattice shifts are not to be understood in a simple-minded way as components of a form.

Whether a more sophisticated point of view might treat that is an interesting problem for future investigations. The recourse to Fourier theory is at the moment a limitation. Still there are a number of spectral geometries allowing such an approach, notably the noncommutative torus which may be used to test the present ideas in a noncommutative setting.

Compared with Connes's axioms [2], this treatment chose not to start out with the Dirac operator and a fixation of the spin structure and can be seen as a possibility to obtain partial geometric information from concepts of quantum field theory.

Finally, let us remark that our work can also be accounted as an attempt to understand the geometry of negative energy solutions of the free Dirac equation. Another attempt on this problem was recently proposed by E. Trübenbacher, who utilizes just the operator 'sign of the energy', see [5].

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A Solutions of the Dirac equation on a static spacetime with flat (possibly toroidal) spatial sections

In the following, low space (resp. spacetime) dimensions are considered. While only 2-dimensional space is relevant to our calculations, comparison with other dimensions explains the status of the involved structure since some striking facts on spinors are just coincidences given by a special choice of dimension while other ones have general significance.

A1 The spin structure and its adjustment to physical requirenemts

The anticommutation relations for the Clifford algebra are assumed in the form

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}, \tag{A.1}$$

where $g_{\mu\nu}$ is the metric on the tangent space and the signature is assumed as

$$\operatorname{sign} g_{\mu\nu} = (-\underbrace{+ + \cdots +}_{n}). \tag{A.2}$$

This allows to set up a Clifford bundle acting on a spin bundle with sections ψ. The Dirac equation [6] for such a section ψ is written in the form

$$(\gamma^\mu \partial_\mu - m)\, \psi = 0. \tag{A.3}$$

Since a number of conventions is available in the literature, it has to be checked that the above settings fit together according to the physical requirements they should satisfy. The signs in the above equation are chosen so that causal propagation is satisfied. This can be checked on flat spacetime, where each solution can be decomposed into plane waves, by showing that the wave vector k_μ of the plane wave $\psi(x) = \psi_0 e^{ik_\mu x^\mu}$ is within the light cone, i.e., $k_\mu k^\mu \leq 0$. This follows from the following calculation:

$$(\gamma^\nu \partial_\nu + m)(\gamma^\mu \partial_\mu - m)\psi = 0 \quad (A.4)$$
$$\left(g^{\nu\mu}\partial_\nu \partial_\mu - m^2\right)\psi_0 e^{ik_\mu x^\mu} = 0 \quad (A.5)$$
$$k_\mu k^\mu = -m^2 \leq 0 \quad (A.6)$$

There are two natural hermitean inner products on spinors given by the intertwinners of up to isomorphism unique irreducible representations of the Clifford algebra [7]:

$$\gamma_\mu^+ A = A\gamma_\mu \qquad -\gamma_\mu^+ B = B\gamma_\mu \quad (A.7)$$

or more explicitly:

$$\gamma_\mu{}^{\bar{A}}{}_{\bar{B}} A_{\bar{A}B} = A_{\bar{B}A}\gamma_\mu{}^A{}_B \qquad -\gamma_\mu{}^{\bar{A}}{}_{\bar{B}} B_{\bar{A}B} = B_{\bar{B}A}\gamma_\mu{}^A{}_B \quad (A.8)$$

The following facts are shown in [7] (our B is their D):

For positive definite g, the A product is positive definite. Since reduction of the spacetime product to the space product changes one type (A, B) into the other (B, A), we have to take B as the correct spacetime spinor product.

For even spacetime dimensions, both A and B exist, in odd spacetime dimensions, just one of them exists. But fortunately in our signature, A always exists for the spatial part while B always exists for the spacetime product. So we can decide for these choices for all $n+1$-dimensional spacetimes. Then the spacetime inner product is B:

$$-\gamma_\mu^+ B = B\gamma_\mu \quad (A.9)$$

and its reduction to space $A = B\gamma^0$ satisfies:

$$\gamma_i^+ B\gamma^0 = B\gamma^0\gamma_i \quad (A.10)$$

We have formal selfadjointness of the operator $\not{D} - m$:

$$B(\phi, (\not{D} - m)\psi) - B((\not{D} - m)\phi, \psi) = \nabla_\mu B(\phi, \gamma^\mu \psi) \quad (A.11)$$

which leads by the application of Stokes' theorem to the invariant inner product on the (phase) space of solutions of the Dirac equation:

$$\langle \phi, \psi \rangle = \int_\Sigma B(\phi, \gamma^\mu \psi)\, d_\mu S = \int_{x^0=0} B(\phi, \gamma^0 \psi)\, d^3\vec{x} \quad (A.12)$$

A2 The Hamiltonian and the positive energy projector

The Dirac equation in flat spacetime (A.3) written as:

$$i\partial_0\psi = \underbrace{(i\gamma^0\gamma^i\partial_i - im\gamma^0)}_{H}\psi, \tag{A.13}$$

allows to read of the Hamiltonian:

$$H = i\gamma^0(\gamma^i\partial_i - m). \tag{A.14}$$

Under Fourier transform:

$$f(x) = \frac{1}{(\sqrt{2\pi})^2}\int \tilde{f}(k)e^{ixk}\, d^n k, \tag{A.15}$$

we have:

$$x^i f(x) \to i\frac{\partial}{\partial k_i}\tilde{f}(k) \qquad \frac{\partial}{\partial x_i}f(x) \to ik_i\tilde{f}(k) \tag{A.16}$$

Denote $E(k) = k^0 = -k_0$, $E(k) = \sqrt{m^2 + \vec{k}^2}$. Then the projectors onto positive and negative frequencies are in the Fourier transform:

$$P_\pm = \frac{1}{2}\left(1 \pm \frac{H}{E}\right) = \frac{1}{2}\left(1 \mp \frac{(\vec{\not k} + im)\gamma^0}{\sqrt{m^2 + \vec{k}^2}}\right). \tag{A.17}$$

This projection (expressed in the Fourier picture) is, unlike the ones given in many textbooks, see, e.g., [8], not only onto orthogonal spaces but also an orthogonal projection.

The positive energy projection of a coordinate x^i is $x^i_P = P_+ x^i P_+$.

$$[x^i_P, x^j_P] = P_+ x^{[i} P_+ x^{j]} P_+ = P_+ \underbrace{x^{[i} x^{j]}}_{0} P_+ - P_+ x^{[i}[x^{j]}, P_+]P_+ = \tag{A.18}$$

$$= -x^{[i}\underbrace{P_+[x^{j]}, P_+]P_+}_{0} + [x^{[i}, P_+][x^{j]}, P_+]P_+ \tag{A.19}$$

and since

$$[x^i, P_+] = i\frac{\partial}{\partial k_i}P_+(k) = \frac{i}{2E}\gamma^0\left(\gamma^i - \frac{\vec{\not k} + im}{E^2}k^i\right) \tag{A.20}$$

we get

$$[x^i_P, x^j_P] = -\frac{1}{4E}\left(\Omega^{ij} - P^i{}_k\Omega^{kj} - \Omega^{ik}P^j{}_k\right), \tag{A.21}$$

where

$$P^i{}_j = \frac{k^i k_j}{E^2} = \frac{k^i k_j}{m^2 + \vec{k}^2} \tag{A.22}$$

$$\Omega^{ij} = [\gamma^i, \gamma^j] \tag{A.23}$$

This projects the spin rotation matrix Ω^{ij} onto the plane orthogonal to $\vec{k}$. This is exactly the case in the high-energy limit (in the massless case), when E is asymptotically equal (exactly equal) to $|\vec{k}|$ and thus $P^i{}_j$ is indeed such a projector. For zero energy, the commutator of the coordinates gives just the spin rotation matrix Ω^{ij}.

In the high-energy limit, one needs at least three dimensions to obtain the spin rotation matrix from the commutators of coordinates. It is easier to get the soldering form directly from the high-energy limit of the positive energy projection in direction $n^i = \frac{k^i}{|\vec{k}|}$ of momentum space. Indeed,

$$\lim_{|\vec{k}|\to\infty} P_+ = \lim_{|\vec{k}|\to\infty} \frac{1}{2}\left(1 - \frac{(\vec{\not k} + im)\gamma^0}{\sqrt{m^2 + \vec{k}^2}}\right) = \frac{1}{2}\left(1 - \vec{\not n}\gamma^0\right) \tag{A.24}$$

That this high energy limit exists, i.e., that the symbol of P has a finite limit in momentum space, is basically the requirement that P should be an order zero pseudo-differential operator.

The corresponding vacuum complex structure $J = iP_+ - iP_-$ has as its limit then

$$\lim_{|\vec{k}|\to\infty} J = -i\vec{\not n}\gamma^0, \tag{A.25}$$

which justifies interpreting (27) as soldering structures.

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