A Lie Algebraic Approach to the Schrödinger Equation for Bound States of Pöschl-Teller Potential

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Abstract: The application of Group theoretical techniques to physical problems has a long and fruitful history. Lie algebraic methods have been useful in the study of problems in physics ever since Lie algebras were introduced by M.Sophus Lie (1842-1899) at the end of the 19th century, especially after the development of quantum mechanics. This is because quantum mechanics makes use of commutators \([x, P_x] = i\hbar\), which are the defining ingredients of Lie algebras. The theory of Lie groups and Lie algebras has become important not only in explaining the behaviour of various physical systems but also in constructing new physical theories. By identifying the suitable Spectrum Generating Algebra (SGA) the problem of interest can be approached. A Spectrum Generating Algebra exists when the Hamiltonian \(H\) can be expressed in terms of generators of the algebra. As a consequence the solution of the Schrödinger equation then becomes an algebraic problem which can be attacked using the tools of group theory. Here in this paper we derive the Schrödinger equation for the bound states of Pöschl-Teller potential using Lie algebra.

Keywords: Schrödinger Equation; Lie Algebra; Pöschl-Teller Potential

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1. Introduction

To understand and analyze a physical problem in its befitting manner quantum mechanics is an area of active interest from many stand points. Having its numerous connections with many other branches, quantum mechanics has been playing an important role both

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in experimental & theoretical approaches. Being fueled by the rapid development of sophisticated mathematical techniques, quantum mechanics is going through an exciting time of renewed interest. The application of group theoretical techniques to physical problems however has a long & fruitful history. Now for addressing the quantum mechanical problems embedded in group theoretical framework, a particularly powerful technique is that of Lie groups & Lie algebras. The matrix or algebraic formulation of the quantum mechanics started to show its greater suitability compared with the differential or wave formulation, at least in regard of matters inherent in symmetry problems. From the early works on the low-lying spectrum of hadrons [1, 2, 3], to the more recent studies of nuclear [4] and molecular [5] many-body problems, the theory of Lie groups and algebras has become important not only in explaining the behavior of various physical systems, but also in constructing new physical theories. Problems of interest can usually be approached in this manner when a spectrum generating algebra (SGA) can be identified. A SGA exists when the Hamiltonian H can be expressed in terms of generators of algebra. As a consequence, the solution of the Schrödinger equation then becomes an algebraic problem, which can be attacked using the tools of group theory.

The systematic use of Lie algebras dates back to the 1930s, with pioneering work by Weyl, Wigner, Racah and others. Lie algebraic methods have been useful in the study of problems in physics ever since Lie algebras were introduced by M. Sophus Lie (1842-1899) at the end of the 19th century, especially after the development of quantum mechanics in the first part of the 20th century. This is because quantum mechanics makes use of commutators \([x, P_x] = i\hbar\), which are the defining ingredients of Lie algebras. The essence of Lie algebraic method can be traced to the Heisenberg formulation of quantum mechanics. The use of Lie algebras as a tool to systematically investigate physical systems (the so called spectrum generating algebra) did not however develop fully until the 1970s when it was introduced in a systematic fashion by F. Iachello and A. Arima in the study of spectra of atomic nuclei (interacting boson model) [6,7]. In 1981 F. Iachello introduced Lie algebraic methods in systematic study of spectra of molecules (vibron model). Soon after the algebraic methods was extended to rotation-vibration spectra of polyatomic molecules. In the intervening years much work was done. Most notable advances were the extension to two coupled one-dimensional oscillators and its generalization to many coupled one-dimensional oscillators [8,11]. From 2002 to till this period many other works were reported by Ramendu Bhattacharjee et al algebraic method gave satisfactory explanations to evaluate the various vibrational energy levels of diatomic, triatomic, polyatomic and bio-molecules [12-21] and to our knowledge no works has been reported yet to use this sophisticated mathematical tool in Schrödinger equation problem, so in this study we aim to find out the Schrödinger equation for the bound states of Pöschl-Teller potential. The Hamiltonian used in this approach is an algebraic one and so are all the operations in this method are unlike the more familiar differential operators of wave mechanics. As it is a new approach and has not developed totally, much work is required to improve the approach.
2. Lie algebra

A set of operators \{X\} is a Lie algebra when it is closed under commutation. That is, for every operator ‘X’ in the algebra ‘G’

\[ [X_a, X_b] = C_{ab}^c X_c, \]

where \( C_{ab}^c = -C_{ba}^c \), \([X_a, X_a] = 0\). and the Jacobi identity

\[ [[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0. \]

3. Derivation

To begin with we consider the group SU(2) generated by three operators \( J_x, J_y, J_z \) which have their usual form obtained from

\[ \vec{J} = r \times (-i \vec{\nabla}). \]

We consider the Hamiltonian as \( H = -J_z^2 \), where the Casimir invariant \( C = J_z^2 \) is a constant. Now one interesting fact about the generators of SU(2) is that they obey Lie algebra as

\[ [J_x, J_y] = i\hbar J_z \]
\[ [J_y, J_z] = i\hbar J_x \]
\[ [J_z, J_x] = i\hbar J_y \]

Also they satisfy the condition of Jacobi identity. Now we have the Schrödinger Eq.

\[ H\Psi = E\Psi, \]
\[ C\Psi = J^2\Psi = j(j + 1)\Psi \]

Now if we realize these operators \( J_x, J_y \) & \( J_z \) in spherical coordinates i.e. \((r, \theta, \phi)\) then we have

\[ J^2 = -\hbar^2 [\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}] \]

and \( J_z = -i\hbar \frac{\partial}{\partial \phi} \)

using Eq.(3) in Eq. (2) we get

\[ J^2\Psi = -\hbar^2 [\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}]\Psi = j(j + 1)\Psi \]

or

\[ \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]\Psi = Q\Psi \]

where $Q = -\frac{j(j+1)}{\hbar^2}$. Writing $\Psi = f(\theta)g(\phi)$ we get

$$\cot \theta \frac{1}{f} \sin^2 \theta \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} - Q \sin^2 \theta = -\frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} \quad (6)$$

The L.H.S. of Eq. (6) is a function of $\theta'$ while R.H.S. is a function of $\phi'$. This is possible only when both sides equal a constant (say $m^2$) i.e.

$$-\frac{1}{g} \frac{\partial^2 g}{\partial \phi^2} = m^2 \quad (7)$$

and

$$\cot \theta \frac{1}{f} \sin^2 \theta \frac{\partial f}{\partial \theta} + \frac{\sin^2 \theta}{f} \frac{\partial^2 f}{\partial \theta^2} - Q \sin^2 \theta = m^2 \quad (8)$$

Solving Eq.(7), we get

$$g = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad (9)$$

For solving Eq.(8) we put $z = \cos \theta$ in Eq.(8) and then we get

$$(1 - z^2) \frac{d^2 f}{dz^2} - 2z \frac{df}{dz} + (-Q - \frac{m^2}{1 - z^2}) f = 0 \quad (10)$$

This is nothing but Associated Legendre’s differential equation whose general solution is given by

$$f(\theta) = \varepsilon \sqrt{\frac{2j + 1(j + |m|)!}{2(j + |m|)!}} P^m_j(\cos \theta) \quad (11)$$

Where $\varepsilon = (-1)^m$ and $\varepsilon = 1$ for $m \leq 0$

As $\Psi = f(\theta)g(\phi)$, hence using Eq.(9) and Eq.(11), we get

$$\Psi = \Psi_{jm}(\theta, \phi) = \varepsilon \sqrt{\frac{2j + 1(j + |m|)!}{2(j + |m|)!}} P^m_j(\cos \theta) e^{im\phi} \quad (12)$$

Substituting this value of $\Psi$ in Schrödinger equation given by Eq. (1), we get the energy as $E = -m^2$.

Again Putting $Z = \cos \theta = \tanh \rho(-\infty < \rho < \infty)$ in Eq.(10) we get

$$\left[ \frac{d^2}{d\rho^2} - \frac{j(j+1)}{\cosh^2 \rho} \right] P^m_j(\rho) = -m^2 P^m_j(\rho) \quad (13)$$

as $Q = -\frac{j(j+1)}{\hbar^2}$

This is the Schrödinger equation for the bound states of Pöschl-Teller potential. The strength of this potential is given by $V_o = j(j+1)$. Here a single representation ‘$j$’ will account for the spectral properties, given by $m^2$. 
Conclusion

The work presented here discusses the derivation of Schrödinger equation for bound states of Pöschl-Teller potential. Although the proportional behaviour of the normalized wave function $\Psi_{jm}(\theta, \phi)$ with associated Legendre’s function $P_j^m(\cos \theta)$ was already established, the mathematical form of the proportionality constant has been derived in the present work using Lie algebra, which will be helpful in understanding the nature & the behaviour of the wave function in greater depth.

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