The Modified Dirac Equation

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Abstract: We consider the behavior of particles at ultra relativistic energies, for both the Klein-Gordon and Dirac equations. We observe that the usual description is valid for energies such that we are outside the particle’s Compton wavelength. For higher energies however, both the Klein-Gordon and Dirac equations get modified and this leads to some new effects for the particles, including the appearance of anti particles with a slightly different energy.

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1. Introduction

It is now being recognized that at very high energies, due to Quantum Gravity effects (non commutative geometry), the usual energy momentum dispersion relation

\[ E^2 = p^2 + m^2 \]  

gets modified (we use natural units, \( c = 1 = \hbar \)). This should result in the modification of the Klein-Gordon and Dirac equations (Cf.ref.[1, 2, 3] and several references therein). We would like to study this aspect in greater detail. Before we do so however, let us reexamine some well known aspects of the Klein-Gordon and Dirac equations.

Our starting point is the well known Klein-Gordon equation \[4, 5]\n
\[ \sum_\mu [D^2_\mu - k^2] \psi = 0, \]  

where

\[ D_\mu = \partial/\partial x_\mu - (ie/\hbar c) A_\mu \quad \text{and} \quad k = mc/\hbar. \]
Equations (2) and (3) are obtained from (1) with the usual substitutions and the introduction of the electromagnetic field.

The difficulties with the Klein-Gordon (K-G) equation are well known. They arise from the fact that equation (2) is second order in the time derivative, unlike the Schrödinger equation. These difficulties however, as was later shown could be circumvented if the K-G equation were to be interpreted in a field theory context.

However valuable insight about (2) can be obtained if it can be recast in the Hamiltonian form.

We follow Feshbach [4] for this purpose and look for an equation like,

$$ H\Psi = \frac{i\hbar}{\Delta t} \left( \partial \Psi / \partial t \right). $$

We then make the substitution

$$ \psi_4 = -k^{-1}D_4\psi $$

to get

$$ D_4\psi + k\psi_4 = 0, $$

$$ \sum_k D_k^2\psi - kD_4\psi_4 - k^2\psi = 0. $$

We further make the substitutions

$$ \psi = 1/\sqrt{2}(\psi + \chi), $$

$$ \psi_4 = 1/\sqrt{2}(\psi - \chi). $$

Thus in effect we have considered the K-G wave function $\Psi$ to be a two component object,

$$ \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, $$

We then get

$$ i\hbar(\partial\psi/\partial t) = (1/2m)(\hbar/\Delta \nabla - eA/c)^2(\psi + \chi) $$

$$ + (e\phi + mc^2\chi) $$

$$ i\hbar(\partial\chi/\partial t) = -(1/2m)(\hbar/\Delta \nabla - eA/c)^2(\psi + \chi) $$

$$ + (e\phi - mc^2\psi). $$

It is interesting to note that in (9) we have the following symmetry:

$$ t \to -t, \chi \to \phi, \phi \to \chi, A_4 \to -A_4 $$

Because of the description (8), we now have to introduce the Pauli matrices for defining suitable current and continuity equations.

Also the normalization of $\Psi$ is now given by

$$ \int \Psi^*\sigma_3\Psi d^3x = \pm 1, $$
As Feshbach and Villars note "... the increase of the 'degrees of freedom' connected with the appearance of a second-order time derivative in the Klein-Gordon equation corresponds to the simultaneous description of a particle of either positive or negative charge; i.e., the value of the charge becomes a degree of freedom of the system. The solution describing a particle of positive charge may be normalized to +1; the charge conjugate solution will automatically be normalized to (−1) and thus describe a negative charge."

However in the absence of an external electromagnetic field, (9) splits into two separate equations, one describing the positive (energy) solution and the other describing the negative solution.

To get better insight into this circumstance let us write

\[ \Psi = \begin{pmatrix} \psi_0(p) \\ \chi_0(p) \end{pmatrix} e^{i/h(p \cdot x - Et)} \]

\[ \Psi = \Psi_0(p) e^{i/h(p \cdot x - Et)} \quad (11) \]

We consider separately the positive and negative values of \( E \) (coming from (1), viz.,

\[ E = ±E_p; \quad E_p = [(cp)^2 + (mc^2)^2]^{1/2}. \quad (12) \]

The solutions associated with these two values of \( E \) are

\[ \phi_0^{(+)} = \frac{E_p + mc^2}{2(mc^2 E_p)^{1/2}} \]

\[ \chi_0^{(+)} = \frac{mc^2 - E_p}{2(mc^2 E_p)^{1/2}}, \]

for \( E = E_p \) and

\[ \phi_0^{(-)} = \frac{mc^2 - E_p}{2(mc^2 E_p)^{1/2}} \]

\[ \chi_0^{(-)} = \frac{E_p + mc^2}{2(mc^2 E_p)^{1/2}}, \]

for \( E = -E_p \).

As is well known the positive solution \( (E = E_p) \) and the negative solution \( (E = -E_p) \) represent solutions of opposite charge. It is also well known that in the non relativistic limit the \( \chi \) components are reduced with respect to the \( \phi \) components, by the factor \((p/mc)^2\). We also mention the well known fact that a meaningful subluminal velocity operator can be obtained only from the wave packets formed by positive energy solutions. However the positive energy solutions alone do not form a complete set, unlike in the non relativistic theory. This also means that a point description in terms of the positive energy solutions alone is not possible for the K-G equation, that is for the position operator,

\[ \delta \left( \hat{X} - \hat{X}_0 \right) \]
In fact the eigen states of this position operator include both positive and negative solutions. All this is well known (Cf.ref.[4, 5, 6]).

The point is that if we approach distances of the order of the Compton wavelength, the negative energy solutions begin to dominate, and we encounter the well known phenomenon of Zitterbewegung. This modifies the coupling of the positive solutions with an external field, particularly if the field varies rapidly over the Compton wavelength. In fact this is the origin of the well known Darwin term in the Dirac theory [6]. The Darwin term is a correction to the interaction of the order

$$\left(\frac{p}{mc}\right)^4 \text{ and } \left(\frac{p}{mc}\right)^2$$

for spin 0 and spin 1/2 particles respectively.

2. Modified Equations

We will now expand our outlook and consider the effects of a non communicative geometry or equivalently a fuzzy space time, all of which is symptomatic of the fact that we cannot go down to point space time. Then as has been argued in detail [3] the dispersion relation (1) gets modified and becomes the Snyder-Sidharth Hamiltonian

$$E^2 = p^2 + m^2 + \alpha l^2 p^4 \quad (c = 1 = \hbar; )$$

(14)

We can see that \( \alpha \sim 0(1) \). This follows by noting that (14) gives an extra term giving effects of Zitterbewegung at the Compton wavelength. Thus it is like the Darwin term \( \sim l^2 p^4 \) encountered in (13) [4]. Hence \( \alpha \sim 0(1) \).

We next observe that effectively

$$m^2 \rightarrow \tilde{m}^2 = m^2 + \alpha l^2 p^4$$

So we can get as in the usual theory of the Dirac equation

$$\left(\gamma^0 p^0 + \Gamma + \tilde{m}\right) \psi = 0$$

(15)

where

$$\tilde{m} \approx m + \frac{1}{2} \alpha l^2 p^4$$

We can demonstrate this as follows. We first rewrite (15) using (14), as

$$\left(\frac{\partial}{\partial t} - \sum_{k=1}^3 \alpha^k \frac{\partial}{\partial x^k} - \tilde{m}\beta - \lambda l^2 p^2 \right) \psi = 0$$

(16)

where \( \lambda \) turns out to be a suitable matrix and \( \alpha^k \) and \( \beta \) are matrices as in the usual theory. Let us now left multiply by

$$\left(\frac{\partial}{\partial t} + \sum \alpha^k \frac{\partial}{\partial x^k} + \tilde{m}\beta + \lambda l^2 p^2 \right)$$
to get
\[(D - \lambda l^2 p^4) \psi = 0\]
where \(D\) is the full \(K - G\) Hamiltonian and where we choose
\[\alpha^k \lambda + \lambda \alpha^k = 0, \quad \beta \lambda + \lambda \beta = 0\]

Going over to the usual \(\gamma\) matrices,
\[\gamma^0 = \beta, \quad \gamma^k = \beta \alpha^k = \gamma^0 \alpha^k,\]
we get, with
\[\Theta = \gamma^0 \lambda, \quad \Theta = \gamma^5\]
as can be easily verified.

Whence the modified Dirac equation (16).

There is a correction for the usual mass in (16), but which is a non-invariant under reflection due to the \(\gamma^5\) factor.

This indicates a decay of the Fermion at ultra high energy – it is somewhat like ”splitting” of mass a la the Zeeman effect, as will be discussed below.

If \(m = 0\) to start with, (16) shows the origin of the neutrino mass. Earlier Chados and others inserted this term ad hoc and interpreted it as a superluminal neutrino. These ideas were subsequently dropped [7].

Let us analyse the ultra high energy Dirac equation (16) [2] further. This can be done best, as for the Klein-Gordon case above or for the usual Dirac equation by considering a reduction to the Pauli equation as is usually done [6].

We get, with
\[\psi = e^{i mc^2 t} \left( \begin{array}{c} \phi \\ \chi \end{array} \right),\]
for the modified Dirac equation, this time
\[\phi = -\frac{\{c(\sigma \cdot \pi) + \omega\}}{2mc^2 + eV} \chi\]
with the Pauli-like equation for \(\chi\) (instead of \(\phi\) as in the usual theory), given by
\[i \hbar \dot{\chi} = [eV - \frac{\{c(\sigma \cdot \pi)\}^2 - \omega^2}{2mc^2 + eV}] \chi\]
where \(\omega = \Theta l p^2\). For a magnetic field \(\vec{B}\), this throws up the spin - magnetic field coupling, this time given by, remembering that we have
\[c(\sigma \cdot \pi)^2 = c^2 \pi^2 - \frac{e \hbar c^2}{c} \vec{\sigma} \cdot \vec{B}\]
\[e \hbar (\vec{\sigma} \cdot \vec{B} + \frac{\omega^2}{e \hbar})\]
instead of $e\hbar(\vec{\sigma} \cdot \vec{B})$ for the usual Dirac equation. We can see from (17) that we now have, additionally a spin half particle with charge $-e$, but the spin magnetic energy shifted due to the additional term involving $\omega$. The other, positive energy solution $\phi$ of the usual theory, represents a spin half particle of charge $e$, but a different spin magnetic coupling energy with the opposite sign of $\omega^2$. In any case this too differs from the usual Dirac theory due to the new effect of the $\omega$ term, that is due to the Hamiltonian (14). The ”splitting” can be seen because of the new $\omega$ term in (16). If $\omega$ were 0 we would be back to the usual theory.

We have here, as noted elsewhere a situation rather like that corresponding [2] to the (neutral) $k^0$ meson in the following sense. The positive and negative solutions are given as in the K-G case, as is well known by [6]

\[
\begin{align*}
\psi^{(+)}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{+\frac{i}{\hbar}(p \cdot x - Et)} \\
\psi^{(+)}_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{+\frac{i}{\hbar}(p \cdot x - Et)} \\
\psi^{(-)}_1 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-\frac{i}{\hbar}(p \cdot x - Et)} \\
\psi^{(-)}_2 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-\frac{i}{\hbar}(p \cdot x - Et)}
\end{align*}
\]

If we consider amplitudes like $\langle \psi^{(+)}_1 \psi^{(+)}_2 \rangle$, they are not real but rather contain the imaginary (Zitterbewegung) terms as can be easily verified, and these terms again vanish on taking averages over time intervals of the order of the Compton time [8, 9]. That is, for energies high enough to penetrate the Compton scale, as we are considering here, we have complex amplitudes, indicative of non conservation of probability and decay.

As can be seen, in this ultra relativistic case in which we are at the Compton wavelength and therefore encounter subsequently the negative energy solutions, there is a new negative energy particle, with a slight shift of energy. This is exactly as for the K-G equation, as noted in Section 1: there is an increase in the number of degrees of freedom with a description for positive charge and another for negative charge. The situation as noted is reminiscent of the spectral splitting in the Zeeman effect [10]. At the relativistic energies, we have the usual Dirac equation. This equation is meaningful only for minimum distances greater than the Compton wavelength. At ultra high energies there is a ”splitting” due to the term $lp^2\gamma^5$. In other words this new effect is characterized by the appearance of an ”anti” particle.
3. Discussion

1. We stress again that the description in terms of positive energy solutions alone, as in conventional theory is possible only outside the Compton wavelength of the particle. At higher energies, as seen above, we encounter negative energy solutions and a new effect namely an anti particle with a slightly different energy as compared to the original particle. To put it another way, the extra term in the SS Hamiltonian (14) is analogous to the magnetic field which leads to the Zeeman “splitting”. It is interesting that in this decay the $\gamma^5$ matrix of the weak interaction plays a role. In this context, we should note once again that the positive energy solutions alone do not form a complete set and the connection between completeness, unitarity and conservation of probability is all too well known [11].

2. If we had started with a mass zero particle, then the Hamiltonian (14) provides a mass. Indeed it has been explicitly shown [12] that the Hamiltonian can be approximated by

$$\frac{p^2}{2m} + m,$$

which shows the mass term explicitly.

In this connection it may be observed that as noted above, Chodos [13] had ad hoc proposed an equation similar to the massless modified Dirac equation. This started off much work on so called superluminal neutrinos, but finally the consensus seems to be that such neutrinos do not exist [14].

3. It may be argued that the mass generating term seen above endows Goldstein bosons with a mass. Indeed the author has argued for many years for a photon mass $\sim 10^{-65} gms$, well within the experimental upper bound [15, 16, 17]. This can be argued from the SS Hamiltonian itself. In any case Rao [18] had experimentally concluded that the photon has a particle, rather than a wave nature. It is proposed to repeat these experiments to verify all this.

4. It is interesting to note the following symmetry holds for the modified Dirac equation:

$$t \rightarrow -t \rightarrow \phi \rightarrow \chi$$

as indeed for the Klein-Gordon equation seen earlier.

5. It may be observed that the two two-component solutions of the Dirac equation seen above, viz., $\phi$ and $\chi$ could be thought of as in an isospin formalism. Thus the four component Dirac wave function consisting of $\phi$ and $\chi$ represents a single particle for the usual energies, much as the proton and neutron could be thought of as the same particle in the absence of the electromagnetic interactions. It is only when the extra term in the Hamiltonian (14) is introduced that the up and down states are distinguishable.

6. Finally it is worth noting that the above discussion would be relevant in the high energy proton-proton collisions which are already being observed in the Large Hadron Collier [19].
References


