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Spinless Relativistic Particle in the Presence of A Minimal Length

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Abstract: In this paper, we propose to study the (1+1)-dimensional Klein-Gordon equation in the presence of a minimal length by two approaches: a method direct in the position space representation and a path integral formalism in energy-momentum space, where a particle is subjected to a mixing of linear vector plus scalar potentials. For a first method, a suitable approximation technique of a non-relativistic quantum mechanics has been applied and the shifts of the relativistic energy levels is determined. For a second method, the Green function is obtained, the energy spectrum together with the normalized wave functions of the bound states are deduced and the limiting case is considered. The results of both methods are compared and we find the same dominant quantities to order 1 on parameter of deformation.

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1. Introduction

The study of the relativistic particle under the action of vector plus scalar potentials raised considerable interest in various domains of physics in order to guarantee the existence of bound states, such as the confining problem in the bag model[1] and hadrons models etc... In addition, it can be considered as an effective mass term in solid state physics for describe impurities in crystals[2], in quantum well and quantum dots. However, to its practical application, the quantum study of this problem requires some precautions relative to the order of operator [3][4].

For this purpose, within the framework, the relativistic Klein Gordon (K-G) equation has been developed and applied to several mixing of vector and scalar potentials,

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for example, the K-G equation is exactly solved, with vector and scalar Hulthen-type potential[5][6], with Coulomb like scalar-plus-vector potentials[7], with vector and scalar exponential type potentials[8], with scalar and vector Rosen-Morse-type potential[9], with the generalized Hulthen potential [10], with linear and exponential potentials [11], some classes of exactly-solvable KG equations are discussed by[12] and the non Hermitian scalar and vector potentials cases in[13].

However, we note that these problems are treated under the usual quantum mechanics, where the position and momentum operators acting on the Hilbert space of states verify the standard Heisenberg algebra. In contrast, if we take into account the effects of quantum fluctuations of the gravitational field in order to incorporate into quantum mechanics, a significant consequence deduced from this unification is the existence of a minimal observable distance on the order of the Planck length[14][15]. This minimal length uncertainty is related to a modification of the Heisenberg algebra by adding small corrections to the canonical commutation relations.

Motivations for the occurrence of a minimal length are multiple, in a string theory [16][17], in a quantum gravity[18], in a Non-commutative geometry[19] and in a black hole physics[20], which yielded correction terms to the uncertainty relations and imply a finite minimal uncertainty in position measurements, e.g. at the Planck scale. In consequence, a study in detail of this minimal length through several areas of modern physics has been exposed, it connects to large extra dimensions, to the running coupling constant, to the physics of black holes production and its acts as a natural regulator for infinities in quantum field theories[21]. In addition, it is remarkable that the presence of the minimal length causes the presence of the UV/IR mode mixing[17] phenomenon connecting the UV and IR scales, which is one of the new features of in non-commutative gauge theories[22] and implies the modification of the dispersion relation.

Recently, various topics were studied in connection with this formalism and a good number of articles were published. Among them let us quote: for example, the Schrodinger equation was exactly solved with the harmonic oscillator[23], the Coulomb potential[24], a one-dimensional box[15], the time-dependent linear potential[25] etc. The relativistic extension of this problem is also of interests within this framework, we found the case of the generalized Dirac equation which was recently considered in[26], the Dirac oscillator in[27], the Bosonic oscillator case of (spin 0 and 1) in one dimension[28] and the $(1+1)$-dimensional Dirac equation with linear potential[29] etc...

In this analysis, we are interested to study the $(1+1)$-dimensional K-G equation in the presence of a minimal length by two approaches, where a particle is subjected to a mixing of linear vector plus scalar potentials. In the first stage, we consider a modified K-G equation with the new form of momentum operator includes higher derivatives in the position space representation. We obtain a differential equation of fourth order, which its analytic solution is complicated in the presence of external fields .In a suitable approximation technique of a non-relativistic quantum mechanics, we find the shifts of the relativistic energy levels, the dominant quantities to order 1 on a parameter of deformation. In the second stage, we introduced Feynman's path-integral formalism in momentum space, which is an alternative formulation of quantum mechanics. The exact form of the Green function is determined, the energy spectrum together with the normalized wave functions of the bound states are deduced and coincide exactly with those of literature[30]. In addition, the limiting cases are also deduced for a small parameter of deformation and compared with the first method from perturbation theory. Although the two methods give the same results shift of the energy levels for a small parameter of deformation. The Feynman one remains more attractive because of its intuitive way of interpreting the basic equations of ordinary quantum mechanics using classical trajectories.

This paper is organized as follows: in section 2, we give a review of minimal length relations. In section 3, we treat the (1+1)-dimensional K-G equation relative to particles under the action of linear vector plus scalar potentials, in the position space representation and in the modified Heisenberg uncertainty relation. In section 4, by following the Feynman approach, we construct the path integral in energy-momentum space representation of the same problem in question. Some concluding remarks are given in the last section.

2. Brief Review of A Minimal Length Relation

The deformed Heisenberg algebra is defined by the following commutation relation with natural units in which $\hbar = 1$[23],

$$[X, P] = i\left(1 + \beta P^2\right), \tag{1}$$

where X, P are position and momentum in one dimension and $\beta \geq 0$ is a parameter of deformation.

This commutation relation (1) implements the minimal length,

$$\Delta X \Delta P \geq \frac{1}{2}\left[1 + \beta\left(\Delta P\right)^2\right], \tag{2}$$

which appears in perturbative string theory and in line with the proposed UV/IR mixing [17]. The relation (2) implies the appearance of a nonzero minimal length in positions $\Delta X_{\min} = \sqrt{\beta}$.

Since, from (1), the X and P are realized in momentum space by[23]

$$\begin{aligned} X &= i\left[\left(1 + \beta P^2\right)\partial_P + \gamma P\right] \\ P &= P, \end{aligned} \tag{3}$$

and in the position space acts as[15][31]

$$\begin{aligned} X &= X, \\ P &= -i\partial_x\left(1 - \frac{\beta}{3}\partial_x^2\right). \end{aligned} \tag{4}$$

The deformed completeness relation and the scalar product of P and P_0 in the relativistic case are given by,

$$\int \frac{dP}{(1+\beta P^2)^{1-\alpha}} \left|P\right\rangle \left\langle P\right| = 1$$
$$\int dP_0 \left|P_0\right\rangle \left\langle P_0\right| = 1, \tag{5}$$

$$\begin{aligned} \left\langle P_j\right| \left. P_{j-1}\right\rangle &= \left(1+\beta P_j^2\right)^{-\frac{\alpha}{2}} \left(1+\beta P_{j-1}^2\right)^{-\frac{\alpha}{2}} \delta\left(\frac{1}{\sqrt{\beta}} arctg\sqrt{\beta}P_j - \frac{1}{\sqrt{\beta}} arctg\sqrt{\beta}P_{j-1}\right) \\ &= \left(1+\beta P_j^2\right)^{\frac{1-\alpha}{2}} \left(1+\beta P_{j-1}^2\right)^{\frac{1-\alpha}{2}} \delta\left(P_j - P_{j-1}\right) \end{aligned} \tag{6}$$

$$\left\langle P_{0j}\right| \left. P_{0j-1}\right\rangle = \delta\left(P_{0j} - P_{0j-1}\right), \tag{7}$$

where it was assumed that the deformation does not affect on the time component P_0.

3. Resolution of the Klein-Gordon Equation in the Position Space Representation

The K-G equation under the action of vector plus scalar potentials in (1 + 1) dimension is given by

$$\left[(P_\mu - eA_\mu)(P^\mu - eA^\mu) - (S+M)^2\right] \Psi\left(x,t\right) = 0, \tag{8}$$

where $eA_\mu = (V(x), 0)$ is the Lorentz vector and $S(x)$ denotes the Lorentz scalar potential with the Minkowski tensor is $g_{\mu\nu} = diag(1, -1)$.

As the potential are independent from the time, we have then to find the stationary states of this equation. Accordingly, let us choose for $\Psi\left(x,t\right)$ the following form $\exp\left(-iEt\right)\Phi\left(x\right)$ and then get the following eigenvalue equation

$$\left[P^2 + \left(E - V\left(x\right)\right)^2 - \left(M + S\left(x\right)\right)^2\right] \Phi\left(x\right) = 0. \tag{9}$$

In order to treat the problem in question (1+1)-dimensional K-G equation in the context of minimal lengths in the position space representation, we replace P by (4) and we propose that the effects of quantum fluctuations of the gravitational field is at order 1 in β. Than, the following modified K-G equation can be written as,

$$\left[-\frac{2}{3}\beta\partial_x^4 + \partial_x^2 + \left(E - V\left(x\right)\right)^2 - \left(M + S\left(x\right)\right)^2\right] \Phi\left(x\right) = 0, \tag{10}$$

which is a differential equation of fourth order whose solution is very complicated in the presence of potentials.

Now, we are proposed that the system subjected to the action of linear vector plus scalar potentials,

$$V(x) = V_0 x \qquad (11)$$
$$S(x) = S_0 x,$$

with S_0 and V_0 being arbitrary constants.

We note that this linear potential is interesting, it envisaged as a quark-confining potential and it describes motion in an uniform gravitational or electrical field[32][33], in Matter-wave diffraction in time[34] and in a light cone QCD inspired model[35].

By replacing $V(x)$ and $S(x)$, the modified Klein-Gordon equation (10) becomes,

$$\left[-\frac{2}{3}\beta\partial_x^4 + \partial_x^2 - \left(S_0^2 - V_0^2\right)x^2 - 2\left(MS_0 + EV_0\right)x + \left(E^2 - M^2\right)\right]\Phi\left(x\right) = 0. \qquad (12)$$

Before applying the approximation method for this equation(12) to determine the energy correction, it is preferable to study the particular case $V\left(x\right) = 0$ and $S\left(x\right) = 0$, it's just for how the physical results are manifested in presence of minimal length, by a direct calculation, it is easy to obtain the following solution

$$\Phi\left(x\right)_{1,2} = C_{\pm}^{\pm}\exp\left(\pm k^{\pm}x\right), \qquad (13)$$

where $C_{\pm}^{\pm}$ are normalization constants, and

$$k^{\pm} = \frac{\sqrt{1 \pm \sqrt{1 + \frac{8}{3}\beta\left(E^2 - M^2\right)}}}{2\sqrt{\frac{\beta}{3}}}. \qquad (14)$$

We remark that the quantities k^{-} is a pure imaginary number which is equal $i\sqrt{(E^2 - M^2)}$ when $\beta \to 0$ and k^{+} is real diverges when $\beta \to 0$, and we can be grouped by this relation

$$\left(E^2 - M^2\right) = \pm\left(k^{\pm}\right)^2 + \frac{2}{3}\beta\left(k^{\pm}\right)^4, \qquad (15)$$

which is nothing but the modified dispersion relation in the presence of the minimal length[36].

Now, returning to the modified Klein-Gordon equation(12), as it is has been said previously that the solution is complicated, we try to find via the usual perturbation method of mechanic quantum the first energy correction at order 1 in β and out how the the introduction of the modified Heisenberg algebra affects the physical results. To do this, let us first suppose in this case that $(S_0^2 - V_0^2) > 0$ so as to avoid complex eigenvalues and we arrange the equation(12) to a sum of two terms, whose one is the perturbative as follows,

$$\left[H^0\left(z, \partial_z\right) + H^{pert}\left(\partial_z\right)\right]\Phi\left(z\right) = 0, \qquad (16)$$

where we have used this transformation

$$z = \left(S_0^2 - V_0^2\right)^{1/4}\left(x + \frac{(MS_0 + EV_0)}{(S_0^2 - V_0^2)}\right), \qquad (17)$$

with

$$H^0 = \partial_z^2 - z^2 + z_1 \tag{18}$$

$$H^{pert} = -\frac{2}{3}\beta\sqrt{(S_0^2 - V_0^2)}\partial_z^4, \tag{19}$$

and

$$z_1 = \frac{(MS_0 + EV_0)^2}{(S_0^2 - V_0^2)^{\frac{3}{2}}} + \frac{(E^2 - M^2)}{\sqrt{(S_0^2 - V_0^2)}}. \tag{20}$$

Now, in case where $H^{pert}(\partial_z)$ is vanish, (otherwise when $\beta \to 0$), the equation(16) becomes the problem of the harmonic oscillator whose solution is known

$$\Phi(z) = C\exp\left(-\frac{1}{2}z^2\right)H_n(z), \tag{21}$$

with z_1 verify,

$$z_1 = 2n + 1. \tag{22}$$

Applying Eqs.(22) into the relation of (20), it is straightforward to show that the energy spectrum $E_{n,\pm}^{\beta=0}$ is

$$E_{n,\pm}^{\beta=0} = -\frac{MV_0}{S_0} \pm \frac{\sqrt{(2n+1)}\,(S_0^2 - V_0^2)^{3/4}}{S_0}. \tag{23}$$

We note, the existence of the two signs in (23) is a characteristic property of energies in relativistic quantum mechanics.

Now, to find the first correction in the energy levels, we take the expectation value of the perturbation operator by using eigenfunctions(21),

$$\Delta z_{n1} = \frac{\langle\Phi(z)|H^{pert}|\Phi(z)\rangle}{\langle\Phi(z)|\Phi(z)\rangle}, \tag{24}$$

consequently, we obtain this result

$$\begin{aligned}\Delta z_{n1} &= \frac{-\frac{2}{3}\sqrt{(S_0^2 - V_0^2)}\int\exp\left(-\frac{1}{2}z^2\right)H_n(z)\,\partial_z^4\exp\left(-\frac{1}{2}z^2\right)H_n(z)\,dz}{\int\exp\left(-\frac{1}{2}z^2\right)H_n(z)\exp\left(-\frac{1}{2}z^2\right)H_n(z)\,dz} \\ &= -\frac{\beta\sqrt{(S_0^2 - V_0^2)}}{2}\left(2n^2 + 2n + 1\right),\end{aligned} \tag{25}$$

where we have used some properties of Hermite polynomials.

From the relation (20), we derive the expression of ΔE_{n1} in function of Δz_{n1}, we obtain

$$\Delta E_{n1} = \frac{(S_0^2 - V_0^2)^{3/2}}{2\left(MS_0V_0 + E_{n,\pm}^{\beta=0}S_0^2\right)}\Delta z_{n1}, \tag{26}$$

and by substituting (23) and (25) in (26), we find

$$\Delta E_{n1} = \pm \frac{\beta}{4S_0} \frac{\left(S_0^2 - V_0^2\right)^{5/4} \left(2n^2 + 2n + 1\right)}{\sqrt{(2n+1)}}, \tag{27}$$

than, the energy spectrum of the system at order 1 in β can be rewritten as

$$E_{n,\pm}^{\beta} = E_{n,\pm}^{\beta=0} + \Delta E_{n1} + O\left(\beta^2\right), \tag{28}$$

which is equal to

$$E_{n,\pm}^{\beta} = -\frac{MV_0}{S_0} \pm \frac{\sqrt{(2n+1)}\left(S_0^2 - V_0^2\right)^{3/4}}{S_0} \pm \frac{\beta}{4S_0} \frac{\left(S_0^2 - V_0^2\right)^{5/4} \left(2n^2 + 2n + 1\right)}{\sqrt{(2n+1)}} + O\left(\beta^2\right). \tag{29}$$

4. The Klein-Gordon Green Function in Momentum Space Representation

In this section, we are interested in solving exactly the same problem the (1+1)-dimensional K-G particle under the action of linear vector plus scalar potentials in energy-momentum space representation and in the presence of the minimal length by introducing the so called Green's function, which is the most important quantity of the path integral formulation.

Let us consider the formal Green function corresponding to the K-G equation for a spinless particle of rest mass M and charge e,

$$G = -\frac{I}{(P_\mu - eA_\mu)(P^\mu - eA^\mu) - (M+S)^2}, \tag{30}$$

Let us first note that, to solve this problem, we can use the relativistic spinless particle path integral formulation [37]. According to the Schwinger proper-time method, the Green's function in energy-momentum space (P_0, P) is expressed as,

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) = i \int_0^\infty d\tau \left\langle P_a, P_{0a}\right| \exp\left(-iH\left(\tau\right)\right) \left|P_b, P_{0b}\right\rangle, \tag{31}$$

where

$$H\left(\tau\right) = -\tau\left[P_0^2 - P^2 - M^2 + V^2\left(x\right) - S^2\left(x\right) - 2P_0 V\left(x\right) - 2MS\left(x\right)\right]. \tag{32}$$

In order to derive a path integral representation for $G\left(P_f, P_i, P_{0f}, P_{0i}\right)$, the standard method is adopted where we discretize time interval τ to $N+1$ infinitesimal equal parts $\varepsilon = \frac{\tau}{N+1}$, and applied the Trotter's formula,

$$\left\langle P_j, P_{0j}\right| \exp\left(-iH\left(\tau\right)\right) \left|P_{j-1}, P_{0j-1}\right\rangle = \lim_{N \longrightarrow \infty} \left\langle P_j, P_{0j}\right| \left[e^{-i\varepsilon H}\right]^{N+1} \left|P_{j-1}, P_{0j-1}\right\rangle. \tag{33}$$

The completeness relations (5) relative to P and P_0 are inserted between each pair of infinitesimal evolution operator and taking at the end the limit $N \to \infty$, we obtain the following expression,

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) = i \int_0^\infty d\tau \lim_{N\to\infty} \prod_{j=1}^{N} \int \frac{dP_j}{\left(1+\beta P_j^2\right)^{1-\alpha}} dP_{0j} \prod_{j=1}^{N+1}$$

$$\langle P_{j,} P_{0j}| \exp\left\{-i\varepsilon\left[P_{0j}^2 - P_j^2 - M^2 + V^2\left(x_j\right) - S^2\left(x_j\right) - 2P_{0j}V\left(x_j\right) - 2MS\left(x_j\right)\right]\right\} |P_{j-1,} P_{0j-1}\rangle, \tag{34}$$

or under this form,

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) = i \lim_{N\to\infty} \int_0^\infty d\tau \prod_{j=1}^{N} \int \frac{dP_j}{\left(1+\beta P_j^2\right)^{1-\alpha}} dP_{0j} \prod_{j=1}^{N+1} \exp\left(-i\varepsilon\left[P_{0j}^2 - P_j^2 - M^2\right]\right)$$

$$\langle P_{j,} P_{0j}| \exp\left\{-i\varepsilon\left[\left(-S^2\left(x_j\right) + V^2\left(x_j\right)\right) - 2\left(MS\left(x_j\right) + P_{0j}V\left(x_j\right)\right)\right]\right\} |P_{j-1,} P_{0j-1}\rangle. \tag{35}$$

Now, we use the form of the vector and scalar potentials(11), than, the expression of the Green function (35) becomes as follows,

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) = i \lim_{N\to\infty} \int_0^\infty d\tau \prod_{j=1}^{N} \int \frac{dP_j}{\left(1+\beta P_j^2\right)^{1-\alpha}} dP_{0j}$$

$$\prod_{j=1}^{N+1} \exp\left(-i\varepsilon\left[P_{0j}^2 - P_j^2 - M^2\right]\right) \langle P_{j,} P_{0j}| \exp\left\{-i\varepsilon\left[\left(V_0^2 - S_0^2\right) x_j^2 - 2\left(MS_0 + P_{0j}V_0\right) x_j\right]\right\} |P_{j-1,} P_{0j-1}\rangle, \tag{36}$$

or under this form,

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) = i \lim_{N\to\infty} \int_0^\infty d\tau \prod_{j=1}^{N} \int \frac{dP_j}{\left(1+\beta P_j^2\right)^{1-\alpha}} dP_{0j} \prod_{j=1}^{N+1} \exp\left(-i\varepsilon\left[P_{0j}^2 - P_j^2 - M^2\right]\right)$$

$$\langle P_{j,} P_{0j}| \exp\left\{-i\varepsilon\left(\left(V_0^2 - S_0^2\right)\left[i\left(\left(1+\beta P_j^2\right)\frac{\partial}{\partial P_j} + \gamma P\right)\right]^2 - \right.\right.$$

$$\left.\left. -2i\left(MS_0 + P_{0j}V_0\right)\left[\left(1+\beta P_j^2\right)\frac{\partial}{\partial P_j} + \gamma P\right]\right)\right\} |P_{j-1,} P_{0j-1}\rangle, \tag{37}$$

where it was used (3).

At this level, we should evaluate this expression (37), it's -easy to demonstrate that the evolution of last term at order 1 in ε is written,

$$\langle P_{j,} P_{0j}| \exp\left\{-i\varepsilon\left(\left(V_0^2 - S_0^2\right)\left[i\left(\left(1+\beta P_j^2\right)\frac{\partial}{\partial P_j} + \gamma P\right)\right]^2 - \right.\right.$$

$$\left.\left. -2i\left(MS_0 + P_{0j}V_0\right)\left[\left(1+\beta P_j^2\right)\frac{\partial}{\partial P_j} + \gamma P\right]\right)\right\} |P_{j-1,} P_{0j-1}\rangle \tag{38}$$

$$\simeq \langle P_j, P_{0j}| \left\{ 1 - i\varepsilon \left(\left(V_0^2 - S_0^2\right) \left[i \left(\left(1 + \beta P_j^2\right) \frac{\partial}{\partial P_j} + \gamma P \right) \right]^2 - \right.\right.$$

$$\left.\left. -2i \left(MS_0 + P_{0j}V_0\right) \left[\left(1 + \beta P_j^2\right) \frac{\partial}{\partial P_j} + \gamma P \right] \right) \right\} |P_{j-1}, P_{0j-1}\rangle, \tag{39}$$

and let's introduce the integral representation of $\langle P_j, P_{0j}|P_{j-1}, P_{0j-1}\rangle$ given by,

$$\langle P_j, P_{0j}|P_{j-1}, P_{0j-1}\rangle = \int\int \frac{dt_j}{2\pi}\frac{d\rho_j}{2\pi} \frac{\exp it_j \left(P_{0j} - P_{0j-1}\right)}{\left(1 + \beta P_j^2\right)^{\frac{\alpha}{2}} \left(1 + \beta P_{j-1}^2\right)^{\frac{\alpha}{2}}} \times$$

$$\exp i\rho_j \left(\frac{1}{\sqrt{\beta}} arctg \sqrt{\beta} P_j - \frac{1}{\sqrt{\beta}} arctg \sqrt{\beta} P_{j-1} \right), \tag{40}$$

where it was used (6), the expression of (37) becomes as follows

$$G\left(P_j, P_{j-1}, P_{0j}, P_{0j-1}\right) = i \int_0^\infty d\tau \lim_{N\to\infty} \prod_{n=1}^{N} \int \frac{dP_j dP_{0j}}{\left(1 + \beta P_j^2\right)^{1-\alpha}} \times$$

$$\prod_{n=1}^{N+1} \int\int \frac{dt_j}{2\pi}\frac{d\rho_j}{2\pi} \frac{\exp it_j \left(P_{0j} - P_{0j-1}\right)}{\left(1 + \beta P_j^2\right)^{\frac{\alpha}{2}} \left(1 + \beta P_{j-1}^2\right)^{\frac{\alpha}{2}}} \tag{41}$$

$$\exp \left\{ -i\varepsilon \left(P_{0j}^2 + \left[\left(S_0^2 - V_0^2\right) \left(\beta\gamma + \gamma^2\right) - 1 \right] P_j^2 - 2i\gamma \left(MS_0 + P_{0j}V_0\right) P_j - M^2 + \left(S_0^2 - V_0^2 \right) \gamma \right) \right\}$$

$$\exp i(\left[\frac{1}{\sqrt{\beta}} \left(arctg\sqrt{\beta}P_j - arctg\sqrt{\beta}P_{j-1} \right) + 2\varepsilon\gamma \left(S_0^2 - V_0^2\right) P_j - 2\varepsilon \left(MS_0 + P_{0j}V_0\right) \right] \rho_j$$
$$+\varepsilon \left(S_0^2 - V_0^2\right) \rho_j^2). \tag{42}$$

The Gaussian integrations over $\{\rho_j\}$ is immediate and the integration over $\{t_j\}$ gives a delta functional, than we obtain the new result

$$G\left(P_f, P_i, P_{0f}, P_{0i}\right) =$$

$$i \left(1 + \beta P_f^2\right)^{\frac{\alpha}{2}} \left(1 + \beta P_i^2\right)^{\frac{\alpha}{2}} \int_0^\infty d\tau \lim_{N\to\infty} \prod_{n=1}^{N} \int \frac{dP_j}{\left(1 + \beta P_j^2\right)} dP_{0j} \prod_{n=1}^{N+1} \frac{\delta\left(P_{0j} - P_{0j-1}\right)}{\sqrt{4\pi i\varepsilon \left(S_0^2 - V_0^2\right)}}$$

$$\exp -i\varepsilon \left\{ P_{0j}^2 + \left(S_0^2 - V_0^2\right)\gamma + \left[-1 + \left(S_0^2 - V_0^2\right)\beta\gamma\right] P_j^2 - M^2 + \frac{\left(mS_0 + P_{0j}V_0\right)^2}{\left(S_0^2 - V_0^2\right)} \right\}$$

$$\exp i \left[\frac{\left(arctg\sqrt{\beta}P_j - arctg\sqrt{\beta}P_{j-1}\right)^2}{4\varepsilon\beta\left(S_0^2 - V_0^2\right)} + \frac{\gamma}{\sqrt{\beta}} \left(arctg\sqrt{\beta}P_j - arctg\sqrt{\beta}P_{j-1} \right) P_j \right], \tag{43}$$

and by the following transformation,

$$k_j = \frac{1}{\sqrt{\beta}} arctg\sqrt{\beta}P_j$$
$$k_{0j} = P_{0j}, \tag{44}$$

where the value of k change in the interval $\left]-\frac{\pi}{2\sqrt{\beta}},+\frac{\pi}{2\sqrt{\beta}}\right[$ according to values of p in the interval $]-\infty,+\infty[$, than, it is easy to obtain that,

$$G\left(k_j,k_{j-1},k_{0j},k_{0j-1}\right)=i\left(1+\tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1+\tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}\int_0^{\infty}d\tau$$

$$\lim_{N\to\infty}\prod_{n=1}^{N}\int dk_{0j}\prod_{n=1}^{N+1}\delta\left(k_{0j}-k_{0j-1}\right)\times$$
$$\exp-i\varepsilon\left\{k_{0j}^2-M^2+\left(S_0^2-V_0^2\right)\gamma+\frac{\left(mS_0+k_{0j}V_0\right)^2}{\left(S_0^2-V_0^2\right)}\right\}\widetilde{G}\left(k_j,k_{j-1},\tau\right), \tag{45}$$

where

$$\widetilde{G}\left(k_j,k_{j-1},\tau\right)=\prod_{n=1}^{N}\int dk_j\prod_{n=1}^{N+1}\frac{1}{\sqrt{4\pi i\varepsilon\left(S_0^2-V_0^2\right)}}$$

$$\exp i\left[\frac{\left(\Delta k_j\right)^2}{4\varepsilon\left(S_0^2-V_0^2\right)}-\frac{\left(MS_0+k_{0j}V_0\right)}{\left(S_0^2-V_0^2\right)}\Delta k_j+\frac{\gamma\Delta k_j}{\sqrt{\beta}}\tan\sqrt{\beta}k_j+\varepsilon\left[-1+\beta\gamma\left(S_0^2-V_0^2\right)\right]\frac{\tan^2\sqrt{\beta}k_j}{\beta}\right], \tag{46}$$

and $\Delta k_j=k_j-k_{j-1}$.

Now, we develop the terms in the infinitesimal action of $\widetilde{G}\left(k_f,k_i,\tau\right)$ (**??**) to the order ε in the vicinity of the mid-point

$$\Delta k_j\tan\sqrt{\beta}k_j\simeq\Delta k_j\tan\sqrt{\beta}\left(\overline{k}_j\right)+\frac{\left(\Delta k_j\right)^2\sqrt{\beta}}{2}\left(1+\tan^2\sqrt{\beta}\overline{k}_j\right), \tag{47}$$

where $\overline{k}_j=\frac{k_j+k_{j+1}}{2}$and we retain only the terms contributing [38]

$$\left\langle\left(\Delta k_j\right)^2\right\rangle=2i\varepsilon\left(S_0^2-V_0^2\right), \tag{48}$$

than the Green function (**??**) takes the following form:

$$G\left(k_f,k_i,k_{0f},k_{0i}\right)=i\left(1+\tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1+\tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}$$

$$\int_0^{\infty}d\tau\lim_{N\to\infty}\prod_{n=1}^{N}\int dk_{0j}\prod_{n=1}^{N+1}\delta\left(k_{0j}-k_{0j-1}\right)\exp-i\varepsilon\left\{k_{0j}^2-M^2+\frac{\left(MS_0+k_{0j}V_0\right)^2}{\left(S_0^2-V_0^2\right)}\right\}\widetilde{G}\left(k_j,k_{j-1},\tau\right), \tag{49}$$

where

$$\widetilde{G}\left(k_f,k_i,\tau\right)=\prod_{n=1}^{N}\int dk_j\prod_{n=1}^{N+1}\frac{1}{\sqrt{4\pi i\varepsilon\left(S_0^2-V_0^2\right)}}\exp i\left\{\frac{\left(\Delta k_j\right)^2}{4\varepsilon\left(S_0^2-V_0^2\right)}-\varepsilon\frac{\tan^2\sqrt{\beta}k_j}{\beta}\right\}. \tag{50}$$

This expression(50) is formally identical with that of the Poschl–Teller potential studied [39], which also written in this form

$$\widetilde{G}\left(k_j, k_{j-1}, \tau\right) = \prod_{n=1}^{N} \int dk_j \prod_{n=1}^{N+1} \frac{1}{\sqrt{4\pi i\varepsilon\left(S_0^2 - V_0^2\right)}} \times$$
$$\exp i \left\{ \frac{\left(\Delta k_j\right)^2}{4\varepsilon\left(S_0^2 - V_0^2\right)} - \varepsilon\beta\left(S_0^2 - V_0^2\right)\lambda\left(\lambda - 1\right)\tan^2\sqrt{\beta}k_j \right\}, \tag{51}$$

with

$$\lambda = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\beta^2\left(S_0^2 - V_0^2\right)}}\right). \tag{52}$$

The solution of this path integral (**??**) can be written as,

$$\widetilde{G}\left(k_f, k_i, \tau\right) =$$
$$\sum_{n=0}^{\infty} N_n \exp\left[i\tau\beta\left(S_0^2 - V_0^2\right)\left(n^2 + (2n+1)\lambda\right)\right]\left(\cos\sqrt{\beta}k_f \cos\sqrt{\beta}k_i\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}k_f\right) C_n^{\lambda}\left(\sin\sqrt{\beta}k_i\right), \tag{53}$$

where C_n^{λ} are Gegenbauer polynomials and the normalization constant N_n is given by

$$N_n = \Gamma\left(\lambda\right)^2 \left[\frac{2^{2\lambda-1} n!\left(n+\lambda\right)\sqrt{\beta}}{\pi\Gamma\left(n+2\lambda\right)}\right]. \tag{54}$$

Performing the integrations over k_{0j},in this case, the Green function (49) reduces to

$$G\left(k_f, k_i, k_{0f}, k_{0i}\right) = i\left(1 + \tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1 + \tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}$$
$$\sum_{n=0}^{\infty} N_n \int_0^{\infty} d\tau\delta\left(k_{0f} - k_{0i}\right) \exp -i\tau\left\{k_{0f}^2 - M^2 + \frac{\left(MS_0 + k_{0f}V_0\right)^2}{\left(S_0^2 - V_0^2\right)} - \beta\left(S_0^2 - V_0^2\right)\left(n^2 + (2n+1)\lambda\right)\right\}$$
$$\left(\cos\sqrt{\beta}k_f \cos\sqrt{\beta}k_i\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}k_f\right) C_n^{\lambda}\left(\sin\sqrt{\beta}k_i\right), \tag{55}$$

or under this form after integration on τ

$$G\left(k_f, k_i, k_{0f}, k_{0i}\right) = i\left(1 + \tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1 + \tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}$$
$$\sum_{n=0}^{\infty} N_n \frac{\delta\left(k_{0f} - k_{0i}\right)\left(\cos\sqrt{\beta}k_f \cos\sqrt{\beta}k_i\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}k_f\right) C_n^{\lambda}\left(\sin\sqrt{\beta}k_i\right)}{k_{0f}^2 - M^2 + \frac{\left(MS_0 + k_{0f}V_0\right)^2}{\left(S_0^2 - V_0^2\right)} - \beta\left(S_0^2 - V_0^2\right)\left(n^2 + (2n+1)\lambda\right)}, \tag{56}$$

we note that the presence of δ reflects the conservation of energy.

In order to evaluate exactly the propagator expression, it is convenient to write the Fourier transformation of (55) for k_{0f} and k_{0i} variables. The first integral on delta is immediate, we obtain

$$G\left(k_f, k_i, t_f, t_i\right) = i\left(1 + \tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1 + \tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}$$

$$\sum_{n=0}^{\infty} N_n \int \frac{dE}{2\pi} e^{-iE(t_f-t_i)} \frac{\left(\cos\sqrt{\beta}k_f \cos\sqrt{\beta}k_i\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}k_f\right) C_n^{\lambda}\left(\sin\sqrt{\beta}k_i\right)}{E^2 - M^2 + \frac{\left(MS_0+EV_0\right)^2}{\left(S_0^2-V_0^2\right)} - \beta\left(S_0^2-V_0^2\right)\left(n^2+\left(2n+1\right)\lambda\right)}. \tag{57}$$

The poles of the $G\left(k_f, k_i, t_f, t_i\right)$ yield the discrete energy spectrum

$$E_{n,\pm}^{\beta} = -\frac{MV_0}{S_0} \pm \omega_n^{\beta}, \tag{58}$$

with ω_n^{β} defined,

$$\omega_n^{\beta} = \frac{\left(S_0^2-V_0^2\right)}{S_0} \sqrt{\beta\left(n^2+n+\frac{1}{2}\right) + \beta\left(n+\frac{1}{2}\right)\sqrt{1+\frac{4}{\beta^2\left(S_0^2-V_0^2\right)}}}. \tag{59}$$

We note that this result coincides exactly with those obtained by[30].

Now, to evaluate the wave functions and energy spectrum, the expression(56) is written in this form,

$$G\left(k_f, k_i, t_f, t_i\right) = i\left(1+\tan^2\sqrt{\beta}k_f\right)^{\frac{\alpha}{2}}\left(1+\tan^2\sqrt{\beta}k_i\right)^{\frac{\alpha}{2}}$$

$$\sum_{n=0}^{\infty} N_n \int \frac{dE}{2\pi} e^{-iE(t_f-t_i)} \frac{\left(\cos\sqrt{\beta}k_f \cos\sqrt{\beta}k_i\right)^{\lambda} C_n^{\lambda}\left(\sin\sqrt{\beta}k_f\right) C_n^{\lambda}\left(\sin\sqrt{\beta}k_i\right)}{\left(E+\frac{MV_0}{S_0}-\omega_n^{\beta}\right)\left(E+\frac{MV_0}{S_0}+\omega_n^{\beta}\right)}, \tag{60}$$

and we use a complex integral along the special contour C, and then using the residue theorem,

$$\oint \frac{dE}{2\pi} \frac{e^{-iE(t_f-t_i)}}{\left(E+\frac{MV_0}{S_0}-\omega_n^{\beta}\right)\left(E+\frac{MV_0}{S_0}+\omega_n^{\beta}\right)} =$$

$$\frac{-i}{2\omega_n^{\beta}}\left[\theta(t_f-t_i)e^{-i\left(\omega_n^{\beta}-\frac{MV_0}{S_0}\right)(t_f-t_i)} + \theta(t_i-t_f)e^{i\left(\omega_n^{\beta}+\frac{MV_0}{S_0}\right)(t_f-t_i)}\right]. \tag{61}$$

Consequently, we obtain this result in the former variable,

$$G\left(P_f, P_i, t_f, t_i\right) =$$

$$\sum_{n=0}^{\infty}\left[\theta(t_f-t_i)\frac{N_n}{2\omega_n^{\beta}}\left(1+\beta P_f^2\right)^{\frac{\alpha-\lambda}{2}}\left(1+\beta P_i^2\right)^{\frac{\alpha-\lambda}{2}} C_n^{\lambda}\left(\frac{\sqrt{\beta}P_f}{\sqrt{1+\beta P_f^2}}\right) C_n^{\lambda}\left(\frac{\sqrt{\beta}P_i}{\sqrt{1+\beta P_i^2}}\right) e^{-i\left(\omega_n^{\beta}-\frac{MV_0}{S_0}\right)(t_f-t_i)}\right.$$

$$\left.+\theta(t_i-t_f)\frac{N_n}{2\omega_n^{\beta}}\left(1+\beta P_f^2\right)^{\frac{\alpha-\lambda}{2}}\left(1+\beta P_i^2\right)^{\frac{\alpha-\lambda}{2}} C_n^{\lambda}\left(\frac{\sqrt{\beta}P_f}{\sqrt{1+\beta P_f^2}}\right) C_n^{\lambda}\left(\frac{\sqrt{\beta}P_i}{\sqrt{1+\beta P_i^2}}\right) e^{i\left(\omega_n^{\beta}+\frac{MV_0}{S_0}\right)(t_f-t_i)}\right], \tag{62}$$

where we have used the following relation,

$$\cos\sqrt{\beta}k = \frac{1}{\sqrt{1+\beta P^2}}, \qquad (63)$$
$$\sin\sqrt{\beta}k = \frac{\sqrt{\beta}P}{\sqrt{1+\beta P^2}}.$$

Finally, we obtain the spectral decomposition of propagator for the (1+1)-dimensional K-G particle under the action of linear vector plus scalar potentials in the presence of minimal length uncertainty,

$$G\left(P_f, P_i, t_f, t_i\right) = \sum_{n=0}^{\infty}\left[\theta(t_f-t_i)\xi\left(P_f\right)_n \xi\left(P_i\right)_n^* e^{-i\left(\omega_n^{\beta}-\frac{MV_0}{S_0}\right)(t_f-t_i)} + \theta(t_i-t_f)\xi\left(P_f\right)_n^* \xi\left(P_i\right)_n e^{i\left(\omega_n^{\beta}+\frac{MV_0}{S_0}\right)(t_f-t_i)}\right], \qquad (64)$$

where $\xi\left(P\right)_n$ are given by,

$$\xi\left(P\right)_n = \sqrt{\frac{2^{2\lambda-1}n!\left(n+\lambda\right)\Gamma\left(\lambda\right)^2\sqrt{\beta}}{2\pi\Gamma\left(n+2\lambda\right)\left(1+\beta P^2\right)^{\lambda-\alpha}\omega_n^{\beta}}}C_n^{\lambda}\left(\frac{\sqrt{\beta}P}{\sqrt{1+\beta P^2}}\right). \qquad (65)$$

In (64), we have two types of propagation, one with positive energy $+\left(\omega_n^{\beta}-\frac{MV_0}{S_0}\right)$ propagating to the future and the other with negative energy $-\left(\omega_n^{\beta}+\frac{MV_0}{S_0}\right)$ propagating to the past.

In the end, it is remarkable if we consider a very small β, the form of (59) can easily be written as

$$\omega_n^{\beta\ll} = \frac{\sqrt{(2n+1)}\left(S_0^2-V_0^2\right)^{3/4}}{S_0} + \frac{\beta}{4S_0}\frac{\left(S_0^2-V_0^2\right)^{\frac{5}{4}}\left(2n^2+2n+1\right)}{\sqrt{(2n+1)}} + O\left(\beta^2\right), \qquad (66)$$

than we get

$$E_{n,\pm}^{\beta} = -\frac{MV_0}{S_0} \pm \frac{\sqrt{(2n+1)}\left(S_0^2-V_0^2\right)^{3/4}}{S_0} \pm \frac{\beta}{4S_0}\frac{\left(S_0^2-V_0^2\right)^{5/4}\left(2n^2+2n+1\right)}{\sqrt{(2n+1)}} + O\left(\beta^2\right). \qquad (67)$$

We can now compare this result (67)with that obtained in the previous section (29) from perturbation theory, though both methods give the same shift of the energy levels.

Conclusion

We have evaluated a (1+1)-dimensional Klein-Gordon particle subjected to the action of linear vector plus scalar potentials in modified Heisenberg uncertainty relation via two methods: by a method direct in the position space representation and by a Feynman formalism in energy-momentum space. For a first method, via a new form of momentum operator, our system converted to a modified K-G equation, a differential equation of fourth order, which its analytic solution is complicated. by using a usual approximation technique of a quantum mechanics, we find the shifts of the relativistic energy levels to order 1 on a parameter of deformation. In contrast, the second method, the Green function is obtained, the energy spectrum together with the normalized wave functions of the bound states are deduced and depend an the deformation parameter β as in noncommutative theory case. The limiting case is then determined and is compared with that obtained in the from perturbation theory, though both methods give the same shift of the energy levels. The merit of Feynman formulation is its intuitive way of interpreting the basic equations of ordinary quantum mechanics using classical trajectories. Furthermore, for pedagogical reasons, it is suitable to treat this problem via these equivalent methods. Finally, let us remark that the same problems in the case of the Dirac equation for spin $\frac{1}{2}$ are under consideration.

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