

# Theory of Dirac Equation without Negative Energies

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**Abstract:** It is shown that the well-known Hermitean operator 'sign of frequency' for the free Dirac equation has the physical meaning of 'sign of charge'. Since the kinetic energy of a free particle should not depend on its charge state, this identification requires a modification of the traditional quantum mechanical 4-momentum operators when used with Dirac spinors. Due to the new 4-momentum operators the Dirac equation has no negative energy solutions and the complex of problems associated with the latter disappears from the theory. The quantum number 'sign of charge' rigorously defines electronic and positronic plane waves. Second quantization of the free Dirac equation does not need the traditional amendments required by the negative energy values. As an example for the application of the theory the relativistic hydrogen ground state wave function is analyzed with respect to the quantum number 'sign of charge'. Since the operator 'sign of charge' does not commute with the Coulomb potential the wave function is only an approximate eigenfunction of the operator 'sign of charge'. It is shown how one can construct 'effective potentials' that commute with the operator 'sign of charge' and thus are able to produce eigenfunctions of charge when used in the Dirac equation.

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## 1. Introductory Remark

This work was initiated by the search for a possibility to better understand the physical nature of the so-called negative energy solutions of the free Dirac equation, and for a possibility to find a theory free from the resulting troubles.

We shall show:

There exists a Hermitean operator  $sgn$ , whose eigenvalues  $\pm 1$  have the physical meaning

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of a quantum number 'sign of particle charge' for Dirac wave functions. Due to this operator, the structure of the Dirac equation, plus the "axiom" that *the kinetic energy of a free particle should not depend on its charge state*, will imply a special theory of this equation which differs from the traditional approach, but is free from negative energies and the associated complex of problems, such as the necessity to introduce the "Dirac sea", defined by the state of completely occupied negative energy solutions, and the hole hypothesis, further the problem with the consistent definition of the positron wave function, and the indefiniteness of the field Hamiltonian in second quantization.

## 2. The Operator *sgn* Sign of Charge

We start with the Dirac equation for a wave function  $\Psi$ ,

$$i\partial_t\Psi = D\Psi \quad (1)$$

$D$  is the Dirac Hamiltonian

$$D = -i\vec{\alpha} \cdot \vec{\partial} + \beta m_0$$

$$\vec{\alpha} = \begin{pmatrix} | & \vec{\sigma} \\ \hline \vec{\sigma} & | \end{pmatrix}, \beta = \begin{pmatrix} e_2 & | \\ \hline | & -e_2 \end{pmatrix} \quad (2)$$

and  $e_2$  is the  $2 \times 2$ -unit matrix  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

With these notions we define the Hermitean operator *sgn* by

$$\boxed{sgn = \frac{D}{\Omega}} \quad (3)$$

The one-component non-local operator  $\Omega$  usually is formally written as

$$\Omega = \sqrt{-\Delta + m_0^2}$$

$\Omega$  may be mathematically rigorously defined as a distribution, sometimes also called pseudodifferential operator [1], [2]. It is most conveniently written in Fourier representation as

$$\Omega(\vec{x} - \vec{x}') = \frac{1}{(2\pi)^3} \int d^3p \omega(\vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \quad (4)$$

$\Omega(\vec{x} - \vec{x}')$  is a non-local operator with a mean range in the order of the Compton wave length  $m_0^{-1}$ :

$$\sqrt{\langle r^2 \rangle} = \sqrt{\frac{\int dx^3 r^2 \Omega(\vec{x})}{\int dx^3 \Omega(\vec{x})}} = \frac{\sqrt{3}}{m_0} \quad (5)$$

A function  $f(\Omega)$  of  $\Omega$  is again a distribution, defined by

$$(f(\Omega))(\vec{x} - \vec{x}') = \frac{1}{(2\pi)^3} \int dk^3 f(\omega(\vec{k})) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}$$

in particular

$$\Omega^{-1}(\vec{x} - \vec{x}') = \frac{1}{(2\pi)^3} \int dk^3 \frac{1}{\omega(\vec{k})} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}$$

We have

$$sgn^2 = \frac{D^2}{\Omega^2} = 1 \quad (6)$$

$\Rightarrow$  the eigenvalues of  $sgn$  are

$$\sigma = \pm 1 \quad (7)$$

$\Omega$  commutes with  $D$ ,

$$[\Omega, D] = 0 \quad (8)$$

and therefore  $sgn$  commutes with  $D$ ,

$$[sgn, D] = 0 \quad (9)$$

Consequently, there exist simultaneous eigenfunctions of  $D$  and  $sgn$ .

We shall show: The operator  $sgn$  has the physical meaning of 'sign of particle charge' for Dirac wave functions.

To this end we make use of the well-known transformation  $C$  called *charge conjugation*.

Let us recapitulate:  $C$  reverses the sign of the charge. This means: If  $\Psi$  is the wave function of a particle carrying charge  $\sigma e$ ,  $\sigma = \pm 1$ , then  $C\Psi$  is a wave function of a particle carrying charge  $-\sigma e$ .

$C\Psi$  is defined by

$$C\Psi = \Psi' = C\Psi^* \quad (10)$$

where the matrix

$$C = \begin{pmatrix} & -i\sigma_y \\ i\sigma_y & \end{pmatrix} = C^* = C^{-1} = \sigma_y \otimes \sigma_y \quad (11)$$

[We shall extensively make use of the so-called direct product  $\otimes$ , defined for two matrices

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

by

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q \\ p_{21}Q & p_{22}Q \end{pmatrix} = \begin{pmatrix} p_{11}q_{11} & p_{11}q_{12} & p_{12}q_{11} & \dots \\ p_{11}q_{21} & p_{11}q_{22} & \dots & p_{12}q_{22} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & p_{22}q_{22} \end{pmatrix}$$

The direct product satisfies the multiplication rule

$$(P \otimes Q)(R \otimes S) = (PR) \otimes (QS) \quad (12)$$

where the product without multiplication symbol means ordinary matrix multiplication. We have

$$\vec{\alpha} = \sigma_x \otimes \vec{\sigma}, \quad \beta = \sigma_z \otimes e_2$$

C satisfies

$$C^2 = 1, \quad C^{-1} = C \quad (13)$$

We are interested in the behaviour of  $sgn$  under the operator C. We have

$$C \, sgn \, C = (CDC)(C\Omega^{-1}C)$$

Using (12) and the identity

$$\sigma_y \vec{\sigma}^* \sigma_y = -\vec{\sigma}, \quad (14)$$

we obtain

$$C \, D \, C = (\sigma_y \otimes \sigma_y) D^* (\sigma_y \otimes \sigma_y) = (\sigma_y \otimes \sigma_y) \left\{ \sigma_x \otimes \vec{\sigma}^* \cdot i\vec{\partial} + \sigma_y \otimes e_2 m_0 \right\} (\sigma_y \otimes \sigma_y) = -D \quad (15)$$

With regard to the second factor  $\frac{1}{\Omega}$  we find for any real function  $f(\Omega)$

$$\begin{aligned} C f(\Omega)(\vec{x} - \vec{x}') C &= C \left( \frac{1}{(2\pi)^3} \int dk^3 f(\omega(\vec{k})) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \right)^* C = \\ &= \left( \frac{1}{(2\pi)^3} \int dk^3 f(\omega(\vec{k})) e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \right)^* = \frac{1}{(2\pi)^3} \int dk^3 f(\omega(\vec{k})) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \end{aligned}$$

Substituting  $\vec{k}' = -\vec{k}$ , we obtain

$$= \frac{1}{(2\pi)^3} \int dk'^3 f(\omega(-\vec{k}')) e^{i\vec{k}' \cdot (\vec{x} - \vec{x}')} \frac{1}{(2\pi)^3} \int dk'^3 f(\omega(\vec{k}')) e^{i\vec{k}' \cdot (\vec{x} - \vec{x}')} = f(\Omega)$$

Therefore finally

$$\boxed{C \operatorname{sgn} C = -D\Omega^{-1} = -\operatorname{sgn}} \quad (16)$$

equivalently

$$\boxed{C \operatorname{sgn} = -\operatorname{sgn} C} \quad (17)$$

Eq.(17) is equivalent to the statement: *C reverses the eigenvalue of an eigenfunction of sgn*, for: Let  $\Phi$  be an eigenfunction of *sgn*, with eigenvalue  $\sigma$ ,

$$\operatorname{sgn}\Phi = \sigma\Phi$$

then from (17) follows

$$\operatorname{sgn}(C\Phi) = -(C \operatorname{sgn})\Phi = -\sigma\Phi$$

*i.e.*  $C\Phi$  is an eigenfunction of *sgn* with eigenvalue  $-\sigma$ .

Since *C reverses the sign of charge* of a wave function - and nothing else - and *C does reverse  $\sigma$  we must interpret the eigenvalues of sgn as the sign of charge.*

Hence:

The operator *sgn* corresponds to the observable *sign of charge*.  
Its eigenvalues  $\pm 1$  are identical with the *sign of the charge*.

Therefore we shall call the operator *sgn* the *sign of charge operator* or sometimes simply the *charge operator*.

### 3. The 4-momentum Operator for Dirac Wave Functions

Solving (3) for  $D$

$$D = \operatorname{sgn} \cdot \Omega = \Omega \cdot \operatorname{sgn} \quad (18)$$

we have

$$i\partial_t = D = \operatorname{sgn} \cdot \Omega = \Omega \cdot \operatorname{sgn} \quad (19)$$

Since the operator *sgn* has the meaning of *sign of charge* our axiom that *the kinetic energy of a free particle should not depend on its charge state*, forbids to interpret  $i\partial_t$  and therefore  $D$  as the kinetic energy of the particle. For the kinetic energy there remains *only the factor  $\Omega$* . Thus *it is the time displacement operator  $i\partial_t$  that depends on the sign of charge, not the energy.*

Actually this is nothing new: eq.(19) confirms the fact that in Feynman diagrams antiparticles have to be represented by arrows running backward in time, because for a

wave function of an antiparticle, *i.e.* an eigenfunction of  $sgn$  with  $\sigma = -1$ , eq.(19) yields  $\Omega = -i\partial_t = i\frac{\partial}{\partial(-t)}$ .

Eq.(19) means for the energy

$$\Omega = sgn \cdot i\partial_t \quad (20)$$

Since according to special relativity the energy and the 3-momentum of a free particle form a 4-vector, and *sign of charge* is a Lorentz invariant, identity (20) must be understood as the zero-component of the 4-vector identity

$$P^\mu = sgn \cdot i\partial^\mu \quad (21)$$

with  $P^0 = \Omega$ .

From definition (21) follows as before

$$P_\mu P^\mu = -(sgn)^2 \partial_\mu \partial^\mu = -\partial_t^2 + \Delta \quad (22)$$

Solving for  $i\partial^\mu$ , (21) reads

$$i\partial^\mu = sgn \cdot P^\mu \quad (23)$$

The definitions (21), (23) *have to be used with Dirac wave functions*, in place of the definition  $P^\mu = i\partial^\mu$  to be used with one-component wave functions.

*E.g.*, for a product  $\varphi(\vec{x}_1)\psi(\vec{x}_2)$  of a one-component Schroedinger wave function  $\varphi(\vec{x}_1)$  and a Dirac wave function  $\psi(\vec{x}_2)$  the total 3-momentum operator is given by

$$\vec{P}_{tot} = -i\frac{\vec{\partial}}{\partial\vec{x}_1} + sgn(\vec{x}_2) \cdot \left(-i\frac{\vec{\partial}}{\partial\vec{x}_2}\right)$$

This situation is completely analogous, for instance, to the situation of a wave function  $Y_\ell^m \chi_{1/2}^\mu$  and the  $z$ -component of the total angular momentum

$$J_z = L_z + S_z$$

(with the familiar meanings of the symbols.)

Definition (21) implies for the 3-momentum operator  $\vec{P}$

$$\vec{P} = sgn \cdot \left(-i\vec{\partial}\right) \quad (24)$$

This raises the question what the commutator of position  $\vec{x}$  with the 3-momentum (24) looks like.

We assume also in the present theory that we can define a basis in  $\vec{x}$ -space representation by the eigenstates  $|\vec{x}\rangle$  of a position operator  $\vec{X}$ ,

$$\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

There is no a priori reason to drop this axiom of standard quantum mechanics.

In this basis the position operator  $\vec{X}$  is diagonal,

$$\langle \vec{x} | \vec{X} | \vec{x}' \rangle = \vec{x} \delta(\vec{x} - \vec{x}') \tag{25}$$

In nonrelativistic quantum mechanics the momentum operator  $\vec{P}$  is diagonal, too,

$$\langle \vec{x} | \vec{P} | \vec{x}' \rangle = -i\vec{\partial} \delta(\vec{x} - \vec{x}')$$

In other cases, however, operators in general need not be diagonal in  $\vec{x}, \vec{x}'$ . In case of definition (24) the momentum  $\vec{P}$  is no longer diagonal in the Dirac picture, due to  $\Omega^{-1}$ :  $\langle \vec{x} | \vec{P} | \vec{x}' \rangle$  is a nonlocal operator, (a distribution), in detail,

$$\langle \vec{x} | \vec{P} | \vec{x}' \rangle = \vec{P}(\vec{x} - \vec{x}') = D \cdot (-i\vec{\partial}) \cdot \Omega^{-1}(\vec{x} - \vec{x}') = \Omega^{-1}(\vec{x} - \vec{x}') \cdot D' \cdot (-i\vec{\partial}')$$

where  $D = \vec{\alpha} \cdot \left(-i\frac{\vec{\partial}}{\partial \vec{x}}\right) + \beta m$ , and  $D' = \vec{\alpha} \cdot \left(-i\frac{\vec{\partial}}{\partial \vec{x}'}\right) + \beta m$ .

It acts on a wave function  $\Psi(\vec{x}) = \langle \vec{x} | \Psi \rangle$  according to

$$\begin{aligned} \vec{P}\Psi(\vec{x}) &= \langle \vec{x} | \vec{P} | \vec{x}' \rangle \langle \vec{x}' | \Psi \rangle = \vec{P}(\vec{x} - \vec{x}') \Psi(\vec{x}') \text{ (summation, i.e. integration over } \vec{x}') \\ &= \text{sgn} \cdot \left(-i\vec{\partial}\right) \Psi(\vec{x}) \end{aligned}$$

( $\Omega^{-1}$  has a mean range of the order of  $1/m = \text{Compton wavelength}$ .)

The commutator of two nonlocal operators  $A(\vec{x}-\vec{x}'), B(\vec{x}-\vec{x}')$  is again a nonlocal operator,

$$[A, B](\vec{x} - \vec{x}') = \int dx''^3 \{A(\vec{x} - \vec{x}'')B(\vec{x}'' - \vec{x}') - B(\vec{x} - \vec{x}'')A(\vec{x}'' - \vec{x}')\}$$

(here the summation over  $\vec{x}''$  is written out).

With this definition we obtain the commutator

$$[X_i, P_j](\vec{x}-\vec{x}') = \int dx''^3 \{x_i \delta(\vec{x} - \vec{x}'') P_j(\vec{x}'' - \vec{x}') - P_j(\vec{x} - \vec{x}'') x_i'' \delta(\vec{x}'' - \vec{x}')\} = (x_i - x_i') P_j(\vec{x} - \vec{x}')$$

This commutator acquires a more physical shape if we use the original definition (24) and the commutator

$$[x_i, \Omega^n] = n \frac{\partial}{\partial x_i} \Omega^{n-2} \tag{26}$$

$$\Rightarrow [x_i, \Omega^{-1}] = -\frac{\partial}{\partial x_i} \Omega^{-3}$$

which follows straightforward from definition (4) (see also first article of ref. [2]). We find

$$\begin{aligned} [x_i, P_j] &= [x_i, \text{sgn} \cdot (-i\partial_j)] = \text{sgn} \cdot [x_i, (-i\partial_j)] + [x_i, \text{sgn}] \cdot (-i\partial_j) = \\ &= \text{sgn} \cdot i\delta_{ij} + \left\{ \left[ x_i, \frac{1}{\Omega} \right] \cdot D + \frac{1}{\Omega} \cdot [x_i, D] \right\} (-i\partial_j) = \text{sgn} \cdot i\delta_{ij} + \frac{1}{\Omega} (\alpha_i - V_i) \partial_j, \end{aligned} \tag{27}$$

where

$$V_i = \frac{P_i}{\Omega}$$

This non-local commutator has a mean range of the Compton wavelength  $1/m$ . It is to be compared with the nonrelativistic commutator written as a distribution:

$$[x_i, P_j](\vec{x} - \vec{x}') = i\delta_{ij} \cdot \delta(\vec{x} - \vec{x}')$$

Let us point out: Usually we shall be operating with wave functions  $\Psi$  of one sign of charge only, *i.e.* with eigenfunctions of the operator  $sgn$ ,  $sgn\Psi = \pm\Psi$ .

With this subsidiary condition we have simply

$$\vec{P} = \pm(-i\vec{\partial})$$

for particle/antiparticle resp..

#### 4. The Plane Wave Solutions

Because of (9) there exist simultaneous eigenfunctions of  $D$  and  $sgn$ .

These are the well-known plane wave solutions of the Dirac equation with positive/negative frequencies,

$$\Psi^{(-,s)}(t, \vec{x}; \vec{p}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \chi^s \\ \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \end{pmatrix} \frac{e^{-i\omega t + i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi^3}} \quad (28a)$$

$$\Psi^{(+,s)}(t, \vec{x}; \vec{p}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \\ \chi^s \end{pmatrix} \frac{e^{+i\omega t - i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi^3}} \quad (28b)$$

$\chi^s$ ,  $s = \pm 1$ , denote the 2-spinors

$$\chi^{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi^{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We shall call the quantum number  $s$  the "rest spin" because it is the spin orientation for  $\vec{p} = 0$ .

Splitting off the time dependence we write

$$\Psi^{(\sigma,s)}(t, \vec{x}; \vec{p}) = \Phi^{(\sigma,s)}(\vec{p}; \vec{x}) e^{-i\sigma\omega t} \quad (29)$$

for  $\sigma = \pm$ , where  $\Phi^{(\sigma,s)}(\vec{p}; \vec{x})$  is the space part of (28a),(28b).

We have



for  $\sigma = +$ :

$$\Phi^{(+,s)}(\vec{p}; \vec{x}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \chi^s \\ \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \end{pmatrix} \frac{e^{i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi}^3} = u^{(+,s)}(\vec{p}) \frac{e^{i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi}^3} \quad (30a)$$

$$u^{(+,s)}(\vec{p}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \chi^s \\ \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \end{pmatrix}$$

for  $\sigma = -$ :

$$\Phi^{(-,s)}(\vec{p}; \vec{x}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \\ \chi^s \end{pmatrix} \frac{e^{-i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi}^3} = u^{(-,s)}(\vec{p}) \frac{e^{-i\vec{p} \cdot \vec{x}}}{\sqrt{2\pi}^3} \quad (30b)$$

$$u^{(-,s)}(\vec{p}) = \sqrt{\frac{\omega + m_0}{2\omega}} \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \\ \chi^s \end{pmatrix}$$

The plane waves (28a), (28b) are eigenfunctions of  $sgn$  with eigenvalues  $\pm 1$  resp.. If we retain the traditional choice of (28a) as electron wave functions then electrons are defined by the eigenvalue  $\sigma = +1$  of  $sgn$ . Consequently the eigenfunctions (28b) for the eigenvalue  $\sigma = -1$  define positrons. This definition of electrons and positrons via the operator  $sgn$  is perfectly symmetric, whereas the definition with the aid of the Dirac sea necessarily distinguishes one sign of the particle/antiparticle pair. The operator  $sgn$  rigorously confirms, though, the traditional definitions of electron/positron waves [3]. Definition (21) solves the two "standard problems" with these wave functions:

1) The *time component* of (21) solves the problem with the negative energies without recourse to the "Dirac sea":

$P^0 = \Omega$  yields *positive values*  $\sqrt{\vec{p}^2 + m_0^2}$  for the energy in both cases (28a) and (28b), *i.e.* for the positive *and* the negative frequency waves:

$$P^0 \Psi^{(\pm, s)}(t, \vec{x}; \vec{p}) = \Omega \Psi^{(\pm, s)}(t, \vec{x}; \vec{p}) = +\omega(\vec{p}) \Psi^{(\pm, s)}(t, \vec{x}; \vec{p}) = sgn \cdot i\partial^0 \Psi^{(\pm, s)}(t, \vec{x}; \vec{p})$$

$$E = \omega(\vec{p}) = +\sqrt{\vec{p}^2 + m_0^2}$$

2) The *space part* of (21) solves the difficulty with the momentum of the positron wave function (28b): Applying the traditional momentum operator  $-i\vec{\partial}$  on (28b) yields the eigenvalue  $-\vec{p}$ , although the function is supposed to describe a positron with momentum  $\vec{p}$ . However, applying the space component of (21) on (28b) yields the correct value  $\vec{p}$ :

$$sgn \cdot (-i\vec{\partial}) \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \\ \chi^s \end{pmatrix} e^{+i\omega t - i\vec{p} \cdot \vec{x}} = = \frac{D}{\Omega}(-\vec{p}) \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{\omega + m_0} \chi^s \\ \chi^s \end{pmatrix} e^{+i\omega t - i\vec{p} \cdot \vec{x}} =$$

$$= (-\omega)(-\vec{p}) \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{\omega+m_0}\chi^s \\ \chi^s \end{pmatrix} e^{+i\omega t - i\vec{p}\cdot\vec{x}} = \vec{p} \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{\omega+m_0}\chi^s \\ \chi^s \end{pmatrix} e^{+i\omega t - i\vec{p}\cdot\vec{x}}$$

Thus the functions (28a), (28b) are identified as simultaneous eigenfunctions of:

- a) momentum  $\vec{P}$ , with eigenvalue  $\vec{p}$ ,
- b) sign of particle charge  $sgn$ , with eigenvalues  $\sigma = +1, -1$ ,
- c) restspin, with eigenvalues  $s = +1/2, -1/2$ ,
- d) the energy in all cases being

$$E = \omega(\vec{p}) = +\sqrt{\vec{p}^2 + m_0^2}$$

## 5. The Operator "Sign of Frequency"

As a matter of fact, the operator  $sgn = D\Omega^{-1}$  is identical with the well-known operator "sign of frequency" [5]. Let us recapitulate:

The functions (28a), (28b) satisfy

$$D\Psi^{(+,s)}(t, \vec{x}; \vec{p}) = +\omega(\vec{p})\Psi^{(+,s)}(t, \vec{x}; \vec{p}),$$

$$D\Psi^{(-,s)}(t, \vec{x}; \vec{p}) = -\omega(\vec{p})\Psi^{(-,s)}(t, \vec{x}; \vec{p})$$

and, because of

$$\Omega\Psi^{(\pm,s)}(t, \vec{x}; \vec{p}) = \omega(\vec{p})\Psi^{(\pm,s)}(t, \vec{x}; \vec{p}),$$

they satisfy the eigenvalue equations for sign of the frequency

$$sgn\Psi^{(+,s)}(t, \vec{x}; \vec{p}) = +\Psi^{(+,s)}(t, \vec{x}; \vec{p}) \quad (31a)$$

$$sgn\Psi^{(-,s)}(t, \vec{x}; \vec{p}) = -\Psi^{(-,s)}(t, \vec{x}; \vec{p}) \quad (31b)$$

hence the name "sign of frequency operator".

Since the operator  $sgn$  turned out to be the observable "particle charge", eqs.(31a), (31b) express: *The sign of the frequency is not "the sign of the energy", but the sign of the particle charge.*

At this point let us insert a remark concerning the so-called "Zitterbewegung" of free electrons. The zitterbewegung originates from the superposition of plane waves with both signs  $\pm$  of frequency. *E.g.* in [3] the expectation value of  $\vec{x}$  is calculated for a general Dirac wave function, in our notation with the basis functions  $\Phi$  from (28a, b) (we omit the inessential index  $s$ )

$$\Psi(\vec{x}) = \int dp^3 (a_+(\vec{p})\Phi^{(+)}(\vec{p}; \vec{x})e^{-i\omega t} + a_-(\vec{p})\Phi^{(-)}(\vec{p}; \vec{x})e^{i\omega t})$$

where  $a_{\pm}(\vec{p})$  are the positive and negative frequency amplitudes.

With this  $\Psi(\vec{x})$  we find

$$\langle \vec{x} \rangle = \int dp^3 \left( a_+^*(\vec{p}) u_+(\vec{p}) e^{i\omega t} - a_-^*(\vec{p}) u_+(\vec{p}) e^{-i\omega t} \right) \left( i \frac{\vec{\partial}}{\partial \vec{p}} \right) \times \\ \left( a_+^*(-\vec{p}) u_+(-\vec{p}) e^{-i\omega t} + a_-^*(-\vec{p}) u_+(-\vec{p}) e^{i\omega t} \right)$$

Zitterbewegung of  $\langle \vec{x} \rangle$  appears when both  $a_+$  and  $a_-$  are present.

The difference between [3] and the present situation consists only *in the interpretation* of the signs of the frequency: traditionally the signs are not attributed an immediate physical meaning, the zitterbewegung is a formal mathematical result. In the present description the signs have the *physical meaning* of sign of charge. Qualitatively it is not surprising that the expectation value of the position  $\vec{x}$  of the particle is not purely linear in  $t$ , since from eq.(19) we know that the time displacement operator  $i\partial_t$  depends on the sign of charge, therefore the charge components of the wave function  $\Psi$  develop according to different time displacement operators. But let us to go one step farther:

Formally from the solutions with different signs of frequency we can construct new solutions by linear superposition. But such a superposition means that the particle which it describes can appear with nonvanishing probabilities as an electron or a positron. Such a phenomenon is unphysical. Therefore, for "physical" superpositions only one sign of frequency is possible. But then there is no zitterbewegung.

In conclusion let us remark: It has always been acknowledged that the solutions (28a,b) correspond to electrons/positrons. This is *absolutely not* a consequence of the present approach. Traditionally this interpretation was supported (though not proven) by the hole hypothesis. The present approach only *confirms* the traditional interpretation of the solutions *rigorously* as electron/positron solutions, due to the physical meaning of the operator *sgn*. Thus the possibility of constructing "unphysical" solutions by linear superposition of different charges has always been present.

A work that extensively discusses the problems connected with the zitterbewegung is given by [4]. It is worth while to put in at least a concise comparison of the methods proposed in [4] with the present approach. In Section 1.7.1 of [4] the question is asked: "Why should a free relativistic particle violate Newton's Second law?" The violation is expected to be caused by an inappropriate definition of the position operator  $\vec{x}$ . Therefore [4] tries to replace  $\vec{x}$  by a new position operator that "does not mix positive and negative energies", in our terminology: that commutes with the operator *sgn*. Two possibilities are discussed. (We use here  $\hbar = c = 1$ ).

1) " $\vec{x}$  is replaced by the part  $\vec{\tilde{x}}$  which commutes with *sgn*".

To separate  $\vec{\tilde{x}}$  amounts to extracting  $\vec{V}$  from  $\dot{\vec{x}}(t)$ , for, in our notation  $H_0 \equiv D$ , and therefore for the term  $\vec{p}H_0^{-1}$  in [4-1.113] we find

$$\vec{p}H_0^{-1} = \frac{(-i\vec{\partial})}{D} = \frac{(-i\vec{\partial}) D}{D^2} = \frac{(-i\vec{\partial}) D}{\Omega^2} = \frac{1}{\Omega} \frac{D}{\Omega} (-i\vec{\partial}) = \frac{1}{\Omega} \text{sgn} \cdot (-i\vec{\partial}) = \frac{\vec{P}}{\Omega} = \vec{V}$$

Therefore [4-1.113] reads  $\vec{F} \equiv \vec{\alpha} - \vec{V}$

$$\Rightarrow \dot{\vec{x}} = \vec{\alpha}(t) = \vec{V} + (\vec{\alpha}(t) - \vec{V}) = \vec{V} + \vec{F}(t)$$

$$\text{Integrating } \Rightarrow \vec{x}(t) - \vec{x}(0) = \vec{V}t + \frac{1}{2iH_0} (\vec{F}(t) - \vec{F}(0))$$

$$\text{Using [4-(1.113)], which correctly must read } \int_0^t dt \vec{F}(t) = \frac{1}{2iH_0} (\vec{F}(t) - \vec{F}(0)),$$

$$\Rightarrow \vec{x}(t) - \frac{1}{2iH_0} \vec{F}(t) = \vec{V}t + \vec{x}(0) - \frac{1}{2iH_0} \vec{F}(0)$$

$$i.e. \quad \vec{\tilde{x}}(t) = \vec{V}t + \vec{\tilde{x}}(0)$$

The disadvantage of this choice of position operator is, as pointed out in [4-(1.154)], "that its components do not commute",

$$[\tilde{x}_i, \tilde{x}_k] \neq 0$$

2) The second possibility consists in the construction of the "Newton-Wigner position operator  $\vec{x}_{NW}$ " satisfying

$$[\vec{x}_{NW}, sgn] = 0$$

To this end one applies exactly the same method we shall use below, in Sect. 7.2, to construct potentials that commute with the operator  $sgn$ .

Whatever choice one makes, replacing the "standard position operator"  $\vec{x}$ , *i.e.* the eigenvalues  $\vec{x}$  of the operator  $\vec{X}$  of Sect.3, by another position operator, makes it difficult to interpret the "standard wave function"  $\psi(\vec{x})$ . Hence in the present approach  $\vec{X}$  is maintained, and the zitterbewegung disappears for physical wave functions, *i.e.* eigenfunctions of the charge operator  $sgn$ .

Finally let us calculate the matrix element of  $\dot{\vec{x}}(t)$  between pairs of eigenstates  $\Psi^{(\pm)}$  of  $sgn$ :

$$\begin{aligned} \langle \Psi^{(\pm)} | \dot{\vec{x}}(t) | \Psi^{(\pm)} \rangle &= \langle \Psi^{(\pm)} | i [D, \vec{x}(t)] | \Psi^{(\pm)} \rangle = \langle \Psi^{(\pm)} | e^{iDt} i [D, \vec{x}] e^{-iDt} | \Psi^{(\pm)} \rangle = \\ &= \langle \Psi^{(\pm)} | e^{i sgn \cdot \Omega t} i (sgn \cdot \Omega \vec{x} - \vec{x} sgn \cdot \Omega) e^{-i sgn \cdot \Omega t} | \Psi^{(\pm)} \rangle \end{aligned}$$

The Hermitean factor  $sgn$  of the first term in the commutator when applied to the bra yields  $\pm$ , so does the factor  $sgn$  of the second term when applied to the ket. Thus we are left with

$$= \langle \Psi^{(\pm)} | e^{\pm i \Omega t} (\pm i [\Omega, \vec{x}]) e^{\mp i \Omega t} | \Psi^{(\pm)} \rangle$$

Using (26), we obtain

$$= \langle \Psi^{(\pm)} | e^{\pm i \Omega t} (\mp i \vec{\partial} \Omega^{-1}) e^{\mp i \Omega t} | \Psi^{(\pm)} \rangle =$$

$$= \langle \Psi^{(\pm)} | e^{\pm i \Omega t} \left( -i \vec{\partial} \Omega^{-1} \cdot sgn \right) e^{\mp i \Omega t} | \Psi^{(\pm)} \rangle = \langle \Psi^{(\pm)} | e^{\pm i \Omega t} \left( \frac{\vec{P}}{\Omega} \right) e^{\mp i \Omega t} | \Psi^{(\pm)} \rangle,$$

which, because of  $[sgn, \frac{\vec{P}}{\Omega}] = 0$ ,

$$= \langle \Psi^{(\pm)} | e^{+i \Omega \cdot sgn t} \left( \frac{\vec{P}}{\Omega} \right) e^{-i \Omega sgn t} | \Psi^{(\pm)} \rangle = \langle \Psi^{(\pm)} | e^{-i D t} \vec{V} e^{-i D t} | \Psi^{(\pm)} \rangle = \langle \Psi^{(\pm)} | \vec{V}(t) | \Psi^{(\pm)} \rangle$$

for both particles *and* antiparticles.

However, a particle with the unphysical property to appear with non-vanishing probabilities as an electron or a positron need not be expected to obey Newton's Second Law. After this excursion on zitterbewegung let us summarize:

The identity (21) is *not* a definition introduced ad hoc to solve the two "standard problems".

It is caused by (16) and the resulting the identification of the operator  $D/\Omega = sgn$  with the observable "sign of particle charge". Thus (16) appears as the "supporting pillar" of the theory. It consistently gives rise to the understanding of the Dirac equation  $i\partial_t = \Omega \cdot sgn$  as the zero-component of (21), essential for the advantage of the present approach.

The second consequence of (16), the identification of frequency with the sign of particle charge, is actually not new. As said before, the present approach only *confirms* the traditional interpretation of the solutions (28a,b) *rigorously* as electron/positron functions. Therefore all the consequences of this latter identification in principle are not consequences of the present approach either. It is true, this identification is not taken into account throughout, see *e.g.* zitterbewegung.

If the operator *sgn* had only the meaning of "sign of frequency", the Dirac equation (19) would *not* force us to interpret  $\Omega$  as the energy. The two signs of the energy would remain a "secret", causing troubles, which - partially, because the problem with the positron function remains - are remedied by the introduction of the "Dirac sea".

After recognizing  $D/\Omega$  as the *sign of charge* we have no choice - unless we doubt the above axiom. Therefore we must take the consequences seriously.

## 6. The S-picture

Before drawing conclusions, let us briefly discuss the theory in a representation in which the "relatively complicated" operator

$$sgn = \left( -i \vec{\alpha} \cdot \vec{\partial} + \beta m_0 \right) / \Omega$$

becomes simply the matrix  $\beta$ . This representation is created by the unitary operator  $S$  which transforms the free Dirac equation into the well-known *Foldy-Wouthuysen picture*

[6], (FW representation or briefly FW picture). The FW-picture is created by the unitary operator  $S$  which we prefer to write as

$$S = \sqrt{\frac{\Omega + m_0}{2\Omega}} (1 + i\sigma_y \otimes S') \quad (32a)$$

$$S^{-1} = \sqrt{\frac{\Omega + m_0}{2\Omega}} (1 - i\sigma_y \otimes S') \quad (32b)$$

$$S' = \frac{\vec{\sigma} \cdot (-i\vec{d})}{\Omega + m_0} \quad (32c)$$

$S$  transforms  $D$  according to

$$SDS^+ = \Omega\beta = \Omega \begin{pmatrix} e_2 & | \\ | & -e_2 \end{pmatrix} = \Omega \sigma_z \otimes e_2 \quad (33)$$

Since  $\Omega$  commutes with  $S$ ,

$$[\Omega, S] = 0$$

the operator  $sgn$  in the  $S$ -picture becomes

$$(sgn)_S = S \frac{D}{\Omega} S^+ = \beta = \sigma_z \otimes e_2 \quad (34)$$

Let us call the picture created by  $S$  the " $S$ -picture" because we shall use it not only for the free Dirac equation, for which it was originally introduced. In this picture the plane wave solutions (28a), (28b) read

$$\Psi_S^{(\sigma,s)}(t, \vec{x}; \vec{p}) = S\Psi_S^{(\pm,s)}(t, \vec{x}; \vec{p}) = \xi^{(\sigma)} \otimes \chi^s \frac{e^{-i\sigma(\omega t - \vec{p} \cdot \vec{x})}}{\sqrt{2\pi^3}} = \Phi_S^{(\sigma)}(\vec{p}; \vec{x}) e^{-i\sigma\omega t}, \sigma = \pm 1 \quad (35)$$

$$\Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) = S\Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) = \xi^{(\sigma)} \otimes \chi^s \frac{e^{i\sigma\vec{p} \cdot \vec{x}}}{\sqrt{2\pi^3}}, \omega = \omega(\vec{p}) = +\sqrt{\vec{p}^2 + m_0^2} \quad (36)$$

For completeness let us note  $C$  in the  $S$ -picture. By multiplying the definition (10)

$$\Psi' = C\Psi^*$$

from the left with  $S$  and inserting  $1 = (S^*)^{-1}S^*$  we get

$$(\Psi')_S = S\Psi' = SC[(S^*)^{-1}S^*]\Psi^* = [SC(S^*)^{-1}](S\Psi)^* = C_S\Psi_S^*$$

$$\Rightarrow C_S = SC(S^*)^{-1} = C = \sigma_y \otimes \sigma_y = C^{-1} \quad (37)$$

(37) confirms

$$C (sgn)_S C = C_S (sgn)_S^* C_S = C_S \beta C_S =$$

$$= (\sigma_y \otimes \sigma_y)(\sigma_z \otimes e_2)(\sigma_y \otimes \sigma_y) = -\sigma_z \otimes e_2 = -\beta = -(sgn)_S$$

In (35) the degrees of freedom of a Dirac plane wave are clearly exhibited: charge  $\sigma$ , momentum  $\vec{p}$ , rest spin  $s$ . The energy is in all cases  $\omega(\vec{p}) > 0$ .

(34) shows that in the  $S$ -picture the upper and lower 2-spinors of a wave function each belong to definite values of the charge operator,

$$(sgn)_S \begin{pmatrix} \Phi_{upper} \\ 0 \end{pmatrix} = + \begin{pmatrix} \Phi_{upper} \\ 0 \end{pmatrix} \quad (38a)$$

$$(sgn)_S \begin{pmatrix} 0 \\ \Phi_{lower} \end{pmatrix} = - \begin{pmatrix} 0 \\ \Phi_{lower} \end{pmatrix} \quad (38b)$$

Hence in order to analyze a wave function with respect to its charge components it is sufficient to transform it into the  $S$ -picture, *i.e.* to apply on it the transformation  $S$ . In the original Dirac picture the upper and lower components of a wave function do not have this convenient property.

In the  $S$ -picture we find an interesting aspect of the relation (21). In this picture (21) reads

$$P^\mu = i\beta\partial^\mu = \begin{pmatrix} i\partial^\mu | 0 \\ 0 | -i\partial^\mu \end{pmatrix} \quad (39)$$

In the shape (39) the definition of the 4-momentum operators may be understood as the equivalence of  $i$  and  $-i$ , *i.e.* of complex conjugate representations of  $P^\mu$ . Since the upper/lower 2-spinors refer to particles/antiparticles the existence of particles and antiparticles appears to be intimately connected with the equivalence of complex conjugate representations of  $P^\mu$ .

With these tools, in the subsequent section we shall discuss the ground state wave function of the relativistic hydrogen atom.

## 7. The Relativistic Hydrogen Atom

### 7.1 Dirac Equation with Coulomb Potential

As an application of the foregoing considerations let us analyze the relativistic hydrogen ground state wave function with respect to the quantum number *charge*. Being a pure electron function, in the  $S$ -picture this function should have only an upper component,

see (38a). We shall find that it is *not* a pure electron function, but contains a non-vanishing admixture of a positron function. This is an unphysical prediction because it would mean a non-vanishing probability to find a positron at the place of the electron. Since the quantitative aspect plays no major role for our line of reasoning we shift the corresponding calculations into the Appendix. The prediction, however, shows that the naive use of the Coulomb potential in the Dirac equation is somehow problematic. At this point one is tempted to remember that in the relativistic theory of *many-electron atoms* the use of the pure Coulomb potential also fails [7].

As a matter of fact, in our case this result is not surprising because a potential  $V \sim 1/r$  does not commute with the operator  $sgn$ ,

$$\left[ sgn, \frac{1}{r} \right] \neq 0$$

The property of  $1/r$  to couple different charge channels becomes still more transparent in the  $S$ -picture where the potential  $1/r$  reads

$$V_S = S \frac{1}{r} S^{-1} = \left( \begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{12}^+ & V_{22} \end{array} \right), V_{12} = 0$$

(each  $V_{ij}$  being a  $2 \times 2$ -matrix in spin space).

In [7] the shortcomings are remedied by using methods of quantum electrodynamics (QED). But within the present framework one can ask as well if it is possible to modify the pure  $1/r$  potential, so that it becomes an "effective potential" that commutes with the operator  $sgn$  and thus allows pure electronic solutions. Indeed, it is easy to construct such potentials. We shall demonstrate this in the next section. Of course, in principle such potentials should also be derivable from QED.

## 7.2 The "Effective" Coulomb Potential

Our aim is to find a potential which satisfies

$$[sgn, V(r)] = 0 \tag{40}$$

We try the ansatz

$$V(r) = \gamma/r + V_{corr} \tag{41}$$

(40) requires

$$\left[ \frac{\gamma}{r} + V_{corr} sgn \right] = 0$$

In the  $S$ -picture (41) reads

$$[S(\frac{\gamma}{r} + V_{corr})S^+, \beta] = 0 \tag{42}$$



The most general  $S$ -picture potential that commutes with  $\beta$  is of the shape

$$V_S^{(gen)} = \begin{pmatrix} V_2^+ & 0 \\ 0 & V_2^- \end{pmatrix}$$

or equivalently

$$V^{(gen)} = \begin{pmatrix} V_0 + V_3 & 0 \\ 0 & V_0 - V_3 \end{pmatrix} = e_2 \otimes V_0 + \sigma_z \otimes V_3 \quad (43)$$

where  $V_2^+, V_2^-$  and  $V_0, V_3$  are arbitrary (of course Hermitean)  $2 \times 2$ -potentials. Therefore  $S(\frac{\gamma}{r} + V_{corr})S^+$  must be of the shape (43):

$$\begin{aligned} S(\frac{\gamma}{r} + V_{corr})S^+ &= e_2 \otimes V_0 + \sigma_z \otimes V_3 \\ \Rightarrow \frac{\gamma}{r} + V_{corr} &= S^+(e_2 \otimes V_0 + \sigma_z \otimes V_3)S = \\ &= \sqrt{\frac{\Omega + m_0}{2\Omega}}(1 - i\sigma_y \otimes S')(e_2 \otimes V_0 + \sigma_z \otimes V_3)\sqrt{\frac{\Omega + m_0}{2\Omega}}(1 + i\sigma_y \otimes S') = \\ &= (1 - i\sigma_y \otimes S')(e_2 \otimes \tilde{V}_0 + \sigma_z \otimes \tilde{V}_3)(1 + i\sigma_y \otimes S') = \\ &= e_2 \otimes \tilde{V}_0 + \sigma_z \otimes \tilde{V}_3 + i\sigma_y \otimes \tilde{V}_0 S' + \sigma_x \otimes \tilde{V}_3 S' + \\ &+ e_2 \otimes S' \tilde{V}_0 S' - \sigma_z \otimes S' \tilde{V}_3 S' - i\sigma_y \otimes S' \tilde{V}_0 + \sigma_x \otimes S' \tilde{V}_3 \end{aligned}$$

where

$$\tilde{V}_i = \sqrt{\frac{\Omega + m_0}{2\Omega}} V_i \sqrt{\frac{\Omega + m_0}{2\Omega}}, \quad i = 0, 3$$

$\frac{\gamma}{r}$  can be obtained from the term  $e_2 \otimes \tilde{V}_0$ , choosing

$$V_0 = \sqrt{\frac{2\Omega}{\Omega + m_0}} V_i \sqrt{\frac{2\Omega}{\Omega + m_0}} \quad (44a)$$

we are left with

$$\begin{aligned} S^+(e_2 \otimes \tilde{V}_0 + \sigma_z \otimes \tilde{V}_3)S &= \\ &= \frac{\gamma}{r} + \begin{pmatrix} S' \frac{\gamma}{r} S' + \tilde{V}_3 - S' V_3 S' & \frac{\gamma}{r} S' - S' \frac{\gamma}{r} + (\tilde{V}_3 S' + S' \tilde{V}_3) \\ -(\frac{\gamma}{r} S' - S' \frac{\gamma}{r}) + (\tilde{V}_3 S' + S' \tilde{V}_3) & S' \frac{\gamma}{r} S' - \tilde{V}_3 + S' V_3 S' \end{pmatrix} \end{aligned}$$

The simplest and most symmetric result is obtained by the choice

$$V_3 = 0 \quad (44b)$$

which yields,

$$V_{corr} = \gamma \begin{pmatrix} S' \frac{1}{r} S' & \frac{1}{r} S' - S' \frac{1}{r} \\ -(\frac{1}{r} S' - S' \frac{1}{r}) & S' \frac{1}{r} S' \end{pmatrix} \quad (45)$$

The non-local correction term  $V_{corr}$  from (45) may be regarded as "small" in the following sense. In  $S' = \frac{\vec{\sigma} \cdot (-i\vec{\partial})}{\Omega + m_0}$  the operator  $-i\vec{\partial}$ , when applied to a plane wave component (28a,b), apart from a sign, yields the momentum eigenvalue  $\vec{p}$ ,

$$\Rightarrow S' \sim \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{p^2 + m_0^2} + m_0} \quad (46)$$

But

$$|S'| = \frac{|\vec{\sigma} \cdot \vec{p}|}{\sqrt{p^2 + m_0^2} + m_0} = \frac{\frac{|\vec{p}|}{m_0}}{1 + \sqrt{1 + \frac{p^2}{m_0^2}}} \leq 1 \quad (47)$$

where the upper limit 1 is attained for  $|\vec{p}| \rightarrow \infty$ . Thus, if in the wave functions involved the essential contributions of momentum are sufficiently small as compared to  $m_0$ ,  $V_{corr}$  will be a small correction compared to  $\frac{\gamma}{r}$ .

Concluding this section, we want to justify the unaltered adoption of the equation

$$i\partial_t \Psi = (D - eV)\Psi, \quad V \sim 1/r \quad (48)$$

used in the literature for the hydrogen atom, within the present framework.

Equation (48) is obtained from the free Dirac equation by "minimal coupling", *i.e.* the replacement

$$i\partial^\mu \rightarrow i\partial^\mu - eA^\mu \quad (49)$$

in the free Dirac equation.

For an electron, in case of absence of a magnetic field, eq.(49) reduces to the replacement

$$i\partial_t \rightarrow i\partial_t - (-eV) = i\partial_t + eV$$

yielding (48).

In the present approach  $i\partial_t$  is no longer the kinetic energy. Since a potential is a potential energy it must not be added to  $D$ , but to  $\Omega$ . Moreover, since the Dirac wave function has the intrinsic degree of freedom of *charge*, we must ensure the correct sign of the potential acting on the charge components of  $\Psi$ . Therefore the potential must have the shape

$$V = -eV(r) \cdot \text{sgn} \quad (50)$$

( $\text{sgn} = +1$  for electrons).

$$\Rightarrow \Omega \rightarrow \Omega - eV(r) \cdot \text{sgn} \quad (51)$$

Making this substitution in the free Dirac equation

$$i\partial_t\Psi = (\Omega \cdot \text{sgn}) \Psi \quad (52)$$

we obtain

$$i\partial_t\Psi = (\Omega - eV(r) \cdot \text{sgn}) \cdot \text{sgn}\Psi = (D - eV(r)) \Psi \quad (53)$$

in agreement with (48). Believing in minimal coupling, this result confirms the correctness of the coupling (51). Other couplings, *e.g.* "scalar coupling" according to  $-e\beta V(r)$ , thus are ruled out.

Eq.(53), due to (15) and  $V^* = V$ , behaves correctly under C:

$$C(i\partial_t\Psi) = -i\partial_t(C\Psi) = (-D - e(CVC)) (C\Psi) = (-D - eV) (C\Psi)$$

$$\Rightarrow i\partial_t(C\Psi) = (D + eV(r)) (C\Psi)$$

(For completeness let us remark: If instead we had substituted (51) into  $i\partial_t\Psi = (\text{sgn} \cdot \Omega) \Psi$  which differs from (52) by the order of the factors  $\Omega$  and  $\text{sgn}$ , we were led to a different equation

$$i\partial_t\Psi = \text{sgn} \cdot (\Omega - eV(r) \cdot \text{sgn}) \Psi = (D - e(\text{sgn}) \cdot V(r) \cdot \text{sgn}) \Psi$$

It also correctly satisfies

$$i\partial_t(C\Psi) = (D + e(\text{sgn}) \cdot V(r) \cdot \text{sgn})(C\Psi)$$

However, because we want the Dirac equation to have solutions of unique sign of charge, *i.e.* solutions which are simultaneous eigenfunctions of  $\text{sgn}$  and  $D + V$  we must impose on  $V$  the condition

$$[\text{sgn}, V(r)] = 0 \quad (40)$$

Under condition (40) the two resulting equations for  $i\partial_t(C\Psi)$  become identical.

## 8. Second Quantization

### 8.1 Second Quantization in the S-Picture

In this section we shall show in detail how (23) makes it possible to carry out the scheme of second quantization of wave equations of the Schroedinger/Dirac type, *i.e.* of first order in  $i\partial_t$ , in perfect agreement with its original physical meaning [8]. Because of the greater transparency of the degrees of freedom ( $\sigma, s$ ) in the  $S$ -picture let us first construct the multiparticle theory corresponding to the free one-particle equation the  $S$ -picture,

$$i\partial_t\Psi_S = (\sigma_z \otimes e_2) \Omega\Psi_S \quad (54)$$

To this end we start by defining creation and annihilation operators

$$a_{\sigma s}^+(\vec{p}), \quad a_{\sigma s}(\vec{p})$$

resp., for the states characterized by the wave functions (36),

$$a_{\sigma s}^+(\vec{p})|0\rangle = \left| \Phi_S^{(\sigma,s)}(\vec{p}, \vec{x}) \right\rangle \hat{=} |\sigma s \vec{p}\rangle \quad (55)$$

The variable  $\vec{p}$  in the creation operator has the meaning of the variable  $\vec{p}$  in the state  $\left| \Phi_S^{(\sigma,s)}(\vec{p}, \vec{x}) \right\rangle \hat{=} |\sigma s \vec{p}\rangle$  which it creates, *i.e.* the eigenvalue of  $\vec{P} = \text{sgn}(-i\vec{\partial})$

We need not include  $\omega(\vec{p})$  in the ket  $|\rangle$  since the energy  $\omega(\vec{p})$  is uniquely defined by  $\vec{p}$ . There are no negative energy values.

$|0\rangle$  is defined as the normalized "no-particle state",

$$\langle 0 | 0 \rangle = 1 \quad (56)$$

$$a_{\sigma s}(\vec{p})|0\rangle = 0 \quad (57)$$

Note that  $|0\rangle$  is a purely algebraic definition.

To ensure Fermi statistics,  $a_{\sigma s}(\vec{p})$ ,  $a_{\sigma s}^+(\vec{p})$  are subject to the anticommutators

$$\{a_{\sigma s}(\vec{p}), a_{\sigma' s'}^+(\vec{p}')\} = \delta_{\sigma\sigma'}\delta_{ss'}\delta(\vec{p}-\vec{p}') \quad (58a)$$

$$\{a_{\sigma s}(\vec{p}), a_{\sigma' s'}(\vec{p}')\} = \{a_{\sigma s}(\vec{p})^+, a_{\sigma' s'}(\vec{p}')^+\} = 0 \quad (58b)$$

Further we define the  $\rho$ -component,  $\rho = \{\sigma, s\} = \{(+, +1), \dots, (-, -1)\}$ , of the 4-component field operators  $\psi(\vec{x})$ ,  $\psi^+(\vec{x})$  as

$$\psi_\rho(\vec{x}) = \sum_{\sigma s} \int dp^3 \left( \Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) \right)_\rho a_{\sigma s}(\vec{p}) \quad (59a)$$

and

$$\psi_\rho^+(\vec{x}) = \sum_{\sigma s} \int dp^3 \left( \Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) \right)_\rho^* a_{\sigma s}^+(\vec{p}) \quad (59b)$$

where  $\left( \Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) \right)_\rho$  is the  $\rho$ -component of the wave function (36).

From (58a), (58b) we obtain

$$\{\psi_\rho(\vec{x}), \psi_{\rho'}^+(\vec{x}')\} = \delta_{\rho\rho'}\delta(\vec{x}-\vec{x}') \quad (60a)$$

$$\{\psi_\rho(\vec{x}), \psi_{\rho'}(\vec{x}')\} = \{\psi_\rho^+(\vec{x}), \psi_{\rho'}^+(\vec{x}')\} = 0 \quad (60b)$$

Note that in the expansion (59a) only the annihilation operators  $a_{\sigma s}(\vec{p})$  appear and in (59b) only the creation operators  $a_{\sigma s}^+(\vec{p})$ , as required by the original meaning of second quantization. This is a profound difference to the traditional expansions, where both creation and annihilation operators appear in each of  $\psi_\rho(\vec{x}), \psi_\rho^+(\vec{x})$ . These operators thus lose their property of well defined creation and annihilation operators in  $\vec{x}$ -space. Using the quantum mechanical definition of a wave function for the  $\rho$ -component of  $\Phi_S^{(\sigma,s)}(\vec{p}; \vec{x})$ ,

$$\left( \Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) \right)_\rho = \langle \rho, \vec{x} | \sigma, s, \vec{p} \rangle \quad (61)$$

we obtain the state

$$\begin{aligned}\psi_\rho^+(\vec{x})|0\rangle &= \sum_{\sigma_s} \int dp^3 \left( \Phi_S^{(\sigma,s)}(\vec{p}; \vec{x}) \right)_\rho^* a_{\sigma_s}^+(\vec{p})|0\rangle = \\ &= \sum_{\sigma_s} \int dp^3 |\sigma_s \vec{p}\rangle \langle \sigma_s \vec{p} | \rho, \vec{x}\rangle = |\rho, \vec{x}\rangle\end{aligned}\quad (62)$$

which is a one-particle state with position  $\vec{x}$  and index  $\rho$  in agreement with the physical meaning of the second quantization of one-particle equations of the Schroedinger/Dirac type.

The operators  $\psi(\vec{x})$ ,  $\psi^+(\vec{x})$  serve to construct multiparticle or field operators corresponding to the one-particle operators in  $\vec{x}$ -space. Here besides the energy and momentum operators  $H_S^{(field)}$ ,  $\vec{P}_S^{(field)}$  resp. we are also interested in the operator  $D_S^{(field)}$  corresponding to the one-particle time displacement operator  $D_S = \beta\Omega$ . We obtain straightforward

$$H_S^{(field)} = \int dx^3 \psi^+(\vec{x}) \Omega \psi(\vec{x}) = \sum_{\sigma_s} \int dp^3 \omega(\vec{p}) a_{\sigma_s}^+(\vec{p}) a_{\sigma_s}(\vec{p}) \quad (63)$$

$$\vec{P}_S^{(field)} = \int dx^3 \psi^+(\vec{x}) \left( -i\beta\vec{\partial} \right) \psi(\vec{x}) = \sum_{\sigma_s} \int dp^3 \vec{p} a_{\sigma_s}^+(\vec{p}) a_{\sigma_s}(\vec{p}) \quad (64)$$

$$\begin{aligned}D_S^{(field)} &= \int dx^3 \psi^+(\vec{x}) \beta\Omega \psi(\vec{x}) = \sum_{\sigma_s} \int dp^3 \sigma \omega(\vec{p}) a_{\sigma_s}^+(\vec{p}) a_{\sigma_s}(\vec{p}) = \\ &= \sum_s \int dp^3 \omega(\vec{p}) \left( a_{+,s}^+(\vec{p}) a_{+,s}(\vec{p}) - a_{-,s}^+(\vec{p}) a_{-,s}(\vec{p}) \right)\end{aligned}\quad (65)$$

One easily confirms

$$\left[ H_S^{(field)}, D_S^{(field)} \right] = 0 \quad (66)$$

We observe

$$H_S^{(field)} |0\rangle = D_S^{(field)} |0\rangle = 0 \quad (67)$$

Finally, with the electron charge  $-e$  we may define a total charge operator  $Q$  corresponding to the one-particle charge operator (34),

$$\begin{aligned}Q &= -e \int dx^3 \psi^+(\vec{x}) (\sigma_z \otimes e_2) \psi(\vec{x}) = -e \sum_{\sigma_s} \int dp^3 \sigma a_{\sigma_s}^+(\vec{p}) a_{\sigma_s}(\vec{p}) = \\ &= -e \sum_{\sigma_s} \int dp^3 \left( a_{+,s}^+(\vec{p}) a_{+,s}(\vec{p}) - a_{-,s}^+(\vec{p}) a_{-,s}(\vec{p}) \right)\end{aligned}\quad (68)$$

To within a normalization factor the eigenstates of  $H_S^{(field)}$  are given by

$$a_{\sigma_1, s_1}^+(\vec{p}_1) \dots a_{\sigma_n, s_n}^+(\vec{p}_n) |0\rangle \quad (69)$$

with eigenvalues  $\omega(\vec{p}_1) + \dots + \omega(\vec{p}_n)$ :

$$H_S^{(field)} \left( a_{\sigma_1, s_1}^+(\vec{p}_1) \dots a_{\sigma_n, s_n}^+(\vec{p}_n) |0\rangle \right) =$$

$$= (\omega(\vec{p}_1) + \dots + \omega(\vec{p}_n)) (a_{\sigma_1, s_1}^+(\vec{p}_1) \dots a_{\sigma_n, s_n}^+(\vec{p}_n) |0\rangle) \quad (70)$$

If we write for  $\sigma = -$ :

$$a_{-, s}(\vec{p}) = b_s(\vec{p}) \quad (71)$$

$H_S^{(field)}$  becomes the familiar Hamiltonian

$$H = \sum_{\sigma s} \int dp^3 \omega(\vec{p}) (a_s^+(\vec{p}) a_s(\vec{p}) + b_s^+(\vec{p}) b_s(\vec{p})) \quad (72)$$

But  $H$  does not determine the evolution in time. The evolution in time is now determined by  $D_S^{(field)}$ . For the wave function  $\Psi_{\rho_1 \dots \rho_n}(t, \vec{x}_1, \dots, \vec{x}_n)$  of the free multiparticle state

$$|\Psi\rangle = \int dx_1^3 \dots dx_n^3 \Psi_{\rho_1 \dots \rho_n}(t, \vec{x}_1, \dots, \vec{x}_n) \psi_{\rho_1}^+(\vec{x}_1) \dots \psi_{\rho_n}^+(\vec{x}_n) \quad (73)$$

the Schrodinger field equation

$$i\partial_t |\Psi\rangle = D_S^{(field)} |\Psi\rangle \quad (74)$$

yields the multiparticle Dirac equation

$$i\partial_t \Psi_{\rho_1 \dots \rho_n}(t, \vec{x}_1, \dots, \vec{x}_n) = ((\beta\Omega)_1 + \dots + (\beta\Omega)_n) \Psi_{\rho_1 \dots \rho_n}(t, \vec{x}_1, \dots, \vec{x}_n) \quad (75)$$

Here,  $(\beta\Omega)_1$  acts on the variables  $\rho_1, \vec{x}_1$ , etc.

As a result we possess a consistent theory of multiparticle systems of free electrons/positrons, i.e. a field theory in agreement with the principles of quantum mechanics and the physical meaning of second quantization.

## 8.2 Second Quantization in the Dirac Picture

In order to facilitate the comparison with the traditional literature where the Dirac picture is used throughout, we shall carry through the program of the preceding section with the standard solutions (30a), (30b).

Indicating the components in the Dirac picture by the indice  $d$ ,

$$(\Phi^{(\sigma, s)}(\vec{p}; \vec{x}))_d = \left( S^+ \Phi_S^{(\sigma, s)}(\vec{p}; \vec{x}) \right)_d = \langle d, \vec{x} | \sigma, s, \vec{p} \rangle \quad (76)$$

we define the  $d$ -component,  $d = 1, \dots, 4$ , of the 4-component field operators  $\psi(\vec{x}), \psi^+(\vec{x})$  as

$$\psi_d(\vec{x}) = \sum_{\sigma s} \int dp^3 (\Phi^{(\sigma, s)}(\vec{p}; \vec{x}))_d a_{\sigma s}(\vec{p}) \quad (77a)$$

$$\psi_d^+(\vec{x}) = \sum_{\sigma s} \int dp^3 (\Phi^{(\sigma, s)}(\vec{p}; \vec{x}))_d^* a_{\sigma s}^+(\vec{p}) \quad (77b)$$

(77a), (77b) are usually written without the indice  $d$  simply as

$$\psi(\vec{x}) = \sum_{\sigma s} \int dp^3 (\Phi^{(\sigma, s)}(\vec{p}; \vec{x})) a_{\sigma s}(\vec{p}) \quad (78a)$$

$$\psi^+(\vec{x}) = \sum_{\sigma s} \int dp^3 (\Phi^{(\sigma,s)}(\vec{p}; \vec{x}))^+ a_{\sigma s}^+(\vec{p}) \quad (78b)$$

As above,

$$a_{\sigma s}^+(\vec{p}) |0\rangle = |\Phi^{(\sigma,s)}(\vec{p}; \vec{x})\rangle \hat{=} |\sigma s \vec{p}\rangle \quad (79)$$

$$\{a_{\sigma s}(\vec{p}), a_{\sigma' s'}^+(\vec{p}')\} = \delta_{\sigma\sigma'} \delta_{ss'} \delta(\vec{p} - \vec{p}') \quad (80a)$$

$$\{a_{\sigma s}(\vec{p}), a_{\sigma' s'}(\vec{p}')\} = \{a_{\sigma s}(\vec{p})^+, a_{\sigma' s'}(\vec{p}')^+\} = 0 \quad (80b)$$

and

$$\{\psi_d(\vec{x}), \psi_{d'}^+(\vec{x}')\} = \delta_{dd'}(\vec{x} - \vec{x}') \quad (81a)$$

$$\{\psi_d(\vec{x}), \psi_{d'}(\vec{x}')\} = \{\psi_d^+(\vec{x}), \psi_{d'}^+(\vec{x}')\} = 0 \quad (81b)$$

Since the one-particle energy operator in the Dirac picture remains  $S^+\Omega S = \Omega$  we obtain for the field energy in the Dirac picture,

$$H^{(field)} = \int dx^3 \psi^+(\vec{x}) \Omega \psi(\vec{x}) = \sum_s \int dp^3 \omega(\vec{p}) (a_s^+(\vec{p}) a_s(\vec{p}) + b_s^+(\vec{p}) b_s(\vec{p})) \quad (82)$$

where we introduced the traditional notations

$$a_{+s}(\vec{p}) = a_s(\vec{p}), \quad a_{-s}(\vec{p}) = b_s(\vec{p}),$$

and for the time displacement field operator

$$D^{(field)} = \int dx^3 \psi^+(\vec{x}) \beta \Omega \psi(\vec{x}) = \sum_s \int dp^3 \omega(\vec{p}) (a_s^+(\vec{p}) a_s(\vec{p}) - b_s^+(\vec{p}) b_s(\vec{p})) \quad (83)$$

The completeness relation

$$\begin{aligned} \sum_s \int dp^3 (|\vec{p}, s, +\rangle \langle \vec{p}, s, +| + |\vec{p}, s, -\rangle \langle \vec{p}, s, -|) = \\ \sum_{\sigma s} \int dp^3 |\vec{p}, \sigma, s\rangle \langle \vec{p}, \sigma, s| = 1 \end{aligned} \quad (84)$$

together with (77a) and (76) yields

$$\psi_d^+(\vec{x}) |0\rangle = |d, \vec{x}\rangle \quad (85)$$

For the Dirac index  $d$  no immediate physical meaning can be specified. It is a great advantage of the  $S$ -picture that the degrees of freedom  $\rho = \{\sigma, s\}$  have a well-defined physical meaning.

The formation of the theory of free fermions based on (23) follows the principles of quantum mechanics. There was no need for any ad hoc amendments.

## Summary

1) We showed that the operator  $= \text{sgn}$  with eigenvalues, which is known in the literature as the operator of "sign of frequency" for the plane wave solutions of the Dirac equation, must be attributed the physical meaning of "*sign of particle charge*".

2) As a consequence, the time displacement operator  $i\partial_t = D$  for free particles appears as a product  $D = \text{sgn} \Omega$ . Since the kinetic energy of a *free* particle must not depend on its charge state the time displacement operator  $i\partial_t = \text{sgn} \Omega$  as a whole cannot be identified with the kinetic energy  $P^0$ , but only the factor  $\Omega : P^0 = \Omega$ . Therefore  $P^0 = \text{sgn} i\partial_t$ .

3) The relativistic equivalence of time and space variables forced us to interpret  $P^0 = \text{sgn} \partial_t$  as the zero-component of  $P^\mu = \text{sgn} i\partial^\mu$ .

4) The plane wave solutions of the Dirac equation  $i\partial_t = \text{sgn} \Omega$  have positive kinetic energy only. They are characterized by the quantum number *sign of charge*  $\sigma = \pm 1$ . Thus the operator  $\text{sgn}$  in a symmetric way defines *electron/positron* wave functions, whereas through the hole hypothesis particles and antiparticles enter the theory in an unsymmetric way.

5) The construction of the corresponding free field theory ("second quantization") proceeds without any of the ad hoc amendments traditionally applied to overcome the difficulties originating from the indefiniteness of the Hamiltonian. These amendments overthrow the original physical meaning of second quantization as a multiparticle theory of identical particles equivalent to a set of wave equations including statistics.

6) As an application of the role of the quantum number  $\sigma$  we discussed, with respect to the quantum number *sign of charge*, the relativistic hydrogen ground state wave function obtained from the Dirac equation with an  $1/r$ - potential. Since this potential does not commute with the operator  $\text{sgn}$  we could not expect the wave function to be a simultaneous eigenfunction of the operators  $D + V$  and  $\text{sgn}$ , *i.e.* to be a pure electron function. A rough estimate confirmed this expectation. We showed how one can construct "effective potentials" that commute with  $\text{sgn}$  and therefore, when used in the Dirac equation, ensure the possibility of pure electron solutions.

Let us remark in conclusion: Everything that the present theory accomplishes certainly somehow can be accomplished with the traditional concepts of the Dirac sea, QED etc. The present considerations showed what type of theory results if one applies elementary physical principles to the Dirac equation, thus staying as close as possible to the concepts of "elementary quantum mechanics".



## Appendix: The Positronic Admixture in the Hydrogen Ground State Wave Function

### (A1) The Hydrogen Ground State Wave Function in the Dirac Picture

The hydrogen solutions of the stationary Dirac equation

$$(D + \gamma/r) \Psi = E\Psi, \quad \text{with some constant } \gamma \quad (\text{A.1})$$

are well known. We shall roughly estimate the magnitude of the admixture of a positron component in the hydrogen ground state wave function. For our analysis we will use the hydrogen wave functions presented in [5], we adopt this author's notation.

We take the general bound state solution in the Dirac picture from [5], p. 243, formula 2, and p. 244, formula 8.

In the case we are interested in with quantum numbers  $n = 1$ ,  $\ell = 0$ ,  $j = 1/2$  and  $z$ -component  $m$  of total angular momentum, the required solution reads:

$$\psi_{\frac{1}{2}m} = \begin{pmatrix} ig(r)\Omega_{\frac{1}{2}0m}(\hat{r}) \\ -f(r)\Omega_{\frac{1}{2}1m}(\hat{r}) \end{pmatrix}, \quad \text{here } m = \pm\frac{1}{2} \quad (\text{A.2})$$

where, with the spherical harmonics  $Y_\ell^m$  and 2-spinors  $\chi_{\frac{1}{2}}^m$ ,

$$\Omega_{\frac{1}{2}0m} = Y_0^0 \chi_{\frac{1}{2}}^m = \frac{1}{\sqrt{4\pi}} \chi_{\frac{1}{2}}^m \quad (\text{A.3a})$$

$$\Omega_{\frac{1}{2}1\frac{1}{2}} = -\frac{1}{\sqrt{3}} Y_1^0 \chi_{\frac{1}{2}}^{\frac{1}{2}} + \sqrt{\frac{2}{3}} Y_1^1 \chi_{\frac{1}{2}}^{-\frac{1}{2}}, \quad \Omega_{\frac{1}{2}1-\frac{1}{2}} = \frac{1}{\sqrt{3}} Y_1^0 \chi_{\frac{1}{2}}^{-\frac{1}{2}} - \sqrt{\frac{2}{3}} Y_1^{-1} \chi_{\frac{1}{2}}^{\frac{1}{2}} \quad (\text{A.3b})$$

$$g(r) = A(2\lambda r)^{\gamma-1} e^{-\lambda r}, \quad -f(r) = B(2\lambda r)^{\gamma-1} e^{-\lambda r} = \sqrt{\frac{1-\lambda}{1+\lambda}} g(r) \quad (\text{A.4})$$

$$\lambda = \frac{m_0 e^2}{\hbar^2}, \quad \alpha = \frac{e^2}{\hbar c}, \quad \gamma = \sqrt{1 - \alpha^2}$$

$$A = \frac{2\sqrt{\lambda^3}}{\sqrt{\Gamma(2\gamma+1)}} \sqrt{1+\gamma}, \quad B = \frac{2\sqrt{\lambda^3}}{\sqrt{\Gamma(2\gamma+1)}} \sqrt{1-\gamma}$$

As said above, in the  $S$ -picture this function should have only an upper component. In the next

subsection we transform (A.2) the into the  $S$ -picture.

### (A2) The Transformation of the Ground State Hydrogen Function into the $S$ -Picture: Approximate Charge Eigenfunction

In [5] the constants  $\hbar$  and  $c$  are kept in evidence, then

$$S = \sqrt{\frac{\Omega + m_0 c^2}{2\Omega}} (1 + i\sigma_y \otimes S') \quad (\text{A.5a})$$

$$S^{-1} = \sqrt{\frac{\Omega + m_0c^2}{2\Omega}} (1 - i\sigma_y \otimes S') \quad (\text{A.5b})$$

$$S' = \frac{c\vec{\sigma} \cdot (-i\hbar\vec{\partial})}{\Omega + m_0c^2} \quad (\text{A.5c})$$

$$\Omega = \sqrt{-\hbar^2c^2\Delta + m_0^2c^4} \hat{\Omega}(\vec{x} - \vec{x}') = \frac{1}{(2\pi)^3} \int dk^3 \sqrt{\hbar^2c^2\vec{k}^2 + m_0^2c^4} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}$$

Let us abbreviate (A.2) as

$$\psi_{\frac{1}{2}m} = \begin{pmatrix} \Psi_{D,2}^{(1)} \\ \Psi_{D,2}^{(2)} \end{pmatrix} = \xi^{(+)} \otimes \Psi_{D,2}^{(1)} + \xi^{(-)} \otimes \Psi_{D,2}^{(2)} \quad (\text{A.6})$$

where the lower index 2 on  $\Psi$  symbolizes a 2-spinor.

Applying (32a) on (A.6) we obtain the corresponding  $S$  wave function.

Introducing explicitly  $c$ ,  $\hbar$  and using  $i\sigma_y\xi^{(\sigma)} = -\sigma\xi^{(-\sigma)}$ , we obtain

$$S\psi_{\frac{1}{2}m} = \psi_{\frac{1}{2}m}^S = \sqrt{\frac{\Omega + m_0c^2}{2\Omega}} \begin{pmatrix} \Psi_{D,2}^{(1)} + \frac{c\vec{\sigma} \cdot (-i\hbar\vec{\partial})}{\Omega + m_0c^2} \Psi_{D,2}^{(2)} \\ \Psi_{D,2}^{(2)} - \frac{c\vec{\sigma} \cdot (-i\hbar\vec{\partial})}{\Omega + m_0c^2} \Psi_{D,2}^{(1)} \end{pmatrix} = \begin{pmatrix} \Psi_{S,2}^{(+)} \\ \Psi_{S,2}^{(-)} \end{pmatrix} \quad (\text{A.7})$$

We are interested in the positron admixture, *i.e.* the lower component. Using

$$\vec{\partial}r = \frac{\vec{r}}{r}, \quad \sigma_x\chi^m = \chi^{-m}, \quad \sigma_y\chi^m = 2im\chi^{-m}, \quad \sigma_z\chi^m = 2m\chi^m \text{ for } m = \pm\frac{1}{2}$$

$$\text{and } Y_1^1 = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}, \quad Y_1^{-1} = -\sqrt{\frac{3}{8\pi}} \frac{x-iy}{r}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

we find

$$c\vec{\sigma} \cdot (-i\hbar\vec{\partial}) \left( ig(r)\Omega_{\frac{1}{2},0m}(\hat{r}) \right) = -\hbar cg'(r)\Omega_{\frac{1}{2},1m}(\hat{r}), \quad m = \pm\frac{1}{2} \quad (\text{A.8})$$

hence, expressing  $-f(r)$  and  $g'(r)$  by  $g(r)$ ,

$$\Psi_{S,2}^{(-)} = \sqrt{\frac{1-\lambda}{1+\gamma}} \sqrt{\frac{\Omega + m_0c^2}{2\Omega}} \left( 1 - \frac{\alpha}{\Omega + m_0c^2} \left( \frac{\hbar c}{r} + \sqrt{\frac{1+\lambda}{1-\gamma}} m_0c^2 \right) \right) \left( g(r)\Omega_{\frac{1}{2},1m}(\hat{r}) \right) \quad (\text{A.9})$$

Note that the Operator  $\frac{\hbar c}{\Omega + m_0c^2}$  in front of the inner bracket also acts on  $g(r)\Omega_{\frac{1}{2},1m}(\hat{r})$ .  $\Psi_{S,2}^{(-)}$  from (A.9) certainly does not vanish identically, as required for a pure charge eigenfunction.

In general applying the nonlocal operator  $\Omega$  involves a non-elementary integral.

However, we can possibly gain a rough insight ("in zeroeth order") by replacing the operator  $\Omega = \sqrt{-\hbar^2c^2\Delta + m_0^2c^4}$  with the rest energy  $m_0c^2$  of the particle. In this approximation we find

$$\Psi_{S,2}^{(-)} \approx \sqrt{\frac{1-\lambda}{1+\gamma}} \left( 1 - \frac{\alpha}{2} \left( \frac{\lambda_c}{r} + \sqrt{\frac{1+\lambda}{1-\gamma}} \right) \right) \left( g(r)\Omega_{\frac{1}{2},1m}(\hat{r}) \right) \quad (\text{A.10})$$

where  $\lambda_c$  is the Compton wavelength of the particle.

Expanding (A.10) in powers of  $\alpha$ , we find

$$\Psi_{S,2}^{\frac{1}{2}1m}(\hat{r}) + \text{higher orders} \quad (\text{A.11})$$

*i.e.* the wave function (A.2) includes a probability

$$P(-) = \int dx^3 \left| \Psi_{S,2}^{(-)} \right|^2 \quad (\text{A.12})$$

in the order of  $\alpha^4 \approx 10^{-9}$  to find the particle as a positron.

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Gratefully dedicated to my wife Inge, and to my children.

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