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Geodesics of Deformed Relativity in Five Dimensions

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Abstract: In a previous paper, we discussed the Killing symmetries of the Kaluza-Klein-like scheme known as Deformed Relativity in five dimensions (DR5), based on a five-dimensional Riemannian space $\Re_5$ in which the four-dimensional space-time metric is deformed (*i.e.* it depends on the energy) and energy plays the role of the fifth dimension. In the present paper, we carry on the investigation of the main mathematical aspects of DR5 by studying the geodesic motions in $\Re_5$. In particular, we consider the case of physical relevance in which the metric coefficients are power functions of the energy (Power Ansatz). The geodesic equations are solved explicitly for all the twelve 5-d. metrics obtained as solutions of the vacuum Einstein equations, and in particular for those describing the four fundamental interactions. It is also shown that it is possible, from the geodesic motion related to one of these Power-Ansatz solutions, to get a time-energy uncertainty relation of the Heisenberg type.

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1. Introduction

In the last twenty years, there has been a growing interest for theories with noncompactified extra dimensions [1-5] as possible candidates for unification of all fundamental forces including gravity. Both theories with space [4] and non-space (*e.g.* mass [3]and charge [5]) dimensions have been considered. Among the latter ones, there is Five-Dimensional Deformed Relativity (DR5), introduced by two of the present authors (F.C. and R.M.),

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together with M. Francaviglia [6-8]. DR5 is a theoretical scheme *a la* Kaluza-Klein (KK) [9], based on a five-dimensional Riemannian space $\Re_5$ with energy as (noncompactified) extra dimension. It represents a covering of Deformed Special Relativity (DSR), which is in turn a generalization of Special Relativity (better, of Lorentzian Relativity) in a deformed Minkowski space-time $\widetilde{M}$ with metric coefficients depending on the energy [8]. DSR allows one, among the others, to give a geometrical description of the four fundamental interactions. We investigated in a previous paper the Killing symmetries of DR5 [10]. In this paper, we carry on the study of the mathematical properties of such a formalism, by discussing the geodesic motions in $\Re_5$.

2. Five-Dimensional Deformed Relativity: A Survey

2.1 The 5-dimensional Space-Time-Energy Manifold $\Re_5$

Deformed Relativity in Five Dimensions is a Kaluza-Klein-like theory based on a 5-dimensional space-time-energy manifold $\Re_5$ endowed with the energy-dependent metric[2] [6-8]:

$$g_{AB,DR5}(E) \equiv diag(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E), f(E)) \overset{\text{ESC off}}{=}$$

$$= \delta_{AB}\left(b_0^2(E)\delta_{A0} - b_1^2(E)\delta_{A1} - b_2^2(E)\delta_{A2} - b_3^2(E)\delta_{A3} + f(E)\delta_{A5}\right), \tag{1}$$

with infinitesimal interval

$$\begin{gathered} ds^2_{DR5}(E) \equiv dS^2(E) \equiv g_{AB,DR5}(E)dx^A dx^B = \\ = b_0^2(E)\left(dx^0\right)^2 - b_1^2(E)\left(dx^1\right)^2 - b_2^2(E)\left(dx^2\right)^2 - b_3^2(E)\left(dx^3\right)^2 + f(E)\left(dx^5\right)^2 = \\ = b_0^2(E)c^2\left(dt\right)^2 - b_1^2(E)\left(dx^1\right)^2 - b_2^2(E)\left(dx^2\right)^2 - b_3^2(E)\left(dx^3\right)^2 + f(E)l_0^2\left(dE\right)^2, \end{gathered} \tag{2}$$

where we have put

$$x^5 \equiv l_0 E \ , \ l_0 > 0. \tag{3}$$

The energy dimension, being a physical — not space — one, is not compactified.

The Riemannian space $\Re_5$ can be considered a covering of the deformed Minkowski space-time $\widetilde{M}(E) = \widetilde{M}(x^5)$ on which Deformed Special Relativity (DSR) [8] is based. We recall that $\widetilde{M}(x^5, g_{DSR})$ is the same vector space on the real field as the Minkowski space

[2] In the following, (lower) Greek and Latin indices label space-time ($\mu = 0, 1, 2, 3$) and space coordinates ($i = 1, 2, 3$), respectively, whereas capital Latin indices take values in the range $\{0, 1, 2, 3, 5\}$, with index 5 labelling the fifth dimension. We choose to label by 5 the extra coordinate, instead of using 4, in order to avoid confusion with the notation often adopted for the (imaginary) time coordinate in a (formally) Euclidean Minkowski space. Moreover, we shall employ the notation "$ESCon$" ("$ESCoff$") to mean that the Einstein sum convention on repeated indices is (is not) used.

M, but with metric g_{DSR} given by

$$g_{DSR}(E) = diag(b_0^2(E), -b_1^2(E), -b_2^2(E)\ , -b_3^2(E))$$

$$\overset{\text{ESC off}}{=} \delta_{\mu\nu}\left[b_0^2(E)\delta_{\mu 0} - b_1^2(E)\delta_{\mu 1} - b_2^2(E)\delta_{\mu 2} - b_3^2(E)\delta_{\mu 3}\right], \tag{4}$$

where the metric coefficients $\{b_\mu^2(E)\ \}(\mu = 0,1,2,3)$ are (dimensionless) positive functions of the energy E of the process considered[3]: $b_\mu^2 = b_\mu^2(E)$. The generalized interval in $\widetilde{M}$ reads therefore

$$\begin{aligned} ds^2 &= b_0^2(E)c^2dt^2 - b_1^2(E)dx^2 - b_2^2(E)dy^2 - b_3^2(E)dz^2 = \\ &= g_{\mu\nu,DSR}dx^\mu dx^\nu = dx * dx \end{aligned} \tag{5}$$

with $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, c being the usual light speed in vacuum. We refer the reader to ref.[8] for a detailed discussion of the physical and mathematical properties of $\widetilde{M}(E)$ and DSR.

One can therefore state that $\Re_5$ has the following *"slicing property"*

$$\Re_5|_{dx^5=0 \Leftrightarrow x^5=\overline{x^5}} = \widetilde{M}(\overline{x^5}) = \left\{\widetilde{M}(x^5)\right\}_{x^5=\overline{x^5}} \tag{6}$$

(where $\overline{x^5}$ is a fixed value of the fifth coordinate) or, at the level of the metric tensor:

$$g_{AB,DR5}(x^5)\Big|_{dx^5=0 \Leftrightarrow x^5=\overline{x^5}\in R_0^+} =$$

$$= diag\left(b_0^2(\overline{x^5}), -b_1^2(\overline{x^5}), -b_2^2(\overline{x^5}), -b_3^2(\overline{x^5}), \pm f(\overline{x^5})\right) = g_{AB,DSR}(\overline{x^5}). \tag{7}$$

Some aspects of the Riemannian structure of $\Re_5$, including the vacuum Einstein equations, are given in App.A.

2.2 5-d. Metrics of Fundamental Interactions

2.2.1 Phenomenological Metrics

The 4-d. phenomenological metrics of the deformed Minkowski spaces $\widetilde{M}(x^5)$ which, in the formalism of DSR, describe the four fundamental interactions (electromagnetic, weak, strong and gravitational) [8] (see App.B) can be embedded in the 5-d. Riemann space

[3] Quantity E is to be understood as the energy measured by the detectors via their electromagnetic interaction in the usual Minkowski space.

$\Re_5$ as follows ($f(x^5) \in R_0^+ \ \forall x^5 \in R_0^+$):

$$g_{AB,DR5,e.m.}(x^5) =$$
$$= diag\left(1, -\left\{1 + \widehat{\Theta}(x^5_{0,e.m.} - x^5)\left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\},\right.$$
$$-\left\{1 + \widehat{\Theta}(x^5_{0,e.m.} - x^5)\left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\},$$
$$\left.-\left\{1 + \widehat{\Theta}(x^5_{0,e.m.} - x^5)\left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\}, \pm f(x^5)\right); \tag{8}$$

$$g_{AB,DR5,weak}(x^5) =$$
$$= diag\left(1, -\left\{1 + \widehat{\Theta}(x^5_{0,weak} - x^5)\left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\},\right.$$
$$-\left\{1 + \widehat{\Theta}(x^5_{0,weak} - x^5)\left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\},$$
$$\left.-\left\{1 + \widehat{\Theta}(x^5_{0,weak} - x^5)\left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\}, \pm f(x^5)\right); \tag{9}$$

$$g_{AB,DR5,strong}(x^5) =$$
$$= diag\left(1 + \widehat{\Theta}(x^5 - x^5_{0,strong})\left[\left(\frac{x^5}{x^5_{0,strong}}\right)^{2} - 1\right], -\left(\frac{\sqrt{2}}{5}\right)^2,\right.$$
$$\left.-\left(\frac{2}{5}\right)^2, -\left\{1 + \widehat{\Theta}(x^5 - x^5_{0,strong})\left[\left(\frac{x^5}{x^5_{0,strong}}\right)^{2} - 1\right]\right\}, \pm f(x^5)\right); \tag{10}$$

$$g_{AB,DR5,grav.}(x^5) =$$
$$= diag\left(1 + \widehat{\Theta}(x^5 - x^5_{0,grav.})\left[\frac{1}{4}\left(1 + \frac{x^5}{x^5_{0,grav.}}\right)^2 - 1\right], -b^2_{1,grav.}(x^5),\right.$$
$$\left.-b^2_{2,grav.}(x^5), -\left\{1 + \widehat{\Theta}(x^5 - x^5_{0,grav.})\left[\frac{1}{4}\left(1 + \frac{x^5}{x^5_{0,grav.}}\right)^2 - 1\right]\right\}, \pm f(x^5)\right). \tag{11}$$

It is possible to show that all the above metrics — derived on a mere phenomenological basis, from the experimental data on some physical phenomena ruled by the four fundamental interactions, at least as far as their space-time part is concerned — can be recovered as solutions of the vacuum Einstein equations in the five-dimensional space $\Re_5$, natural covering of the deformed Minkowski space $\widetilde{M}(x^5)$ [6-8].

2.2.2 Power Ansatz

Since the space-time metric coefficients are dimensionless, it can be assumed that they are functions of the ratio E/E_0, where E_0 is an energy scale characteristic of the interaction (and the process) considered (for instance, the energy threshold in the phenomenological metrics of App.B). The coefficients of the metric $g_{AB,DR5}(E)$ can be therefore expressed as

$$\left\{b_\mu\left(\frac{E}{E_0}\right)\right\} \equiv \left\{b_\mu\left(\frac{x^5}{x_0^5}\right)\right\} = \left\{b_\mu(x^5)\right\}, \forall\mu = 0,1,2,3;$$
$$f\left(\frac{E}{E_0}\right) = f\left(\frac{x^5}{x_0^5}\right) = f(x^5), \tag{12}$$

where we put

$$x_0^5 \equiv l_0 E_0 \ . \tag{13}$$

In particular, following also the hints from phenomenology, it is possible to assume that the metric coefficients are powers of the energy, namely to put

$$g_{AB,DR5power}(\widetilde{\mathbf{q}}, x^5) =$$
$$= diag\left(\left(\frac{x^5}{x_0^5}\right)^{q_0}, -\left(\frac{x^5}{x_0^5}\right)^{q_1}, -\left(\frac{x^5}{x_0^5}\right)^{q_2}, -\left(\frac{x^5}{x_0^5}\right)^{q_3}, \pm\left(\frac{x^5}{x_0^5}\right)^{r}\right) \tag{14}$$

($q_0, q_1, q_2, q_3, r \in Q$, $A, B = 0,1,2,3,5$), in which the double sign of the energy coefficient has been made clear and the (fake) 5-vector $\widetilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$ introduced[4]. In the following we shall refer to the form (14) as the *"Power Ansatz"*.

In the Power Ansatz, the 5-d. phenomenological metrics for the four interactions,

[4] In the following, we shall use the tilded-bold notation $\widetilde{\mathbf{v}}$ for a (true or fake) vector in $\Re_5$, in order to distinguish it from a vector $\mathbf{v}$ in the usual 3-d.space.

Eqs.(8)-(11), can be written in the form

$$g_{AB,DR5,e.m.,weak}(x^5) =$$

$$= \begin{cases} diag\begin{pmatrix} 1, -\left(\dfrac{x^5}{x^5_{0,e.m.,weak}}\right)^{1/3}, -\left(\dfrac{x^5}{x^5_{0,e.m.,weak}}\right)^{1/3}, \\ -\left(\dfrac{x^5}{x^5_{0,e.m.,weak}}\right)^{1/3}, \pm\left(\dfrac{x^5}{x^5_{0,e.m.,weak}}\right)^{r} \end{pmatrix}, \\ 0 < x^5 < x^5_{0,e.m.,weak}; \\ diag\left(1, -1, -1, -1, \pm\left(\dfrac{x^5}{x^5_{0,e.m.,weak}}\right)^{r}\right), \\ x^5 \geq x^5_{0,e.m.,weak}; \end{cases} \tag{15}$$

$$g_{AB,DR5,strong}(x^5) =$$

$$= \begin{cases} diag\begin{pmatrix} \left(\dfrac{x^5}{x^5_{0,strong}}\right)^2, -\left(\dfrac{\sqrt{2}}{5}\right)^2, -\left(\dfrac{2}{5}\right)^2, \\ -\left(\dfrac{x^5}{x^5_{0,strong}}\right)^2, \pm\left(\dfrac{x^5}{x^5_{0,strong}}\right)^{r} \end{pmatrix}, \\ x^5 > x^5_{0,strong}; \\ diag\left(1, -\left(\dfrac{\sqrt{2}}{5}\right)^2, -\left(\dfrac{2}{5}\right)^2, -1, \pm\left(\dfrac{x^5}{x^5_{0,strong}}\right)^{r}\right), \\ 0 < x^5 \leq x^5_{0,strong}; \end{cases} \tag{16}$$

$$g_{AB,DR5,grav.}(x^5) =$$
$$= \begin{cases} diag\begin{pmatrix} \frac{1}{4}\left(1+\frac{x^5}{x^5_{0,grav.}}\right)^2, -b^2_{1,grav.}(x^5), -b^2_{2,grav.}(x^5), \\ \frac{1}{4}\left(1+\frac{x^5}{x^5_{0,grav.}}\right)^2, \pm\left(\frac{x^5}{x^5_{0,grav.}}\right)^r \end{pmatrix}, \\ \qquad x^5 > x^5_{0,grav.}; \\ diag\left(1, -b^2_{1,grav.}(x^5), -b^2_{2,grav.}(x^5), -1, \pm\left(\frac{x^5}{x^5_{0,grav.}}\right)^r\right), \\ \qquad 0 < x^5 \leq x^5_{0,grav.}. \end{cases} \tag{17}$$

where their piecewise structure has been made explicit. In the gravitational metric $g_{AB,DR5,grav.}(x^5)$ the expressions of the two space coefficients $b^2_{1,grav.}(x^5)$ and $b^2_{2,grav.}(x^5)$ have not been specified, due to their indeterminacy at experimental level.

The phenomenological 5-d. metrics in the Power Ansatz are therefore characterized by the parameter sets

$$\widetilde{\mathbf{q}}_{e.m./weak} = \begin{cases} \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, r\right), & 0 < x^5 < x^5_{0,e.m./weak}; \\ \left(0, 0, 0, 0, r\right), & x^5 \geq x^5_{0,e.m./weak}; \end{cases} ; \tag{18}$$

$$\widetilde{\mathbf{q}}_{strong} = \begin{cases} \left(2, (0,0), 2, r\right), & x^5 > x^5_{0,strong}; \\ \left(0, (0,0), 0, r\right), & 0 < x^5 \leq x^5_{0,strong}; \end{cases} ; \tag{19}$$

$$\widetilde{\mathbf{q}}_{grav.} = \begin{cases} \left(\underline{2}, ?, ?, \underline{2}, r\right), & x^5 > x^5_{0,grav.}; \\ \left(0, ?, ?, 0, r\right), & 0 < x^5 \leq x^5_{0,grav.} \end{cases} , \tag{20}$$

where the question marks "?" reflect the unknown nature of the two gravitational spatial coefficients.

Let us clarify the notation adopted for $\widetilde{\mathbf{q}}_{strong}$ and $\widetilde{\mathbf{q}}_{grav.}$. The zeros in brackets in $\widetilde{\mathbf{q}}_{strong}$ reflect the fact that such exponents do not refer to the metric tensor $g_{AB,DR5power}(x^5)$ (Eq.(14)), but to the more general tensor

$$g_{AB,DR5power-conform}(x^5) =$$
$$= diag\left(\vartheta_0\left(\frac{x^5}{x^5_0}\right)^{q_0}, -\vartheta_1\left(\frac{x^5}{x^5_0}\right)^{q_1}, -\vartheta_2\left(\frac{x^5}{x^5_0}\right)^{q_2}, -\vartheta_3\left(\frac{x^5}{x^5_0}\right)^{q_3}, \pm\vartheta_5\left(\frac{x^5}{x^5_0}\right)^{r}\right) , \tag{21}$$

with $\widetilde{\boldsymbol{\vartheta}} = (\vartheta_A)$ being a constant 5-vector. Eq.(21) can be written in matrix form as

$$\mathbf{g}_{DR5power-conform}(x^5) = \mathbf{g}_{DR5power}(x^5)\widetilde{\boldsymbol{\vartheta}} \tag{22}$$

where $\widetilde{\vartheta}$ is meant to be a column vector. The passage from the metric tensor $g_{AB,DR5power}(x^5)$ to $g_{AB,DR5power-conform}(x^5)$ is obtained by means of the tensor transformation law in $\Re_5$ (ESC on)

$$g_{AB,DR5power-conform}(x^5) = \frac{\partial x^K}{\partial x'^A}\frac{\partial x^L}{\partial x'^B} g_{KL,DR5power}(x^5) \tag{23}$$

induced by the following *5-d. anisotropic rescaling* of the coordinates of $\Re_5$:

$$dx^A = \sqrt{\vartheta_A} dx'_A \leftrightarrow x^A = \sqrt{\vartheta_A} x'_A. \tag{24}$$

Such a transformation allows one to get, in the Power Ansatz, metric coefficients *constant* (i.e. independent of the energy) *but different*. This is just the case of the two constant space coefficients $b_1^2(x^5)$, $b_2^2(x^5)$ in the strong metric. In this case, the vector $\widetilde{\boldsymbol{\vartheta}}$ explicitly reads

$$\widetilde{\boldsymbol{\vartheta}}_{strong} = \left(0, \left(\frac{\sqrt{2}}{5}\right)^2, \left(\frac{2}{5}\right)^2, 0, ?\right), \tag{25}$$

where the question mark "?" reflects again the unknown nature of $\pm f(x^5)$.

The underlined 2, $\underline{2}$, in $\widetilde{\mathbf{q}}_{grav.}$ are due to the fact that actually the functional form of the related metric coefficients is not $\left(\frac{x^5}{x^5_{0,grav.}}\right)^2$ but $\frac{1}{4}\left(1+\frac{x^5}{x^5_{0,grav.}}\right)^2$. Again, it is possible to recover the phenomenological 5-d. metric $g_{AB,DR5,grav.}(x^5)$ from the Power Ansatz form $g_{AB,DR5power}(x^5)$ by a rescaling and a translation of the energy. In fact, one has

$$x^5 \longrightarrow x^{5\prime} = x^5 - \overline{x_0^5} \Leftrightarrow dx^{5\prime} = dx^5. \tag{26}$$

Such a translation in energy is allowed because we are just working in the framework of DR5. Therefore

$$b_0^2(x^5) = b_0^2(x^{5\prime} + \overline{x_0^5}) = b^2_{0,new}\left(x^{5\prime}\right) = \left(\frac{x^{5\prime} + \overline{x_0^5}}{x_0^5}\right)^2 = \left(\frac{x^{5\prime}}{x_0^5} + \frac{\overline{x_0^5}}{x_0^5}\right)^2. \tag{27}$$

By rescaling the threshold energy (in a physically consistent way, because it amounts to a redefinition of the scale of measure of energy)

$$x_0^5 \longrightarrow x_0^{5\prime} = x_0^5\left(\frac{\widetilde{x_0^5}}{\overline{x_0^5}}\right), \tag{28}$$

one gets

$$b^2_{0,new}\left(x^{5\prime}\right) = \left(\frac{x^{5\prime}}{x_0^{5\prime}}\frac{\widetilde{x_0^5}}{\overline{x_0^5}} + \frac{\widetilde{x_0^5}}{x_0^{5\prime}}\right)^2 = \left(\frac{\widetilde{x_0^5}}{x_0^{5\prime}}\right)^2\left(1 + \frac{x^{5\prime}}{\overline{x_0^5}}\right)^2. \tag{29}$$

This metric time coefficient is of the gravitational type, except for the factor $\left(\frac{\widetilde{x_0^5}}{x_0^{5\prime}}\right)^2$, which however can be got rid of by the following rescaling of the time coordinate:

$$x^0 \longrightarrow x'^0 = \frac{\widetilde{x_0^5}}{x_0^{5\prime}} x^0. \tag{30}$$

This is a conformal transformation corresponding to a redefinition of the scale of measure of time. Notice that the above rescaling procedure of energy and time does not account for the factor 1/4 in front of $\left(1+\frac{x^5}{x^5_{0,grav.}}\right)^2$. This can be dealt with by the method followed for $\widetilde{\mathbf{q}}_{strong}$, namely by considering the generalized metric $g_{AB,DR5power-conform}(x^5)$, where now the vector $\widetilde{\boldsymbol{\vartheta}}$ is given by $\widetilde{\vartheta}=\left(\frac{1}{4},?,?,\frac{1}{4},?\right)$ (as before, the question marks reflect the unknown nature of the related metric coefficients).

Notice that both in Eqs.(15)-(17) and in Eqs.(18)-(20) it was *assumed* that

$$q_{\mu.int.}(x^5_{0,int.})=0,\quad \mu=0,1,2,3\ ,\ int.=e.m.,weak,strong,grav., \tag{31}$$

for simplicity reasons, since in general nothing can be said on the behavior of the metrics at the energy thresholds.

Let us introduce the *left and right specifications* $\widehat{\Theta}_L(x)$*,* $\widehat{\Theta}_R(x)$ *of the Heaviside theta function*, defined respectively by

$$\widehat{\Theta}_L(x)\equiv\begin{cases}1, x>0\\ 0, x\leq 0\end{cases}, \tag{32}$$

$$\widehat{\Theta}_R(x)\equiv\begin{cases}1, x\geq 0\\ 0, x<0\end{cases}, \tag{33}$$

and satisfying *the complementarity relation*

$$1-\widehat{\Theta}_R(x)=\widehat{\Theta}_L(x)\,. \tag{34}$$

Then, the exponent sets (18)-(20) can be written in compact form as

$$\begin{aligned}\widetilde{\mathbf{q}}_{e.m./weak}=&\left(0,\frac{1}{3}\widehat{\Theta}_L\left(x^5_{0,e.m./weak}-x^5\right),\right.\\ &\left.\frac{1}{3}\widehat{\Theta}_L\left(x^5_{0,e.m./weak}-x^5\right),\frac{1}{3}\widehat{\Theta}_L\left(x^5_{0,e.m./weak}-x^5\right),r\right),\end{aligned} \tag{35}$$

$$\widetilde{\mathbf{q}}_{strong}=\left(2\widehat{\Theta}_L\left(x^5-x^5_{0,strong}\right),(0,0),2\widehat{\Theta}_L\left(x^5-x^5_{0,strong}\right),r\right), \tag{36}$$

$$\widetilde{\mathbf{q}}_{grav.}=\left(2\underline{\widehat{\Theta}_L}\left(x^5-x^5_{0,grav.}\right),?,?,2\underline{\widehat{\Theta}_L}\left(x^5-x^5_{0,grav.}\right),r\right), \tag{37}$$

where the underlining and the question marks have the same meaning as above.

2.2.3 Solutions of Einstein's Equations in the Power Ansatz

In the Power Ansatz, the vacuum Einstein equations in $\Re_5$, Eqs.(A.1), for $\Lambda_{(5)} = 0$ reduce to the following algebraic equations in the five exponents q_0, q_1, q_2, q_3, r (cfr. Eq.(14)):

$$\begin{cases} (2+r)(q_3+q_1+q_2) - q_1^2 - q_2^2 - q_3^2 - q_1q_2 - q_1q_3 - q_2q_3 = 0; \\ (2+r)(q_3+q_0+q_2) - q_2^2 - q_3^2 - q_0^2 - q_2q_3 - q_2q_0 - q_3q_0 = 0; \\ (2+r)(q_3+q_0+q_1) - q_1^2 - q_3^2 - q_0^2 - q_1q_3 - q_1q_0 - q_3q_0 = 0; \\ (2+r)(q_0+q_1+q_2) - q_1^2 - q_2^2 - q_0^2 - q_1q_2 - q_1q_0 - q_2q_0 = 0; \\ q_1q_2 + q_1q_3 + q_1q_0 + q_2q_3 + q_2q_0 + q_3q_0 = 0. \end{cases} \tag{38}$$

It can be shown [6-8] that they admit twelve possible classes of solutions, which can be classified according to the values of the five-dimensional set $\widetilde{\mathbf{q}} \equiv (q_0, q_1, q_2, q_3, r)$. Explicitly one has:

- **Class (I):** $\widetilde{\mathbf{q}}_I = \left(q_0 = q_2 = n, q_1 = -n\left(\dfrac{2p+n}{2n+p}\right), n, q_3 = p, r_I\right)$, with
$$r_I = \frac{p^2 - 2p + 2np - 4n + 3n^2}{2n+p};$$
- **Class (II):** $\widetilde{\mathbf{q}}_{II} = (0, q_1 = m, 0, 0, r = m-2)$;
- **Class (III):** $\widetilde{\mathbf{q}}_{III} = (q_0 = q_2 = q_3 = n, q_1 = -q_2 = -n, n, n, r = -2(1-n))$;
- **Class (IV):** $\widetilde{\mathbf{q}}_{IV} = (0, 0, 0, q_3 = p, r = p-2)$;
- **Class (V):** $\widetilde{\mathbf{q}}_V = (q_0 = q_1 = q_2 = -q_3 = -p, -p, -p, p, r = -(1+p))$;
- **Class (VI):** $\widetilde{\mathbf{q}}_{VI} = (q_0 = q, 0, 0, 0, r = q-2)$;
- **Class (VII):** $\widetilde{\mathbf{q}}_{VII} = (q_0 = q, -q, -q, -q, r = -q-2)$;
- **Class (VIII):** $\widetilde{\mathbf{q}}_{VIII} = (0, 0, 0, 0, r)$;
- **Class (IX):** $\widetilde{\mathbf{q}}_{IX} = (0, 0, q_2 = n, 0, r = n-2)$;
- **Class (X):** $\widetilde{\mathbf{q}}_X = \left(q_0 = q, q_1 = -\dfrac{pq+np+nq}{n+p+q}, q_2 = n, q_3 = p, r_X\right)$**,** with
$$r_X = \frac{(n+p+q)(n+p+q-2) - (pq+np+nq)}{n+p+q} = (n+p+q-2) + q_1;$$

- **Class (XI):** $\widetilde{\mathbf{q}}_{XI} = \left(q_0 = q, q_1 = -\dfrac{n(2q+n)}{2n+q}, q_2 = n, q_3 = q_2 = n, r_{XI}\right)$, with

$$r_{XI} = \frac{3n^2 - 4n + 2nq - 2q + q^2}{2n+q};$$

- **Class (XII):** $\widetilde{\mathbf{q}}_{XII} = \left(q_0 = q, q_1 = q_2 = n, n, q_3 = -\dfrac{n(2q+n)}{2n+q}, r_{XII}\right)$, with

$$r_{XII} = \frac{p^2 + pq - 2p + np - 2n + nq + n^2 - 2q + q^2}{n+p+q}.$$

Moreover, it was shown that it is possible to recover, as special cases of such classes of solutions, *all* the 5-d. phenomenological, deformed metrics for the four interactions given in Subsubsect.2.2.1. We refer the reader to refs.[6-8] for a detailed discussion.

3. Geodesic Motions in $\Re_5$

It is well known that, in a Riemann space, the dynamic laws are actually geometrical laws. As familiar from General Relativity, the motion of a body in a gravitational field is described by the geodesic equations, which are in turn related to the affine and metric properties of the Riemann space-time. Analogously, we expect that, in the framework of DR5, the *local* dynamics of particles ruled by the four fundamental interactions — described by the 5-d embeddings of the DSR phenomenological metrics: see Subsubsect.2.2.1 — is embodied in the 5-dimensional geodesic equations in $\Re_5$.

3.1 Proper time and geodesics in DSR

This has to be compared with the case of Deformed Special Relativity, where — as in any special-relativistic theory — the geodesics equations are trivially given by

$$\frac{d^2 x^\mu(\tau_{DSR})}{d\tau_{DSR}^2} = 0. \tag{39}$$

Here τ_{DSR} is the proper time in DSR, defined, in analogy with the SR case, by

$$(d\tau_{DSR}(E))^2 = \frac{1}{c^2}(ds_{DSR}(E))^2 =$$
$$= \frac{1}{c^2}\left[b_0^2(E)c^2dt^2 - b_1^2(E)\left(dx^1\right)^2 - b_2^2(E)\left(dx^2\right)^2 - b_3^2(E)\left(dx^3\right)^2\right]. \tag{40}$$

Since $d\tau$ is an invariant, it is possible to choose a suitable inertial frame in order to simplify its expression. As is well known, in SR one takes the rest frame of the particle, *i.e.* $dx^1 = dx^2 = dx^3 = 0$ (*SR natural frame*). In DSR, the natural frame corresponds to the frame where the particle is at rest *with fixed energy* $\overline{E}$, namely with $dx^1 = dx^2 = dx^3 = dE = 0$:

$$\left(x^0, x^1, x^2, x^3\right)_{DSR,nat} = \left(x^0, \overline{x^1}, \overline{x^2}, \overline{x^3}\right)_{E=\overline{E}} \tag{41}$$

(*DSR natural frame*). Then

$$\left(d\tau_{DSR}(\overline{E})\right)^2 = \frac{1}{c^2}\left(ds_{DSR}(\overline{E})\right)^2\Big|_{nat} = b_0^2(\overline{E}))dt_0^2. \tag{42}$$

Finally, omitting the dependence on $\overline{E}$, one gets for the infinitesimal proper time in DSR

$$d\tau_{DSR} = b_0 dt_0 \tag{43}$$

or, in finite form [8]:

$$\tau_{DSR} = b_0 t_0. \tag{44}$$

Such relations are analogous to those between proper time and coordinate time found in General Relativity ($d\tau_{GR} = \sqrt{g_{00}}dt$)[5]. In the DSR case, too, the proper time *does not coincide* with the time measured by the local observer in the particle frame. Like in GR, therefore, one has to distinguish between the *real (proper) time* τ and the coordinate (or *universe*) time t[6].

As for SR, the geodesic equations (39) in the deformed Minkowski space $\widetilde{M}$ yield solutions which are straight word-lines:

$$x^\mu(\tau_{DSR}) = \alpha_{\mu 1}\tau_{DSR} + \alpha_{\mu 2} \tag{45}$$

($\mu = 0, 1, 2, 3$).

On the contrary, we shall see that the geodesic equations in $\Re_5$ do possess a non-trivial structure, corresponding to an *extrinsic dynamic behavior*[7].

3.2 Proper Time in DR5

The proper time in DR5 can be found by a procedure analogous to that followed in Special Relativity and in DSR. By definition, it is

$$\begin{aligned}\left(d\tau_{DR5}(x^5)\right)^2 &= \frac{1}{c^2}\left(ds_{DR5}(x^5)\right)^2 = \frac{1}{c^2}\left(dS(x^5)\right)^2 = \\ &= \frac{1}{c^2}\left[b_0^2(x^5)c^2dt^2 - \sum_i b_i^2(x^5)\left(dx^i\right)^2 \pm f(x^5)\left(dx^5\right)^2\right].\end{aligned} \tag{46}$$

As in DSR, the DR5 natural frame is the frame where the particle is at rest with fixed energy[8] [8], namely

$$\left(x^0, x^1, x^2, x^3, x^5\right)_{DR5,nat} = \left(x^0, \overline{x^1}, \overline{x^2}, \overline{x^3}, \overline{x^5}\right). \tag{47}$$

5 Indeed, since $g_{DSR00} = b_0^2$, the general-relativistic relation for dτ becomes exactly the DSR relation (43).

6 As is well known, such a distinction is fundamental, within GR, for the analysis of gravitational phenomena (like gravitational collapse). Something analogous occurs in the DSR framework for interactions described by asynchronous metrics (like the strong one: see App.A), for which this fact may have deep physical implications. We refer the reader to refs.[8] for more details.

7 Actually, it can be shown that *the non-trivial dynamics of DSR is related to the intrinsic geometrical structure of the deformed Minkowski space $\widetilde{M}$ as Generalized Lagrange space* (see second ref.[8]).

8 However, let us notice that, in the DR5 framework, the natural frame is in general non-inertial, due to the Riemannian structure of $\Re_5$.

In other words, due to the slicing structure of $\Re_5$ (cfr. Eq.(6)), the natural frame in DR5 is the five-dimensional, *local* (since the fifth metric coordinate is fixed) generalization of the four-dimensional, *global* inertial, natural frame of DSR.

Then

$$\begin{aligned}\left(d\tau_{DR5}(\overline{x^5})\right)^2 &= \frac{1}{c^2}\left(dS(\overline{x^5})\right)^2\Big|_{nat} = b_0^2(\overline{x^5})dt^2 \Longrightarrow \\ &\Longrightarrow d\tau_{DR5} = b_0 dt,\end{aligned} \tag{48}$$

or, in finite form:

$$\tau_{DR5}(\overline{x^5}) = b_0(\overline{x^5})t \tag{49}$$

(where t is the coordinate time in the natural frame (47)). One gets therefore the same result of DSR, as expected on physical and mathematical grounds, on account of the embedding $\widetilde{M}(x^5) \subset \Re_5$.

3.3 Geodesic Equations

Let us now consider the geodesics in the five-dimensional space-time-energy Riemannian space $\Re_5$, in order to clarify their possible physical meaning (see ref.[11] for a thorough discussion of the geodesic equation of motion in a general Kaluza-Klein model).

The geodesic equations are

$$\frac{d^2x^A}{d\tau^2} + \Gamma^A_{BC}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} = 0, \tag{50}$$

where, for massive particles, $\tau = \tau_{DR5}$ is the proper time in $\Re_5$ (or another affine parameter — not necessarily invariant — for massless particles[9]). The compatibility condition of the definition (49) of τ_{DR5} with the geodesic equations (50) in $\Re_5$ is straightforward (the components of the connection Γ^A_{BC} vanish for $x^5 = \overline{x^5}$: see Eqs.(B.4)).

On account of the explicit form (B.4) of the affine connection, one gets the following

[9] Indeed, let us recall that, in a Riemannian space, the geodesic equations — although formally identical in any reference frame — actually do depend on the chosen frame, due to the non-tensor nature of the affine connection Γ^A_{BC}.
For instance, it is possible to parametrize the space-time trajectory of a massless particle in terms of x^0, determined by the null-interval condition as follows:

$$\begin{gathered} ds^2_{DR5} = 0 \Longleftrightarrow \\ \Longleftrightarrow b_0^2\left(dx^0\right)^2 - \sum_i b_i^2(x^5)\left(dx^i\right)^2 \pm f(x^5)\left(dx^5\right)^2 = 0 \Longleftrightarrow \\ \Longleftrightarrow dx^0 = \frac{1}{b_0}\sqrt{\left[\sum_i b_i^2(x^5)\left(dx^i\right)^2 \mp f(x^5)\left(dx^5\right)^2\right]}. \end{gathered}$$

system of five coupled differential equations

$$\begin{cases} \dfrac{d^2x^\mu(\tau)}{d\tau^2} + 2\dfrac{b'_\mu(x^5(\tau))}{b_\mu(x^5(\tau))}\dfrac{dx^\mu(\tau)}{d\tau}\dfrac{dx^5(\tau)}{d\tau} = 0, \\ \mu = 0,1,2,3 \quad (ESC\ off); \\ \\ \dfrac{d^2x^5(\tau)}{d\tau^2} = \\ = \pm\dfrac{1}{f(x^5(\tau))}\left[g_{\alpha\beta}b_\alpha(x^5(\tau))b'_\beta(x^5(\tau))\left(\dfrac{dx^\alpha(\tau)}{d\tau}\right)^2 \mp f'(x^5(\tau))\left(\dfrac{dx^5(\tau)}{d\tau}\right)^2\right] \\ (ESC\ on), \end{cases} \tag{51}$$

where the prime denotes derivation with respect to x^5 and $g_{\alpha\beta}$ is the Minkowskian metric tensor.

System (51) does not admit solutions in general. In the following, we shall confine ourselves to look for physically significant solutions in the simpler case of the Power Ansatz for the metric coefficients.

4. Solution of the Geodesic Equations in the Power Ansatz

It has been already noticed that the phenomenological metrics for the electromagnetic, weak, gravitational and strong interactions, derived in the context of DSR, can be recovered as 5-d. metrics found, in the Power Ansatz, as solutions of the Einstein equations in vacuum and with cosmological constant $\Lambda_{(5)} = 0$ [6-8]. In this Section we will solve the geodesic equations in the Power Ansatz for the 12 classes of the vacuum Einstein equations.

4.1 General Solution

In the Power Ansatz for the metric coefficients (see Subsubsect.2.2.2), the system of geodesic equations (51) takes the form

$$\frac{d^2x^\mu}{d\tau^2} + \frac{q_\mu}{x^5}\frac{dx^\mu}{d\tau}\frac{dx^5}{d\tau} = 0, \mu = 0,1,2,3 \quad (ESC\ off); \tag{52}$$

$$\frac{d^2x^5}{d\tau^2} \mp \frac{1}{2}\left[g_{\alpha\beta}q_\alpha\left(\frac{x^5}{x^5_0}\right)^{q_\alpha - r - 1}\frac{1}{x^5_0}\left(\frac{dx^\alpha}{d\tau}\right)^2 \mp \frac{r}{2x^5}\left(\frac{dx^5}{d\tau}\right)^2\right] = 0 \quad (ESC\ on), \tag{53}$$

with $g_{\alpha\beta}$ denoting, as usual, the Minkowskian metric tensor.

The solution of Eq.(52) in terms of x^5 reads (ESC off)

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2}\int d\tau\left(x^5(\tau)\right)^{-q_\mu}\ , \tag{54}$$

where $C_{\mu 1}$, $C_{\mu 2}$ are real integration constants. Replacing (54) in (53) yields (ESC on)

$$\frac{d^2x^5}{d\tau^2} \mp \frac{1}{2}\left(x_0^5\right)^r \left(x^5\right)^{-r-1} g_{\alpha\beta} C_{\alpha 2}^2 q_\alpha \left(\frac{x^5}{x_0^5}\right)^{-q_\beta} + \frac{r}{2x^5}\left(\frac{dx^5}{d\tau}\right)^2 = 0. \tag{55}$$

By solving this equation one gets the following implicit functional relation for $x^5(\tau)$:

$$\tau + A_1 \mp \left(x_0^5\right)^{-\frac{r}{2}} \int d\zeta F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)\Big|_{\zeta = x^5(\tau)} = 0, \tag{56}$$

(A_1, $A_2 \in R$), where (ESC on)

$$F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2) \equiv \left\{\mp\left[g_{\mu\nu}\left(x_0^5\right)^{q_\mu} C_{\mu 2}^2 \zeta^{-r-q_\nu} \mp \left(x_0^5\right)^{-r} A_2 \zeta^{-r}\right]\right\}^{-\frac{1}{2}} \tag{57}$$

and $\widetilde{\mathbf{q}} = (q_0, q_1, q_2, q_3, r)$ is the parametric set of exponents of the metric coefficients (in the Power Ansatz) for the class of solutions considered. Of course, the explicit form of the function $F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)$, and therefore of the indefinite integral in ζ in Eq.(56), depends on the set $\widetilde{\mathbf{q}}$ (and on the integration constant A_2); moreover, it determines the geodesic motions in $\Re_5$ for the class of solutions characterized by the exponent set $\widetilde{\mathbf{q}}$. This is why we shall refer to it as *the geodesic generating function*. The integral in (56) then becomes (ESC on)

$$\begin{aligned} &\int d\zeta F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2) = \\ &= \int d\zeta \left[\alpha_{\mu,\pm}(q_\mu, C_{\mu 2}^2, \left(x_0^5\right))\delta_{\mu\nu}\zeta^{-r-q_\nu} + \alpha_5(r, A_2, \left(x_0^5\right))\zeta^{-r}\right]^{-\frac{1}{2}}, \end{aligned} \tag{58}$$

with

$$\begin{aligned} \alpha_{0,\pm}(q_0, C_{02}^2, \left(x_0^5\right)) &= \mp \left(x_0^5\right)^{q_0} C_{02}^2; \\ \alpha_{1,\pm}(q_1, C_{12}^2, \left(x_0^5\right)) &= \pm \left(x_0^5\right)^{q_1} C_{12}^2; \\ \alpha_{2,\pm}(q_2, C_{22}^2, \left(x_0^5\right)) &= \pm \left(x_0^5\right)^{q_2} C_{22}^2; \\ \alpha_{3,\pm}(q_3, C_{32}^2, \left(x_0^5\right)) &= \left(x_0^5\right)^{q_3} C_{32}^2; \\ \alpha_5(r, A_2, \left(x_0^5\right)) &= \left(x_0^5\right)^{-r} A_2. \end{aligned} \tag{59}$$

Such an integral is of the type (ESC on)

$$\int dx \frac{x^{\frac{r}{2}}}{\sqrt{a + \delta_{\mu\nu} c_\mu x^{-q_\nu}}} \tag{60}$$

with a, c_μ ($\mu = 0, 1, 2, 3$) real constants. For $r \neq -2$, putting $y = x^{\frac{r}{2}+1}$ yields

$$\int dx \frac{x^{\frac{r}{2}}}{\sqrt{a + \delta_{\mu\nu} c_\mu x^{-q_\nu}}} = \frac{2}{r+2} \int \frac{dy}{\sqrt{a + \delta_{\mu\nu} c_\mu y^{\gamma_\nu(q_\nu, r)}}} \tag{61}$$

where $\gamma_\nu(q_\nu, r) = -\dfrac{2q_\nu}{r+2}$. For $r = -2$ one gets instead, by the substitution $y = \ln x$ ($x \geq 0$):

$$\int \frac{dx}{x\sqrt{a + \delta_{\mu\nu} c_\mu x^{-q_\nu}}} = \int \frac{dy}{\sqrt{a + \delta_{\mu\nu} c_\mu e^{-q_\nu y}}}. \tag{62}$$

The Riemann integrals (61) and (62) do not exist in literature for generic values of the constants. Then, the general solution of the geodetic equations (52), even in the Power Ansatz, can only be expressed in terms of such integrals by means of Eqs.(57)-(62), and therefore takes the implicit form

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2} \int d\tau \left(x^5(\tau)\right)^{-q_\mu} \qquad (ESC\ off); \tag{63}$$

$$\tau + A_1 \mp \left(x_0^5\right)^{-\frac{r}{2}} \int d\zeta F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)\Big|_{\zeta = x^5(\tau)} = 0; \tag{64}$$

with the generating function given by Eq.(57). Eq.(64) for the energy coordinate explicitly reads

$$\begin{aligned} &\tau + A_1 = \\ &\overset{ESC\ on}{=} \pm \frac{2\left(x_0^5\right)^{-\frac{r}{2}}}{r+2} \int\int \frac{dy}{\sqrt{\alpha_{\mu,\pm}(q_\mu, C_{\mu 2}^2, (x_0^5))\delta_{\mu\nu} y^{\gamma_\nu(q_\nu, r)} + \alpha_5(r, A_2, (x_0^5))}} = 0, \\ &\gamma_\nu(q_\nu, r) = -\frac{2q_\nu}{r+2}, \quad y = (x_5(\tau))^{\frac{r}{2}+1}, \quad r \neq -2; \end{aligned} \tag{65}$$

$$\begin{aligned} &\tau + A_1 = \\ &\overset{ESC\ on}{=} \pm \left(x_0^5\right)^{-\frac{r}{2}} \int\int^{x_5(\tau)} \frac{dx}{\sqrt{\alpha_{\mu,\pm}(q_\mu, C_{\mu 2}^2, (x_0^5))\delta_{\mu\nu} x^{-q_\nu} + \alpha_5(r, A_2, (x_0^5))}} = 0, \\ &r = 2; \end{aligned} \tag{66}$$

where we put

$$\begin{aligned} \alpha_{0,\pm}(q_0, C_{02}^2, \left(x_0^5\right)) &= \mp \left(x_0^5\right)^{q_0} C_{02}^2; \\ \alpha_{1,\pm}(q_1, C_{12}^2, \left(x_0^5\right)) &= \pm \left(x_0^5\right)^{q_1} C_{12}^2; \\ \alpha_{2,\pm}(q_2, C_{22}^2, \left(x_0^5\right)) &= \pm \left(x_0^5\right)^{q_2} C_{22}^2; \\ \alpha_{3,\pm}(q_3, C_{32}^2, \left(x_0^5\right)) &= \left(x_0^5\right)^{q_3} C_{32}^2; \\ \alpha_5(r, A_2, \left(x_0^5\right)) &= \left(x_0^5\right)^{-r} A_2. \end{aligned} \tag{67}$$

4.2 Explicit and Implicit Forms of Geodesics for the 12 Classes of Solutions of Einstein's Equations in Vacuum in the Power Ansatz

From the general solution (63), (64) it is possible to get the (explicit or implicit) solutions of the geodesic equations corresponding to the 12 classes of metrics, obtained from the 5-d. Einstein equations in vacuum in the Power Ansatz (see Subsubsect.2.2.3). The reader is referred to ref.[12] for calculation details.

4.2.1 Class (I)

The geodesic generating function $F_{\pm,I}(\zeta; n, p, A_2)$ (57) is given by

$$F_{\pm,I}(\zeta; n, p, A_2) =$$
$$= \Bigg[\pm(x_0^5)^{-\frac{2np+n^2}{2n+p}} C_{12}^2 \zeta^{-\frac{p^2-2p-4n+2n^2}{2n+p}} \pm (x_0^5)^n (C_{22}^2 - C_{02}^2)\zeta^{-\frac{p^2-2p+3np-4n+5n^2}{2n+p}} +$$
$$\pm(x_0^5)^p C_{32}^2 \zeta^{-\frac{2p^2-2p+4np-4n+3n^2}{2n+p}} + A_2 (x_0^5)^{-\frac{p^2-2p+2np-4n+3n^2}{2n+p}} \zeta^{-\frac{p^2-2p+2np-4n+3n^2}{2n+p}}\Bigg]^{-\frac{1}{2}}. \tag{68}$$

Therefore, the integral (58) can be put in the form ($2n + p \neq 0$):

$$\frac{4n+p}{3n^2+4np+2p^2} \int \frac{dy}{\sqrt{c_0 y^{\alpha(n,p)} + c_1 y^{\beta(n,p)} + c_3 y^{\gamma(n,p)} + c_2}}, \tag{69}$$

with

$$\begin{cases} \alpha(n,p) = \dfrac{2p^2+2n^2+8np}{3n^2+4np+2p^2}; \\ \beta(n,p) = \dfrac{2p^2-4n^2+2np}{3n^2+4np+2p^2}; \\ \gamma(n,p) = \dfrac{2p^2+4np}{3n^2+4np+2p^2}; \\ y = (x_5(\tau))^{\frac{3n^2}{4n+2p}+p}, x_5(\tau) > 0; \end{cases} \tag{70}$$

$$\begin{cases} c_0 = \pm(x_0^5)^{-\frac{2np+n^2}{2n+p}} C_{12}^2; \\ c_1 = \pm(x_0^5)^n (C_{22}^2 - C_{02}^2); \\ c_2 = \pm(x_0^5)^p C_{32}^2; \\ c_3 = A_2 (x_0^5)^{-\frac{p^2-2p+2np-4n+3n^2}{2n+p}}, \end{cases} \tag{71}$$

and cannot be evaluated for arbitrary values of the parameters. Therefore, the solution for Class (I) can only be given in implicit form by replacing Eqs.(69)-(71) in the general solution (63)-(64).

4.2.2 Class (II)

The function $F_{\pm}(\zeta; q, A_2)$ reads, in this case

$$F_{\pm,II}(\zeta; m, A_2) =$$
$$= \left\{\pm\left[C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2 (x_0^5)^{-m+2}\right]\zeta^{-m+2} \pm (x_0^5)^m C_{12}^2 \zeta^{-2m+2}\right\}^{-\frac{1}{2}}. \tag{72}$$

The Riemann indefinite integrals of the generating functions $F_{\pm,i}(\zeta; \widetilde{\mathbf{q}}, A_2)$ ($i = II, IV, VI, IX$) for the classes (II), (IV), (VI) and (IX) can be put in the following form (ESC off)

$$\int d\zeta F_{\pm,i}(\zeta; \widetilde{\mathbf{q}}, A_2) = \int d\zeta \left[a_i \zeta^{-k_i+2} + b_i \zeta^{-2k_i+2}\right]^{-\frac{1}{2}},$$
$$k_{II} = m, k_{IV} = p, k_{VI} = q, k_{IX} = n, \tag{73}$$

where $a_i = a_i(C_{02}^2, C_{12}^2, C_{22}^2, C_{32}^2, \widetilde{\mathbf{q}}, A_2, x_0^5)$, $b_i = b_i(C_{02}^2, C_{12}^2, C_{22}^2, C_{32}^2, \widetilde{\mathbf{q}}, A_2, x_0^5)$. They can be evaluated as follows ($y = x^k$)

$$\int \frac{dx}{\sqrt{ax^{-k+2} + bx^{-2k+2}}} = \begin{cases} \dfrac{2\sqrt{b}}{ka}\sqrt{\dfrac{a}{b}y + 1}, & a \neq 0, b \neq 0; \\[2ex] \dfrac{1}{\sqrt{b}}\dfrac{x^k}{k}, & a = 0, b \neq 0; \\[2ex] \dfrac{2}{\sqrt{a}}\dfrac{x^{k/2}}{k}, & a \neq 0, b = 0; \\[2ex] \dfrac{1}{\sqrt{a+b}} \ln|x|, & k = 0. \end{cases} \tag{74}$$

The solution therefore reads

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2}\tau, \quad \mu = 0, 2, 3; \tag{75}$$

$$x^1(\tau) = C_{11} + C_{12}\int d\tau \left(x^5(\tau)\right)^{-m}; \tag{76}$$

As to $x^5(\tau)$, we have the following cases:

1) $m \neq 0$:

1.1) $C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2} \neq 0$, $C_{12} \neq 0$:

$$x^5(\tau) = \left\{ \frac{(x_0^5)^m C_{12}^2}{[C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2}]} \times \right.$$
$$\left. \times \left[\frac{m^2\left(C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2}\right)^2}{4(x_0^5)^2 C_{12}^2} (\tau + A_1)^2 - 1 \right] \right\}^{1/m}; \tag{77}$$

1.2) $C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2} = 0$, $C_{12} \neq 0$:

$$x^5(\tau) = \left[\pm m\sqrt{\pm(x_0^5)^m C_{12}^2}(x_0^5)^{\frac{m-2}{2}}(\tau + A_1) \right]^{1/m}; \tag{78}$$

1.3) $C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2} \neq 0$, $C_{12} = 0$:

$$x^5(\tau) =$$
$$= \left[\pm m\frac{\sqrt{\pm\left[C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2}\right]}}{2}(x_0^5)^{\frac{m-2}{2}}(\tau + A_1) \right]^{2/m}. \tag{79}$$

2) $m = 0$:

$$x^5(\tau) =$$
$$= \exp\left\{ \pm\sqrt{\pm\left[C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-m+2} + (x_0^5)^m C_{12}^2\right]}(x_0^5)^{\frac{m-2}{2}}(\tau + A_1) \right\}. \tag{80}$$

4.2.3 Class (III)

One gets

$$F_{\pm,III}(\zeta;n,A_2) = \left\{\pm[C_{22}^2 + C_{32}^2 - C_{02}^2](x_0^5)^n\zeta^{-3n+2} + \pm A_2(x_0^5)^{-2n+2}\zeta^{-2n+2} \pm (x_0^5)^{-n}C_{12}^2\zeta^{-n+2}\right\}^{-\frac{1}{2}}. \tag{81}$$

The integral of $F_{\pm,III}(\zeta;n,A_2)$ is then of the kind

$$\int \frac{dx}{\sqrt{ax^{2-n} + bx^{2-2n} + cx^{2-3n}}} \tag{82}$$

which admits an explicit solution at least for some values of the parameters.

The geodesic solution therefore reads

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2}\int d\tau \left(x^5(\tau)\right)^{-n}, \quad \mu = 0, 2, 3; \tag{83}$$

$$x^1(\tau) = C_{11} + C_{12}\int d\tau \left(x^5(\tau)\right)^n; \tag{84}$$

where $x^5(\tau)$ is given (explicitly or implicitly) by:

1) $n \neq 0$:

1.1) $C_{22}^2 + C_{32}^2 - C_{02}^2 = 0$:

1.1.1 - $C_{12} \neq 0$, $A_2 \neq 0$:

$$x^5(\tau) = \left\{\pm\frac{A_2(x_0^5)^{-n+2}}{C_{12}}\left[\pm(x_0^5)^{2n-4}\frac{n^2}{4}\frac{C_{12}^4}{A_2}(\tau + A_1)^2 - 1\right]\right\}^{1/n}; \tag{85}$$

1.1.2 - $C_{12} = 0$, $A_2 \neq 0$:

$$x^5(\tau) = \left[\pm n\sqrt{A_2}\,(\tau + A_1)\right]^{1/n}; \tag{86}$$

1.1.3 - $C_{12} \neq 0$, $A_2 = 0$:

$$x^5(\tau) = \left[\pm\frac{nC_{12}\sqrt{\pm 1}}{2}(x_0^5)^{\frac{n}{2}-1}(\tau + A_1) - 1\right]^{2/n}. \tag{87}$$

1.2) $A_2 = 0$:

1.2.1 - $C_{12} \neq 0$, $C_{22}^2 + C_{32}^2 - C_{02}^2 \neq 0$, $C_{12}^2 \neq (C_{22}^2 + C_{32}^2 - C_{02}^2)\,(x_0^5)^{2n}$:

1.2.1 a) - $(C_{22}^2 + C_{32}^2 - C_{02}^2) < 0$:

$$(\tau + A_1) = \pm\frac{(x_0^5)^{-\frac{n}{2}+1}}{nC_{12}\sqrt{\mp 1}}\left\{\widehat{\Theta}\left((x_0^5)^n\sqrt{\frac{C_{02}^2 - C_{22}^2 - C_{32}^2}{C_{12}^2}} - \left(x^5(\tau)\right)^n\right)\widehat{\Theta}\left(\left(x^5(\tau)\right)^n\right) \times\right.$$

$$\times\left\{\begin{array}{c}\left[\begin{array}{c}\sqrt{2}(x_0^5)^{\frac{n}{2}}\sqrt[4]{\dfrac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\int\int_0^{\sqrt{\frac{2}{\left(\frac{x_0^5}{x^5(\tau)}\right)^n\sqrt{\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}-1}}}dt\dfrac{\sqrt{1-t^2}}{\sqrt{1-\dfrac{t^2}{2}}}+\\ -2\sqrt{(x^5(\tau))^n\dfrac{\sqrt{\dfrac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}-(x^5(\tau))^n}{\sqrt{\dfrac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}+(x^5(\tau))^n}}\end{array}\right]+\\ \text{or}\\ \sqrt{2}(x_0^5)^{\frac{n}{2}}\sqrt[4]{\dfrac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\int\int_0^{\sqrt{1-\sqrt{\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}}dt\dfrac{\sqrt{1-t^2}}{\sqrt{1-\dfrac{t^2}{2}}}\end{array}\right.$$

$$\mp i\widehat{\Theta}\left(\left(x^5(\tau)\right)^n-(x_0^5)^n\sqrt{\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\right)\times$$

$$\times\left[-\sqrt{2}(x_0^5)^{\frac{n}{2}}\sqrt[4]{\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\int\int_0^{\sqrt{1-\sqrt{\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}}dt\frac{\sqrt{1-t^2}}{\sqrt{1-\frac{t^2}{2}}}+\right.$$

$$\left.\left.+2\sqrt{(x^5(\tau))^n-\frac{C_{02}^2-C_{22}^2-C_{32}^2}{C_{12}^2}\left(\frac{x_0^5}{x^5(\tau)}\right)^n(x_0^5)^n}\right]\right\}. \tag{88}$$

1.2.1 b) - $(C_{22}^2+C_{32}^2-C_{02}^2)>0$:

In this case, the integral

$$\int^{x^5(\tau)}\frac{dx}{\sqrt{\pm(x_0^5)^{-n}C_{12}^2x^{2-n}\pm\left(C_{22}^2+C_{32}^2-C_{02}^2\right)(x_0^5)^nx^{2-3n}}} \tag{89}$$

is unknown, and therefore no solution can be given even in implicit form.

1.2.2 - $C_{12}\neq 0$, $C_{22}^2+C_{32}^2-C_{02}^2\neq 0$, $C_{12}^2=\left(C_{22}^2+C_{32}^2-C_{02}^2\right)(x_0^5)^{2n}$:

$$\left(x^5(\tau)\right)^n\mp n^2C_{12}^2(x_0^5)^{n-2}\left(\tau+A_1\right)^2+\left(x^5(\tau)\right)^{-n}=0. \tag{90}$$

1.2.3 - $C_{12}\neq 0$, $C_{22}^2+C_{32}^2-C_{02}^2=0$:

$$x^5(\tau)=\left[\pm\sqrt{\pm 1}\frac{n}{2}C_{12}(x_0^5)^{\frac{n}{2}-1}\left(\tau+A_1\right)\right]^{2/n}. \tag{91}$$

1.2.4 - $C_{12}=0$, $C_{22}^2+C_{32}^2-C_{02}^2\neq 0$:

$$x^5(\tau)=\left[\pm\frac{3n}{2}\sqrt{\pm\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}(x_0^5)^{\frac{3n}{2}-1}\left(\tau+A_1\right)\right]^{2/3n}. \tag{92}$$

1.3) $C_{12} = 0$:

1.3.1 - $A_2 \neq 0$, $C_{22}^2 + C_{32}^2 - C_{02}^2 \neq 0$:

$$\tau + A_1 = \pm (x_0^5)^{1-n} \left\{ \frac{\left(C_{22}^2+C_{32}^2-C_{02}^2\right)(x_0^5)^{4n-3}}{nA_2^{3/2}} \widehat{\Theta}\left(\mp \frac{A_2}{C_{22}^2+C_{32}^2-C_{02}^2}\right) \times \right.$$

$$\times \widehat{\Theta}\left(\mp \frac{A_2 (x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^n - 1\right) \frac{A_2 (x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^n \times$$

$$\times \ {}_2F_1\left(\frac{1}{2}, -1; 0; \mp \left(\frac{x_0^5}{x^5(\tau)}\right)^n \frac{C_{22}^2 + C_{32}^2 - C_{02}^2}{A_2} (x_0^5)^{2n-2}\right) +$$

$$+ \widehat{\Theta}\left(\pm \frac{A_2 (x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^n + 1\right) \widehat{\Theta}\left(\mp \frac{A_2}{C_{22}^2+C_{32}^2-C_{02}^2}\right) \times$$

$$\times \left\{ \begin{array}{c} \frac{2}{3} \frac{A_2 (x_0^5)^{2-\frac{3}{2}n}}{C_{22}^2 + C_{32}^2 - C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^{\frac{3}{2}n} \times \\ \times \ {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; \mp \left(\frac{x^5(\tau)}{x_0^5}\right)^n \frac{A_2 (x_0^5)^{2-2n}}{C_{22}^2 + C_{32}^2 - C_{02}^2}\right) \\ \\ \text{or} \\ \\ \frac{A_2 (x_0^5)^{2-\frac{3}{2}n}}{C_{22}^2 + C_{32}^2 - C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^n \times \\ \times \ {}_2F_1\left(\frac{1}{2}, -1; 0; \mp \left(\frac{x_0^5}{x^5(\tau)}\right)^n \frac{C_{22}^2 + C_{32}^2 - C_{02}^2}{A_2} (x_0^5)^{2-2n}\right) \end{array} \right. +$$

$$
\pm\frac{\left(C_{22}^2+C_{32}^2-C_{02}^2\right)(x_0^5)^{4n-3}}{nA_2^{3/2}}\widehat{\Theta}\left(\pm\frac{A_2}{C_{22}^2+C_{32}^2-C_{02}^2}\right)\times
$$

$$
\times\left\{\begin{array}{c}
\left[\begin{array}{c}\pm\frac{2}{3}\frac{A_2(x_0^5)^{2-\frac{3}{2}n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^{\frac{3}{2}n}\times\\ \times\ {}_2F_1\left(\frac{1}{2},\frac{3}{2};\frac{5}{2};\mp\left(\frac{x^5(\tau)}{x_0^5}\right)^n\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\right)\end{array}\right]\Bigg\},\\
\text{or}\\
\left[\begin{array}{c}\pm\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\times\\ \times\ {}_2F_1\left(\frac{1}{2},-1;0;\mp\left(\frac{x_0^5}{x^5(\tau)}\right)^n\frac{C_{22}^2+C_{32}^2-C_{02}^2}{A_2}(x_0^5)^{2n-2}\right)\end{array}\right]\Bigg\},\\
\text{or}\\
\left[\sqrt{\pm\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\left[\pm\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n+1\right]}+\right.\\
\left.-\ln\left|\begin{array}{c}\sqrt{\pm\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}+\\ +\sqrt{\pm\frac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n+1}\end{array}\right|\right]\Bigg\}.
\end{array}\right.
\tag{93}
$$

Here, ${}_2F_1$ is the generalized hypergeometric function of class (2,1), defined by

$$
{}_2F_1(\alpha,\beta;\gamma;z)\equiv\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}
\tag{94}
$$

with $(\alpha)_k$ being the Pochhammer symbols

$$
(\alpha)_k\equiv\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}
\tag{95}
$$

and

$$
\Gamma(z)\equiv\int_0^{\infty}e^{-t}t^{z-1}dt\qquad(Re(z)>0)
\tag{96}
$$

denotes the Euler gamma function (or Euler integral of second kind).

1.3.2 - $A_2\neq 0$, $C_{22}^2+C_{32}^2-C_{02}^2=0$:

$$
x^5(\tau)=\left[\pm n\sqrt{A_2}\,(\tau+A_1)\right]^{1/n}.
\tag{97}
$$

1.3.3 - $A_2=0$, $C_{22}^2+C_{32}^2-C_{02}^2\neq 0$:

$$
x^5(\tau)=\left[\pm\frac{3n}{2}\sqrt{\pm\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}(x_0^5)^{\frac{3n}{2}-1}\,(\tau+A_1)\right]^{2/3n}.
\tag{98}
$$

1.4) C_{12}, A_2, $C_{22}^2 + C_{32}^2 - C_{02}^2 \neq 0$:

1.4.1 - $C_{12}^2 \neq (C_{22}^2 + C_{32}^2 - C_{02}^2)\,(x_0^5)^{2n}$:

1.4.1.a) - $(C_{22}^2 + C_{32}^2 - C_{02}^2) < 0$:

$$(\tau + A_1) = \pm (x_0^5)^{1-n} \left\{ \widehat{\Theta}\left(\pm \tfrac{A_2}{C_{22}^2+C_{32}^2-C_{02}^2}\right) \left[\mp \frac{1}{n} \frac{\sqrt{A_2}}{C_{12}^2} x_0^5 \times \right.\right.$$

$$\times \widehat{\Theta}\left(\begin{array}{c} -\frac{A_2^2 (x_0^5)^{4-4n}}{2C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)} \left(1 + \sqrt{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}\right) + \\ \mp \frac{A_2^2 (x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^n \end{array} \right) \times$$

$$\times \left\{ \begin{array}{c} \left[\begin{array}{c} 2\sqrt[4]{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}\, E(A,B) + \\ -\dfrac{-1 + \sqrt{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}}{\sqrt[4]{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}} F(A,B) + \\ -2\sqrt{\mp \frac{A_2^2 (x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}} \left(\frac{x^5(\tau)}{x_0^5}\right)^n \times \\ \times \dfrac{(x^5(\tau))^n \pm \dfrac{A_2^2 (x_0^5)^{2-n}}{2C_{12}^2} \left[1 + \sqrt{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}\right]}{(x^5(\tau))^n \pm \dfrac{A_2^2 (x_0^5)^{2-n}}{2C_{12}^2} \left[1 - \sqrt{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}\right]} \end{array} \right] + \\ \\ \text{or} \\ \\ \left[\begin{array}{c} -2\sqrt[4]{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}\, E(C,B) + \\ +\dfrac{-1 + \sqrt{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}}{\sqrt[4]{1 - 4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2 (x_0^5)^{4-4n}}}} F(C,B) \end{array} \right] + \end{array} \right.$$

$$
-\frac{i}{n}\frac{\sqrt{A_2}}{C_{12}^2}x_0^5\widehat{\Theta}\left(\begin{array}{c}\dfrac{A_2^2(x_0^5)^{4-4n}}{2C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}\left(1+\sqrt{1-4\dfrac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right)\\ \pm\dfrac{A_2^2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^n\end{array}\right)\times
$$

$$
\times\left[\begin{array}{c}2\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}E\left(D.G\right)+\\ +\dfrac{-1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}F\left(D,G\right)+\end{array}\right.
$$

$$
\left.-2\sqrt{\frac{-\dfrac{C_{12}^2(x_0^5)^{3n-2}}{A_2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{2n}-\dfrac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{n}\mp 1}{\dfrac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{n}}}\right]
$$

$$
+\widehat{\Theta}\left(\mp\frac{A_2}{C_{22}^2+C_{32}^2-C_{02}^2}\right)\times
$$

$$
\times\left\{\left[-\tfrac{i}{n}\tfrac{\sqrt{A_2}}{C_{12}^2}x_0^5\widehat{\Theta}\left(\tfrac{A_2^2(x_0^5)^{4-4n}}{2C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}\left(-1+\sqrt{1-4\tfrac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right)\mp\tfrac{A_2^2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\tfrac{x^5(\tau)}{x_0^5}\right)^n\right)\right]\times\right.
$$

$$
\times\left[\begin{array}{c}\dfrac{-1+\sqrt{1-4\dfrac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{\sqrt[4]{1-4\dfrac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}F(H,B)+\\ -2\sqrt[4]{1-4\dfrac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}E(H,B)+\\ +2\sqrt{\dfrac{\dfrac{C_{12}^2(x_0^5)^{3n-2}}{A_2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{2n}-\dfrac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{n}\pm 1}{\dfrac{A_2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^{n}}}\end{array}\right]+
$$

$$+\frac{i}{n}\frac{\sqrt{A_2}}{C_{12}^2}x_0^5\widehat{\Theta}\left(\begin{array}{c}-\frac{A_2^2(x_0^5)^{4-4n}}{2C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}\left(-1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right)\\ \pm\frac{A_2^2(x_0^5)^{2-2n}}{C_{22}^2+C_{32}^2-C_{02}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\end{array}\right)\times$$

$$\times\left\{\begin{array}{c}\left[\begin{array}{c}2\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}E\left(L,B\right)+\\ -\frac{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}F\left(L,B\right)\end{array}\right]\\ \\ \text{or}\\ \\ \left[\begin{array}{c}-2\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}E\left(M,G\right)+\\ +\frac{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{\sqrt[4]{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}F\left(M,G\right)+\\ +2\sqrt{\frac{\pm\frac{C_{12}^2(x_0^5)^{2n-2}}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^{2n}+\frac{1}{2}\left[1-\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]}{-1\mp\frac{A_2(x_0^5)^{2-2n}}{2C_{12}^2}\left(\frac{x_0^5}{x^5(\tau)}\right)^n\left[1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]}}\end{array}\right]\end{array}\right. \tag{99}$$

where $F(z,k)$ and $E(z,k)$ are the elliptic integrals of first and second kind, respectively, in the normal Legendre form

$$F(z,k)\equiv\int_0^z\frac{dx}{\sqrt{1-k^2\sin^2 x}}=\int_0^{\sin z}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}},\qquad z,k\in C; \tag{100}$$

$$E(z,k)\equiv\int_0^z dx\sqrt{1-k^2\sin^2 x}=\int_0^{\sin z}\frac{dt\sqrt{1-k^2t^2}}{\sqrt{1-t^2}},\qquad z,k\in C, \tag{101}$$

and we put

$$A = \arcsin\left\{\left[\pm 4\frac{C_{12}^2}{A_2(x_0^5)^{2n-2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]\times\right.$$
$$\left.\times\left[1-\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]^{-1}\left[1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{C_{12}^2}{A_2(x_0^5)^{2n-2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\right]^{-1}\right\}^{\frac{1}{2}}; \tag{102}$$

$$B = \sqrt{\frac{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{2\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}}; \tag{103}$$

$$C = \arcsin\sqrt{\frac{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{C_{12}^2}{A_2(x_0^5)^{2n-2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}}; \tag{104}$$

$$D = \arcsin\sqrt{\frac{1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{C_{12}^2}{A_2(x_0^5)^{2n-2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}{\pm 2\frac{(x_0^5)^{n-1}C_{12}^2}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}}; \tag{105}$$

$$G = \sqrt{\frac{-1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}{2\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}}; \tag{106}$$

$$H = \arcsin\sqrt{\frac{\left[1-\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{(x_0^5)^{2n-2}C_{12}^2}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\right]}{\pm 2\frac{(x_0^5)^{2n-2}C_{12}^2}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n}}; \tag{107}$$

$$L = \arcsin\sqrt{\frac{\left[1-\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{(x_0^5)^{2n-2}C_{12}^2}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\right]}{1-\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}}}; \tag{108}$$

$$M = \arcsin\left\{\left[\pm 4\frac{C_{12}^2(x_0^5)^{2n-2}}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]\left[-1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\right]^{-1}\times\right.$$
$$\left.\times\left[1+\sqrt{1-4\frac{C_{12}^2\left(C_{22}^2+C_{32}^2-C_{02}^2\right)}{A_2^2(x_0^5)^{4-4n}}}\pm 2\frac{C_{12}^2(x_0^5)^{2n-2}}{A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^n\right]^{-1}\right\}^{\frac{1}{2}}. \tag{109}$$

$$\tag{110}$$

1.4.1.b) - $(C_{22}^2 + C_{32}^2 - C_{02}^2) > 0$:

In this case, the Riemann integral

$$\int^{x^5(\tau)} \frac{dx}{\sqrt{\pm(x_0^5)^{-n} C_{12}^2 x^{2-n} + A_2 (x_0^5)^{2-2n} x^{2-2n} \pm (C_{22}^2 + C_{32}^2 - C_{02}^2)\,(x_0^5)^n x^{2-3n}}} \tag{111}$$

is unknown, and therefore no solution can be obtained for $x^5(\tau)$.

1.4.2 - $C_{12}^2 = (C_{22}^2 + C_{32}^2 - C_{02}^2)\,(x_0^5)^{2n}$:

$$\left(x^5(\tau)\right)^n \mp C_{12}^2 (x_0^5)^{n-2} (\tau + A_1)^2 + \left(x^5(\tau)\right)^{-n} \pm \frac{A_2 (x_0^5)^{2n-2}}{C_{12}^2} = 0. \tag{112}$$

2) $n = 0$:

$$x^5(\tau) = \exp\left\{\pm\sqrt{\pm\,(C_{22}^2 + C_{32}^2 - C_{02}^2)\,(x_0^5)^{3n-2} \pm C_{12}^2 (x_0^5)^{n-2} + A_2}\,(\tau + A_1)\right\}. \tag{113}$$

4.2.4 Class (IV)

One gets:

$$F_{\pm,IV}(\zeta; p, A_2) =$$
$$= \left\{\pm[C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2 (x_0^5)^{2-p}]\zeta^{2-p} \pm (x_0^5)^p C_{32}^2 \zeta^{2-2p}\right\}^{-\frac{1}{2}}. \tag{114}$$

From the results obtained for Class II, we can write the geodesic solution for Class IV as

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2}\tau, \quad \mu = 0, 1, 2; \tag{115}$$

$$x^3(\tau) = C_{31} + C_{32} \int d\tau \left(x^5(\tau)\right)^{-p}, \tag{116}$$

with $x^5(\tau)$ given by:

1) $p \neq 0$:

1.1) $C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2 (x_0^5)^{2-p} \neq 0$, $C_{32} \neq 0$:

$$x^5(\tau) = \left\{\frac{(x_0^5)^p C_{32}^2}{C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2 (x_0^5)^{2-p}} \times \right.$$
$$\left. \times \left[\frac{p^2\left[C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2 (x_0^5)^{2-p}\right]^2}{4 (x_0^5)^2 C_{32}^2} (\tau + A_1)^2 - 1\right]\right\}^{1/p}; \tag{117}$$

1.2) $C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2 (x_0^5)^{2-p} = 0$, $C_{32} \neq 0$:

$$x^5(\tau) = \left[\pm p \sqrt{\pm (x_0^5)^p C_{32}^2} (x_0^5)^{\frac{p-2}{2}} (\tau + A_1)\right]^{1/p}; \tag{118}$$

1.3) $C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-p} \neq 0$, $C_{32} = 0$:

$$x^5(\tau) = \left[\pm p\sqrt{\frac{\pm\left[C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-p}\right]}{2}}(x_0^5)^{\frac{p-2}{2}}(\tau + A_1)\right]^{2/p}; \tag{119}$$

2) $p = 0$:

$$\begin{aligned} x^5(\tau) = \exp\Big[&\pm p(x_0^5)^{\frac{p-2}{2}}(\tau + A_1) \times \\ &\times\sqrt{\pm\left[C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-p} + (x_0^5)^p C_{32}^2\right]}\Big]. \end{aligned} \tag{120}$$

4.2.5 Class (V)

One has

$$\begin{aligned} &F_{\pm,V}(\zeta; p, A_2) = \\ &= \left\{\pm(x_0^5)^{-p}[C_{12}^2 + C_{22}^2 - C_{02}^2]\zeta^{1+2p} + A_2(x_0^5)^{1+p}\zeta^{1+p} \pm (x_0^5)^p C_{32}^2\zeta^{1+p}\right\}^{-\frac{1}{2}}. \end{aligned} \tag{121}$$

The solution writes

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2}\int d\tau\left(x^5(\tau)\right)^p, \quad \mu = 0, 1, 2; \tag{122}$$

$$x^3(\tau) = C_{31} + C_{32}\int d\tau\left(x^5(\tau)\right)^{-p}, \tag{123}$$

with $x^5(\tau)$ given by:

1) $p \neq 0$:

1.1) $C_{12}^2 + C_{22}^2 - C_{02}^2 = 0$:

1.1.1 - $C_{32} \neq 0$, $A_2 \neq 0$:

$$\begin{aligned} 0 = \tau + A_1 \mp (x_0^5)^{\frac{1+p}{2}} \Bigg\{ &(\pm 1)^{\frac{1-p}{2p}} \frac{C_{32}^{\frac{1-p}{p}}}{pA_2^{1/2p}} (x_0^5)^{-\frac{p^2+1}{2p}} \times \\ &\times\left\{\delta\left(\frac{1}{2p} + m\right)\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p + 1\right)\times\right. \\ \times\Bigg[&\frac{2(-1)^{\frac{1}{2p}-1}\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p + 1}}{2\left|\frac{1}{2p} - 1\right| - 1}\times \\ &\times\sum_{k=1}^{\left|\frac{1}{2p}-1\right|-1}\frac{\left(2\left|\frac{1}{2p}-1\right|-1\right)\left(2\left|\frac{1}{2p}-1\right|-3\right)\left(2\left|\frac{1}{2p}-1\right|-5\right)\ldots\left(2\left|\frac{1}{2p}-1\right|-2k+1\right)}{2^k\left(\left|\frac{1}{2p}-1\right|-1\right)\left(\left|\frac{1}{2p}-1\right|-2\right)\ldots\left(\left|\frac{1}{2p}-1\right|-k\right)\left[\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)\right]^{p\left|\frac{1}{2p}-1\right|-k}} + \\ &+\frac{\left(2\left|\frac{1}{2p}-1\right|-3\right)!!}{2^{\left|\frac{1}{2p}-1\right|}\left(\left|\frac{1}{2p}-1\right|-1\right)!}\ln\left|\frac{1+\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{1-\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}\right| + \Bigg] \end{aligned}$$

or

$$\times 2(-1)^{\frac{1}{2p}-1}\widehat{\Theta}\left(1-\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right)\times$$

$$\times\left[\begin{array}{c}\dfrac{\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{2\left|\frac{1}{2p}-1\right|-1}\times\\ \times\sum_{k=1}^{\left|\frac{1}{2p}-1\right|-1}\frac{\left(2\left|\frac{1}{2p}-1\right|-1\right)\left(2\left|\frac{1}{2p}-1\right|-3\right)\left(2\left|\frac{1}{2p}-1\right|-5\right)...\left(2\left|\frac{1}{2p}-1\right|-2k+1\right)}{2^k\left(\left|\frac{1}{2p}-1\right|-1\right)\left(\left|\frac{1}{2p}-1\right|-2\right)...\left(\left|\frac{1}{2p}-1\right|-k\right)\left[\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)\right]^{p\left|\frac{1}{2p}-1\right|-k}}+\\ +\frac{\left(2\left|\frac{1}{2p}-1\right|-3\right)!!}{2^{\left|\frac{1}{2p}-1\right|}\left(\left|\frac{1}{2p}-1\right|-1\right)!}arctgh\left[\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]+\end{array}\right]$$

$$\pm 2i\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\left\{\delta\left(\frac{1}{2p}+\frac{1}{2}\right)\sqrt{\frac{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}{\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p}}+\right.$$

$$+\delta\left(\frac{1}{2p}+\frac{2m+1}{2}\right)\frac{\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{2m-1}\left[\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{-m+\frac{1}{2}}\times$$

$$\left.\times\left[1+\sum_{k=1}^{m-1}\frac{2^k(m-1)(m-2)...(m-k)}{(2m-3)(2m-5)...(2m-2k-1)}\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^k\right]\right\}+$$

$$+\widehat{\Theta}(p)\widehat{\Theta}\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p-1\right)\widehat{\Theta}\left(\frac{1}{2p}-\frac{1}{2}\right)\frac{2p}{1-p}\times$$

$$\times\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-\frac{1}{2}}\times$$

$$\times\ {}_2F_1\left(\frac{1}{2},-\left(\frac{1}{2p}-\frac{1}{2}\right);-\left(\frac{1}{2p}-\frac{3}{2}\right);\left(\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right)+$$

$$+\widehat{\Theta}\left(\mp\frac{A_2}{C_{32}^2}\right)\widehat{\Theta}\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$

$$\times\left\{\begin{array}{c}2p\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{1/2p}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2p};\frac{1}{2p}+1;\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)+\\ \text{or}\\ \left\{\begin{array}{c}\widehat{\Theta}\left(\frac{1}{2p}-\frac{1}{2}\right)\frac{2p}{1-p}\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-\frac{1}{2}}\\ \times\ {}_2F_1\left(\frac{1}{2},-\left(\frac{1}{2p}-\frac{1}{2}\right);-\left(\frac{1}{2p}-\frac{3}{2}\right);\left(\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right)\end{array}\right.+\end{array}\right.$$

or

$$+\widehat{\Theta}\left(\pm\frac{A_2}{C_{32}^2}\right)\begin{cases} 2p\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{1/2p}\times \\ \times\ {}_2F_1\left(\frac{1}{2},\frac{1}{2p};\frac{1}{2p}+1;\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right) \\ \text{or} \\ \begin{cases} \widehat{\Theta}\left(\frac{1}{2p}-\frac{1}{2}\right)\frac{2p}{1-p}\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-\frac{1}{2}}\times \\ \times\ {}_2F_1\left(\frac{1}{2},-\left(\frac{1}{2p}-\frac{1}{2}\right);-\left(\frac{1}{2p}-\frac{3}{2}\right);\left(\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right) \end{cases} \end{cases} +$$

$$\left.\begin{array}{c}\mp 2i\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\delta\left(\frac{1}{2p}-\frac{1}{2}\right)\times \\ \times\arcsin\left[-\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\end{array}\right\};$$

or

$$\left.+2\delta\left(\frac{1}{2p}-1\right)\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right\};$$

or

$$\begin{array}{c}\mp i\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\delta\left(\frac{1}{2p}-\frac{3}{2}\right)\times \\ \times\left\{\sqrt{\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]}+\right. \\ \left.\left.+\arcsin\left[-\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\right\}\right\};\end{array}$$

or

$$\begin{array}{c}\pm 2i\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\delta\left(\frac{1}{2p}-m-\frac{3}{2}\right)\times \\ \times\frac{\sqrt{\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]}}{2(m+1)}\times \\ \times\left\{\left[\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^m+\right. \\ \left.+\sum_{k=1}^{m-1}\frac{(2m+1)(2m-1)\ldots(2m-2k+1)}{2^{k+1}m(m-1)(m-2)\ldots(m-k)}\left(\mp\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{m-k-1}\right\}+ \\ \left.-\frac{(2m+1)!!}{2^{m+1}(m+1)!}\arcsin\left[-\sqrt{\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\right\};\end{array}$$

or

$$\pm 2(-1)^{\frac{1}{2p}-1}\widehat{\Theta}_R\left(\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\delta\left(\frac{1}{2p}-m-1\right)\times$$

$$\left.\times\sum_{k=0}^{\frac{1}{2p}-1}\begin{pmatrix}\frac{1}{2p}-1\\ k\end{pmatrix}(-1)^k\frac{\left[\pm\frac{A_2(x_0^5)^{1+p}}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]^{\frac{2k+1}{2}}}{2k+1}\right\};\tag{124}$$

1.1.2 - $C_{32}=0,\ A_2\neq 0$:

1.1.2.1 - $p\neq 1$:

$$x^5(\tau)=\left[\pm\frac{2}{1-p}\sqrt{A_2}\,(\tau+A_1)\right]^{\frac{1-p}{2}};\tag{125}$$

1.1.2.2 - $p=1$:

$$x^5(\tau)=\exp\left[\pm\sqrt{A_2}\,(\tau+A_1)\right];\tag{126}$$

1.1.3- $C_{32}\neq 0,\ A_2=0$:

$$x^5(\tau)=\pm\frac{1}{4}\frac{C_{32}^2}{x_0^5}(\tau+A_1).$$

1.2) $A_2=0$:

1.2.1 - $C_{32}\neq 0,\ C_{12}^2+C_{22}^2-C_{02}^2\neq 0$:

$$0=\tau+A_1\mp\left(x_0^5\right)^{\frac{1+p}{2}}\left\{\widehat{\Theta}\,(p)\times\right.$$

$$\times\left\{\begin{array}{c}\left[-\widehat{\Theta}\left(-\frac{1}{2p}+1\right)\frac{\left(x^5(\tau)\right)^{\frac{1}{2}-p}\left(x_0^5\right)^{\frac{p}{2}}}{p\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\frac{\Gamma\left(-\frac{1}{2p}+1\right)}{\Gamma\left(-\frac{1}{2p}\right)}\right]\left[\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)+\right.\\ \left.+\widehat{\Theta}\left(C_{02}^2-C_{12}^2-C_{22}^2\right)\widehat{\Theta}\left(\left(x^5(\tau)\right)^p-\left(x_0^5\right)^p\sqrt{\frac{C_{32}^2}{C_{02}^2-C_{12}^2-C_{22}^2}}\right)\right]\times\\ \times\ {}_3F_2\left(\frac{1}{2},-\frac{1}{4p}+\frac{1}{2},-\frac{1}{4p}+1;-\frac{1}{4p}+1,-\frac{1}{4p}+\frac{3}{2};\frac{C_{32}^2}{C_{02}^2-C_{12}^2-C_{22}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^{2p}\right)\times\end{array}\right.$$

or

$$\times\left\{\begin{array}{c}\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)\frac{\sqrt{x^5(\tau)}\left(x_0^5\right)^{-\frac{p}{2}}}{pC_{32}\sqrt{\pm 1}}B\left(\frac{1}{2p},1\right)\times\\ \times\ {}_3F_2\left(\frac{1}{2},\frac{1}{4p},\frac{1}{4p}+\frac{1}{2};\frac{1}{4p}+\frac{1}{2},\frac{1}{4p}+1;\frac{C_{02}^2-C_{12}^2-C_{22}^2}{C_{32}^2}\left(\frac{x^5(\tau)}{x_0^5}\right)^{2p}\right)\times\end{array}\right.$$

$$\times\delta\left(\frac{1}{2p}-1\right)\frac{\left(x_0^5\right)^{\frac{p}{2}}}{p\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times$$

$$\times\left\{\begin{array}{c}\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)\operatorname{arcsin}h\left[\sqrt{\frac{C_{12}^2+C_{22}^2-C_{02}^2}{C_{32}^2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]+\\ +\widehat{\Theta}\left(C_{02}^2-C_{12}^2-C_{22}^2\right)\times\\ \times\ln\left\{\left(x^5(\tau)\right)^p\left[1+\sqrt{1+\frac{C_{32}^2}{C_{12}^2+C_{22}^2-C_{02}^2}\left(\frac{x_0^5}{x^5(\tau)}\right)^{2p}}\right]\right\}\end{array}\right\}+$$

or

$$\times\delta\left(\frac{1}{2p}-m-1\right)\times$$

$$\times\left\{\begin{array}{c}\dfrac{\sqrt{\left(x^5(\tau)\right)^{2p}+\frac{\left(x_0^5\right)^{2p}C_{32}^2}{C_{12}^2+C_{22}^2-C_{02}^2}}}{p\sqrt{\pm\left(x_0^5\right)^{-p}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times\\ \times Q\left(x^5(\tau);C_{02},C_{12},C_{22},C_{32},p,x_0^5\right)+\frac{\kappa\left(x_0^5\right)^{p/2}}{p\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times\\ \times\left\{\begin{array}{c}\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)\operatorname{arcsin}h\left[\sqrt{\frac{C_{12}^2+C_{22}^2-C_{02}^2}{C_{32}^2}}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]+\\ +\widehat{\Theta}\left(C_{02}^2-C_{12}^2-C_{22}^2\right)\times\\ \times\ln\left\{\left(x^5(\tau)\right)^p\left[1+\sqrt{1+\frac{C_{32}^2}{C_{12}^2+C_{22}^2-C_{02}^2}\left(\frac{x_0^5}{x^5(\tau)}\right)^{2p}}\right]\right\}\end{array}\right\}\end{array}\right.+$$

$$+\left\{2\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)+\right.$$
$$\left.+\widehat{\Theta}\left(C_{02}^2-C_{12}^2-C_{22}^2\right)\widehat{\Theta}\left(\left(x^5(\tau)\right)^p-\left(x_0^5\right)^p\sqrt{\frac{C_{32}^2}{C_{02}^2-C_{12}^2-C_{22}^2}}\right)\right\}\times$$
$$\times\frac{\left(x^5(\tau)\right)^{\frac{1}{2}-p}\left(x_0^5\right)^{\frac{p}{2}}}{2p^2\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times$$
$$\times\ {}_3F_2\left(\tfrac{1}{2},-\tfrac{1}{4p}+\tfrac{1}{2},-\tfrac{1}{4p}+1;-\tfrac{1}{4p}+1,-\tfrac{1}{4p}+\tfrac{3}{2};\tfrac{C_{32}^2}{C_{02}^2-C_{12}^2-C_{22}^2}\left(\tfrac{x^5(\tau)}{x_0^5}\right)^{2p}\right),$$

or

$$+\delta\left(\frac{1}{2p}+m\right)\left\{-\frac{\sqrt{\dfrac{C_{32}^2}{C_{12}^2+C_{22}^2-C_{02}^2}\left(\dfrac{x_0^5}{x^5(\tau)}\right)^{2p}-1}}{p\sqrt{\pm\left(x_0^5\right)^{-p}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times\right.$$
$$\times\widetilde{Q}\left(x^5(\tau);C_{02},C_{12},C_{22},C_{32},p,x_0^5\right)-\frac{\lambda\left(x_0^5\right)^{p/2}}{p\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}}\times$$
$$\times\left[\widehat{\Theta}\left(C_{12}^2+C_{22}^2-C_{02}^2\right)\left(x_0^5\right)^{-p/2}\sqrt{\frac{C_{12}^2+C_{22}^2-C_{02}^2}{C_{32}^2}}\times\right.$$
$$\times\operatorname{arcsin}h\left(\sqrt{\frac{C_{32}^2}{C_{12}^2+C_{22}^2-C_{02}^2}}\left(\frac{x_0^5}{x^5(\tau)}\right)^{2p}\right)+$$
$$\left.-\begin{array}{c}\widehat{\Theta}\left(C_{02}^2-C_{12}^2-C_{22}^2\right)\sqrt{\frac{C_{02}^2-C_{12}^2-C_{22}^2}{C_{32}^2}}\times\\ \times\arcsin\left(\sqrt{\frac{C_{32}^2}{C_{02}^2-C_{12}^2-C_{22}^2}}\left(\frac{x_0^5}{x^5(\tau)}\right)^{2p}\right)\end{array}\right\},\tag{127}$$

where ${}_3F_2$ is the generalized hypergeometric function of class (3,2)

$$ {}_3F_2(\alpha,\beta,\gamma;\delta,\varepsilon;z) \equiv \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k\,(\gamma)_k}{(\delta)_k\,(\varepsilon)_k} \frac{z^k}{k!}, \tag{128} $$

$B(x,y)$ is the Euler beta function (or Euler integral of first kind), defined by

$$ B(x,y) \equiv \int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt, \quad Re(x)>0 \;\;,\;\; Re(y)>0 \tag{129} $$

and we put

$$ \begin{gathered} \widetilde{Q}\left(x^5(\tau); C_{02}, C_{12}, C_{22}, C_{32}, p, x_0^5\right) \equiv \\ \equiv q_{-\frac{1}{2p}-1}\left(x^5(\tau)\right)^{\frac{1}{2}+p} + \sum_{n=2}^{-\frac{1}{2p}-1} q_{-\frac{1}{2p}--n-1}\left(x^5(\tau)\right)^{\frac{1}{2}+(n+1)p}; \\ q_{-\frac{1}{2p}-1} \equiv -\frac{2p\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{2p} C_{32}^2}; \\ q_{-\frac{1}{2p}--n-1} \equiv -\frac{\left(C_{12}^2+C_{22}^2-C_{02}^2\right)\left(\dfrac{1}{2p}+n+1\right)}{\left(x_0^5\right)^{2p} C_{32}^2 \left(\dfrac{1}{2p}+n\right)} q_{-\frac{1}{2p}--n+1}, \\ n = 2, ..., -\frac{1}{2p}-1; \\ \lambda = -q_1. \end{gathered} \tag{130} $$

1.2.2 - $C_{32} \neq 0$, $C_{12}^2 + C_{22}^2 - C_{02}^2 = 0$:

$$ x^5(\tau) = \pm\frac{1}{4}\frac{C_{32}^2}{x_0^5}\left(\tau + A_1\right). \tag{131} $$

1.2.3 - $C_{32} = 0$, $C_{12}^2 + C_{22}^2 - C_{02}^2 \neq 0$:

1.2.3.1 - $p \neq \frac{1}{2}$:

$$ x^5(\tau) = \left[\pm\left(-p+\frac{1}{2}\right)\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}\left(x_0^5\right)^{-\frac{1+2p}{2}}\left(\tau+A_1\right)\right]^{\frac{2}{1-2p}}. \tag{132} $$

1.2.3.2 - $p = \frac{1}{2}$:

$$ x^5(\tau) = \exp\left[\pm\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}\left(x_0^5\right)^{-\frac{1+2p}{2}}\left(\tau+A_1\right)\right]. \tag{133} $$

1.3) $C_{32} = 0$:

1.3.1 - $A_2 \neq 0,\ C_{12}^2 + C_{22}^2 - C_{02}^2 \neq 0$:

$$0 = \tau + A_1 \mp \left(x_0^5\right)^{\frac{1+p}{2}} \left\{ \frac{\left(x_0^5\right)^{-\frac{1+p}{2}}}{p\sqrt{A_2}} \left(\frac{A_2 \left(x_0^5\right)^{1+2p}}{\pm\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right)} \right)^{\frac{1}{2p}-\frac{1}{2}} \times \right.$$

$$\times \left\{ 2\left(-1\right)^{\left(\frac{1}{2p}-\frac{3}{2}\right)} \delta\left(\frac{1}{2p} + m - \frac{1}{2}\right) \widehat{\Theta}_R \left[\frac{\pm\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right)}{\left(x_0^5\right)^{1+p} A_2} \left(\frac{x^5(\tau)}{x_0^5}\right)^p + 1 \right] \times \right.$$

$$\times \left\{ \begin{array}{c} \left[\begin{array}{c} \dfrac{\sqrt{\dfrac{\pm\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right)}{\left(x_0^5\right)^{1+p} A_2} \left(\dfrac{x^5(\tau)}{x_0^5}\right)^p + 1}}{2\left|\dfrac{1}{2p} - \dfrac{3}{2}\right| - 1} \times \\ \times \displaystyle\sum_{k=1}^{\left|\frac{1}{2p}-\frac{3}{2}\right|-1} \frac{\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-3\right)\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-5\right)\ldots\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-2k+1\right)}{2^k\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-2\right)\ldots\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-k\right)\left[\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\left|\frac{1}{2p}-\frac{3}{2}\right|-k}} + \\ + \dfrac{\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-3\right)!!}{2^{\left|\frac{1}{2p}-\frac{3}{2}\right|}\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)!} \ln\left| \dfrac{1+\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{1-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}} \right| + \end{array} \right] \\ \text{or} \\ \times \widehat{\Theta}\left(1 - \sqrt{\dfrac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^p + 1}\right) \times \\ \times \left[\begin{array}{c} \dfrac{\sqrt{\dfrac{\pm\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right)}{\left(x_0^5\right)^{1+p} A_2} \left(\dfrac{x^5(\tau)}{x_0^5}\right)^p + 1}}{2\left|\dfrac{1}{2p} - \dfrac{3}{2}\right| - 1} \times \\ \times \displaystyle\sum_{k=1}^{\left|\frac{1}{2p}-\frac{3}{2}\right|-1} \frac{\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-3\right)\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-5\right)\ldots\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-2k+1\right)}{2^k\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-2\right)\ldots\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-k\right)\left[\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\left|\frac{1}{2p}-\frac{3}{2}\right|-k}} + \\ + \dfrac{\left(2\left|\frac{1}{2p}-\frac{3}{2}\right|-3\right)!!}{2^{\left|\frac{1}{2p}-\frac{3}{2}\right|-1}\left(\left|\frac{1}{2p}-\frac{3}{2}\right|-1\right)!} arctgh\left[\sqrt{\dfrac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\dfrac{x^5(\tau)}{x_0^5}\right)^p + 1}\right] + \end{array} \right] \end{array} \right.$$

$$\pm 2i\delta\left(\frac{1}{2p} + m\right) \widehat{\Theta}_R \left[\frac{\pm\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right)}{\left(x_0^5\right)^{1+p} A_2} \left(\frac{x^5(\tau)}{x_0^5}\right)^p + 1 \right] \times$$

$$\times \frac{\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{(2m-1)} \left[\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p \right]^{-m+\frac{1}{2}} \times$$

$$\times \left\{ 1 + \sum_{k=1}^{m-1} \frac{2^k(m-1)(m-2)\ldots(m-k)}{(2m-3)(2m-5)\ldots(2m-2k-1)} \left[\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p \right]^k \right\} +$$

$$-2\delta\left(\frac{1}{2p}-\frac{1}{2}\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$

$$\times\begin{cases}\frac{1}{2}\ln\left|\frac{1+\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}{1-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}}\right|+\\ \\ \text{or}\\ \\ \widehat{\Theta}\left(1-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right)\times\\ \times arctgh\left[\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]+\end{cases}$$

$$+\delta\left(\frac{1}{2p}-\frac{1}{2}\right)\times$$

$$
\times\left\{\begin{array}{c}
\widehat{\Theta}\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p-1\right)\widehat{\Theta}\left(\frac{1}{2p}-1\right)\times\\
\times\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-1}}{\left(\frac{1}{2p}-1\right)}\times\\
\times\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2p}+1;-\frac{1}{2p}+2;\left(\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right)+\\
+\widehat{\Theta}\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\widehat{\Theta}\left(\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{A_2}\right)\times\\
\times\left\{\begin{array}{c}
\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-\frac{1}{2}}}{\left(\frac{1}{2p}-\frac{1}{2}\right)}\times\\
\times\ {}_2F_1\left(\frac{1}{2},\frac{1}{2p}-\frac{1}{2};\frac{1}{2p}+\frac{1}{2};\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)+\\
\text{or}\\
\widehat{\Theta}\left(\frac{1}{2p}-1\right)\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-1}}{\left(\frac{1}{2p}-1\right)}\times\\
\times\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2p}+1;-\frac{1}{2p}+2;\left(\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right)+
\end{array}\right.\\
+\widehat{\Theta}\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{A_2}\right)\times\\
\times\left\{\begin{array}{c}
\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-\frac{1}{2}}}{\left(\frac{1}{2p}-\frac{1}{2}\right)}\times\\
\times\ {}_2F_1\left(\frac{1}{2},\frac{1}{2p}-\frac{1}{2};\frac{1}{2p}+\frac{1}{2};\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)+\\
\text{or}\\
\widehat{\Theta}\left(\frac{1}{2p}-1\right)\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{\frac{1}{2p}-1}}{\left(\frac{1}{2p}-1\right)}\times\\
\times\ {}_2F_1\left(\frac{1}{2},-\frac{1}{2p}+1;-\frac{1}{2p}+2;\left(\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right)^{-1}\right)+
\end{array}\right.
\end{array}\right.
$$

$$
\left.\begin{array}{c}
\mp2\delta\left(\frac{1}{2p}-1\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times\\
\times2i\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]
\end{array}\right\}
$$

$$\mp 2\delta\left(\frac{1}{2p}-1\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times 2i\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\Bigg\}$$

$$\mp 2\delta\left(\frac{1}{2p}-1\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times 2i\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\Bigg\}$$

$$\mp 2\delta\left(\frac{1}{2p}-1\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times 2i\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\Bigg\}$$

$$\mp 2\delta\left(\frac{1}{2p}-1\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times 2i\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\Bigg\};$$

or

$$\pm 2\delta\left(\frac{1}{2p}-\frac{3}{2}\right)\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\Bigg\};$$

or

$$\mp i\delta\left(\frac{1}{2p}-2\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times\left\{\sqrt{\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]}+\right.$$
$$\left.+\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\right\}\Bigg\};$$

or

$$\pm 2i\delta\left(\frac{1}{2p}-m-2\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\times\frac{\sqrt{\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]}}{(2m+1)}\times$$
$$\times\left\{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^m+\right.$$
$$\left.+\sum_{k=0}^{m-1}\frac{(2m+1)(2m-1)...(2m-2k+1)}{2^{k+1}m(m-1)(m-2)...(m-k)}\left[\frac{\mp\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p\right]^{m-k-1}\right\}+$$
$$\left.-\frac{(2m+1)!!}{2^{m+1}(m+1)!}\arcsin\left[-\sqrt{\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1}\right]\right\};$$

or

$$+2\left(-1\right)^{\left(\frac{1}{2p}-\frac{3}{2}\right)}\delta\left(\frac{1}{2p}-m-\frac{3}{2}\right)\widehat{\Theta}_R\left(\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right)\times$$
$$\left.\left.\times\sum_{k=0}^{\frac{1}{2p}-\frac{3}{2}}\begin{pmatrix}\frac{1}{2p}-\frac{3}{2}\\ k\end{pmatrix}(-1)^k\frac{\left[\frac{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}{\left(x_0^5\right)^{1+p}A_2}\left(\frac{x^5(\tau)}{x_0^5}\right)^p+1\right]^{\frac{2k+1}{2}}}{(2k+1)}\right\}\right\}. \tag{134}$$

1.3.2 - $A_2\neq 0,\ C_{12}^2+C_{22}^2-C_{02}^2=0$:

1.3.2.1 - $p\neq 1$:

$$x^5(\tau)=\left[\pm\frac{1-p}{2}\sqrt{A_2}\left(\tau+A_1\right)\right]^{\frac{2}{1-p}}. \tag{135}$$

1.3.2.2 - $p=1$:

$$x^5(\tau)=\exp\left[\pm\sqrt{A_2}\left(\tau+A_1\right)\right]. \tag{136}$$

1.3.3 - $A_2=0,\ C_{12}^2+C_{22}^2-C_{02}^2\neq 0$:

1.3.3.1 - $p\neq\frac{1}{2}$:

$$x^5(\tau)=\left[\pm\left(-p+\frac{1}{2}\right)\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}\left(x_0^5\right)^{-\frac{1+2p}{2}}\left(\tau+A_1\right)\right]^{\frac{2}{1-2p}}. \tag{137}$$

1.3.3.2 - $p=\frac{1}{2}$:

$$x^5(\tau)=\exp\left[\pm\sqrt{\pm\left(C_{12}^2+C_{22}^2-C_{02}^2\right)}\left(x_0^5\right)^{-\frac{1+2p}{2}}\left(\tau+A_1\right)\right]. \tag{138}$$

1.4) $C_{32},\ \ A_2,\ C_{12}^2+C_{22}^2-C_{02}^2\neq 0$:

1.4.1 - $\left(\frac{1}{2p}-1\right)\in N \Longleftrightarrow p=\frac{1}{2(m+1)}$, $m\in N$:

$$
\begin{gathered}
0=\tau+A_1\mp\left(x_0^5\right)^{\frac{1+p}{2}}\times\\
\times\left\{\sqrt{\frac{c}{b}\left(x^5(\tau)\right)^{2p}+\frac{b}{a}\left(x^5(\tau)\right)^{p}+1}Q(x^5(\tau);a,b,c,p)+\right.\\
+\lambda\left[\widehat{\Theta}\left(\frac{ac}{b^2}\right)\widehat{\Theta}\left(\frac{1}{4}-\frac{ac}{b^2}\right)\frac{b}{\sqrt{ac}}\times\right.\\
\times\ln\left|2\sqrt{\left(\frac{ac^2}{b^2}\left(x^5(\tau)\right)^{2p}+\frac{c}{b}\left(x^5(\tau)\right)^{p}+\frac{ac}{b^2}\right)}+2\frac{c}{b}\left(x^5(\tau)\right)^{p}+1\right|+\\
+\delta\left(\frac{ac}{b^2}-\frac{1}{4}\right)2\ln\left|\frac{b}{2a}\left(x^5(\tau)\right)^{p}+1\right|+\\
+\widehat{\Theta}\left(\frac{ac}{b^2}-\frac{1}{4}\right)\frac{b}{\sqrt{ac}}\operatorname{arcsin}h\left(\frac{2\frac{c}{b}\left(x^5(\tau)\right)^{p}+1}{\sqrt{4\frac{ac}{b^2}-1}}\right)+\\
\left.\left.-\widehat{\Theta}\left(-\frac{ac}{b^2}\right)\frac{b}{\sqrt{-ac}}\arcsin\left(\frac{2\frac{c}{b}\left(x^5(\tau)\right)^{p}+1}{\sqrt{-4\frac{ac}{b^2}-1}}\right)\right]\right\},
\end{gathered}
\tag{139}
$$

where

$$Q(x^5(\tau); a, b, c, p) \equiv$$
$$\equiv q_{\frac{1}{2p}-2}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}-2\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}-2p\right)} + q_{\frac{1}{2p}-3}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}-3\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}-3p\right)} +$$
$$+\sum_{n=2}^{\frac{1}{2p}-2} q_{\frac{1}{2p}-n-2}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}-n-2\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}-(n+2)p\right)};$$

$$q_{\frac{1}{2p}-2} = \frac{b^2}{ac\left(\frac{1}{2p}-1\right)};$$

$$q_{\frac{1}{2p}-3} = -\frac{b^4\left(\frac{1}{2p}-\frac{3}{2}\right)}{ac\left(\frac{1}{2p}-1\right)\left(\frac{1}{2p}-2\right)};$$

$$q_{\frac{1}{2p}-n-2} = -\frac{b^2\left(\frac{1}{2p}-n-\frac{1}{2}\right)q_{\frac{1}{2p}-n-1} + \left(\frac{1}{2p}-n\right)q_{\frac{1}{2p}-n}}{ac\left(\frac{1}{2p}-1-n\right)},$$
$$n = 2, ..., \frac{1}{2p}-2;$$

$$\lambda = -\frac{q_0}{2} - q_1, \tag{140}$$

with

$$\begin{aligned} a &\equiv \pm\left(x_0^5\right)^p C_{32}^2; \\ b &\equiv A_2\left(x_0^5\right)^{1+p}; \\ c &= \pm\left(x_0^5\right)^{-p}\left(C_{12}^2 + C_{22}^2 - C_{02}^2\right). \end{aligned} \tag{141}$$

1.4.2 - $p = \dfrac{1}{2}$:

$$\begin{gathered} 0 = \tau + A_1 \mp \left(x_0^5\right)^{\frac{1+p}{2}} \times \\ \times \left[\widehat{\Theta}\left(\frac{ac}{b^2}\right) \widehat{\Theta}\left(\frac{1}{4} - \frac{ac}{b^2}\right) \frac{b}{\sqrt{ac}} \times \right. \\ \times \ln \left| 2\sqrt{\frac{c}{b}\left(\frac{ac}{b}\left(x^5(\tau)\right)^{2p} + \left(x^5(\tau)\right)^p + \frac{a}{b}\right)} + 2\frac{c}{b}\left(x^5(\tau)\right)^p + 1 \right| + \\ + \delta\left(\frac{ac}{b^2} - \frac{1}{4}\right) 2\ln\left|\frac{b}{2a}\left(x^5(\tau)\right)^p + 1\right| + \\ + \widehat{\Theta}\left(\frac{ac}{b^2} - \frac{1}{4}\right) \frac{b}{\sqrt{ac}} \operatorname{arcsin} h \left(\frac{2\frac{c}{b}\left(x^5(\tau)\right)^p + 1}{\sqrt{4\frac{ac}{b^2} - 1}} \right) + \\ \left. - \widehat{\Theta}\left(-\frac{ac}{b^2}\right) \frac{b}{\sqrt{-ac}} \arcsin \left(\frac{2\frac{c}{b}\left(x^5(\tau)\right)^p + 1}{\sqrt{-4\frac{ac}{b^2} - 1}} \right) \right], \end{gathered} \tag{142}$$

with a, b, c given by Eq.(141).

1.4.3 - $\dfrac{1}{2p} - 1 = -2, -3, \ldots \Longleftrightarrow p = -\dfrac{1}{2m}$, $m \in N$:

$$\begin{gathered} 0 = \tau + A_1 \mp \left(x_0^5\right)^{\frac{1+p}{2}} \times \\ \times \left\{ -\sqrt{\frac{a}{b}\left[\left(x^5(\tau)\right)^{-2p} + \left(x^5(\tau)\right)^{-p} + \frac{c}{b}\right]} \widetilde{Q}(x^5(\tau); a, b, c, p) + \right. \\ -\kappa \left[\widehat{\Theta}\left(\frac{ac}{b^2} - \frac{1}{4}\right) \operatorname{arcsin} h \left(\frac{2\frac{a}{b}\left(x^5(\tau)\right)^{-p} + 1}{\sqrt{4\frac{ac}{b^2} - 1}} \right) + \right. \\ + \delta\left(\frac{ac}{b^2} - \frac{1}{4}\right) \ln\left|\frac{2a}{b}\left(x^5(\tau)\right)^{-p} + 1\right| + \end{gathered} \tag{143}$$

$$+ \widehat{\Theta}\left(\frac{1}{4} - \frac{ac}{b^2}\right) \left(\begin{array}{c} \ln \left| \begin{array}{c} 2\sqrt{\frac{a}{b}\left(\left(x^5(\tau)\right)^{-2p} + \left(x^5(\tau)\right)^{-p} + \frac{c}{b}\right)} + \\ +2\frac{a}{b}\left(x^5(\tau)\right)^{-p} + 1 \end{array} \right| = \\ = arctgh \left(\frac{2 + \frac{b}{a}\left(x^5(\tau)\right)^p}{2\sqrt{\frac{c}{b}\left(x^5(\tau)\right)^{2p} + \frac{b}{a}\left(x^5(\tau)\right)^p + 1}} \right) \end{array} \right) \left.\left.\vphantom{\int}\right]\right\}, \tag{144}$$

where

$$\widetilde{Q}(x^5(\tau); a, b, c, p) \equiv$$
$$\equiv q_{-\frac{1}{2p}-1}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}+1\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}+p\right)} + q_{-\frac{1}{2p}-2}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}+2\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}+2p\right)} +$$
$$+\sum_{n=2}^{-\frac{1}{2p}-1} q_{-\frac{1}{2p}-n-1}\left(\frac{b}{a}\right)^{\left(\frac{1}{2p}+n+1\right)}\left(x^5(\tau)\right)^{\left(\frac{1}{2}+(n+1)p\right)};$$
$$q_{-\frac{1}{2p}-1} = -2p;$$
$$q_{-\frac{1}{2p}-2} = \frac{2p(1+p)}{(1+2p)};$$
$$q_{-\frac{1}{2p}-n-1} = \frac{\left(-\frac{1}{2p}-n+\frac{1}{2}\right)q_{-\frac{1}{2p}-n} + \frac{ac}{b^2}\left(-\frac{1}{2p}-n-1\right)q_{-\frac{1}{2p}-n+1}}{\frac{1}{2p}+n},$$
$$n = 2, ..., -\frac{1}{2p}-1;$$
$$\kappa = -\frac{1}{2}q_0 - \frac{ac}{b^2}q_1, \tag{145}$$

and a, b, c are given by Eq.(141).

2) $p = 0$:

$$x^5(\tau) = \frac{(\tau + A_1)^2\left[A_2 x_0^5 \pm (C_{32}^2 + C_{12}^2 + C_{22}^2 - C_{02}^2)\right]}{4x_0^5}. \tag{146}$$

4.2.6 Class (VI)

One has

$$F_{\pm,VI}(\zeta; q, A_2) =$$
$$= \left\{\pm[C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q}]\zeta^{2-q} \mp (x_0^5)^q C_{02}^2 \zeta^{2-2q}\right\}^{-\frac{1}{2}}. \tag{147}$$

The solution writes

$$x^0(\tau) = C_{01} + C_{02}\int d\tau \left(x^5(\tau)\right)^{-q}; \tag{148}$$

$$x^i(\tau) = C_{i1} + C_{i2}\tau, \quad i = 1, 2, 3, \tag{149}$$

with $x^5(\tau)$ given by:

1) $q \neq 0$:

1.1) $C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q} \neq 0$, $C_{02} \neq 0$:

$$x^5(\tau) = \left\{-\frac{(x_0^5)^q C_{02}^2}{C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q}} \times\right.$$
$$\left.\times\left[-\frac{q^2\left(C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q}\right)^2}{4\left(x_0^5\right)^2 C_{02}^2}(\tau + A_1)^2 - 1\right]\right\}^{\frac{1}{q}}; \tag{150}$$

1.2) $C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q} = 0$, $C_{02} \neq 0$:

$$x^5(\tau) = \left[\pm q \sqrt{\mp (x_0^5)^q \, C_{02}^2} \left(x_0^5\right)^{\frac{q-2}{2}} (\tau + A_1) \right]^{\frac{1}{q}}; \tag{151}$$

1.3) $C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q} \neq 0$, $C_{02} = 0$:

$$x^5(\tau) = \left[\pm q \frac{\sqrt{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q}\right)}}{2} \left(x_0^5\right)^{\frac{q-2}{2}} (\tau + A_1) \right]^{\frac{2}{q}}; \tag{152}$$

2) $q = 0$:

$$\begin{aligned} & x^5(\tau) = \\ & = \exp\left[\pm \sqrt{\pm \left(C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2(x_0^5)^{2-q} - (x_0^5)^q \, C_{02}^2\right)} \left(x_0^5\right)^{\frac{q-2}{2}} (\tau + A_1) \right]. \end{aligned} \tag{153}$$

4.2.7 Class (VII)

One has

$$\begin{aligned} & F_{\pm,VII}(\zeta; q, A_2) = \\ & = \left\{ \pm (x_0^5)^{-q} [C_{12}^2 + C_{22}^2 + C_{32}^2] \zeta^{2+2q} \mp (x_0^5)^q C_{02}^2 \zeta^2 + A_2 (x_0^5)^{2+q} \right\}^{-\frac{1}{2}}. \end{aligned} \tag{154}$$

The solution reads

$$x^0(\tau) = C_{01} + C_{02} \int d\tau \left(x^5(\tau)\right)^{-q}; \tag{155}$$

$$x^i(\tau) = C_{i1} + C_{i2} \int d\tau \left(x^5(\tau)\right)^{q}, \quad i = 1, 2, 3, \tag{156}$$

with $x^5(\tau)$ given by:

1) $q \neq 0$:

1.1) $C_{12}^2 + C_{22}^2 + C_{32}^2 = 0$:

1.1.1 - $C_{02} \neq 0$, $A_2 \neq 0$:

$$\begin{aligned} & 2 + 2\sqrt{\mp \frac{A_2 (x_0^5)^{2+q}}{C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^q + 1} + \\ & \mp \left\{ 1 + \exp\left[\mp (\tau + A_1) \frac{q C_{02} \sqrt{\pm 1}}{x_0^5} \right] \right\} \frac{A_2 (x_0^5)^{2+q}}{C_{02}^2} \left(\frac{x^5(\tau)}{x_0^5}\right)^q = 0; \end{aligned} \tag{157}$$

or

$$x^5(\tau) = \sqrt[q]{\mp \frac{C_{02}^2}{A_2 (x_0^5)^2} \left\{ tgh^2 \left[\mp (\tau + A_1) \frac{q C_{02} \sqrt{\pm 1}}{x_0^5} - 1 \right] \right\}}; \tag{158}$$

1.1.2 - $C_{02} = 0$, $A_2 \neq 0$:

$$x^5(\tau) = \left[\mp \frac{q}{2} \sqrt{A_2} \, (\tau + A_1) \right]^{-\frac{2}{q}}; \tag{159}$$

1.1.3 - $C_{02} \neq 0,\ A_2 = 0$:

$$x^5(\tau) = \exp\left[\pm\sqrt{\mp 1}\frac{C_{02}}{x_0^5}(\tau + A_1)\right]; \tag{160}$$

1.2) $A_2 = 0$:

1.2.1 - $C_{02} \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

$$x^5(\tau) = \left[\mp q\sqrt{\pm(C_{12}^2 + C_{22}^2 + C_{32}^2)(x_0^5)^{-1-q}}\,(\tau + A_1)\right]^{-\frac{1}{q}}; \tag{161}$$

1.3) $C_{02} = 0$:

1.3.1 - $A_2 \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

$$x^5(\tau) = \left[\frac{A_2(x_0^5)^{2+2q}}{\frac{q^2 A_2^2}{4}(x_0^5)^{2+2q}(\tau + A_1)^2 \mp (C_{12}^2 + C_{22}^2 + C_{32}^2)}\right]^{\frac{1}{q}}; \tag{162}$$

1.3.2 - $A_2 \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 = 0$:

$$x^5(\tau) = \left[\mp\frac{q}{2}\sqrt{A_2}\,(\tau + A_1)\right]^{-\frac{2}{q}}; \tag{163}$$

1.3.3 - $A_2 = 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 = 0$:

$$x^5(\tau) = \left[\mp q\sqrt{\pm(C_{12}^2 + C_{22}^2 + C_{32}^2)(x_0^5)^{-1-q}}\,(\tau + A_1)\right]^{-\frac{1}{q}}; \tag{164}$$

1.4) $C_{02},\ A_2,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

$$0 = \tau + A_1 \mp (x_0^5)^{\frac{2+q}{2}}\left\{-\frac{1}{q\sqrt{a}}\widehat{\Theta}\left(\frac{ac}{b^2} - \frac{1}{4}\right)\times\right.$$

$$\times\left[\begin{array}{c} \operatorname{arcsin} h\dfrac{2 + \dfrac{b}{a}(x^5(\tau))^q}{\dfrac{b}{a}(x^5(\tau))^q\sqrt{4\dfrac{ac}{b^2} - 1}} = \\ = \ln\left|2\dfrac{2 + \dfrac{b}{a}(x^5(\tau))^q}{\dfrac{b}{a}(x^5(\tau))^q\sqrt{4\dfrac{ac}{b^2} - 1}}\right| = \\ = -\ln\left|\dfrac{\dfrac{b}{a}(x^5(\tau))^q\sqrt{4\dfrac{ac}{b^2} - 1}}{2 + \dfrac{b}{a}(x^5(\tau))^q + 2\sqrt{\frac{c}{a}(x^5(\tau))^{2q} + \dfrac{b}{a}(x^5(\tau))^q + 1}}\right| \end{array}\right] +$$

$$
\begin{gathered}
+\frac{1}{q\sqrt{a}}\delta\left(\frac{ac}{b^2}-\frac{1}{4}\right)\times \\
\times\begin{cases}
\ln\left|\dfrac{\frac{b}{a}\left(x^5(\tau)\right)^q}{2+\frac{b}{a}\left(x^5(\tau)\right)^q}\right|+ \\
\text{or} \\
-\widehat{\Theta}\left(\frac{b}{a}\left(x^5(\tau)\right)^q\right)\widehat{\Theta}\left(-2-\frac{b}{a}\left(x^5(\tau)\right)^q\right)\times \\
\times 2arccotgh\left(\frac{b}{a}\left(x^5(\tau)\right)^q+1\right)+
\end{cases} \\
-\frac{1}{q\sqrt{a}}\widehat{\Theta}\left(\frac{ac}{b^2}-\frac{1}{4}\right)\times \\
\times\ln\left|\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q+2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}{\frac{b}{a}\left(x^5(\tau)\right)^q}\right|,
\end{gathered}
$$

or

$$
\times\left\{
\begin{gathered}
\widehat{\Theta}\left(1-\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q}{2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right)\times \\
\times\widehat{\Theta}\left(1+\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q}{2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right)\times \\
\times\left[arctgh\left(\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q}{2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right)+\frac{1}{2}\ln\left|\frac{4ac}{b^2}-1\right|\right]= \\
=\widehat{\Theta}\left(1-\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q}{2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right)\times \\
\times\widehat{\Theta}\left(1+\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q}{2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right)\times \\
\times\left[\frac{1}{2}\ln\left|\frac{2+\frac{b}{a}\left(x^5(\tau)\right)^q+2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}{-\left(2+\frac{b}{a}\left(x^5(\tau)\right)^q\right)+2\sqrt{\frac{c}{a}\left(x^5(\tau)\right)^{2q}+\frac{b}{a}\left(x^5(\tau)\right)^q+1}}\right|+\frac{1}{2}\ln\left|\frac{4ac}{b^2}-1\right|\right]
\end{gathered}
\right\}, \tag{165}
$$

where

$$
\begin{aligned}
&a\equiv\mp\left(x_0^5\right)^q C_{02}^2; \\
&b\equiv A_2\left(x_0^5\right)^{2+q}; \\
&c=\pm\left(x_0^5\right)^{-q}\left(C_{12}^2+C_{22}^2+C_{32}^2\right);
\end{aligned} \tag{166}
$$

2) $q=0$:

$$
\begin{gathered}
x^5(\tau)= \\
\exp\left[\pm\sqrt{\pm\left(x_0^5\right)^{-q}\left(C_{12}^2+C_{22}^2+C_{32}^2\right)+A_2(x_0^5)^{2+q}\mp\left(x_0^5\right)^q C_{02}^2}\left(x_0^5\right)^{\frac{2+q}{2}}(\tau+A_1)\right].
\end{gathered} \tag{167}
$$

4.2.8 Class (VIII)

One has

$$F_{\pm,VIII}(\zeta;A_2) = \left\{\pm\left[C_{12}^2 + C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-r}\right]\right\}^{-\frac{1}{2}}\zeta^{\frac{r}{2}}. \tag{168}$$

The solution writes

$$x^\mu(\tau) = C_{\mu1} + C_{\mu2}(\tau+\chi_\mu) = \widetilde{C}_{\mu1} + C_{\mu2}\tau\ ,\ \ \mu = 0,1,2,3, \tag{169}$$

where the χ_μ's are integration constant, and $\widetilde{C}_{\mu1} = C_{\mu1} + C_{\mu2}\chi_\mu$, $x^5(\tau)$ being given by

$$x^5(\tau) =$$

$$= \left[\pm(r+2)\sqrt{\pm\left[C_{12}^2 + C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{-r}\right]}\left(x_0^5\right)^{\frac{r}{2}}(\tau+A_1)\right]^{\frac{2}{r+2}}. \tag{170}$$

4.2.9 Class (IX)

One gets

$$F_{\pm,IX}(\zeta;n,A_2) =$$

$$= \left\{\pm\left[C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n}\right]\zeta^{2-n} \pm (x_0^5)^n C_{22}^2\zeta^{2-2n}\right\}^{-\frac{1}{2}}. \tag{171}$$

The solution writes

$$x^\mu(\tau) = C_{\mu1} + C_{\mu2}\tau\ ,\ \ \mu = 0,1,3, \tag{172}$$

$$x^2(\tau) = C_{21} + C_{22}\int d\tau\left(x^5(\tau)\right)^{-n}, \tag{173}$$

with $x^5(\tau)$ given by:

1) $n \neq 0$:

1.1) $C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n} \neq 0$, $C_{22} \neq 0$:

$$x^5(\tau) =$$

$$= \left\{\frac{(x_0^5)^n C_{22}^2}{C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n}}\left[\frac{n^2\left(C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n}\right)}{4(x_0^5)^2 C_{22}^2}(\tau+A_1)^2 - 1\right]\right\}^{1/n}; \tag{174}$$

1.2) $C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n} = 0$, $C_{22} \neq 0$:

$$x^5(\tau) = \left[\pm nC_{22}\sqrt{\pm1}(x_0^5)^{n-1}(\tau+A_1)\right]^{1/n}; \tag{175}$$

1.3) $C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n} \neq 0$, $C_{22} = 0$:

$$x^5(\tau) = \left[\pm\frac{n}{2}\sqrt{\pm\left(C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n}\right)}(x_0^5)^{\frac{n-2}{2}}(\tau+A_1)\right]^{2/n}; \tag{176}$$

2) $n = 0$:

$$x^5(\tau) =$$

$$= \exp\left[\pm\sqrt{\pm\left(C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2(x_0^5)^{2-n} + (x_0^5)^n C_{22}^2\right)}(x_0^5)^{\frac{n-2}{2}}(\tau+A_1)\right]. \tag{177}$$

4.2.10 Class (X)

One gets

$$F_{\pm,X}(\zeta;n,p,q,A_2)=$$
$$=\zeta^{\frac{r}{2}}\left\{\mp\left[(x_0^5)^{q_0}C_{02}^2-(x_0^5)^{q_1}C_{12}^2\zeta^{-q_1}-(x_0^5)^{q_2}C_{22}^2-(x_0^5)^{q_3}C_{32}^2\right]+A_2(x_0^5)^{-r}\right\}^{-\frac{1}{2}}. \quad (178)$$

The solution reads

$$x^\mu(\tau)=C_{\mu 1}+C_{\mu 2}\int d\tau\left(x^5(\tau)\right)^{-q_\mu}\ ,\ \ \mu=0,1,2,3\quad (ESC\ off), \quad (179)$$

with $x^5(\tau)$ given by:

$$\tau+A_1=$$
$$=\pm(x_0^5)^{-r}\frac{2(n+p+q)}{2p(n+p+q)+nq+n^2+q^2}\int\frac{dy}{\sqrt{\sum\limits_{K=0,1,2,3,5}c_K y^{\alpha_K(n,p,q)}}}, \quad (180)$$

where $y=(x^5(\tau))^{\frac{2p^2+2pq+2np+nq+n^2+q^2}{2(n+p+q)}}$, and

$$\begin{aligned} c_0 &\equiv \mp(x_0^5)^q C_{02}^2,\\ c_i &\equiv \pm(x_0^5)^{q_i}C_{i2}^2,\ \ i=1,2,3,\\ c_5 &\equiv A_1(x_0^5)^r, \end{aligned} \quad (181)$$

$$\begin{aligned} \alpha_0(n,p,q) &= \frac{2p^2+2np-2nq-2q^2}{2p\,(n+p+q)+nq+n^2+q^2},\\ \alpha_1(n,p,q) &= \frac{2p^2++4pq+4np+2nq}{2p\,(n+p+q)+nq+n^2+q^2},\\ \alpha_2(n,p,q) &= \frac{2p^2+2pq-2nq-2n^2}{2p\,(n+p+q)+nq+n^2+q^2},\\ \alpha_3 &= 0,\\ \alpha_5(n,p,q) &= \frac{2p^2+2np+2pq}{2p\,(n+p+q)+nq+n^2+q^2}. \end{aligned} \quad (182)$$

The Riemann integral in Eq.(180) is unknown, and therefore not even an implicit solution can be obtained for $x^5(\tau)$.

4.2.11 Class (XI)

The generating function is

$$F_{\pm,XI}(\zeta;n,q,A_2)=$$
$$=\zeta^{\frac{r}{2}}\left\{\mp\left[(x_0^5)^{q}C_{02}^2-(x_0^5)^{q_1}C_{12}^2\zeta^{-q_1}-(x_0^5)^{n}\left(C_{22}^2+C_{32}^2\right)\zeta^{-n}\right]+A_2(x_0^5)^{-r}\right\}^{-\frac{1}{2}}. \quad (183)$$

The solution reads

$$x^\mu(\tau)=C_{\mu 1}+C_{\mu 2}\int d\tau\left(x^5(\tau)\right)^{-q_\mu}\ ,\ \ \mu=0,1,2,3\quad (ESC\ off), \quad (184)$$

with $x^5(\tau)$ given by:

$$0 = \tau + A_1 \mp (x_0^5)^{-r} \frac{2(2n+q)}{5n^2+3nq+q^2} \int \frac{dy}{\sqrt{\sum_{K=0,1,2,3,5} c_K y^{\alpha_K(n,q)}}}, \tag{185}$$

where $y = (x^5(\tau))^{\frac{5n^2+3nq+q^2}{2n+q}}$, and

$$\begin{aligned} c_0 &\equiv \mp(x_0^5)^q C_{02}^2, \\ c_i &\equiv \pm(x_0^5)^{q_i} C_{i2}^2, \;\; i = 1,2,3, \\ c_5 &\equiv A_1 (x_0^5)^{-r}, \end{aligned} \tag{186}$$

$$\begin{aligned} \alpha_0(n,q) &= \frac{4n^2-2nq-2q^2}{5n^2+3nq+q^2}, \\ \alpha_1(n,q) &= \frac{6n^2+6nq}{5n^2+3nq+q^2}, \\ \alpha_2(n,q) &= \alpha_3(n,q) = 0, \\ \alpha_5 &= \frac{4n^2+2nq}{5n^2+3nq+q^2}. \end{aligned} \tag{187}$$

The Riemann integral in Eq.(185) is unknown, and therefore not even an implicit solution can be obtained for $x^5(\tau)$.

4.2.12 Class (XII)

One gets

$$\begin{gathered} F_{\pm,XII}(\zeta; n,p,q,A_2) = \\ = \zeta^{\frac{r}{2}} \left\{ \mp \left[(x_0^5)^q C_{02}^2 - (x_0^5)^n (C_{12}^2 + C_{22}^2)\zeta^n - (x_0^5)^{q_3} C_{32}^2 \right] + A_2 (x_0^5)^{-r} \right\}^{-\frac{1}{2}}. \end{gathered} \tag{188}$$

The solution reads

$$x^\mu(\tau) = C_{\mu 1} + C_{\mu 2} \int d\tau \left(x^5(\tau)\right)^{-q_\mu} \;\;,\;\; \mu = 0,1,2,3 \quad (ESC\ off), \tag{189}$$

with $x^5(\tau)$ given by:

$$0 = \tau + A_1 \mp (x_0^5)^{-r} \tfrac{2(n+p+q)}{p^2+pq+2np+2nq+2n^2+q^2} \int \frac{dy}{\sqrt{\sum_{K=0,1,2,3} c_K y^{\alpha_K(n,p,q)}}}, \tag{190}$$

where $y = (x^5(\tau))^{\frac{p^2+pq+2np+2nq+2n^2+q^2}{2(n+p+q)}}$, and

$$\begin{aligned} c_0 &\equiv \mp(x_0^5)^q C_{02}^2, \\ c_i &\equiv \pm(x_0^5)^{q_i} C_{i2}^2, \;\; i = 1,2,3, \\ c_5 &\equiv A_1 (x_0^5)^{-r}, \end{aligned} \tag{191}$$

$$\begin{aligned}
\alpha_0(n,q) &= \frac{-2pq+2np+2n^2-2q^2}{p^2+pq+2np+2nq+2n^2+q^2}, \\
\alpha_1(n,q) &= \alpha_2(n,q) = 0, \qquad (192) \\
\alpha_3(n,q) &= \frac{2np+6nq+4n^2}{p^2+pq+2np+2nq+2n^2+q^2}, \\
\alpha_5 &= \frac{2np+2nq+2n^2}{p^2+pq+2np+2nq+2n^2+q^2}. \qquad (193)
\end{aligned}$$

As in the two previous cases, the Riemann integral in Eq.(189) is unknown, and therefore not even an implicit solution can be obtained for $x^5(\tau)$.

5. Geodesic Motions for the 5-d. Metrics of Fundamental Interactions

5.1 Generating Function for Electromagnetic and Weak Metrics

The metrics for electromagnetic and weak interactions are characterized by the power dependence $\left(\frac{x^5}{x_0^5}\right)^{1/3}$. Then, they can be obtained from the following classes of solutions of the algebraic Einstein equations (38):

1) Class II for $m = \frac{1}{3}$, characterized therefore by the coefficient set $\widetilde{\mathbf{q}}_{II,int.} = (0,1/3,0,0,-5/3)$, $int. = e.m., weak$. One gets:

$$g_{AB,DR5,II,m=\frac{1}{3}}(x^5) = \\ = diag\left(1, -\left(\frac{x^5}{x_0^5}\right)^{\frac{1}{3}}, -1, -1, \pm\left(\frac{x^5}{x_0^5}\right)^{-\frac{5}{3}}\right). \qquad (194)$$

The geodesic generating function $F_{\pm}$ (57) takes the form

$$F_{\pm,II}\left(\zeta; m=\frac{1}{3}, A_2\right) = \\ = \left\{\pm\left[C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2\left(x_0^5\right)^{\frac{5}{3}}\right]\zeta^{\frac{5}{3}} \pm C_{12}^2\left(x_0^5\right)^{\frac{1}{3}}\zeta^{\frac{4}{3}}\right\}^{-\frac{1}{2}}. \qquad (195)$$

2) Class IV for $p = \frac{1}{3}$ ($\widetilde{\mathbf{q}}_{IV,int.} = (0,0,0,1/3,-5/3)$):

$$g_{AB,DR5,IV,p=\frac{1}{3}}(x^5) = diag\left(1,, -1, -1, -\left(\frac{x^5}{x_0^5}\right)^{\frac{1}{3}}, \pm\left(\frac{x^5}{x_0^5}\right)^{-\frac{5}{3}}\right); \qquad (196)$$

$$F_{\pm,IV}\left(\zeta; p=\frac{1}{3}, A_2\right) = \\ = \left\{\pm\left[C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2\left(x_0^5\right)^{\frac{5}{3}}\right]\zeta^{\frac{5}{3}} \pm C_{32}^2\left(x_0^5\right)^{\frac{1}{3}}\zeta^{\frac{4}{3}}\right\}^{-\frac{1}{2}}. \qquad (197)$$

3) Class IX for $n = \frac{1}{3}$ $(\widetilde{\mathbf{q}}_{IX,int.} = (0, 0, 1/3, 0, -5/3))$:

$$g_{AB,DR5,IX,n=\frac{1}{3}}(x^5) = diag\left(1, , -1, -\left(\frac{x^5}{x_0^5}\right)^{\frac{1}{3}}, -1, , \pm\left(\frac{x^5}{x_0^5}\right)^{-\frac{5}{3}}\right); \tag{198}$$

$$F_{\pm,IX}\left(\zeta; n = \frac{1}{3}, A_2\right) =$$
$$= \left\{\pm\left[C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2\left(x_0^5\right)^{\frac{5}{3}}\right]\zeta^{\frac{5}{3}} \pm C_{22}^2\left(x_0^5\right)^{\frac{1}{3}}\zeta^{\frac{4}{3}}\right\}^{-\frac{1}{2}}. \tag{199}$$

It is easily seen that, at level of both the metric structure $g_{AB,DR5}(x^5)$ and of the integrand function $F_{\pm}(\zeta; \widetilde{\mathbf{q}}, A_2)$, the following relations hold:

$$II)|_{m=\frac{1}{3}} \underset{x^1 \leftrightarrow x^3 (C_{12} \leftrightarrow C_{32})}{\Leftrightarrow} IV)|_{p=\frac{1}{3}} \Leftrightarrow$$
$$\underset{x^2 \leftrightarrow x^3 (C_{22} \leftrightarrow C_{32})}{\Leftrightarrow} IX)|_{n=\frac{1}{3}} \underset{x^1 \leftrightarrow x^2 (C_{12} \leftrightarrow C_{22})}{\Leftrightarrow} II)|_{m=\frac{1}{3}}. \tag{200}$$

This essentially means that (as it is also seen by the expressions of the coefficient sets) the three cases are equivalent — apart from a ridenomination of the spatial axes — and it is therefore possible to consider any of them without loss of generality.

The forms (195), (197), (199) of the generating function $F_{\pm,int.}(\zeta; \widetilde{\mathbf{q}}, A_2)$ ($int. = e.m.$, $weak$) for the three classes can be summarized as

$$F_{\pm,int.}(\zeta; A_2, K_{1,\pm,int.}, K_{2,int.}) = \left[\pm\left(K_{1,\pm,int.}\zeta^{\frac{5}{3}} + K_{2,int.}\zeta^{\frac{4}{3}}\right)\right]^{-\frac{1}{2}}, \tag{201}$$

where the constants $K_{1,\pm,int.}$, $K_{2,int.}$ are given by

$$\begin{aligned} & II)|_{m=\frac{1}{3}} : \\ K_{1,\pm,int.} &= C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2\left(x_{0,int.}^5\right)^{\frac{5}{3}}, \\ K_{2,int.} &= C_{12}^2\left(x_{0,int.}^5\right)^{\frac{1}{3}}; \end{aligned} \tag{202}$$

$$\begin{aligned} & IV)|_{p=\frac{1}{3}} : \\ K_{1,\pm,int.} &= C_{12}^2 + C_{22}^2 - C_{02}^2 \pm A_2\left(x_{0,int.}^5\right)^{\frac{5}{3}}, \\ K_{2,int.} &= C_{32}^2\left(x_{0,int.}^5\right)^{\frac{1}{3}}; \end{aligned} \tag{203}$$

$$\begin{aligned} & IX)|_{n=\frac{1}{3}} : \\ K_{1,\pm,int.} &= C_{12}^2 + C_{32}^2 - C_{02}^2 \pm A_2\left(x_{0,int.}^5\right)^{\frac{5}{3}}, \\ K_{2,int.} &= C_{22}^2\left(x_{0,int.}^5\right)^{\frac{1}{3}}. \end{aligned} \tag{204}$$

5.2 Generating Function for Strong and Gravitational Metrics

The metrics for strong and gravitational interactions are characterized by the power dependence $\left(\frac{x^5}{x_0^5}\right)^2$, and can be therefore obtained from the following classes of solutions:

1) Class I for $n = 2$ $(p = 0)$ $(\widetilde{\mathbf{q}}_{I,int.} = (2, -1, 2, 0, 1)$, $int. = strong, grav.)$. One gets:

$$g_{AB,DR5,I,n=2}(x^5) =$$
$$= diag\left(\left(\frac{x^5}{x_0^5}\right)^2, -\left(\frac{x^5}{x_0^5}\right)^{-\frac{4+4p}{4+p}}, -\left(\frac{x^5}{x_0^5}\right)^2, -\left(\frac{x^5}{x_0^5}\right)^p, \pm\left(\frac{x^5}{x_0^5}\right)^{\frac{p^2+2p+4}{4+p}}\right); \tag{205}$$

$$g_{AB,DR5,I,n=2,p=0}(x^5) =$$
$$= diag\left(\left(\frac{x^5}{x_0^5}\right)^2, -\left(\frac{x^5}{x_0^5}\right)^{-1}, -\left(\frac{x^5}{x_0^5}\right)^2, -1, \pm\left(\frac{x^5}{x_0^5}\right)\right); \tag{206}$$

$$F_{\pm,I}(\zeta; n = 2, p, A_2) =$$
$$= \left[\pm\left(x_0^5\right)^{-\frac{4+4p}{4+p}} C_{12}^2\zeta^{-\frac{p^2-2p}{4+p}} \pm \left(x_0^5\right)^2\left(C_{22}^2 - C_{02}^2\right)\zeta^{-\frac{p^2+4p+12}{4+p}} + \right.$$
$$\left. \pm\left(x_0^5\right)^p C_{32}^2\zeta^{-\frac{2p^2+6p+4}{4+p}} \pm A_2\left(x_0^5\right)^{-\frac{p^2+2p+4}{4+p}}\zeta^{-\frac{p^2+2p+4}{4+p}}\right]^{-1/2}, \tag{207}$$

whence the following forms of the generating function:

$$F_{\pm,I}(\zeta; n = 2, p = 0, A_2) =$$
$$= \left\{\pm C_{12}^2\left(x_0^5\right)^{-1} + \left(\pm C_{32}^2 + A_2\left(x_0^5\right)^{-1}\right)\zeta^{-1} \pm \left(x_0^5\right)^2\left(C_{22}^2 - C_{02}^2\right)\zeta^{-3}\right\}^{-\frac{1}{2}}. \tag{208}$$

The last equation can be written as

$$F_{\pm,I,int.}(\zeta; K_{1,\pm,int.}, K_{2,\pm,int.}, K_{3,\pm,int.}) =$$
$$= \left[K_{1,\pm,int.} + K_{2,\pm,int.}\zeta^{-1} + K_{3,\pm,int.}\zeta^{-3}\right]^{-\frac{1}{2}}, \tag{209}$$

with

$$\begin{aligned} K_{1,\pm,int.} &= \pm C_{12}^2\left(x_0^5\right)^{-1}, \\ K_{2,\pm,int.} &= \pm C_{32}^2 + A_2\left(x_0^5\right)^{-1}, \\ K_{3,\pm,int.} &= \pm\left(x_0^5\right)^2\left(C_{22}^2 - C_{02}^2\right). \end{aligned} \tag{210}$$

2) Class X for $q = 2$, $n = p = 0$ $(\widetilde{\mathbf{q}}_{X,int.} = (2, 0, 0, 0, 0))$:

$$g_{AB,DR5,X,q=2,n=p=0}(x^5) = diag\left(\left(\frac{x^5}{x_0^5}\right)^2, -1, -1, -1, \pm 1\right); \tag{211}$$

$$F_{\pm,X}(\zeta; q=2, n=p=0, A_2) =$$
$$= \left\{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right) + A_2 \mp \left(x_0^5\right)^2 C_{02}^2 \zeta^{-2}\right\}^{-\frac{1}{2}}. \tag{212}$$

3) Class XI for $q = 2$, $n = 0$ ($\widetilde{\mathbf{q}}_{XI,int.} = (2, 0, 0, 0, 0)$):

$$g_{AB,DR5,XI,q=2,n=0}(x^5) = diag\left(\left(\frac{x^5}{x_0^5}\right)^2, -1, -1, -1, \pm 1\right); \tag{213}$$

$$F_{\pm,XI}(\zeta; q=2, n=0, A_2) =$$
$$= \left\{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right) + A_2 \mp \left(x_0^5\right)^2 C_{02}^2 \zeta^{-2}\right\}^{-\frac{1}{2}}. \tag{214}$$

4) Class XII for $q = 2$, $n = 0$ ($\widetilde{\mathbf{q}}_{X,int.} = (2, 0, 0, 0, 0)$):

$$g_{AB,DR5,XII,q=2,n=0}(x^5) = diag\left(\left(\frac{x^5}{x_0^5}\right)^2, -1, -1, -1, \pm 1\right) \quad ; \tag{215}$$

$$F_{\pm,XII}(\zeta; q=2, n=0, A_2) =$$
$$= \left\{\pm\left(C_{12}^2 + C_{22}^2 + C_{32}^2\right) + A_2 \mp \left(x_0^5\right)^2 C_{02}^2 \zeta^{-2}\right\}^{-\frac{1}{2}} \quad . \tag{216}$$

The last three cases are perfectly equivalent at either level of the metric structure $g_{AB,DR5}(x^5)$ and of the generating function $F_{\pm}(\zeta; \widetilde{\mathbf{q}}, A_2)$:

$$X)|_{q=2,n=p=0} = XI)|_{q=2,n=0} = XII)|_{q=2,n=0}\,. \tag{217}$$

The function $F_{\pm}(\zeta)$ for all three cases can be therefore written in the compact form

$$F_{\pm,int.}(\zeta; K_{1,\pm,int.}, K_{2,int.}) = \left[\pm\left(K_{1,\pm,int.} + K_{2,int.}\zeta^{-2}\right)\right]^{-\frac{1}{2}} \quad , \tag{218}$$

where we put

$$\begin{aligned} K_{1,\pm,int.} &= \left(C_{12}^2 + C_{22}^2 + C_{32}^2\right) \pm A_2, \\ K_{2,int.} &= -\left(x_{0,int.}^5\right)^2 C_{02}^2. \end{aligned} \tag{219}$$

5.3 Geodesics for Electromagnetic and Weak Interactions

It is now possible, on account of the results of Subsect.4.1 and Subsect.5.1, to write the explicit expressions of the geodesics in $\Re_5$ corresponding to the electromagnetic and weak metrics. Eq.(64) for the energy coordinate reads, in this case:

$$\tau + A_1 =$$
$$= \pm\left(x_0^5\right)^{\frac{5}{6}} \int d\zeta F_{\pm,e.m.,weak}(\zeta; A_2, K_{1,\pm}, K_2)\Big|_{\zeta=x^5(\tau)}, \tag{220}$$

whence, from Eq.(201):

$$x^5_{\pm,int.}(\tau) = \pm\frac{1}{K^3_{1,\pm,int.}}\left[\frac{K^2_{1,\pm,int.}}{36\left(x^5_{0,int.}\right)^{\frac{5}{3}}}\left(\tau^2 + 2A_2\tau + A_2^2\right) \mp K_{2,int.}\right]^3 . \tag{221}$$

This equation can be also put in the form

$$x^5_{\pm,int.}(\tau) = \pm a_{1,\pm,int.}\left(a_{2,\pm,int.}\tau^2 + a_{3,\pm,int.}\tau + a_{4,\pm,int.}\right)^3 , \tag{222}$$

with

$$\begin{aligned} a_{1,\pm,int.} &= \frac{1}{K^3_{1,\pm,int.}}; \quad a_{2,\pm,int.} = \frac{K^2_{1,\pm,int.}}{36\left(x^5_{0,int.}\right)^{\frac{5}{3}}};\\ a_{3,\pm,int.} &= \frac{K^2_{1,\pm,int.}A_2}{18\left(x^5_{0,int.}\right)^{\frac{5}{3}}}; \; a_{4,\pm,int.} = \frac{K^2_{1,\pm,int.}A_2^2}{36\left(x^5_{0,int.}\right)^{\frac{5}{3}}} \mp K_{2,int.}. \end{aligned} \tag{223}$$

As noted in Subsect.5.1, one can, without loss of generality, consider any of the three classes (which only differ by the name of the spatial axes). Taking *e.g.* class II, we get, for the space-time coordinates of the geodesics (cfr. Eq.(63)):

$$\begin{aligned} x^\mu_{\pm,int.}(\tau) &= C_{\mu 1} + C_{\mu 2}\int d\tau\left(x^5(\tau)\right)^{-q_\mu} = \\ &= C_{\mu 1} + C_{\mu 2}\left(\tau + \chi_\mu\right) = \widetilde{C}_{\mu 1} + C_{\mu 2}\tau, \\ \widetilde{C}_{\mu 1} &= C_{\mu 1} + C_{\mu 2}\chi_\mu, \qquad \mu = 0, 2, 3; \end{aligned} \tag{224}$$

$$\begin{aligned} x^1_{\pm,int.}(\tau) &= C_{11} + C_{12}\int d\tau\left(x^5(\tau)\right)^{-q_1} = C_{11} + C_{12}\int d\tau\left(x^5(\tau)\right)^{-\frac{1}{3}} = \\ &= C_{11} + C_{12}\int d\tau\left[\pm a_{1,\pm,int.}\left(a_{2,\pm,int.}\tau^2 + a_{3,\pm,int.}\tau + a_{4,\pm,int.}\right)^3\right]^{-\frac{1}{3}} = \\ &= \begin{cases} C_{11} + C_{12}\left(\pm a_{1,\pm,int.}\right)^{-\frac{1}{3}}\dfrac{2}{\sqrt{|\Delta_\pm|}} arctg\left(\dfrac{2a_{2,\pm,int.}\tau + a_{3,\pm,int.}}{\sqrt{|\Delta_\pm|}}\right), \\ \qquad \Delta_\pm < 0, \\ C_{11} + C_{12}\left(\pm a_{1,\pm,int.}\right)^{-\frac{1}{3}}\dfrac{1}{\sqrt{\Delta_\pm}} arctg\left(\dfrac{2a_{2,\pm,int.}\tau + a_{3,\pm,int.} - \sqrt{\Delta_\pm}}{2a_{2,\pm,int.}\tau + a_{3,\pm,int.} + \sqrt{\Delta_\pm}}\right), \\ \qquad \Delta_\pm > 0, \\ C_{11} + C_{12}\left(\pm a_{1,\pm,int.}\right)^{-\frac{1}{3}}\dfrac{2}{2a_{2,\pm,int.}\tau + a_{3,\pm,int.}}, \\ \qquad \Delta_\pm = 0, \end{cases} \end{aligned} \tag{225}$$

where we put

$$\Delta_\pm = a^2_{3,\pm,int.} - 4a_{2,\pm,int.}a_{4,\pm,int.}. \tag{226}$$

5.4 Geodesics for Strong and Gravitational Interactions

Analogously to what done in the previous section for the electromagnetic and the weak interaction, one can exploit the results of Subect.4.1 and Subsect.5.2 in order to get the expressions of the geodesics in $\Re_5$ for particles subjected either to the strong force or to the gravitational one. On account of the fact that the last three metrics discussed in Subsect.5.2 are equivalent (they differ only for labelling of the spatial axes), it is possible to consider only two cases (and the related subcases).

5.4.1 **Case 1)** *(Class I for $n = 2$, $p = 0$)*

In this case the energy coordinate is determined by the equation ($int. = strong$, $grav.$)

$$\begin{gathered} \tau + A_1 = \\ = \pm \left(x^5_{0,int.}\right)^{-\frac{r}{2}} \int d\zeta F_{\pm,I,int.}(\zeta; K_{1,\pm,int.}, K_{2,\pm,int.}, K_{3,\pm,int.})\Big|_{\zeta = x^5(\tau)} = \\ = \pm \left(x^5_{0,int.}\right)^{-\frac{1}{2}} \int d\zeta \left[K_{1,\pm,int.} + K_{2,\pm,int.}\zeta^{-1} + K_{3,\pm,int.}\zeta^{-3}\right]^{-\frac{1}{2}}\Big|_{\zeta = x^5(\tau)} \end{gathered} \tag{227}$$

(cfr. Eqs.(64), (210)). Let us consider the following subcases:

i) - $K_{3,\pm,int.} = 0$:

i.1) - $K_{1,\pm,int.} = 0$, $K_{2,\pm,int.} \neq 0$:

$$x^5_{\pm,int.}(\tau) = \left[\pm\frac{2}{3}\sqrt{x^5_{0,int.}K_{2,\pm,int.}}\,(\tau + A_1)\right]^{\frac{2}{3}}; \tag{228}$$

i.2) - $K_{1,\pm,int.} \neq 0$, $K_{2,\pm,int.} = 0$:

$$x^5_{\pm,int.}(\tau) = \pm\sqrt{x^5_{0,int.}K_{1,\pm,int.}}\,(\tau + A_1); \tag{229}$$

i.3) - $K_{1,\pm,int.} \neq 0$, $K_{2,\pm,int.} \neq 0$: The function $x^5_{\pm,int.}(\tau)$ is determined implicitly by the equation

$$\tau + A_1 = \pm \left(x^5_{0,int.}\right)^{-\frac{1}{2}} \left\{ \frac{x^5_{\pm,int.}(\tau)}{\sqrt{K_{1,\pm,int.}}} \sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}} + \right.$$
$$- \frac{K_{2,\pm,int.}}{2K^{3/2}_{1,\pm,int.}} \left[\widehat{\Theta}\left(K_{1,\pm,int.}\right) \ln \left| \frac{1 + \sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}}}{1 - \sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}}} \right| + \right.$$
$$+ 2\widehat{\Theta}\left(-K_{1,\pm,int.}\right) \widehat{\Theta}\left(1 - \sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}}\right) \times$$
$$\left. \left. \times \widehat{\Theta}\left(1 + \sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}}\right) arctgh \left(\sqrt{1 + \frac{K_{2,\pm,int.}}{x^5_{0,int.} K_{1,\pm,int.}}}\right) \right] \right\} ; \tag{230}$$

ii) - $K_{2,\pm,int.} = 0$:

ii.1) - $K_{1,\pm,int.} \neq 0$, $K_{3,\pm,int.} \neq 0$: The Riemann integral at the right-hand side of Eq.(227) is unknown in this case;

ii.2) - $K_{1,\pm,int.} \neq 0$, $K_{3,\pm,int.} = 0$:

$$x^5_{\pm,int.}(\tau) = \pm \sqrt{x^5_{0,int.} K_{1,\pm,int.}} \left(\tau + A_1\right); \tag{231}$$

ii.3) - $K_{1,\pm,int.} = 0$, $K_{3,\pm,int.} \neq 0$:

$$x^5_{\pm,int.}(\tau) = \left[\pm \frac{5}{2} \sqrt{x^5_{0,int.} K_{3,\pm,int.}} \left(\tau + A_1\right) \right]^{\frac{2}{5}}; \tag{232}$$

iii) - $K_{1,\pm,int.} = 0$:

iii.1) - $K_{2,\pm,int.} \neq 0$, $K_{3,\pm,int.} \neq 0$:

iii.1.1 - $\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}} > 0$: The Riemann integral at the right-hand side of Eq.(227) is unknown;

iii.1.2 - $\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}} < 0$: The function $x^5_{\pm,int.}(\tau) = x^5(\tau)$ is determined implicitly by the equation

$$\tau + A_1 =$$
$$= \pm \left(x^5_{0,int.}\right)^{-\frac{1}{2}} \left\{ \pm i \frac{2\sqrt{K_{3,\pm,int.}}}{3K_{2,\pm,int.}} \widehat{\Theta} \left(\sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}} - \frac{1}{x^5_{\pm,int.}(\tau)} \right) \times \right.$$
$$\times \left[\left(-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}\right)^{-\frac{1}{4}} \int_0^{\sqrt{1-\frac{1}{x^5_{\pm,int.}(\tau)}\sqrt{-\frac{K_{3,\pm,int.}}{K_{2,\pm,int.}}}}} \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} + \right.$$
$$\left. + \sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}} \left(x^5_{\pm,int.}(\tau)\right)^3 - x^5_{\pm,int.}(\tau)} \right] +$$
$$- \frac{2\sqrt{K_{3,\pm,int.}}}{3K_{2,\pm,int.}} \widehat{\Theta} \left(\frac{1}{x^5_{\pm,int.}(\tau)} - \sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}} \right) \times$$
$$\times \left\{ \left[- \left(-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}\right)^{-\frac{1}{4}} \int_0^{\sqrt{1-x^5_{\pm,int.}(\tau)\sqrt{-\frac{K_{3,\pm,int.}}{K_{2,\pm,int.}}}}} \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} + \right. \right.$$
$$\left. \left. - \sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}} \left(x^5_{\pm,int.}(\tau)\right)^3 - x^5_{\pm,int.}(\tau)} \right] \right\}, \tag{233}$$

or

$$\times \left\{ \left[\left(-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}\right)^{-\frac{1}{4}} \int_0^{\sqrt{\frac{2\sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}}}{\frac{1}{x^5_{\pm,int.}(\tau)}+\sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}}}}}} \frac{dt}{\sqrt{(1-t^2)(2-t^2)}} + \right. \right.$$
$$\left. \left. - \sqrt{-\frac{K_{2,\pm,int.}}{K_{3,\pm,int.}} \left(x^5_{\pm,int.}(\tau)\right)^3 - x^5_{\pm,int.}(\tau)} \right] \right\}; \tag{234}$$

iii.2) - $K_{2,\pm,int.} \neq 0$, $K_{3,\pm,int.} = 0$:

$$x^5_{\pm,int.}(\tau) = \left[\pm \frac{2}{3} \sqrt{x^5_{0,int.} K_{3,\pm,int.}} (\tau + A_1) \right]^{\frac{2}{3}}; \tag{235}$$

iii.3) - $K_{2,\pm,int.} = 0$, $K_{3,\pm,int.} \neq 0$:

$$x^5_{\pm,int.}(\tau) = \left[\pm \frac{5}{2} \sqrt{x^5_{0,int.} K_{3,\pm,int.}} (\tau + A_1) \right]^{\frac{2}{5}}; \tag{236}$$

iv) - $K_{1,\pm,int.}, K_{2,\pm,int.}, K_{3,\pm,int.} \neq 0$: The Riemann integral at the right-hand side of Eq.(227) is unknown.

The space-time coordinates x^μ are obtained from Eq.(63) by substituting the expressions (228)-(236) of $x^5_{int.}(\tau)$ obtained in the various subcases.

5.4.2 **Case 2)** *(Classes* $X)|_{q=2,n=p=0} = XI)|_{q=2,n=0} = XII)|_{q=2,n=0})$

As stressed above, these three cases are equivalent. The fifth coordinate is determined by the equation ($int. = strong,\ grav.$)

$$\begin{aligned} \tau + A_1 &= \pm \left(x^5_{0,int.}\right)^{-\frac{\tau}{2}} \int d\zeta F_{\pm,,int.}(\zeta; K_{1,\pm,int.}, K_{2,,int.}) \Bigg|_{\zeta = x^5(\tau)} = \\ &= \pm \int d\zeta \left[\pm K_{1,\pm,int.} + K_{2,int.}\zeta^{-2}\right]^{-\frac{1}{2}} \Bigg|_{\zeta = x^5(\tau)}, \end{aligned} \tag{237}$$

whose explicit solution is

$$x^5_{\pm,int.}(\tau) = \sqrt{\pm \frac{1}{K_{1,\pm,int.}} \left[K^2_{1,\pm,int.} \left(\tau + A_1\right)^2 \mp K_{2,int.}\right]} \tag{238}$$

(we discarded the solution $x^5_{\pm,int.}(\tau) < 0$, on account of the physical meaning of the fifth coordinate, energy, in DR5).

Solution (238) can be also written as

$$x^5_{\pm,int.}(\tau) = \sqrt{\pm\alpha_{1,\pm,int.}\tau^2 \pm \alpha_{2,\pm,int.}\tau + \alpha_{3,\pm,int.}}, \tag{239}$$

where we put

$$\begin{aligned} \alpha_{1,\pm,int.} &= K_{1,\pm,int.}; \quad \alpha_{2,\pm,int.} = 2A_1 K_{1,\pm,int.}; \\ \alpha_{3,\pm,int.} &= \pm A_1^2 K_{1,\pm,int.} - \frac{K_{2,int.}}{K_{1,\pm,int.}}. \end{aligned} \tag{240}$$

The space-time coordinates are obtained by replacing (239) in Eq.(63) and using the expressions (240) and (219) of the constants. One gets, for the time coordinate:

$$x^0_{\pm,int.}(\tau) = C_{01} + C_{02}\int d\tau \left(x^5(\tau)\right)^{-q_0} = C_{01} + C_{02}\int d\tau \left(x^5(\tau)\right)^{-2} =$$
$$= C_{01} + C_{02}\int \frac{d\tau}{\pm\alpha_{1,\pm,int.}\tau^2 \pm \alpha_{2,\pm,int.}\tau + \alpha_{3,\pm,int.}} =$$
$$= \begin{cases} C_{01} + C_{02}\dfrac{2}{\sqrt{|\Delta_\pm|}} arctg\left(\dfrac{\pm 2\alpha_{1,\pm,int.}\tau \pm \alpha_{2,\pm,int.}}{\sqrt{|\Delta_\pm|}}\right), \\ \Delta_\pm < 0, \\ C_{01} + C_{02}\dfrac{1}{\sqrt{\Delta_\pm}}\ln\left|\dfrac{\pm 2\alpha_{1,\pm,int.}\tau \pm \alpha_{2,\pm,int.} - \sqrt{\Delta_\pm}}{\pm 2\alpha_{1,\pm,int.}\tau \pm \alpha_{2,\pm,int.} + \sqrt{\Delta_\pm}}\right|, \\ \Delta_\pm > 0, \\ C_{01} - C_{02}\dfrac{2}{\pm 2\alpha_{1,\pm,int.}\tau \pm \alpha_{2,\pm,int.}}, \\ \Delta_\pm = 0, \end{cases} \tag{241}$$

where

$$\Delta_\pm = \alpha^2_{2,\pm,int.} \mp 4\alpha_{1,\pm,int.}\alpha_{3,\pm,int.}. \tag{242}$$

Finally, the space coordinates read

$$x^i_{\pm,int.}(\tau) = C_{i1} + C_{i2}\int d\tau \left(x^5(\tau)\right)^{-q_i} =$$
$$= C_{i1} + C_{i2}\left(\tau + \chi_i\right) = \widetilde{C}_{i1} + C_{i2}\tau,$$
$$\widetilde{C}_{i1} = C_{i1} + C_{i2}\chi_i, \quad i = 1,2,3. \tag{243}$$

6. Gravitational Metric of the Einstein Type

Let us consider the class VI of solutions of the vacuum Einstein equations in the Power Ansatz, characterized by the coefficient set $\widetilde{\mathbf{q}}_{VI} = (q, 0, 0, 0, q-2)$, to which correspond the 5-d. metric

$$g_{AB,DR5,VI}(x^5) = diag\left(\left(\frac{x^5}{x^5_0}\right)^q, -1, -1, -1, \pm\left(\frac{x^5}{x^5_0}\right)^{q-2}\right) \tag{244}$$

and the function

$$F_{\pm,VI}(\zeta; q, A_2) =$$
$$= \left\{\pm\left[C^2_{12} + C^2_{22} + C^2_{32} \pm A_2\left(x^5_0\right)^{2-q}\right]\zeta^{2-q} \mp C^2_{02}\left(x^5_0\right)^q \zeta^{2-2q}\right\}^{-\frac{1}{2}}. \tag{245}$$

By putting $q = 1$, these equations become

$$g_{AB,DR5,VI,q=1}(x^5) = diag\left(\frac{x^5}{x_0^5}, -1, -1, -1, \pm\left(\frac{x^5}{x_0^5}\right)^{-1}\right); \tag{246}$$

$$F_{\pm,VI}(\zeta; q = 1, A_2) = \left\{\pm\left[C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2 x_0^5\right]\zeta \mp C_{02}^2 x_0^5\right\}^{-\frac{1}{2}}. \tag{247}$$

Metric (244) has standard Minkowskian structure for the spatial part, whereas the time metric coefficient is linear in the energy coordinate x^5. Then, as far as the space-time sector is concerned, it is similar to the 4-d. gravitational metric

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{248}$$

(with ϕ being the Newtonian gravitational potential), introduced by Einstein in order to account for the slowing down of clocks in a (weak) gravitational field.

Let us derive the geodesic equations for such a metric. The function $F_{\pm,VI}$ can be written as

$$F_{\pm,VI}(\zeta; K_1, K_{2,\pm.}) = \left[\pm\left(K_1 + K_{2,\pm}\zeta\right)\right]^{-\frac{1}{2}}, \tag{249}$$

where

$$\begin{aligned} K_1 &= -C_{02}^2 x_0^5; \\ K_{2,\pm} &= C_{12}^2 + C_{22}^2 + C_{32}^2 \pm A_2 x_0^5. \end{aligned} \tag{250}$$

On account of the results of Sect.4.1, one gets therefore, for the fifth coordinate of the geodesic equation:

$$x_\pm^5(\tau) = \pm\frac{K_{2,\pm}}{4x_0^5}\tau^2 \pm \frac{K_{2,\pm}A_1}{2x_0^5}\tau \pm \frac{K_{2,\pm}A_1^2}{4x_0^5} - K_1, \tag{251}$$

or also

$$x_\pm^5(\tau) = \pm a_{1,\pm}\tau^2 \pm a_{2,\pm}\tau \pm a_{3,\pm}, \tag{252}$$

where

$$a_{1,\pm} = \frac{K_{2,\pm}}{4x_0^5} \quad ; \quad a_{2,\pm} = \frac{K_{2,\pm}A_1}{2x_0^5}; \quad a_{3,\pm} = \frac{K_{2,\pm}A_1^2}{4x_0^5} \mp K_1. \tag{253}$$

The time coordinate is given by

$$x^0_{\pm}(\tau) = C_{01} + C_{02}\int d\tau\left(x^5(\tau)\right)^{-q_0} = C_{01} + C_{02}\int d\tau\left(x^5(\tau)\right)^{-1} =$$
$$= C_{01} + C_{02}\int \frac{d\tau}{\pm a_{1,\pm}.\tau^2 \pm a_{2,\pm}\tau + a_{3,\pm}.} =$$
$$= \begin{cases} C_{01} + C_{02}\dfrac{2}{\sqrt{|\Delta_{\pm}|}} arctg\left(\dfrac{\pm 2a_{1,\pm}.\tau \pm a_{2,\pm}}{\sqrt{|\Delta_{\pm}|}}\right), \\ \Delta_{\pm} < 0, \\ C_{01} + C_{02}\dfrac{1}{\sqrt{\Delta_{\pm}}}\ln\left|\dfrac{\pm 2a_{1,\pm}\tau \pm a_{2,\pm} - \sqrt{\Delta_{\pm}}}{\pm 2a_{1,\pm}\tau \pm a_{2,\pm} + \sqrt{\Delta_{\pm}}}\right|, \\ \Delta_{\pm} > 0, \\ C_{01} - C_{02}\dfrac{2}{\pm 2a_{1,\pm}\tau \pm a_{2,\pm}}, \\ \Delta_{\pm} = 0, \end{cases} \tag{254}$$

where

$$\Delta_{\pm} = a^2_{2,\pm} \mp 4a_{1,\pm}a_{3,\pm}. \tag{255}$$

Finally, the space coordinates read

$$x^i_{\pm,int.}(\tau) = C_{i1} + C_{i2}\int d\tau\left(x^5(\tau)\right)^{-q_i} =$$
$$= C_{i1} + C_{i2}(\tau + \chi_i) = \widetilde{C}_{i1} + C_{i2}\tau,$$
$$\widetilde{C}_{i1} = C_{i1} + C_{i2}\chi_i, \quad i = 1, 2, 3. \tag{256}$$

The result obtained is therefore identical to that of classes $X)_{q=2,n=p=0} = XI)|_{q=2,n=0} = XII)|_{q=2,n=0}$ for the strong and gravitational interactions (as expected on physical grounds).

7. Class VIII and the Heisenberg Time-Energy Uncertainty

As a final case we shall consider the solution of Eqs.(63), (64) for the class VIII of solutions ($\widetilde{\mathbf{q}}_{VIII} = (0, 0, 0, 0, r)$). The corresponding metric is

$$g_{AB,DR5,VIII}(x^5) = diag\left(1, -1, -1, -1, \pm\left(\frac{x^5}{x^5_0}\right)^r\right), \tag{257}$$

whose four-dimensional space-time sector is a standard Minkowski space (which, as by now familiar, represents, in the DSR framework, the electromagnetic interaction), whereas the energy exponent is undetermined.

The generating function $F_{\pm,VIII}$ reads

$$F_{\pm,VI}(\zeta; r, A_2) = \left\{\pm\left[C^2_{12} + C^2_{22} + C^2_{32} - C^2_{02} \pm A_2\left(x^5_0\right)^{-r}\right]\right\}^{-\frac{1}{2}}\zeta^{\frac{r}{2}} =$$
$$= K_{1,\pm}\zeta^{\frac{r}{2}}, \tag{258}$$

where we put

$$K_{1,\pm} = \left\{ \pm \left[C_{12}^2 + C_{22}^2 + C_{32}^2 - C_{02}^2 \pm A_2 \left(x_0^5 \right)^{-r} \right] \right\}^{-\frac{1}{2}} . \tag{259}$$

It is therefore easily got for the fifth coordinate

$$x_{\pm}^5(\tau; r) = \left[\pm \frac{r+2}{2K_{1,\pm}} \left(x_0^5 \right)^{\frac{r}{2}} (\tau + A_1) \right]^{\frac{2}{r+2}} \tag{260}$$

or

$$x_{\pm}^5(\tau; r) = \lambda_{\pm} \frac{r+2}{2} (\tau + A_1)^{\frac{2}{r+2}} \tag{261}$$

where

$$\lambda_{\pm} = \left[\pm \frac{(x_0^5)^{\frac{r}{2}}}{K_{1,\pm}} \right]^{\frac{2}{r+2}} . \tag{262}$$

As to the space-time coordinates, they read (ESC off)

$$\begin{aligned} x_{\pm}^{\mu}(\tau) &= C_{\mu 1} + C_{\mu 2} \int d\tau \left(x^5(\tau) \right)^{-q_{\mu}} = \\ &= C_{\mu 1} + C_{\mu 2} (\tau + \chi_{\mu}) = \widetilde{C}_{\mu 1} + C_{\mu 2}\tau, \\ &\mu = 0, 1, 2, 3, \quad \widetilde{C}_{\mu 1} = C_{\mu 1} + C_{\mu 2}\chi_{\mu}. \end{aligned} \tag{263}$$

Let us consider the expression (261) of the energy coordinate. Putting $A_1 = 0$ and $r = -4$, one gets

$$x_{\pm}^5(\tau; r = 4) = -\lambda_{\pm} \left(C_{02}^2, C_{12}^2, C_{22}^2, C_{32}^2, A_2, \left(x_0^5 \right), r = -4 \right) \tau^{-1} \tag{264}$$

where the parametric dependence of $\lambda_{\pm}$ has been made explicit.

Taking

$$\lambda_{\pm} = -\hbar \ell_0, \tag{265}$$

with $\hbar$ being the Planck constant, one obtains

$$x^5(\tau) = \hbar \ell_0 \tau^{-1} \Longleftrightarrow \frac{x^5}{\ell_0} \tau = \hbar \Longleftrightarrow E\tau = \hbar. \tag{266}$$

From Eq.(263) one gets

$$x^0(\tau) = ct(\tau) = \widetilde{C}_{01} + C_{02}\tau \Leftrightarrow \tau(t) = \frac{1}{C_{02}} \left(ct - \widetilde{C}_{01} \right) \tag{267}$$

where t is the time coordinate. Therefore, by putting $\widetilde{C}_{01} = 0$, $C_{02} = c$,

$$Et = \hbar \tag{268}$$

namely *Eq.(261) takes a form which reminds the quantum-mechanical, Heisenberg uncertainty relation for time and energy.* Otherwise stated, we can say that *the geodesics in a five-dimensional space-time, endowed with the 5-d. metric (257)* (with suitable values of the coefficient r and of the constants), *embedding a standard four-dimensional Minkowski space* (i.e. whose 4-d slices at $dx^5 = 0$ coincide with M), *correspond to trajectories (5-d. world lines) of minimal time-energy uncertainty.* This result seemingly indicates that the five-dimensional scheme of DR5 may play a role toward understanding certain aspects of quantum mechanics *in purely classical (geometrical) terms*. Similar conclusions on the connection between Heisenberg's principle and Kaluza-Klein theory can be drawn also in the context of the Space-Time-Matter model [2-3].

8. Complete Solutions of Geodesic Equations for the 5-d. Metrics

We discussed in the previous Section the solutions of the geodesic equations for the four phenomenological metrics of the fundamental interactions, obtained as special cases of the classes of solutions of the vacuum Einstein equations in the Power Ansatz. However, it is easily seen that they hold only for the energy ranges where the metrics are not Minkowskian (namely below threshold for the electromagnetic and weak metrics, and above threshold for the strong and gravitational ones). Moreover, in most cases the value of the parameter r was fixed (as functions of the other coefficients q_μ, $\mu = 0, 1, 2, 3$) by the structure of the Einstein equations. We want now to give the general solutions of the geodesic equations for the four interactions, starting from the general form of the metrics (8)-(11), obtained by the 5-d embedding of the 4-d. DSR phenomenological metrics in the DR5 framework. As already stressed, such a procedure leaves undetermined the fifth metric coefficient $f(x^5)$, and therefore yields r-parametrized metrics if the power forms (15)-(17) are used.

The general expression of the geodesic generating function $F_\pm(\zeta; \widetilde{\mathbf{q}}, A_2)$, which determines the geodesic motions in the Power Ansatz, is given by Eq.(57). On account of it, and of the exponent sets $\widetilde{\mathbf{q}}_{int.}$ (int.=e.m.,weak,strong,grav.), Eqs.(35)-(37), one gets, in correspondence to the 5-d. metrics of the four fundamental interactions in the Power Ansatz:

$$
\begin{aligned}
&F_{\pm,e.m./weak}(\zeta; r, A_2, x^5_{0,e.m./weak}) = \\
&= \zeta^{\frac{r}{2}} \left\{ A_2 \left(x^5_{0,e.m./weak}\right)^{-r} \mp C_{02}^2 + \right. \\
&\left. \pm \left[\left(x^5_{0,e.m./weak}\right)\zeta^{-1}\right]^{\frac{1}{3}\widehat{\Theta}_L\left(x^5_{0,e.m./weak}-x^5\right)} \left(C_{12}^2 + C_{32}^2\right) \right\}^{-\frac{1}{2}};
\end{aligned}
\tag{269}
$$

$$
\begin{aligned}
&F_{\pm,strong}(\zeta; r, A_2, x^5_{0,strong}) = \\
&= \zeta^{\frac{r}{2}} \left\{ A_2 \left(x^5_{0,strong}\right)^{-r} \pm \left(C_{12}^2 + C_{22}^2\right) + \right. \\
&\left. \pm \left[\left(x^5_{0,strong}\right)\zeta^{-1}\right]^{2\widehat{\Theta}_L\left(x^5-x^5_{0,strong}\right)} \left(C_{32}^2 - C_{02}^2\right) \right\}^{-\frac{1}{2}};
\end{aligned}
\tag{270}
$$

$$F_{\pm,grav.}(\zeta; r, A_2, x^5_{0,grav.}) =$$
$$= \zeta^{\frac{r}{2}} \left\{ \left(x^5_{0,grav.}\right)^{-r} A_2 \pm \left(x^5_{0,grav.}\right)^{?} \zeta^{-?} \left(C^2_{12} + C^2_{22}\right) + \right.$$
$$\left. \pm \left[\left(x^5_{0,grav.}\right) \zeta^{-1}\right]^{2\widetilde{\widehat{\Theta}}_L\left(x^5 - x^5_{0,grav.}\right)} \left(C^2_{32} - C^2_{02}\right) \right\}^{-\frac{1}{2}}, \tag{271}$$
$$\tag{272}$$

where the tilde and the question marks in $F_{\pm,grav.}(\zeta; r, A_2, x^5_{0,grav.})$ have the meaning clarified in Subsubsect.2.2.2.

The solutions for the geodesic equations are still given by Eqs.(63), (64). Let us distinguish the two cases of Minkowskian and non-Minkowskian behavior.

8.1 Minkowskian Behavior

This is the case of the electromagnetic and weak interactions above threshold, i.e. for $x^5 \geq x^5_{0,e.m./weak}$, and of the strong and gravitational interactions below threshold, i.e. for $x^5 \leq x^5_{0,strong}$ and $x^5 \leq x^5_{0,grav.}$ In these cases, the exponent sets of the metrics reduce to

$$\widetilde{\mathbf{q}}_{e.m./weak}\left(x^5 \geq x^5_{0,e.m./weak}\right) = (0, 0, 0, 0, r); \tag{273}$$

$$\widetilde{\mathbf{q}}_{strong}\left(0 < x^5 \leq x^5_{0,strong}\right) = (0, (0, 0), 0, r); \tag{274}$$

$$\widetilde{\mathbf{q}}_{grav.}\left(0 < x^5 \leq x^5_{0,grav.}\right) = (0, (0, 0), 0, r). \tag{275}$$

These coefficient sets are identical to that of Class VIII we already discussed in Sect.7. The corresponding solution for the fifth coordinate is therefore

$$x^5_{\pm,int.}(\tau; x^5_{0,int.}, r_{int}) = \left[\pm \frac{r_{int} + 2}{2K_{1,\pm,int.}} \left(x^5_{0,int.}\right)^{\frac{r_{int}}{2}} (\tau + A_1)\right]^{\frac{2}{r_{int}+2}}, \tag{276}$$

$$K_{1,\pm,int.} = \left\{\pm \left[C^2_{12} + C^2_{22} + C^2_{32} - C^2_{02} \pm A_2 \left(x^5_{0,int.}\right)^{-r_{int}}\right]\right\}^{-\frac{1}{2}}, \tag{277}$$

whereas the space-time coordinates are given by (ESC off)

$$x^\mu_{\pm,int.}(\tau) = C_{\mu 1} + C_{\mu 2} \int d\tau \left(x^5(\tau)\right)^{-q_\mu} =$$
$$= C_{\mu 1} + C_{\mu 2}(\tau + \chi_\mu) = \widetilde{C}_{\mu 1} + C_{\mu 2}\tau,$$
$$\mu = 0, 1, 2, 3, \quad \widetilde{C}_{\mu 1} = C_{\mu 1} + C_{\mu 2}\chi_\mu. \tag{278}$$

8.2 Non-Minkowskian Behavior

Let us consider separately the different cases.

8.2.1 a) Electromagnetic and Weak Interactions under Threshold

a.1 - $\mp C_{02}^2 + A_2 \left(x^5_{0,e.m./weak} \right)^{-r} = 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

a.1.1 - $3r+7 \neq 0$:

$$x^5_{\pm,e.m./weak.}(\tau) = \\ = \left[\pm \frac{3r+7}{6} \sqrt{\pm \left(C_{12}^2 + C_{22}^2 + C_{32}^2 \right)} \left(x^5_{0,e.m./weak} \right)^{\frac{3r+1}{6}} (\tau + A_1) \right]^{\frac{6}{3r+7}}; \tag{279}$$

a.1.2 - $r = -\frac{7}{3}$:

$$x^5_{\pm,e.m./weak.}(\tau) = \\ = \exp \left[\pm \sqrt{\pm \left(C_{12}^2 + C_{22}^2 + C_{32}^2 \right)} \left(x^5_{0,e.m./weak} \right)^{\frac{3r+1}{6}} (\tau + A_1) \right]; \tag{280}$$

a.2 - $\mp C_{02}^2 + A_2 \left(x^5_{0,e.m./weak} \right)^{-r} \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 = 0$:

a.2.1 - $r \neq -2$:

$$x^5_{\pm,e.m./weak.}(\tau) = \\ = \left[\pm \frac{r+2}{2} \sqrt{\mp C_{02}^2 + A_2 \left(x^5_{0,e.m./weak} \right)^{-r}} \left(x^5_{0,e.m./weak} \right)^{\frac{r}{2}} (\tau + A_1) \right]^{\frac{2}{r+2}}; \tag{281}$$

a.2.2 - $r = -2$:

$$x^5_{\pm,e.m./weak.}(\tau) = \\ = \exp \left[\pm \sqrt{\mp C_{02}^2 + A_2 \left(x^5_{0,e.m./weak} \right)^{-r}} \left(x^5_{0,e.m./weak} \right)^{\frac{r}{2}} (\tau + A_1) \right]; \tag{282}$$

a.3 - $\mp C_{02}^2 + A_2 \left(x^5_{0,e.m./weak} \right)^{-r} \neq 0,\ C_{12}^2 + C_{22}^2 + C_{32}^2 \neq 0$:

a.3.1 - $3r+7 \neq 0$:

$$\tau + A_1 = \pm \frac{6 \left(x^5_{0,e.m./weak} \right)^{\frac{-3r+2}{9}}}{(3r+7) \sqrt{\pm \left(C_{12}^2 + C_{22}^2 + C_{32}^2 \right)}} \times \\ \times \left[\frac{-\left(C_{12}^2 + C_{22}^2 + C_{32}^2 \right)}{C_{02}^2 \mp A_2 \left(x^5_{0,e.m./weak} \right)^{-r}} \right]^{\frac{3r+7}{6}} \int \frac{dt}{\sqrt{t^{\frac{2}{3r+7}} + 1}}, \tag{283}$$

where

$$t = \left[\frac{-C_{02}^2 \pm A_2 \left(x^5_{0,e.m./weak} \right)^{-r}}{\left(x^5_{0,e.m./weak} \right)^{1/3} \left(C_{12}^2 + C_{22}^2 + C_{32}^2 \right)} \right]^{\frac{3r+7}{2}} \left(x^5_{\pm,e.m./weak.}(\tau) \right)^{\frac{3r+7}{6}}; \tag{284}$$

a.3.2 - $r = -\frac{7}{3}$:

$$x^5_{\pm,e.m./weak.}(\tau) =$$
$$= x^5_{0,e.m./weak} \left[\frac{C^2_{12} + C^2_{22} + C^2_{32}}{-C^2_{02} \pm A_2 \left(x^5_{0,e.m./weak} \right)^{-r}} \right]^3 \times \tag{285}$$
$$\times \sinh^{-6} \left[\pm \frac{1}{6} \sqrt{\pm \left(C^2_{12} + C^2_{22} + C^2_{32} \right)} \left(x^5_{0,e.m./weak} \right)^{\frac{3r+1}{6}} (\tau + A_1) \right]. \tag{286}$$

8.2.2 b) Strong Interaction above Threshold

b.1 - $\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r} = 0$, $C^2_{02} - C^2_{32} \neq 0$:

b.1.1 - $r \neq -4$:

$$x^5_{\pm,strong}(\tau) = \left[\pm \frac{r+4}{2} (\tau + A_1) \left(x^5_{0,strong} \right)^{\frac{r+2}{2}} \sqrt{\mp \left(C^2_{02} - C^2_{32} \right)} \right]^{\frac{2}{r+4}}; \tag{287}$$

b.1.2 - $r = -4$:

$$x^5_{\pm,strong}(\tau) = \exp \left[\pm (\tau + A_1) \left(x^5_{0,strong} \right)^{\frac{r+2}{2}} \sqrt{\mp \left(C^2_{02} - C^2_{32} \right)} \right]; \tag{288}$$

b.2 - $\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r} \neq 0$, $C^2_{02} - C^2_{32} = 0$:

b.2.1 - $r \neq -2$:

$$x^5_{\pm,strong}(\tau) =$$
$$= \left[\pm \frac{r+2}{2} \sqrt{\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r}} \left(x^5_{0,strong} \right)^{\frac{r}{2}} (\tau + A_1) \right]^{\frac{2}{r+2}}; \tag{289}$$

b.2.2 - $r = -2$:

$$x^5_{\pm,strong}(\tau) =$$
$$= \exp \left[\pm \sqrt{\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r}} \left(x^5_{0,strong} \right)^{\frac{r}{2}} (\tau + A_1) \right]; \tag{290}$$

b.3 - $\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r} \neq 0$, $C^2_{02} - C^2_{32} \neq 0$:

b.3.1 - $r \neq -4$:

$$\tau + A_1 = \pm \frac{2 \left[\mp \left(C^2_{02} - C^2_{32} \right) \right]^{\frac{r-2}{4}}}{(r+4) \left[\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r} \right]^{\frac{r+4}{4}}} \int \frac{dt}{\sqrt{t^{\frac{4}{r+4}} + 1}}, \tag{291}$$

where

$$t = \left[\frac{\pm \left(C^2_{12} + C^2_{22} \right) + A_2 \left(x^5_{0,strong} \right)^{-r}}{\left(x^5_{0,strong} \right)^2 \left(\mp C^2_{02} \pm C^2_{32} \right)} \right]^{\frac{r+4}{4}} \left(x^5_{\pm,strong}(\tau) \right)^{\frac{r+4}{2}}; \tag{292}$$

b.3.2 - $r = -4$:

$$x^5_{\pm,strong}(\tau) = x^5_{0,strong}\sqrt{\frac{-C_{02}^2 + C_{32}^2}{C_{12}^2 + C_{22}^2 \pm A_2\left(x^5_{0,strong}\right)^{-r}}}\times$$
$$\times\sinh^{-1}\left[\pm\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\left(x^5_{0,strong}\right)^{\frac{r+2}{2}}(\tau + A_1)\right]. \tag{293}$$

8.2.3 c) Gravitational Interaction above Threshold

I) ? = 0:

c.I.1 - $\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r} = 0,\ C_{02}^2 - C_{32}^2 \neq 0$:

c.I.1.1 - $r \neq -4$:

$$x^5_{\pm,grav}(\tau) = \left[\pm\frac{r+4}{2}(\tau + A_1)\left(x^5_{0,grav}\right)^{\frac{r+2}{2}}\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\right]^{\frac{2}{r+4}}; \tag{294}$$

c.I.1.2 - $r = -4$:

$$x^5_{\pm,grav}(\tau) = \exp\left[\pm(\tau + A_1)\left(x^5_{0,grav}\right)^{\frac{r+2}{2}}\sqrt{\mp\left(C_{02}^2 - C_{32}^2\right)}\right]; \tag{295}$$

c.I.2 - $\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r} \neq 0,\ C_{02}^2 - C_{32}^2 = 0$:

c.I.2.1 - $r \neq -2$:

$$x^5_{\pm,grav}(\tau) =$$
$$= \left[\pm\frac{r+2}{2}\sqrt{\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r}}\left(x^5_{0,grav}\right)^{\frac{r}{2}}(\tau + A_1)\right]^{\frac{2}{r+2}}; \tag{296}$$

c.I.2.2 - $r = -2$:

$$x^5_{\pm,grav}(\tau) =$$
$$= \exp\left[\pm\sqrt{\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r}}\left(x^5_{0,grav}\right)^{\frac{r}{2}}(\tau + A_1)\right]; \tag{297}$$

c.I.2 - $\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r} \neq 0,\ C_{02}^2 - C_{32}^2 \neq 0$:

c.I.2.1 - $r \neq -4$:

$$\tau + A_1 = \pm\frac{2\left[\mp\left(C_{02}^2 - C_{32}^2\right)\right]^{\frac{r-2}{4}}}{(r+4)\left[\pm\left(C_{12}^2 + C_{22}^2\right) + A_2\left(x^5_{0,grav}\right)^{-r}\right]^{\frac{r+4}{4}}}\int\frac{dt}{\sqrt{t^{\frac{4}{r+4}} + 1}} \tag{298}$$

where

$$t=\left[\frac{\pm\left(C_{12}^2+C_{22}^2\right)+A_2\left(x_{0,grav}^5\right)^{-r}}{\left(x_{0,grav}^5\right)^2\left(\mp C_{02}^2\pm C_{32}^2\right)}\right]^{\frac{r+4}{4}}\left(x_{\pm,grav}^5(\tau)\right)^{\frac{r+4}{2}}; \tag{299}$$

c.I.2.2 - $r=-4$:

$$x_{\pm,grav}^5(\tau)=x_{0,grav}^5\sqrt{\frac{-C_{02}^2+C_{32}^2}{C_{12}^2+C_{22}^2\pm A_2\left(x_{0,grav}^5\right)^{-r}}}\times$$
$$\times\sinh^{-1}\left[\pm\sqrt{\mp\left(C_{02}^2-C_{32}^2\right)}\left(x_{0,grav}^5\right)^{\frac{r+2}{2}}(\tau+A_1)\right]. \tag{300}$$

II) $? = \tilde{2}$:

c.II.1 - $A_2\left(x_{0,grav}^5\right)^{-r}=0,\ \mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)\neq 0$:

c.II.1.1 - $r\neq-4$:

$$x_{\pm,grav}^5(\tau)=$$
$$=\left[\pm\frac{r+4}{2}(\tau+A_1)\left(x_{0,grav}^5\right)^{\frac{r+2}{2}}\sqrt{\mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)}\right]^{\frac{2}{r+4}}; \tag{301}$$

c.II.1.2 - $r=-4$:

$$x_{\pm,grav}^5(\tau)=\exp\left[\pm(\tau+A_1)\left(x_{0,grav}^5\right)^{\frac{r+2}{2}}\sqrt{\mp\left(C_{02}^2-C_{32}^2\right)}\right]; \tag{302}$$

c.II.2 - $A_2\left(x_{0,grav}^5\right)^{-r}\neq 0,\ \mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)=0$:

c.II.2.1 - $r\neq-2$:

$$x_{\pm,grav}^5(\tau)=\left[\pm\frac{r+2}{2}\sqrt{A_2\left(x_{0,grav}^5\right)^{-r}}\left(x_{0,grav}^5\right)^{\frac{r}{2}}(\tau+A_1)\right]^{\frac{2}{r+2}}; \tag{303}$$

c.II.2.2 - $r=-2$:

$$x_{\pm,grav}^5(\tau)=\exp\left[\pm\sqrt{A_2\left(x_{0,grav}^5\right)^{-r}}\left(x_{0,grav}^5\right)^{\frac{r}{2}}(\tau+A_1)\right]; \tag{304}$$

c.II.3 - $A_2\left(x_{0,grav}^5\right)^{-r}\neq 0,\ \mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)\neq 0$:

c.II.3.1 - $r\neq-4$:

$$\tau+A_1=$$
$$=\pm\left(x_{0,grav}^5\right)^{-\frac{r}{2}}\frac{2}{(r+4)\sqrt{\mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)}}\times$$
$$\times\left[\frac{\mp\left(x_{0,grav}^5\right)^{\tilde{2}}\left(C_{02}^2-C_{12}^2-C_{22}^2-C_{32}^2\right)}{A_2\left(x_{0,grav}^5\right)^{-r}}\right]^{\frac{r+4}{4}}\int\frac{dt}{\sqrt{t^{\frac{4}{r+4}}+1}}, \tag{305}$$

where

$$t=\left[\frac{A_2\left(x^5_{0,grav}\right)^{-r}}{\mp\left(x^5_{0,grav}\right)^{\tilde{2}}\left(C^2_{02}-C^2_{12}-C^2_{22}-C^2_{32}\right)}\right]^{\frac{r+4}{4}}\left(x^5_{\pm,grav}(\tau)\right)^{\frac{r+4}{2}}; \tag{306}$$

c.II.3.2 - $r=-4$:

$$x^5_{\pm,grav}(\tau)=\sqrt{\frac{\mp\left(x^5_{0,grav}\right)^{\tilde{2}}\left(C^2_{02}-C^2_{12}-C^2_{22}-C^2_{32}\right)}{A_2\left(x^5_{0,grav}\right)^{-r}}}\times$$
$$\times\sinh^{-1}\left[\pm\sqrt{\mp\left(x^5_{0,grav}\right)^{\tilde{2}}\left(C^2_{02}-C^2_{12}-C^2_{22}-C^2_{32}\right)}\left(x^5_{0,grav}\right)^{\frac{r}{2}}(\tau+A_1)\right]. \tag{307}$$

Let us stress that, in both special cases $? = 0, \tilde{2}$, the treatment is quite analogous to those of the strong interaction above energy threshold.

9. Conclusions

It must be by now clear, from the examples discussed in the previous Sections, that the explicit form of the geodesics in $\Re_5$ (namely, its dynamics) strictly depends on the sets of metric exponents $\widetilde{\mathbf{q}}_{int.}$ (Eqs.(18)-(20)), which determine $x^A(\tau)$ through the knowledge of the generating function $F_{\pm}(\zeta;\widetilde{\mathbf{q}},A_2)$.

But — as is easily seen from their expressions — the exponent sets $\widetilde{\mathbf{q}}_{int.}$ are discontinuous at the threshold energy $x^5_{0,int.}$:

$$\lim_{x^5\longrightarrow x^{5+}_{0,int.}}\widetilde{\mathbf{q}}_{int.}(x^5)\neq\lim_{x^5\longrightarrow x^{5-}_{0,int.}}\widetilde{\mathbf{q}}_{int.}(x^5), \tag{308}$$

namely, for a given interaction, different sets are obtained in the two different energy ranges (below and above threshold). This entails among the others that, as done in Subsect.2.2.2, it is necessary to use the right and left specifications of the Heaviside function in order to write $\widetilde{\mathbf{q}}_{int.}$ in the compact form (35)-(37), valid on the whole energy range. In turn, such a discontinuity in $\widetilde{\mathbf{q}}_{int.}$ at $x^5_{0,int}$ causes an analogous behavior in the geodesic motions. In fact, let $x^5_{int.,<}(\tau)$, $x^5_{int.,>}(\tau)$ denote the solutions (65) of the geodesic equation for the fifth coordinate under and above threshold, respectively. Then, it is possible to impose *e.g.* $x^5_{int.,<}(\overline{\tau})=x^5_{0,int.}$ and find the corresponding value $\overline{\tau}\in R$. However, if such a value is replaced in the geodesic solution corresponding to the other energy range, one finds in general $x^5_{int.,>}(\overline{\tau})=\overline{x^5_{int.}}\neq x^5_{0,int.}$.

The situation is exactly analogous to that one encounters in the case of the Killing symmetries (see ref.[10]). The non-trivial "bifurcation of dynamics" in the two energy ranges is clearly related to the nature change (from parameter to coordinate) the variable x^5 undergoes in the passage DSR→DR5. Therefore, dynamic structures present in an energy range in which the space-time sector is standard Minkowskian — or at least its metric coefficients are constant — may no longer occur when (in a different energy range) the space-time of $\Re_5$ becomes Minkowskian deformed, and *viceversa*.

Such a change of role of energy in the geometrical embedding of $\widetilde{M}$ in $\Re_5$ implies also, in full analogy with the case of the Killing isometries, that the dynamics in a given 4-d. space $\widetilde{M}\left(x^5 = \overline{x^5}\right)$ is *different* from the dynamics obtained for the slice of $\Re_5$ at constant energy $x^5 = \overline{x^5}$ with space-time sector coinciding with $\widetilde{M}\left(x^5 = \overline{x^5}\right)$. Symbolically one has:

$$\text{Dynamics in } \Re_5|_{dx^5=0 \Leftrightarrow x^5=\overline{x^5}} \neq \text{Dynamics in } \widetilde{M}\left(x^5 = \overline{x^5}\right). \qquad (309)$$

In fact the change of role of x^5 causes the destruction of the non-homogeneous linearity in τ of the geodesic motions in DSR, which is no longer recovered in the inverse process of slicing of $\Re_5$ at $dx^5 = 0$. This is again at variance with the metric level, where (see Eq.(7)) the constant-energy sections of $\Re_5$ at $x^5 = \overline{x^5}$ are endowed with the same metric structure of $\widetilde{M}\left(\overline{x^5}\right)$. Again, as in the case of the Killing symmetries, it is possible to understand this point by remembering that one is considering sections of a genuine Riemannian space, which therefore do keep memory of the fifth coordinate.

An explicit example of the key dynamic role played by the fifth coordinate in the embedding process is provided by the results of Sect.7 for the geodesics relevant to class VIII of solutions of the 5-d. Einstein's equations. As already noted, the 5-d. metric (257) corresponding to the exponent set $\widetilde{\mathbf{q}}_{VIII} = (0, 0, 0, 0, r)$ has a standard Minkowski structure for its space-time sector. In spite of this, the embedding of such a Minkowski space in $\Re_5$ (*i.e.* the presence and the form of the fifth metric coefficient) makes the dynamic behavior genuinely non-trivial, because to the standard geodesic motion of M it is added the further condition that the geodesics must correspond to a minimal value of the time-energy uncertainty.

We can therefore conclude that not only Killing isometries (as discussed in ref.[10], but dynamics, too, depends on the geometrical framework. This further supports the deep physical (not only mathematical) significance of the geometrical embedding of $\widetilde{M}$ in $\Re_5$.

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A Riemannian Structure of $\Re_5$

The vacuum Einstein equations in the Riemannian space $\Re_5$ are [6-8]

$$R_{AB} - \frac{1}{2} g_{DR5} R = \Lambda_{(5)} g_{AB,DR5}, \qquad (A.1)$$

where R_{AB} and $R = R_A^A$ are the five-dimensional Ricci tensor and scalar curvature, respectively, and $\Lambda_{(5)}$ is the "cosmological" constant, which may, in principle, depend on both the energy E and the space-time coordinates x: $\Lambda_{(5)} = \Lambda_{(5)}(x, E)$. As is well known, the Ricci tensor explicitly reads

$$R_{AB} = \partial_I \Gamma^I_{AB} - \partial_B \Gamma^I_{AI} + \Gamma^I_{AB} \Gamma^K_{IK} - \Gamma^K_{AI} \Gamma^I_{BK}, \qquad (A.2)$$

with the second-kind Christoffel symbols $\Gamma^{I}_{AB} = \left\{ \begin{matrix} I \\ AB \end{matrix} \right\}$ given by

$$2\Gamma^{I}_{AB} = g^{IK}_{DR5}(\partial_B g_{KA,DR5} + \partial_A g_{KB,DR5} - \partial_K g_{AB,DR5}). \tag{A.3}$$

For reference, let us give the explicit expression of the main geometric quantities in $\Re_5$ [6-8].

- Connection $\Gamma^{A}_{BC}(x^5)$ (the prime denotes derivation with respect to x^5):

$$\begin{cases} \Gamma^{0}_{05} = \Gamma^{0}_{50} = \dfrac{b_0'}{2b_0} \; ; \; \Gamma^{1}_{15} = \Gamma^{1}_{51} = \dfrac{b_1'}{2b_1}; \\[2ex] \Gamma^{2}_{25} = \Gamma^{2}_{52} = \dfrac{b_2'}{2b_2} \; ; \; \Gamma^{3}_{35} = \Gamma^{3}_{53} = \dfrac{b_3'}{2b_3}; \\[2ex] \Gamma^{5}_{00} = -\dfrac{b_0 b_0'}{2f} \; ; \; \Gamma^{5}_{11} = \dfrac{b_1 b_1'}{2f} \; ; \; \Gamma^{5}_{22} = \dfrac{b_2 b_2'}{2f}; \\[2ex] \Gamma^{5}_{33} = \dfrac{b_3 b_3'}{2f} \; ; \; \Gamma^{5}_{55} = \dfrac{f'}{2f} \; . \end{cases} \tag{A.4}$$

The Riemann-Christoffel (curvature) tensor in $\Re_5$ is given by

$$R^{A}_{BCD}(x^5) = \partial_C \Gamma^{A}_{BD} - \partial_D \Gamma^{A}_{BC} + \Gamma^{A}_{KC}\Gamma^{K}_{BD} - \Gamma^{A}_{KD}\Gamma^{K}_{BC}. \tag{A.5}$$

- Riemann-Christoffel tensor $R_{ABCD}(x^5)$:

$$R_{0101} = \frac{b_0' b_1'}{4f}; \; R_{0202} = \frac{b_0' b_2'}{4f}; \; R_{0303} = \frac{b_0' b_3'}{4f}; \tag{A.6}$$

$$R_{0505} = \frac{b_0' f' b_0 + (b_0')^2 - 2b_0'' b_0 f}{4b_0 f}; \tag{A.7}$$

$$R_{1212} = -\frac{b_1' b_2'}{4f}; \; R_{1313} = -\frac{b_1' b_3'}{4f}; \; R_{1515} = \frac{b_1' f' b + (b_1')^2 - 2b_1'' b_1 f}{4b_1 f}; \tag{A.8}$$

$$R_{2323} = -\frac{b_2' b_3'}{4f}; \; R_{2525} = \frac{b_2' f' b_2 + (b_2')^2 - 2b_2'' b_2 f}{4cf}; \tag{A.9}$$

$$R_{3535} = \frac{b_3' f' b_3 + (b_3')^2 - 2b_3'' b_3 f}{4b_3 f}. \tag{A.10}$$

- Ricci tensor $R_{AB}(x^5)$:

$$R_{00} = -\frac{1}{2}\frac{b_0''}{f} - \frac{b_0'}{4f}\left(-\frac{b_0'}{b_0} + \frac{b_1'}{b} + \frac{b_2'}{b_2} - \frac{f'}{f}\right); \tag{A.11}$$

$$R_{11} = \frac{1}{2}\frac{b_1''}{f} + \frac{b_1'}{4f}\left(\frac{b_0'}{b_0} - \frac{b_1'}{b_1} + \frac{b_2'}{b_2} + \frac{b_3'}{b_3} - \frac{f'}{f}\right); \tag{A.12}$$

$$R_{22} = \frac{1}{2}\frac{b_2''}{f} + \frac{b_2'}{4f}\left(\frac{b_0'}{b_0} + \frac{b_1'}{b_1} - \frac{b_2'}{b_2} + \frac{b_3'}{b_3} - \frac{f'}{f}\right); \tag{A.13}$$

$$R_{33} = \frac{1}{2}\frac{b_3''}{f} + \frac{b_3'}{4f}\left(\frac{b_0'}{b_0} + \frac{b_1'}{b_1} + \frac{b_2'}{b_2} - \frac{b_3'}{b_3} - \frac{f'}{f}\right); \tag{A.14}$$

$$\begin{aligned} R_{44} = &-\frac{1}{2}\left(\frac{b_0'}{b_0} + \frac{b_1'}{b_1} + \frac{b_2'}{b_2} + \frac{b_3'}{b_3}\right)' + \frac{f'}{4f}\left(\frac{b_0'}{b_0} + \frac{b_1'}{b_1} + \frac{b_2'}{b_2} + \frac{b_3'}{b_3}\right) - \\ &-\frac{1}{4}\left[\left(\frac{b_0'}{b_0}\right)^2 + \left(\frac{b_1'}{b_1}\right)^2 + \left(\frac{b_2'}{b_2}\right)^2 + \left(\frac{b_3'}{b_3}\right)^2\right]. \end{aligned} \tag{A.15}$$

- Scalar curvature $R(x^5)$:

$$\begin{aligned} R(x^5) = &\frac{b_1 b_1' f(b_2' b_3 b_0 + b_3' b_2 b_0 + b_0' b_2 b_3) + b_2 b_3\left[2b_1'' b_1 f - b_0 (b_1')^2 f - b_0 b_1' f' b_1\right]}{4b_1^2 f^2 b_2 b_3 b_0} + \\ &+\frac{b_2' b_2 f(b_1' b_3 b_0 + b_3' b_0 b_1 + b_0' b_3 b_1) + b_0 b_1 b_3\left[2b_2'' b_2 f - (b_2')^2 f - b_2' f' b_2\right]}{4b_2^2 f^2 b_3 b_0 b_1} + \\ &+\frac{b_3' b_3 f(b_1' b_0 b_2 + b_2' b_0 b_1 + b_0' b_2 b_1) + b_0 b_1 b_2\left[2b_3'' b_3 f - (b_3')^2 f - b_3' f' b_3\right]}{4b_3^2 f^2 b_2 b_0 b_1} + \\ &+\frac{b_0' b_0 f\,(b_1' b_2 b_3 + b_2' b_3 b_1 + b_3' b_2 b_1) + b_1 b_2 b_3\left[2b_0'' b_0 f - (b_0')^2 f - b_0' f' b_0\right]}{4b_0^2 f^2 b_2 b_3 b_1} + \\ &+\frac{1}{4f^2 b_1^2 b_2^2}\left\{b_2^2\left[2b_1'' b_1 f - (b_1')^2 f - b_1' f' b_1\right] + b_1^2\left[2b_2'' b_2 f - (b_2')^2 f - b_2' f' b_2\right]\right\} + \\ &+\frac{1}{4f^2 b_0^2 b_3^2}\left\{b_0^2\left[2b_3'' b_3 f - (b_3')^2 f - b_3' f' b_3\right] + b_3^2\left[2b_0'' b_0 f - (b_0')^2 f - b_0' f' b_0\right]\right\}. \end{aligned} \tag{A.16}$$

B Energy-Dependent Phenomenological Metrics in DSR

A detailed derivation and discussion of the DSR energy-dependent phenomenological metrics for all the four interactions can be found in refs. [8]. Here, we confine ourselves to recall their basic features:

1) Both the electromagnetic and the weak metric show the same functional behavior, namely

$$g_{DSR,e.m,weak.}(E) = diag\left(1, -b^2_{e.m.,weak}(E), -b^2_{e.m.,weak}(E), -b^2_{e.m.,weak}(E)\right); \tag{B.1}$$

$$b^2_{e.m.,weak}(E) = \begin{cases} (E/E_{0e.m.,weak})^{1/3}, 0 \leq E \leq E_{0e.m.,weak} \\ 1, E_{0e.m.,weak} \leq E \end{cases} =$$
$$= 1 + \Theta(E_{0e.m.,weak} - E)\left[\left(\frac{E}{E_{0e.m.,weak}}\right)^{1/3} - 1\right], E > 0. \tag{B.2}$$

(where $\Theta(x)$ is the Heaviside theta function, stressing the piecewise structure of the metric), with the only difference between them being the threshold energy E_0, the energy value at which the metric parameters are constant, *i.e.* the metrics become Minkowskian; the fits to the experimental data yield

$$E_{0e.m.} = (4.5 \pm 0.2)\,\mu eV\,;\ E_{0weak} = (80.4 \pm 0.2)\,GeV. \tag{B.3}$$

The metric parameter exhibits a *"sub-Minkowskian"* behavior, *i.e.* $b(E)$ approaches 1 from below as energy increases.

2) For the strong interaction, the metric reads:

$$g_{DSR,strong}(E) =$$
$$= diag\left(b^2_{strong}(E), -b^2_{1,strong}(E), -b^2_{2,strong}(E), -b^2_{strong}(E)\right); \tag{B.4}$$

$$b^2_{strong}(E) = \begin{cases} 1, 0 \leq E \leq E_{0strong} \\ (E/E_{0strong})^2, E_{0strong} \leq E \end{cases} =$$
$$= 1 + \Theta(E - E_{0,strong})\left[\left(\frac{E}{E_{0,strong}}\right)^2 - 1\right], E > 0; \tag{B.5}$$

$$b^2_{1,strong}(E) = \left(\sqrt{2}/5\right)^2; \tag{B.6}$$

$$b^2_{2,strong} = (2/5)^2. \tag{B.7}$$

with

$$E_{0,strong} = (367.5 \pm 0.4)\,GeV. \tag{B.8}$$

Let us stress that, in this case, contrarily to the electromagnetic and the weak ones, *a deformation of the time coordinate occurs*; moreover, *the three-space is anisotropic,* with two spatial parameters constant (but different in value) and the third one variable with energy in an *"over-Minkowskian" way.*

3) In the gravitational case, the form of the metric is:

$$g_{DSR,grav}(E) = diag\left(b^2_{grav}(E), -b^2_{1,grav}(E), -b^2_{2,grav}(E), -b^2_{grav}(E)\right); \tag{B.9}$$

$$b^2_{grav}(E) = \begin{cases} 1 \ , 0 \leq E \leq E_{0grav} \\ \frac{1}{4}(1 + E/E_{0grav})^2, E_{0grav} \leq E \end{cases} =$$

$$= 1 + \Theta(E - E_{0,grav.}) \left[\frac{1}{4} \left(1 + \frac{E}{E_{0,grav.}} \right)^2 - 1 \right], E > 0 \tag{B.10}$$

(the coefficients $b^2_{1,grav}(E)$ and $b^2_{2,grav}(E)$ are presently undetermined at phenomenological level), with

$$E_{0,grav} = (20.2 \pm 0.1)\, \mu eV. \tag{B.11}$$

Analogously to the strong case, the gravitational metric (B.9), (B.10) is time-deformed and over-Minkowskian.

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