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On the Noncommutative Space-time Bianchi I Universe and Particles Pair Creation Process

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Abstract: Using an approach of modified Euler-Lagrange field equations obtained from an invariant action under infinitesimal modified general coordinates, local Lorentz and $U_*(1)$ gauge transformations together with the corresponding Seiberg-Witten maps of the dynamical fields, a generalized Dirac equation in the presence of a constant electric field and a noncommutative cosmological anisotropic Bianchi I universe is derived and the particles pair creation process is studied.

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1. Introduction

The standard concept of space-time as a geometric manifold is based on the notion of a manifold whose points are locally labeled by a finite number of real coordinates. However, it is generally believed that this picture of space-time as a manifold should break down at very short distances of the order of the Planck length. This implies that the mathematical concepts of High energy physics has to be changed or more precisely our classical geometric concepts may not be well suited for the description of physical phenomenon at short distances. Noncommutativity is a mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables such as position and momentum. The most concrete motivation for space-time noncommutativity, where

the commutation relations for the canonical variables (coordinate-momentum operators)

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were generalized to non trivial ones of coordinate operators came from the work of ref.[1] ,where it was shown that combining Heisenberg's uncertainty principle with Einstein's theory of classical gravity leads to the conclusion that ordinary space-time loses any operational meaning at short distances; that is a space-time coordinate with a great accuracy ε causes an uncertainty in momentum or energy of the order of $\frac{1}{\epsilon}$ which is transmitted to the system and concentrated at some time in the localization region. The associated energy-momentum tensor $T_{\mu\nu}$ generates a gravitational field according to Einstein equations.The smaller the uncertainty Δx_{μ}, the stronger will be the gravitational field generated by the measurement of coordinates. When this field becomes so strong as to prevent light from leaving the region in question, an operational meaning can no longer be attached to the localization. Exploring the limitations of localization measurements which are due to the possible black hole creation by concentration of energy, one arrives at uncertainty relations among space-time coordinates which can be traced back to the commutation relations:

$$[\widehat{x}^{\mu}, \widehat{x}^{\nu}] = i\theta^{\mu\nu} = \frac{i}{\Lambda_{NC}^{2}} C^{\mu\nu} \tag{1}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix and $C^{\mu\nu}$ are dimensionless parameters which are presumably of order unity, and $.\Lambda_{NC}$ is a scale energy where the noncommutative effects of the space-time become relevant. From a theoretical point of view and model building, it is a challenge to formulate a theory of gravitation like general relativity on noncommutative manifold and there are different approaches in the literature. The main problem is that it is difficult to implement symmetries such as general coordinate covariance and local Lorentz invariance and to define derivatives which are torsion free and satisfy the metricity condition. In a flat space-time, to get non commutative local gauge theories, but with Lorentz violation symmetry, a formulation within the enveloping algebra approach has been proposed $[2]-[3]$. Following a similar idea a gauge formulation of gravity is proposed. It is a theory of general relativity on curved space-time with canonical preservation of noncommutative space-time commutative relations and based partially in implementing symmetries on flat noncommutative space-time $[4]-[6]$. The resulting theory appears to be a noncommutative extension of unimodular theory of gravitation. Following ref.[7], and contrary to the approach of ref.[8], the goal of this paper is to use a Lagrangian formalism, derive a modified Dirac equation in a noncommutative curved space-time and in the presence of a constant electric field and compute the number density of particles pair creation in a noncommutative cosmological anisotropic Bianchi I universe. The paper is organized as follows: In section 2, we derive the modified Dirac equation and compute the number density of particle creation. We discuss the weak and strong field limits. Finally, in section 3, we draw our conclusions.

2. Formalism

The noncommutative space-time is characterized by the operators $\widehat{x}^{\mu}$ satisfying the relation (1). In order to preserve this relation, generalized infinitesimal general coordinates and local Lorentz transformations at the level the noncommutative space-time manifold are derived (see ref.[7]). Moreover, the Seiberg-Witten maps of the various dynamical fields are also obtained:

$$\begin{aligned}
\widehat{\psi} &= \widetilde{\psi} + \frac{1}{2}\theta^{\mu\nu} A_{\nu}\partial_{\mu}\widetilde{\psi} + \frac{i}{8}\theta^{\mu\nu}\left[A_{\mu}, A_{\nu}\right]\widetilde{\psi} + O\left(\theta^{2}\right)\\
\widehat{A}_{\xi} &= A_{\xi} + \frac{1}{4}\theta^{\mu\nu}\left\{A_{\nu}, \partial_{\mu}A_{\xi}\right\} + \frac{1}{4}\theta^{\mu\nu}\left\{F_{\mu\xi}, A_{\nu}\right\} + O\left(\theta^{2}\right)\\
\widehat{\omega}_{\mu} &= \omega_{\mu} - \frac{1}{4}\theta^{\alpha\beta}\left\{\omega_{\alpha}, \partial_{\beta}\omega_{\mu} + \Pi_{\beta\mu}\right\} + O\left(\theta^{2}\right).\\
\widehat{e}^{a}_{\mu} &= e^{a}_{\mu} + \frac{1}{4}\theta^{\rho\sigma}\varepsilon^{a}_{bcd}\left(\partial_{\sigma}e^{b}_{\mu}\omega^{cd}_{\rho} - \frac{1}{2}\omega^{b}_{\rho k}e^{k}_{\mu}\omega^{cd}_{\sigma}\right) + O\left(\theta^{2}\right)
\end{aligned}$$

where ψ, A_{ξ}, ω_{μ} and e^{a}_{μ} are the spinor, gauge, spin connection and vierbein fields respectively and

$$\begin{aligned}
\widetilde{\psi} &= \psi + \frac{1}{2}\theta^{\mu\nu}\omega_{\nu}\partial_{\mu}\psi + \frac{i}{8}\theta^{\mu\nu}\left[\omega_{\mu}, \omega_{\nu}\right]\psi + O\left(\theta^{2}\right)\\
\Pi_{\beta\mu} &= e^{a}_{\mu}e^{b}_{\nu}\Pi_{ab} \qquad (2)\\
\Pi_{ik} &= i\left[\Delta_{i}, \Delta_{k}\right] = \frac{1}{2}R^{ab}_{ik}\Sigma_{ab} + T^{c}_{ik}\Delta_{c}\\
R^{ab}_{ik} &= \partial_{a}\omega^{ik}_{b} - \partial_{b}\omega^{ik}_{a} + \omega^{i}_{ac}\omega^{ck}_{b} - \omega^{i}_{bc}\omega^{ck}_{b}\\
T^{c}_{ab} &= (\Delta_{a}e^{\mu}_{b} - \Delta_{b}e^{\mu}_{a})e^{c}_{\mu}\\
F_{\rho\mu} &= \partial_{\rho}A_{\mu} - \partial_{\mu}A_{\rho} - i\left[A_{\rho}, A_{\mu}\right] \qquad (3)\\
\omega_{\mu} &= \omega^{ab}_{\mu}\Sigma_{ab} = \omega^{ab}_{k}e^{k}_{\mu}\Sigma_{ab}\\
\omega^{ab}_{\mu} &= \omega^{ab}_{k}e^{k}_{\mu} \qquad (4)
\end{aligned}$$

$$\widehat{A}_{\mu} = \widehat{e}^{k}_{\mu} * \widehat{A}_{k} \qquad (5)$$

Δ_{μ} is the covariant derivative in the commutative curved space time defined as:

$$\Delta_{\mu} = \partial_{\mu} + \omega^{ab}_{\mu}\Sigma_{ab} \qquad (6)$$

and with a torsion free space-time one has:

$$\omega^{ba}_{\mu} = e_{\mu i}(C^{abi} - C^{aib} - C^{iab}), \qquad (7)$$

with

$$C^{d}_{ab} = e^{\mu}_{a}\ e^{\nu}_{b}\partial_{[\mu}e^{d}_{\nu]} \tag{8}$$

and

$$\partial_{[\mu}e^{d}_{\nu]} = \frac{1}{2}[\partial_{\mu}e^{d}_{\nu} - \partial_{\nu}e^{d}_{\mu}] \tag{9}$$

For a non commutative action S of the quantum electrodynamics (Q.E.D) in a curved non commutative space-time (where gravity is treated as a gauge theory), we have proposed the following action [7]:

$$S = \frac{1}{2\kappa^2}\int d^4x\ (\mathcal{L}_G + \mathcal{L}_M) \tag{10}$$

Where $\mathcal{L}_G$ and $\mathcal{L}_M$ are the pure gravity and matter scalar densities respectively in the non commutative curved space-time and are given in the holonomic coordinates by:

$$\mathcal{L}_G = \widehat{e} * \widehat{R} \tag{11}$$

and

$$\mathcal{L}_M = \widehat{e} * \widehat{\overline{\psi}} * \widetilde{\gamma}^{\mu} * \widehat{D}_{\mu} * \widehat{\psi} \tag{12}$$

with:

$$\begin{aligned}
\widehat{e} &= \det{}_*((\widehat{e}^{a}_{\mu})) \equiv \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\widehat{e}^{a}_{\mu} * \widehat{e}^{b}_{\nu} * \widehat{e}^{c}_{\rho} * \widehat{e}^{d}_{\sigma} \\
\widehat{R} &= \widehat{e}^{\mu}_{*a} * \widehat{e}^{\nu a}_{*} * \widehat{e}^{i}_{\mu} * \widehat{e}^{k}_{\nu} * \widehat{R}_{ik} \\
\widehat{R}_{ab} &= R_{ab} + \frac{1}{2}\theta^{cd}\left\{R_{ac}, R_{bd}\right\} - \frac{1}{4}\theta^{cd}\left\{\omega_c, (\partial_d + \Delta_d)R_{ab}\right\} + O\left(\theta^2\right) \\
\widehat{D}_{\mu} &\equiv \widehat{\Delta}_{\mu} - iQ\widehat{A}_{\mu} \\
\widehat{\Delta}_{\mu} &= \partial_{\mu} + \widehat{\omega}_{\mu}
\end{aligned} \tag{13}$$

where $\varepsilon^{\mu\nu\rho\sigma}$, ε_{abcd} and Q are the completely antisymmetric tensors in curved and flat space-time and the charge of the particle respectively. we denote by $\widehat{e}^{\mu}_{*a}$ the *-inverse of the noncommutative vierbein $\widehat{e}^{a}_{\mu}$ defined as:

$$\widehat{e}^{b}_{\mu} * \widehat{e}^{\mu}_{*a} = \delta^{b}_{a} \tag{14}$$

and

$$\widehat{e}^{a}_{\mu} * \widehat{e}^{\nu}_{*a} = \delta^{\nu}_{\mu} \tag{15}$$

In what follows we consider a symmetric metric $\widehat{g}_{\mu\nu}$such that:

$$\widehat{g}_{\mu\nu} = \frac{1}{2}(\widehat{e}^{b}_{\mu} * \widehat{e}_{\nu b} + \widehat{e}^{b}_{\nu} * \widehat{e}_{\mu b}) \tag{16}$$

Now, because of the star Moyal product, the scalar density $\mathcal{L}$ is a function of the fields $\widehat{\varphi}^A$, their first and second derivatives up to the $O(\theta^2)$ i.e.

$$\mathcal{L}= \mathcal{L}\left(\widehat{\varphi}^A, \partial_\mu\widehat{\varphi}^A, \partial_\mu\partial_\nu\widehat{\varphi}^A\right) + O(\theta^2) \tag{17}$$

and thus, the principle of the least action leads to the following modified field equations[7] :

$$\frac{\partial\mathcal{L}}{\partial\widehat{\varphi}^A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\widehat{\varphi}^A\right)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\partial_\nu\widehat{\varphi}^A\right)} + O(\theta^2) = 0 \tag{18}$$

An interesting scenario for discussing the particle creation process is the non commutative version of the Bianchi I anisotropic universe associated with the metric:

$$ds^2 = -dt^2 + t^2\left(dx^2 + dy^2\right) + dz^2. \tag{19}$$

with dimensionless space-time coordinates. Since the metric presents a space like singularity at $t = 0$, it is difficult to define the particle states within the adiabatic approach [9]. To do so, we follow the quasi classical approach of ref.[10]. It consists, first to identify the positive and negative frequency modes by solving the classical Hamilton-Jacobi equation and lok for the asymptotic behaviour of the solutions when $t \to 0$ and $t \to \infty$.Second , we solve the Dirac equation by comparing with the above quasi classical limits, we identify the positive and negative frequency states. Third, using Bogoliubov" transformations we compute the particle number of the created particles. It turns out that in the quasi classical limit, the positive and negative frequency modes have positive (resp.negative) eigenvalues. Now, regarding the Dirac equation in the Bianchi I curved non commutative space-time and in the presence of an electromagnetic field and using the modified field equations of eq.(18), with the generic field $\widehat{\widehat{\psi}}$ one can finds in the co-moving coordinates the following modified Dirac equation:

$$\left[\widehat{\widetilde{D}} - m + \frac{i}{2}\theta^{\alpha\beta}\left\{\begin{array}{c}[(\partial_\alpha\tilde{\gamma}^\mu)(\partial_\beta\widetilde{D}_\mu) + \tilde{\gamma}^\mu(\partial_\alpha\widetilde{D}_\mu)\partial_\beta] + \\ +(\partial_\alpha\ln\widehat{e})[\partial_\beta(\tilde{\gamma}^\mu\mathring{A}_\mu) + \tilde{\gamma}^\mu\mathring{A}_\mu\partial_\beta + m\ \partial_\beta]\end{array}\right\} + O(\theta^2)\right]\widehat{\Psi} = 0, \tag{20}$$

with

$$\widetilde{D} = \tilde{\gamma}^\mu * \widetilde{D}_\mu \tag{21}$$

and

$$\mathring{A}_\mu = \tilde{\omega}_\mu + iQ\widetilde{A}_\mu \tag{22}$$

and for a convenience in the notations, we denote by: $\tilde{\gamma}^\mu$, $\widetilde{\Delta}_\mu$, $\tilde{\omega}_\mu$, and $\widetilde{A}_\mu$ etc.., the

γ^{μ}, $\Delta_{\mu}, \omega_{\mu}$, and A_{μ} etc..in the curved space-time. The matrices $\tilde{\gamma}^{\mu}$ satisfy the Clifford algebra:

$$\{\widetilde{\gamma}^{\mu} \overset{*}{,} \widetilde{\gamma}^{\nu}\} = 2g^{\mu\nu} \tag{23}$$

where $\{.\overset{*}{,}.\}$ stands for the anti-commutator with the star product. Notice that, in ref.[8],

and for the same phenomena, the authors use another approach where they have derived the Dirac equation in a noncommutative curved space-time by just taking the Dirac equation in a noncommutative flat space-time used in ref.[11] and replacing the ordinary derivative ∂_{μ} by the covariant one D_{μ} . Of course, this unjustified correspondance in a noncommutative space-time with a star product does not lead to the same Dirac equation as ours. This means in a curved noncommutative space-time, the derivation of the field equation from the action is more founded. Now, regarding our modified Dirac equation of eq.(20), It is worth to mention that since the metric of eq.(19) is diagonal, and for further simplifications, we choose to work with a diagonal tetrad too. Moreover, the vector potential $\widetilde{A}_{\mu}$ is choosen such that:

$$\widetilde{A}_{\mu} = (0, 0, 0, -Et) \tag{24}$$

which corresponds to a constant electric field along the z-direction. This means that the system pocesses a rotational symmetry along the z-axis, preserving the invariance of the metric of eq.(19). To visualize the influence of the non commutativity, we use the parametrization that determines the elements of the θ-matrix from the direction of the background electromagnetic field [12]:

$$C^{\mu\nu} = \begin{pmatrix} 0 & \sin\alpha\cos\beta & \cos\alpha\sin\beta & \cos\alpha \\ -\sin\alpha\cos\beta & 0 & \sin\gamma & -\cos\gamma\sin\beta \\ -\cos\alpha\sin\beta & -\sin\gamma & 0 & -\sin\gamma\cos\beta \\ -\cos\alpha & \cos\gamma\sin\beta & \sin\gamma\cos\beta & 0 \end{pmatrix}. \tag{25}$$

and again keep the background electric field parallel to the z axis to take benefit from the existed rotational symmetry in order to be sure that the particles number density remains unchanged under this symmetry. Geometrically, we have $\alpha = \beta = \gamma = 0$ to obtain $\theta^{03} = -\theta^{30} = \theta$, and set all the remaining components equal to zero. Now, using the fact that:

$$\mathring{A}_0 = 0,\ \mathring{A}_3 = -iQEt,\ \mathring{A}_1 = \frac{1}{2}\gamma^0\gamma^1,\ \mathring{A}_2 = \frac{1}{2}\gamma^0\gamma^2, \tag{26}$$

$$\begin{aligned} \theta^{\alpha\beta}(\partial_{\alpha}\tilde{\gamma}^{\mu})(\partial_{\beta}\widetilde{D}_{\mu}) &= -\frac{\theta}{t^2}\left(\gamma^2\partial_2\partial_3 + \gamma^1\partial_1\partial_3\right) \\ \theta^{\alpha\beta}\tilde{\gamma}^{\mu}(\partial_{\alpha}\widetilde{D}_{\mu})\partial_{\beta} &= i\theta QE\gamma^3\partial_3 \end{aligned} \tag{27}$$

and

$$\theta^{\rho\sigma}\partial_\rho \ln \widehat{e}\, \partial_\sigma = \frac{2\theta}{t}\partial_3 \tag{28}$$

the Dirac equation and up to the O(θ^2) takes the form:

$$[\gamma^0 \left(\partial_0 - i\frac{\theta}{t^2}\partial_3\right) + \gamma^3 \left(e^{\frac{Q}{2}\theta E}\partial_3 + iQEte^{-\frac{1}{2}\theta E}\right) + \frac{1}{t}\left(\gamma^1\partial_1 + \gamma^2\partial_2\right)$$

$$-i\frac{\theta}{2t^2}\left(\gamma^2\partial_2\partial_3 + \gamma^1\partial_1\partial_3\right) - m(1 - \frac{1}{t}i\theta\partial_3)]t\ \widehat{\Psi}_0 = 0 \tag{29}$$

where

$$\widehat{\Psi} = t\ \widehat{\Psi}_0 \tag{30}$$

The factor t was introduced in order to cancel the contribution due to the spin connection in the term proportional to γ^0. It is important to mention that eqs.(29) can be rewritten as a sum of two first order differential equations as follows [13]:

$$\left(\widehat{K}_1 + \widehat{K}_2\right)\widehat{\Phi} = 0 \tag{31}$$

with

$$\begin{aligned} \widehat{K}_2\widehat{\Phi} &= k\widehat{\Phi} \\ \widehat{K}_1\widehat{\Phi} &= -k\widehat{\Phi} \end{aligned} \tag{32}$$

where

$$\widehat{\Phi} = \gamma^3\gamma^0\widehat{\Psi}_0 \tag{33}$$

and k is a separation constant. The operators $\widehat{K}_1$ and $\widehat{K}_2$ have as expressions:

$$\widehat{K}_1 = t\begin{bmatrix} [\gamma^3\left(\partial_0 - i\frac{\theta}{t^2}\partial_3\right) + \gamma^0\left(e^{\frac{Q}{2}\theta E}\partial_3 + iQEte^{-\frac{1}{2}\theta E}\right) \\ -i\frac{\theta}{2t^2}\left(\gamma^2\partial_2\partial_3 + \gamma^1\partial_1\partial_3\right)\gamma^3\gamma^0 - \gamma^3\gamma^0 m(1 - \frac{1}{t}i\theta\partial_3)] \end{bmatrix}, \tag{34}$$

and

$$\widehat{K}_2 = (\gamma^1\partial_1 + \gamma^2\partial_2)\gamma^3\gamma^0, \tag{35}$$

Notice that in order to keep our results compact and transparents, the approximation

$$1 + \theta g \approx e^{\theta g} \tag{36}$$

was made (g is an arbitrary regular function). The spinor $\widehat{\Phi}$ can be written as:

$$\widehat{\Phi} = \widehat{\Phi}_0 \exp i\ (k_x x + k_y y + k_z z) \tag{37}$$

Where $\widehat{\Phi}_0$ is a bispinor. Working in the representation where the Dirac matrices in the tangent space are given by:

$$\gamma^0 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \gamma^3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \tag{38}$$

and the σ_i 's denote the $2x2$ Pauli matrices, we deduce that eqs.(32) are simplified to algebraic equations that permit to determine the relation between the components of the bispinor $\widehat{\Phi}_0$ such that:

$$\widehat{\Phi}_0 = \begin{pmatrix} \widehat{\Phi}_1 \\ \widehat{\Phi}_2 \end{pmatrix} = \begin{pmatrix} \widehat{\Phi}_1 \\ \sigma_2 \frac{ik_y - k}{k_x} \widehat{\Phi}_1 \end{pmatrix} = \begin{pmatrix} \widehat{\Phi}_1 \\ \sigma_2 \frac{k_x}{ik_y + k} \widehat{\Phi}_1 \end{pmatrix} \tag{39}$$

and the eigenvalue k is given by:

$$k = ik_\perp \equiv i\sqrt{k_x^2 + k_y^2} \tag{40}$$

By using the representation in eq.(38) and taking into account the spinor structure of eq.(39), we deduce that for $k_z = 0$, eqs.(32) are reduced to the following coupled system of first order differential equations:

$$\frac{d\widehat{\Phi}_1}{dt} + \frac{k}{t}\widehat{\Phi}_1 - \left(m + iQEte^{-\frac{1}{2}E\theta}\right)\widehat{\Phi}_2 = 0,$$

$$-\frac{d\widehat{\Phi}_2}{dt} + \frac{k}{t}\widehat{\Phi}_2 - \left(m - iQEte^{-\frac{1}{2}E\theta}\right)\widehat{\Phi}_1 = 0, \tag{41}$$

In this way, we have reduced the problem of solving eqs.(32) to that of finding the solutions of eqs.(41). From eqs.(41), we obtain the second order differential equation:

$$\left[\frac{d^2}{dt^2} - \frac{k^2}{t^2} + \frac{2k}{t^2} - \frac{3/4}{\left(t - \frac{im}{QE}e^{\frac{1}{2}E\theta}\right)^2} + m^2 + Q^2E^2t^2e^{-\frac{1}{2}E\theta}\right]\widehat{\Psi}_2 = 0, \tag{42}$$

where the variable $\widehat{\Psi}_2$ such that:

$$\widehat{\Psi}_2 = \left(t + \frac{im}{QE}e^{\frac{1}{2}E\theta}\right)^{\frac{-1}{2}}\widehat{\Phi}_2 \tag{43}$$

was introduced. Now, if we neglect the mass in the first order variation of $\widehat{\Phi}_2$, eq.(42) becomes:

$$\left[\frac{d^2}{dt^2} - \frac{k^2 - 2k + 3/4}{t^2} + m^2 + Q^2E^2t^2e^{-\frac{1}{2}E\theta}\right]\widehat{\Psi}_2 = 0, \tag{44}$$

Following ref.[10], the solution $\widehat{\Psi}_2$ of eq.(44) can be expressed as a conbination of Whittaker functions $M_{\lambda,\mu}$ and $W_{\lambda,\mu}$:

$$\widehat{\Psi}_2 = C_1M_{\lambda,\mu}(iQEt^2e^{-\frac{1}{2}E\theta}) + C_2W_{\lambda,\mu}(iQEt^2e^{-\frac{1}{2}E\theta}) \tag{45}$$

where

$$\lambda = +\frac{imt}{4QE}e^{\frac{1}{2}E\theta} \tag{46}$$

and

$$\mu = \frac{k-1}{2} \tag{47}$$

(C_1 and C_2 are normalization constants). Similarly, after introducing the new variable $\widehat{\Psi}_1$ such that:

$$\widehat{\Psi}_1 = \left(t + \frac{im}{QE}e^{\frac{1}{2}E\theta}\right)^{\frac{-1}{2}}\widehat{\Phi}_1 \tag{48}$$

and neglecting the mass in the first order variation of $\widehat{\Phi}_{1,}$ we obtain the following second order differential equation:

$$\left[\frac{d^2}{dt^2} - \frac{k^2 + 2k + 3/4}{t^2} + m^2 + Q^2E^2t^2e^{-\frac{1}{2}E\theta}\right]\widehat{\Psi}_1 = 0, \tag{49}$$

where the solution is:

$$\widehat{\Psi}_1 = C_3M_{\lambda,\mu+1}(iQEt^2e^{-\frac{1}{2}E\theta}) + C_4W_{\lambda,\mu+1}(iQEt^2e^{-\frac{1}{2}E\theta}) \tag{50}$$

(C_3 and C_4 are normalization constants). In order to construct the positive and negative frequency modes, we use the asymptotic behavior of the solutions $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$and compare the result with that obtained by solving the Hamilton-Jacobi relativistic equation. In fact, for $t \to 0$, one can show that the positive and negative frequency solutions $\widehat{\Psi}^+(t \to 0)$ and $\widehat{\Psi}^-(t \to 0)$ respectively are given by the following asymptotic forms:

$$\widehat{\Psi}^+(t \to 0) \approx C_0^+M_{\lambda,\mu}(iQEt^2e^{-\frac{1}{2}E\theta}) \tag{51}$$

and

$$\widehat{\Psi}^-(t \to 0) \approx C_0^+(-1)^{-\mu+1/2}M_{\lambda,-\mu}(iQEt^2e^{-\frac{1}{2}E\theta}) \tag{52}$$

where the Whittaker function $M_{\lambda,\mu}(z)$ has the asymptotic behaviour:

$$M_{\lambda,\mu}(z) \approx e^{-z/2}z^{\mu+1/2}\ ;\ z \ll 1 \tag{53}$$

and C_0^+ is a normalization function. Similarly, for $t \to \infty$, the corresponding positive and negative frequency modes are:

$$\widehat{\Psi}^+(t \to \infty) \approx C_\infty^+ W_{\lambda,\mu}(iQEt^2 e^{-\frac{1}{2}E\theta}) \tag{54}$$

and

$$\widehat{\Psi}^-(t \to \infty) \approx C_\infty^- W_{-\lambda,\mu}(-iQEt^2 e^{-\frac{1}{2}E\theta}) \tag{55}$$

($C_0^\pm$ are normalization functions). Now, using the fact that [14]:

$$M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+1/2)} e^{-i\pi\lambda} W_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\lambda+1/2)} e^{-i\pi(\lambda-\mu-1/2)} W_{\lambda,\mu}(z) \tag{56}$$

and

$$W_{-\lambda,\mu}(-z) = (W_{\lambda,\mu}(z))^* \tag{57}$$

(Γ (x) is the Euler Gamma function) we deduce that:

$$\widehat{\Psi}^-(t \to 0) = \frac{\Gamma(2\mu+1)}{\Gamma(\mu-\lambda+1/2)} e^{-i\pi\lambda} \widehat{\Psi}^-(t \to \infty) \tag{58}$$

$$+ \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\lambda+1/2)} (-1)^{1/4} e^{-i\pi(\lambda-\mu-1/2)} (\widehat{\Psi}^-(t \to \infty))^* \tag{59}$$

Now, since we are able to obtain the single particle states in the vicinity of $t \to 0$ and $t \to \infty$, we can then compute the density of the created particles $\widehat{n}$ by the noncommutative curved space-time and electromagnetic field. In fact, with the help of the Bogouliugov transformations[15] – [17]:

$$\widehat{\Psi}^-(t \to 0) = \widehat{\alpha} \widehat{\Psi}^-(t \to \infty) + \widehat{\beta} \widehat{\Psi}^+(t \to \infty) \tag{60}$$

and the use of eqs.(54) – (55), as well as the normalization condition:

$$|\widehat{\alpha}|^2 + \left|\widehat{\beta}\right|^2 = 1 \tag{61}$$

we obtain:

$$\widehat{n} = \left[\left(\frac{\left|\widehat{\beta}\right|^2}{|\widehat{\alpha}|^2} \right)^{-1} + 1 \right]^{-1} \tag{62}$$

where

$$\frac{\left|\widehat{\beta}\right|^2}{\left|\widehat{\alpha}\right|^2}=e^{2i\pi\mu}\frac{\left|\Gamma\left(\frac{1}{2}+\mu-\lambda\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}+\mu+\lambda\right)\right|^2} \tag{63}$$

Using the relation:

$$\left|\Gamma\left(ix\right)\right|^2=\frac{\pi}{x\sinh\pi x} \tag{64}$$

Direct simplifications lead to:

$$\frac{\left|\widehat{\beta}\right|^2}{\left|\widehat{\alpha}\right|^2}=e^{-\pi k_\perp}\frac{\left(\frac{k_\perp}{2}-\frac{m^2}{4QEe^{-\frac{1}{2}E\theta}}\right)}{\left(\frac{k_\perp}{2}+\frac{m^2}{4QEe^{-\frac{1}{2}E\theta}}\right)}.\frac{\sinh\pi\left(\frac{k_\perp}{2}-\frac{m^2}{4QEe^{-\frac{1}{2}E\theta}}\right)}{\sinh\pi\left(\frac{k_\perp}{2}+\frac{m^2}{4QEe^{-\frac{1}{2}E\theta}}\right)}, \tag{65}$$

and the number density of the created particles $\widehat{n}$ reads

$$\widehat{n}=n+n_\theta \tag{66}$$

where n denotes the ordinary number density of created particles in the presence of an electric field and has as expression:

$$n=e^{-\pi k_\perp}\frac{\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)\sinh\pi\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)}{\left(\frac{k_\perp}{2}+\frac{m^2}{4QE}\right)\sinh\pi\left(\frac{k_\perp}{2}+\frac{m^2}{4QE}\right)+\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)e^{-\pi k_\perp}\sinh\pi\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)}, \tag{67}$$

and n_θ is the generated noncommutative correction of order θ given by:

$$n_\theta=-\theta\frac{\pi m^2}{8Q}e^{-\pi k_\perp}\left[\frac{\frac{k_\perp}{2}e^{\pi k_\perp}+\frac{m^2}{2QE}\sinh\pi\left(\frac{m^2}{2QE}\right)-\frac{k_\perp}{2}\cosh\pi\left(\frac{m^2}{2QE}\right)}{\left(\left(\frac{k_\perp}{2}+\frac{m^2}{4QE}\right)\sinh\pi\left(\frac{k_\perp}{2}+\frac{m^2}{4QE}\right)+\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)e^{-\pi k_\perp}\sinh\pi\left(\frac{k_\perp}{2}-\frac{m^2}{4QE}\right)\right)^2}\right], \tag{68}$$

Notice that as it is expected, our result differs from the one obtained in ref.[8] at the order θ.In fact, if one takes $k_z=0$ in eq.(51) of this reference (as in our case), the noncommutative correction vanishes . The reason lies in the fact that the authors in ref.[8] use a different approach. It is also very important to consider the weak and strong electric field limits and see the behavior of the number density and derive some of the related thermodynamical quantities.

2.1 The weak field approximation:

In this limit, if we set:

$$x = \frac{2QE}{\pi m^2}(1 - \frac{E\ \theta}{2}) \tag{69}$$

such that:

$$|x| < 1 \text{ and } \frac{E\ \theta}{2} < 1 \tag{70}$$

It is easy to show that the number density $\widehat{n}$ takes the form:

$$\widehat{n} = \frac{1}{1 + e^{2\pi(1+x)k_\perp}} \tag{71}$$

This density is thermal and looks like a two dimensional Fermi-Dirac distribution with a temperature correction by a factor of $1/(1+x)$.

To get the total number $\widehat{N}$ of the created particles per a unit volume, we have to integrate the particle density $\widehat{n}$ over the momentum space . Taking into account the fact that $\widehat{n}$ does not depend on k_z $[12-13]$, the total number $\widehat{N}$ reads:

$$\widehat{N} \approx \frac{QE}{4\pi^4 T}.\Gamma(2)\ \zeta(2)/(1+x)^2 \tag{72}$$

where T is the time the external field interaction, $\Gamma(b)$ the Gamma function and $\zeta(b)$ has the following serie expansian:

$$\zeta(b) = \sum_{l=1}^{\infty} \frac{1}{l^{\ b}} \tag{73}$$

with

$$\zeta(2l) = \frac{2^{2l}-1}{2l}\pi^{2l}B_l \tag{74}$$

for l integer and B_l reads for the Bernouli numbers $(B_1 = 1/6, B_2 = 1/30, B_3 = 1/46, etc...)$.

Regarding the noncommutative internal energy $\widehat{U}$ and the grand potential $\widehat{\Omega}$ per a unit volume, it is straightforward to show that::

$$\widehat{U} = \frac{QE}{8\pi^5 T}.\Gamma(3)\ \zeta(3)/(1+x)^3 \tag{75}$$

and

$$\widehat{\Omega} \propto \frac{QE}{8\pi^5 T}.\Gamma(3)\ \zeta(3)/(1+x)^2 \tag{76}$$

Notice that, since the gravitational density is proportional to $1/T^{\ 2}$it decreases faster than $\widehat{N}$ and consequently the particle creation mechanism effectively isotropizes also in the presence of a constant electric field of the anisotropic Bianchi I universe of the non commutative space-time. To get an idea obout the qualitative effect of the noncommutativity, we have displayed the ratios $R_1 = \frac{\widehat{N}}{N}$ and $R_2 = \frac{\widehat{U}}{U}$ as a function of the dimensionless variable x (see Fig.1 and Fig.2).Here N (resp. U) represents the total number density (resp. internal energy) per a unit volume without the space-time noncommutativity. Notice that if the noncommutative parameter θ increases and consequently x decreases (with a negative sign because of the electron charge Q) the ratios R_1 and R_2 increases.

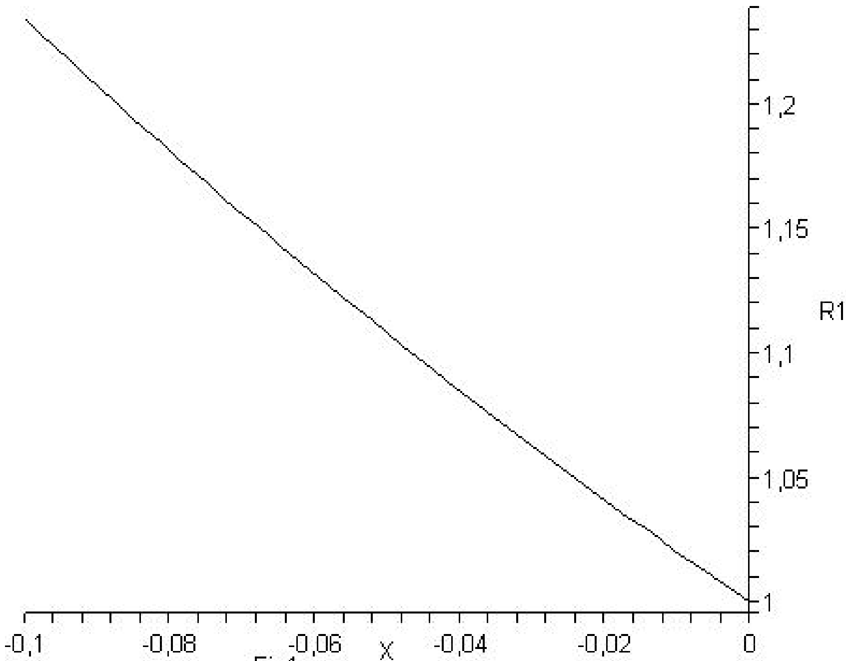

Fig1

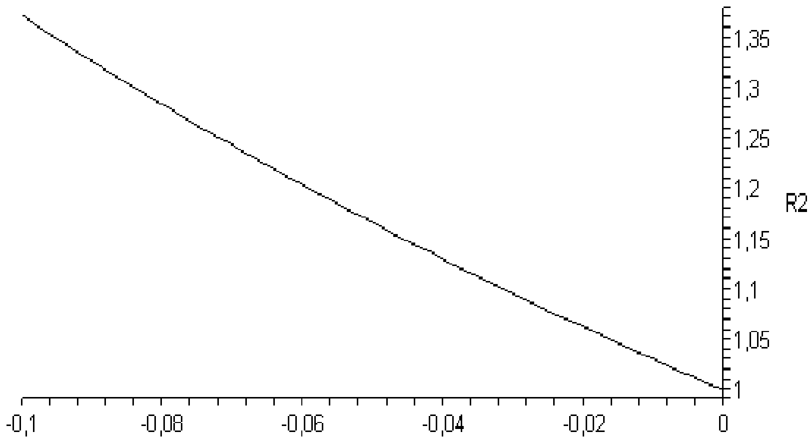

Fig2

2.2 The Strong Field Approximation

In this limit, if we set:

$$y = \frac{m^2}{4QE}(1 + \frac{E\ \theta}{2}) \tag{77}$$

such that:

$$|y| < 1 \text{ and } \frac{E\ \theta}{2} < 1 \tag{78}$$

Direct simplifications show that the number density $\widehat{n}$ takes the form:

$$\widehat{n} = \frac{1}{1 + f(k, y)e^{\pi k_\perp}} \tag{79}$$

where

$$f(k, y) = \frac{v}{w}\frac{e^v - 1}{1 - e^{-w}} \tag{80}$$

with

$$v = 2\pi(\frac{k_\perp}{2} + y) \tag{81}$$

and

$$w = 2\pi(\frac{k_\perp}{2} - y) \tag{82}$$

Notice that at very high energies (ultra relativistic limit), the number density takes the form:

$$\widehat{n} \approx e^{-2\pi(\frac{k_\perp}{2}+y)} \approx \frac{1}{1 + e^{2\pi y}e^{\pi k_\perp}} \tag{83}$$

and it looks like a two dimensional Fermi-Dirac distribution with a fugacity z and a chemical potential μ given by:

$$z = e^{-2\pi y} \tag{84}$$

and

$$\mu = -2y \tag{85}$$

Conclusion

Throught out this work, by applying the generalized field equations, we have derived a modified Dirac equation in the presence of an electromagnetic field and study the particle creation process in a noncommutative space-time anisotropic Bianchi I universe. After straightforward calculations, using the Bogoliubov transformations and the quasi classical limit for identifying the positive and negative frequency modes, we have deduced the corresponding number density as a function of θ.We have studied both the weak and strong field limits and obtained under certain circumstances a Fermi-Dirac like distribution. In the weak electric field approximation certains thermodynamical quantities like the total number of the created particles, internal and grand potential per a unit volume were calculated. Moreover, and as in the ordinary case [10], we have found that in this limit the gravitational density decreases faster than the total number $\widehat{N}$. Consequently, the particle creation mechanism effectivelly isotropizes in the non commutative space-time of an anisotropic Bianchi I universe in the presence of an electric field. It is worth to mention

that in the limit $\theta \to 0$ and high energies our results coincide with those or ref.[10]. As a conclusion, the noncommutativity plays the same role as the electric field and gravity [18] and contribute to the pair creation process.

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