Exactly solved potentials generated from the Manning-Rosen potential using extended transformation method

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Abstract: Generation of exactly solvable quantum systems in non-relativistic quantum mechanics from an already analytically solved quantum system is presented using extended transformation method. The bound state quantized energy spectra and the corresponding wavefunctions of the generated potentials are obtained. It is also shown that eigenfunctions of the new quantum systems can easily be normalized.
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1. Introduction

In non-relativistic quantum mechanics maximum information of a quantum system can be had when one knows the exact solutions of the corresponding Schrodinger equation. However, exact analytic solutions are possible only for a few set of quantum systems. The next best thing to do is to find approximate analytical solutions of a given potential by appropriate approximation techniques. In such situations, to work efficiently, it is always desirable to have an exactly solvable potential (ESP) similar to the given potential whose solution is required. A mapping procedure [1,2] is developed to map an already exactly solved quantum system (QS) to various exactly solved quantum systems. The method is based on a transformation called extended transformation (ET) which includes a coordinate transformation (CT) followed by functional transformation (FT) and a set of plausible ansatz to restore the transformed equation to a standard Schrodinger equation

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form. In our scheme we start from a known analytically solved QS and transformed it to generate a class of new exactly solvable quantum systems. When the original QS has a multiterm potential, then in general it is possible to generate a number of different solvable QS, depending on the number of ways the various term(s) of the original QS is grouped, which specifies a certain transformation function $g_B(r)$ (equation (8)) but more often than not we get Sturmian QSs. In transformation method the normalizability of the eigenfunctions of the generated QS can easily be verified in most cases. We discuss a procedure to regroup this set of Sturmian QS to a normal/ physical QS.

The plan of the paper is as follows: In section 2 the formalism of ET is given briefly. In section 3 generation of exactly solvable quantum systems from the Manning-Rosen (M-R) potential is discussed that includes first and second generation quantum systems. The important question of normalizability of the generated eigenfunctions are discussed in section 4. Discussion is given in section 5 with reference to case for $\alpha = 0$ and/or $\alpha = 1$. Concluding remarks are given section 6 .

2. Formalism

The radial part of the s- wave Schrodinger equation for the potential $V_A(r)$ in D_{A}-dimensional Euclidean space ($\hbar = 1 = 2m$):

$$\Psi_A^{''}(r) + \frac{D_A - 1}{r}\Psi_A^{'}(r) + \left[E_n^A - V_A(r)\right]\Psi_A(r) = 0 \qquad (1)$$

where r is a dimensionless modulus of radius vector in the D_A-dimensional space.

The normalized eigenfunctions $\Psi_A(r)$ and energy eigenvalues E_n^A are known for the given $V_A(r)$. Prime denotes the differentiation of the wavefunction with respect to its argument.

We now invoke a coordinate transformation

$$r \rightarrow g_B(r),$$

followed by a functional transformation

$$\Psi_B(r) = f_B^{-1}(r)\Psi_A(g_B(r)), \qquad (2)$$

where the transformation function $g_B(r)$ and the modulated amplitude $f_B(r)$ have get specified within the framework of ET.

The transformed B-QS after implementing ET on the already solved A-QS takes the form:

$$\Psi_B^{''}(r) + \left(\frac{d}{dr} \ln \frac{f_B^2 g_B^{D_A-1}}{g_B^{'}} \right) \Psi_B^{'}(r)$$

$$+ \left[\left(\frac{d}{dr} \ln f_B \right) \left(\frac{d}{dr} \ln \frac{f_B^{'} g_B^{D_A-1}}{g_B^{'}} \right) + g_B^{'2} \left(E_n^A - V_A \left(g_B(r) \right) \right) \right] \Psi_B(r) = 0. \qquad (3)$$

The dimension of the Euclidean spaces of the transformed quantum system, henceforth called the B- quantum system (B-QS) can be chosen arbitrarily, let it be denoted by D_B.

This requires that the co-efficient of $\Psi_B^{'}(r)$ in equation (3), must be:

$$\frac{d}{dr} \ln \frac{f_B^2 g_B^{D_A-1}}{g_B^{'}} = \frac{D_B - 1}{r} = \frac{d}{dr} \ln r^{D_B-1} \qquad (4)$$

which fixes $f_B(r)$ as a function of $g_B(r)$ and its derivative.

From equations (2) and (4) ,we obtain

$$\Psi_B(r) = g_B^{'-\frac{1}{2}} g_B^{\frac{D_A-1}{2}} r^{-\frac{D_B-1}{2}} \Psi_A \left(g_B(r) \right). \qquad (5)$$

The transformation function $g_B(r)$ is at least three times differentiable.

The corresponding second order equation for the B-QS in D_B- dimensional space becomes:

$$\Psi_B^{''}(r) + \frac{D_B - 1}{r} \Psi_B^{'}(r) + \left[\frac{1}{2} \{g_B, r\} + g_B^{'2} \left(E_n^A - V_A \left(g_B(r) \right) \right) \right] \Psi_B(r) = 0 \qquad (6)$$

with Schwartzian derivative symbol

$$\{g_B, r\} = \frac{g_B^{'''}(r)}{g_B^{'}(r)} - \frac{3}{2} \left(\frac{g_B^{''}(r)}{g_B^{'}(r)} \right)^2. \qquad (7)$$

In case mutiterm A-QS to implement ET we have to select one or more term(s) of $V_A \left(g_B(r) \right)$ as working potential and is designated by $V_A^W \left(g_B(r) \right)$.

In order to mould equation (6) to the standard Schrodinger equation form, following plausible ansatz have to be made which are an integral part of the transformation method:

$$g_B^{'2} V_A^W \left(g_B(r) \right) = -E_n^B, \qquad (8)$$

$$g_B^{'2} E_n^A = -V_B^{(1)}(r), \qquad (9)$$

$$-g_B^{'2} \left(V_A \left(g_B(r) \right) - V_A^W \left(g_B(r) \right) \right) = -V_B^{(2)}(r), \qquad (10)$$

$$\frac{1}{2} \{g_B, r\} = -V_B^{(3)}(r). \qquad (11)$$

These ansatz leads the B-QS potential $V_B(r)$ as:

$$V_B(r) = V_B^{(1)}(r) + V_B^{(2)}(r) + V_B^{(3)}(r).$$ (12)

Further, it is to be noted that, the equation (8) specifies the functional form of the transformation function $g_B(r)$.

The familiar radial Schrodinger equation for B-QS in D_B-dimensional spaces takes the form

$$\Psi_B''(r) + \frac{D_B - 1}{r}\Psi_B'(r) + \left[E_n^B - V_B(r)\right]\Psi_B(r) = 0.$$ (13)

3. Generation of exactly solvable quantum systems from the Manning -Rosen potential

We have considered the Manning-Rosen quantum system [3-8] as a typical representative of an exactly solvable non-powerlaw quantum system whose only s-wave bound state solutions are possible/ available.

The potential is denoted as $V_A(r)$and is:

$$V_A(r) = \frac{1}{b^2}\left[\frac{\alpha(\alpha - 1)\exp\left(-\frac{2r}{b}\right)}{\left(1 - \exp\left(-\frac{r}{b}\right)\right)^2} - \frac{A\exp\left(-\frac{r}{b}\right)}{\left(1 - \exp\left(-\frac{r}{b}\right)\right)}\right]$$ (14)

where A and b are constants but the screeing parameter b has dimension of length which has a potential range $\frac{1}{b}$.

It is found that the potential $V_A(r)$ at $r = r_0$ has a relative minimum for $\alpha > 1$ where $r_0 = b\ln\left[1 + 2\alpha(\alpha - 1)/A\right]$. For $\alpha = 0$ and/or $\alpha = 1$ the potential reduces to the Hulthen potential.

The discrete energy eigenvalues are

$$E_n^A = -\frac{1}{b^2}\left[\frac{A - \alpha}{2(\alpha + n)} - \frac{n(n + 2\alpha)}{2(\alpha + n)}\right]^2$$ (15)

with

$$n = 0, 1, 2, 3, ...n_{max.}, \quad n_{max} = \sqrt{A + \alpha(\alpha - 1)} - \alpha.$$

The general quantum condition is given by

$$\alpha + \lambda_A - \sqrt{A - \alpha + \alpha^2 + \lambda_A^2} = -n$$ (16)

where

$$\lambda_A = \sqrt{b^2(-E_n^A)}.$$

The corresponding radial wavefunctions are given by

$$\Psi_A(r) = N(1 - z)^\alpha z^{\lambda_A} {}_2F_1(a, b, c, z)$$ (17)

where

$$z = \exp\left(-\frac{r}{b}\right), \tag{18}$$

$$a = \alpha + \lambda_A - \sqrt{A - \alpha + \alpha^2 + \lambda_A^2}, \tag{19}$$

$$b = \alpha + \lambda_A + \sqrt{A - \alpha + \alpha^2 + \lambda_A^2}, \tag{20}$$

and

$$c = 1 + 2\lambda_A. \tag{21}$$

3.1 First Order transformation

Applying ET on the A-QS potential, one obtains the transformed equation for the B-QS for $(D_B = D)$ as follows:

$$\Psi_B''(r) + \frac{D-1}{r}\Psi_B'(r) + \left[\frac{1}{2}\{g_B, r\} + g_B'^2\left(E_n^A - V_A\left(g_B\left(r\right)\right)\right)\right]\Psi_B(r) = 0. \tag{22}$$

Choosing

$$V_A^w\left(g_B\left(r\right)\right) = -\frac{A\exp\left(-\frac{g_B(r)}{b}\right)}{b^2\left(1 - \exp\left(-\frac{g_B(r)}{b}\right)\right)} \tag{23}$$

as working potential and utilizing equation (8), the transformation function $g_B(r)$ is found as

$$g_B(r) = 2b \ \ln\sec\left(\eta_n r\right) \tag{24}$$

where

$$\eta_n = \frac{1}{2b}\sqrt{\frac{-E_n^B}{\left(-\frac{A}{b^2}\right)}}. \tag{25}$$

Equations (9)and (24) yield:

$$V_B^{(1)}(r) = C_B^2 \tan^2\left(\eta_n r\right) \tag{26}$$

where C_B^2 is the characteristic constant of the generated B-QS obtained from the transformation of A-QS and is

$$C_B^2 = 4b^2\eta_n^2\left(-E_n^A\right). \tag{27}$$

The equation (27) subsequently provides us the B-system energy eigenvalues.

The equations (10) and (24) lead to

$$V_B^{(2)}(r) = 4\eta_n^2\alpha(\alpha - 1)\cot^2(\eta_n r). \tag{28}$$

Again equations (11) and (24) yield

$$V_B^{(3)}(r) = -\eta_n^2\sec^2\eta_n r + 3\eta_n^2\csc^2 2\eta_n r. \tag{29}$$

The multiterm B-QS potential $V_B(r)$ is found from equation (12)as :

$$V_B(r) = 3\eta_n^2\csc^2(2\eta_n r) - \eta_n^2\sec^2\eta_n r + C_B^2\tan^2\eta_n r + 4\eta_n^2\alpha(\alpha - 1)\cot^2\eta_n r \tag{30}$$

which specifies a Sturmian QS and is denoted by B-SQS.
The characteristic constant C_B^2 of the B-QS can also be written as

$$C_B^2 = \frac{E_n^B}{A}\left[\frac{A - \alpha}{2(\alpha + n)} - \frac{n(n + 2\alpha)}{2(\alpha + n)}\right]^2. \tag{31}$$

The expression (31) is utilized to find the energy eigenvalues of the B-SQS and is

$$E_n^B = AC_B^2\left[\frac{A - \alpha}{2(\alpha + n)} - \frac{n(n + 2\alpha)}{2(\alpha + n)}\right]^{-2}. \tag{32}$$

The exact eigenfunction $\Psi_B(r)$ of the generated B-SQS comes out from equation (5) as:

$$\Psi_B(r) = N_B r^{-\frac{D-1}{2}}(\cot\eta_n r)^{\frac{1}{2}}\sin^{2\alpha}(\eta_n r)\cos^{2\lambda_A}(\eta_n r)\,_2F_1\left(-n,\,b,\,c,\,\cos^2\eta_n r\right). \tag{33}$$

As always the potential $V_B(r)$ is $n-$ dependent through $n-$dependence of η_n . This special type of energy dependent potential is equipped with only a single normalized eigenstate. The Sturmian form of B-QS comprises of a finite set of quantum systems. This Sturmian form of B-QS can be converted to a normal quantum system by a case specific regrouping technique, where we have to redefine the parameters of A-QS preserving the type of constraint equation(s).

To make η_n n-independent we make $A \to A_n$ by setting $A = \left(\frac{E_n^B}{4s^2}\right)$ by using the constraint equation (16), where a scale factor s is introduced. This leads to $\eta_n \to \eta = s$.

Therefore, the normal/physical form of the newly constructed B-QS potential comes out to be:

$$V_B(r) = 3s^2\csc^2(2sr) - s^2\sec^2 sr + C_B^2\tan^2 sr + 4s^2\alpha(\alpha - 1)\cot^2 sr. \tag{34}$$

As a consequence the expression for quantized energy eigenvalues and energy level spacing of the constructed B-QS are found as:

$$E_n^B = 4s^2\left[n^2 + 2(\alpha\lambda_B + \alpha n + \lambda_B n) + \alpha\right] \tag{35}$$

and

$$\triangle E_n^B = 4s^2 \left[2 \left(n + \lambda_B + \alpha \right) + 1 \right], \tag{36}$$

where

$$\lambda_B = \frac{C_B}{2s}.$$

The constraint equation for B-QS is obtained as:

$$2\alpha\, s + C_B - \sqrt{E_B + C_B^2 + 4s^2\alpha\left(\alpha - 1\right)} = -2sn.$$

The corresponding s-wave exact energy eigenfunctions of the constructed B-QS comes out to be:

$$\Psi_B(r) = N_B r^{-\frac{D-1}{2}} \left(\cot sr\right)^{\frac{1}{2}} \sin^{2\alpha}(sr) \cos^{2\lambda_B}(sr)\, {}_2F_1\left(-n,\, 2\left(\alpha + \lambda_B\right) + n,\, 1 + 2\lambda_B,\, \cos^2 sr\right). \tag{37}$$

The familiar Schrodinger equation in D- dimensional Euclidean space for $l = 0$ is found as:

$$\Psi_B''(r) + \frac{D-1}{r}\Psi_B'(r) + \left[E_n^B - V_B(r) \right] \Psi_B(r) = 0. \tag{38}$$

From the multiterm B-QS potential, the working potential can be chosen in $(2^4 - 1)$ i.e. 15 different ways. But we consider the single term working potential only for simplicity. It appears that the choice of $C_B^2 \tan^2 \eta_n r$ as the working potential reverts it back to the parent QS.

3.2 Second order transformation

Application of extended transformation on the B-SQS as obtained from equation (30) we can generate another new Sturmian quantum system and is designated by C-QS, by the above procedure.

From the multiterm B-SQS , we have chosen $-\eta_n^2 \sec^2 \eta_n r$ as the working potential so that

$$g_C'^2 V_A^w\left(g_C(r)\right) = -E_n^C, \tag{39}$$

with

$$V_A^w\left(g_C(r)\right) = -\eta_n^2 \sec^2 \eta_n g_C(r). \tag{40}$$

The set of ansatz required to write equation (6) in the form of the standard Schrodinger equation form can be written as:

$$\frac{1}{2}\left\{g_C, r\right\} + g_C'^2 \left(E_n^B - V_B\left(g_C(r)\right)\right) = E_n^C - V_C(r). \tag{41}$$

The transformation function is found as,

$$g_C\left(r\right) = \frac{1}{\eta}\arcsin\left[\tanh\left(\sqrt{E_n^C}\ r\right)\right]. \tag{42}$$

The Sturmian form of the newly generated C-SQS potential comes out to be:

$$V_C\left(r\right) = \left(\frac{C_B^2 E_n^C}{\eta^2} - C_C^2 - \frac{E_C}{4}\right)\tanh^2\sqrt{E_n^C}r + \frac{3}{4}E_n^C\left(\sinh\sqrt{E_n^C}r\right)^{-2} +$$
$$16\alpha\left(\alpha - 1\right)E_n^C\left(\sinh 2\sqrt{E_n^C}r\right)^{-2}. \tag{43}$$

Where C_C^2 is the characteristic constant of C-SQS.

The energy eigenvalues E_n^C of the C-SQS is found as

$$E_n^C = -\frac{C_C^2}{4\left[n^2 + 2\left(\alpha\lambda_B + \alpha n + \lambda_B n\right) + \alpha\right]}. \tag{44}$$

The normalized energy eigenfunction $\Psi_C\left(r\right)$ of the generated C-SQS is found for $\left(D_B = 1, D_C = D\right)$ as follows:

$$\Psi_C\left(r\right) = N_C r^{-\frac{D-1}{2}}\left(\tanh\sqrt{E_n^C}r\right)^{2\alpha - \frac{1}{2}}\left(\frac{1}{\cosh\left(\sqrt{E_n^C}r\right)}\right)^{2\lambda_B} \times$$
$${}_2F_1\left(-n, 2\left(\alpha + \lambda_B\right) + n, 1 + 2\lambda_B, \frac{1}{\cosh^2\sqrt{E_n^C}r}\right). \tag{45}$$

However, the present C-QS is Sturmian in nature which can not be made normal / physical.

Our choice of $3\eta_n^2\csc^2 2\eta_n r$ as working potential will also lead to a different form of Sturmian potential which can be found merely by inspection. This is also an example of a stubborn potential which can not be made normal / physical by any case- specific regrouping procedure.

4. Normalizability of the generated quantum systems

The normalizability condition of the wave function for the generated bound state QS obtain by the ET method can be proved under fairly general conditions, as it seems to preserve the normalizability property to quite a good extent. Normalizability condition for D_B- dimensional B-QS eigenfunction is

$$|N_B|^2\int_0^\infty\left|\Psi_B^{(n)}\right|^2 r^{D_B - 1}dr = finite. \tag{46}$$

The $\Psi_B^{(n)}$ is given by equation (37) and it can be reduced to

$$|N_B|^2 \frac{<V_A(r)>}{-E_n^B} = 1 \qquad (47)$$

since

$$g_B^{'2}(r) = \frac{-E_n^B}{V_A(g_B(r))}. \qquad (48)$$

Hence all the $\Psi_B^{(n)}(r)$ are normalizable for which $E_n^B \neq 0$. For any physical/ real QS, $<V_A(r)>$ exists ensuring normalizibility of daughter (B-QS). As such, the wavefunction of the generated QS are always normalizable corresponding to non-null eigenvalues, when the wavefunctions of the parent QS are normalizable. Hence the wavefunctions of the C-QS is also normalizable. The above expressions shows that it is a positive feature of the transformation procedure, that more often than not the wavefunction of the newly generated QS is normalizable. This is true that unless the transformation function $g_B(r)$ is not badly behaved so far its local and asymptotic properties are concerned, the transformation carries over the normalizability of the parent QS to the daughter QS.

5. Discussion

It is found that the Manning-Rosen potential can be reduced to the Hulthen potential [9] by setting $\alpha = 0$and/or $\alpha = 1$. Implementing ET on that Hulthen potential the constructed physical / normal first generation potential, quantized energy eigenvalues and the corresponding s-wave energy eigenfunctions in D- dimensional spaces are found for simplicity $\alpha = 0$ in following forms:

$$V_B(r) = 3s^2 \csc^2 2sr - s^2 \sec^2 sr + C_B^2 \tan^2 sr, \qquad (49)$$

$$E_n^B = 4s^2 n(n + 2\lambda_B) \qquad (50)$$

with energy level spacing

$$\Delta E_n^B = 8s^2 \left(n + \lambda_B + \frac{1}{2}\right) \qquad (51)$$

and

$$\Psi_B(r) = N_B r^{-\frac{D-1}{2}} (\cot sr)^{\frac{1}{2}} \cos^{2\lambda_B} sr \, {}_2F_1\left(-n, \, n + 2\lambda_B, \, 1 + 2\lambda_B, \, \cos^2 sr\right). \qquad (52)$$

Conclusions

We have presented a method of generation of exactly solved quantum systems in non relativistic quantum mechanics using the extended transformation method (ET) in D_B-dimensional Euclidean spaces from a given non-powerlaw potential in $1-D$ space. In ET, D_B can be chosen arbitrary in any integer dimensions and also in fractional dimensions

although we have yet to have a meaningful QS in fractional dimensions. It is evident that the ET may be applied successively any number of times to generate new QSs, when we are considering a non-powerlaw potential. The Manning Rosen potential shown in equation (14) is invariant under the interchange of $\alpha \leftrightarrow 1 - \alpha$, which is reflected in the exactly solvable daughter potentials. Although the potential is symmetric under $\alpha \leftrightarrow 1 - \alpha$, the symmetry is broken in the energy eigenvalues spectrum. But an important point to be noted is that in order to generate third order exactly solved potential from the initial exactly solved system we are required to go through second order generated potentials, ET does not have the group composition law as the transformation depends on the working potential (WP) of the parent QS. Therefore the ET does not have a group structure in general. The inverse transformation is obtained when WP is chosen as the term containing the characteristic constant, coming from the energy term of the parent QS. Here Manning-Rosen quantum system is taken as typical representative of non-powerlaw potential whose exact analytic solutions for s-wave ($l = 0$) are only possible. In addition approximate l- state solutions of the $D-$dimensional Schrodinger equation for the MR potential [10] is possible. In case of non-power law potential the transformed quantum system is always comes out as Sturmian quantum system, the transformation function is non factorizable[2] unlike the power law type potentials. The Sturmian form of quantum systems can be made normal/ physical by case-specific regrouping procedure. The first order application of the ET on the Manning -Rosen potential generates a potential similar to the Harmonic plus linear type potential with finite number of eigenstates. It is also to be noted that the wavefunctions of the generated quantum systems are almost always normalizable.

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[2] where n- dependent part and r dependent part can't be separated.