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# An Interruption in the Highway: New Approach to Modeling the Car-Traffic

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**Abstract:** A very common phenomena in car-traffic system is investigated in this article. The problem is one-dimensional. We try to find the wave equation of the traffic and then, we'll talk more about the simulation of the system using Matlab7.6.


*Keywords: Car-Traffic Modeling; Microscopic Model for Traffic Flow ; Wave Equation*
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## 1. Introduction

Several articles and investigations haves shown an increasing realization of the need for a scientific approach to various aspects of road traffic, a field that seemed formerly to lie only at the periphery of physics. Nowadays, most of us experience the car traffic during our daily movement. Car traffic usually happens inside a city, that is, some streets get filled with cars and therefore the traffic path is something similar to a loop. A part from the loop-shaped traffic which is not a new phenomena, the car traffic in a highway or ring road is actually something new. I'm not sure if you have experienced such a traffic, but in rush hours I sometimes get stuck in a large traffic jam in the Hemmat highway, including thousands of cars in each line. Therefore, it was interesting to me to model this kind of car traffic. Some sources have described road traffic as a "stream" instead of focusing their attention on individual vehicle [1].

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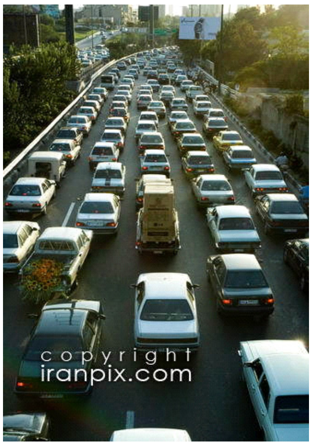



**Fig. 1** Hemmat Highway during the rush hour.

# 2. Theory

## 2.1 Parameters Definitions

Simply, I just focus on one of the lanes of the highway. Since the traffic is almost heavy, the cars would not be able to change their lanes easily and so we can consider them all, driving in one dimension (for example in the fastest lane).
To start our work, suppose there are $n$ cars driving at speed $V_0$ in one lane and the safe distance between each car at the rated speed is $l$ (see Fig. 2). For convenience, let us neglect the delay time in which the driver can react.
An interruption is applied to the system at t=0 and that is the first car suddenly stops and since we have ignored the delay time of reaction, the drivers can control the car immediately. Each car should get to the stationary state in a safe distance [2] $l'$ from the car in front of it.

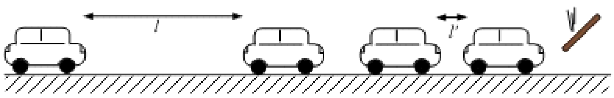



**Fig. 2** An Interruption at t=0 makes all cars to get stopped.

## 2.2 Speed Control Modeling

As mentioned above, at rated speed ($V_0$) all cars should maintain the safe distance $l$ from the car in front. Since each car should get stopped at the minimum distance $l$' from the previous car, it is conveniently useful to model the speed control by a linear function of the distance between two cars. That is, the velocity of car ranked "k" is dependent directly to how much it is far from the car in front (ranked k-1). We can formulate it mathematically as,

$$V_k(t) = V_0 + \alpha\,(l_k(t) - l) \tag{1}$$

Where $l_k$ is the distance between the k-th car and k-1.
And $\alpha$ is a constant which is determined by taking $V_k = 0$ at $l_k = l'$. $\alpha$ can be different from car to car [3]. But here we assume conveniently that all cars have a same safe distance.

$$0 = V_0 + \alpha\,(l' - l)$$

Therefore,

$$\alpha = \frac{V_0}{l - l'} \tag{2}$$

From definition of $l_k$ we have,

$$\frac{dl_k\,(t)}{dt} = V_{k-1}\,(t) - V_k\,(t) \tag{3}$$

Substituting Eq. (3) in Eq. (1) we get to a new equation versus $V_k(t)$.

$$\frac{dV_k(t)}{dt} = \alpha\,(V_{k-1}\,(t) - V_k\,(t)) \tag{4}$$

It can be interpret as two effective forces [4]: motor reaction and a force proportional to the velocity (for example drag force due to the air resistance).

## 2.3 Propagating Wave

Eq. (4) describes a discrete wave that propagates over the cars. The information of the carrier wave is the *interruption* that the first car has made it in the system. In order to find the velocity of the wave, suppose that the velocity of the (k-1)-th car is equal to the velocity of the k-th car after passing time $\tau$. That is,

$$V_{k-1}\,(t) = V_k(t + \tau) \tag{5}$$

Assume $\tau$ is very small compared to t. Thus, it is possible to expand the Eq.(5) to the first order of $\tau$.

$$V_k\,(t + \tau) \approx V_k\,(t) + \frac{dV_k\,(t)}{dt}\tau \tag{6}$$

Substituting Eq. (6) in Eq.(4) we have,

$$\frac{dV_k\,(t)}{dt} = \alpha\left(\frac{dV_k\,(t)}{dt}\tau\right) \tag{7}$$

And therefore,

$$\tau = \frac{1}{\alpha} = \frac{l - l'}{V_0} \tag{8}$$

The velocity function of the first car ($V_1(t)$) affects the all motion of other cars. In another word, Eq. (4) has too many answers as a propagating wave. These answers are responses to the $V_1(t)$ and thus, we can consider $V_1(t)$ as an input to the car traffic system. Let us put the all variables $V_k(t)$ in a vector V, i.e.

$$V = \begin{pmatrix} V_2 \\ V_3 \\ \vdots \\ V_n \end{pmatrix},$$

Therefore, Eq.(4) is changed to an ordinary differential equation of vectors and matrices.

$$\dot{V} = A.V + U \tag{9}$$

Where,

$$A = \begin{pmatrix} -\alpha & 0 & 0 & \cdots & 0 \\ \alpha & -\alpha & 0 & \cdots & 0 \\ 0 & \alpha & -\alpha & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \alpha & -\alpha \end{pmatrix}$$

And

$$U = \begin{pmatrix} \alpha V_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is the input vector.

As you see, we have considered $V_1(t)$ as an input to the car system, this is because if the function $V_1(t)$ is determined, then all $V_k(t)$ are known.

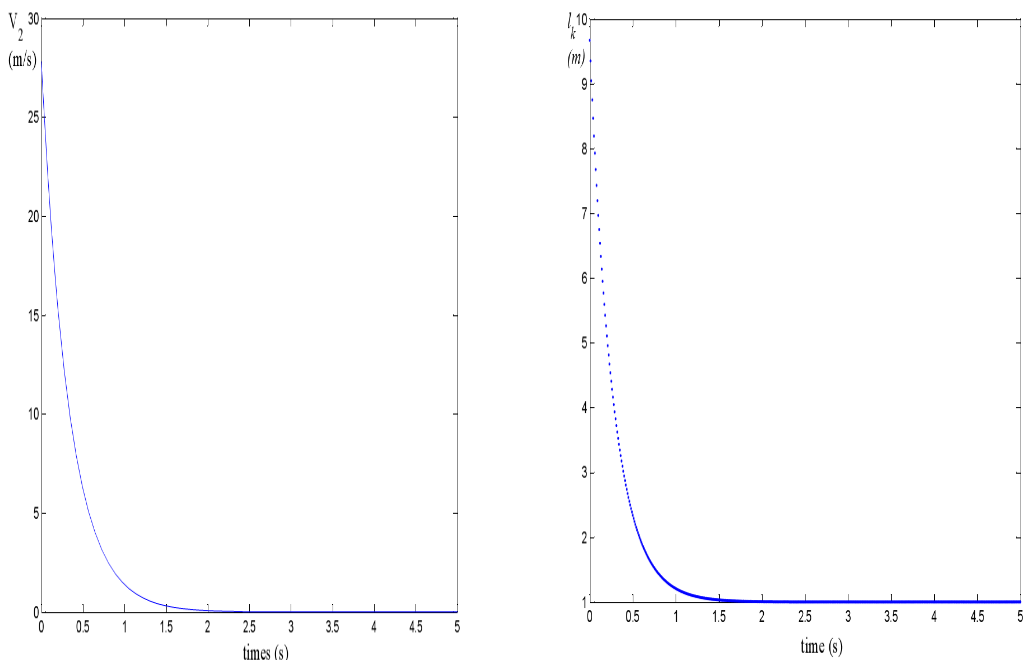
Eq. (9) is a first order differential matrix equation. Thus, it easy to solve and get the answers. One method for solution is applying Laplace Transform to the both sides of the equation. We are not going to solve the Eq. (9) but also the solution is going to be observed by computer simulation.

# 3. MatLab® Simulation

## 3.1 Braking Process

We are interested to sketch the wave function V(k,t), as described in Eq. (4), versus cars (k's). Let us take $V_1(t) = 0$ for all t and that means all cars should brake and get ready to stop.

Since the first car is stopped, the safe distance $l_2(t)$ decreases continuously and therefore the second car pushes the brake pedal instantaneously. Fig. (3-a) shows velocity of the second car $V_2(t)$, versus t. And Fig. (3-b) depicts the safe distance $l_2(t)$ versus time. Notice that the values of parameters used in Fig. (3) are gathered in Table 1. We will consider these values in all part of this article. These values are gained by real observation and experiments at the Hemmat Highway mentioned before.




(a) (b)

Fig. 3 (a) Plot of $V_2(t)$ versus time, (b) Plot of safe distance $l_2(t)$ from the first car, versus time.

| Parameter | Value |
|---|---|
| $l$ | 10 m |
| $l$’ | 1 m |
| V0 | 100km/h $\approx$ 28 m/s |
| n | 200 |
| $\alpha = \frac{V_0}{l-l'}$ | 3.08 $s^{-1}$ |
| $\tau = \frac{1}{\alpha}$ | 0.324 s |

Table 1. Values of parameters.

Now, let us have a look at the wave function V(k,t). It is useful to illustrate the V(k,t) function versus k, at a specific time t. This plot shows the velocity of each car (k) at a fixed time $t = t_0$. Fig. (4) explains this idea. According to the parameters gathered in Table 1, at t=5s almost 10 cars have stopped, nearly 20 cars are in the middle of braking and are ready to stop. The other 170 cars are driving at the rated speed V0.
Fig. (5) shows the safe distance function$l_k$(t) for each car (k) at the time t=5s. As you notice, this distance is 1m (=$l$’) for those stationary cars and 10m (=$l$) for other cars driving at the rated speed.

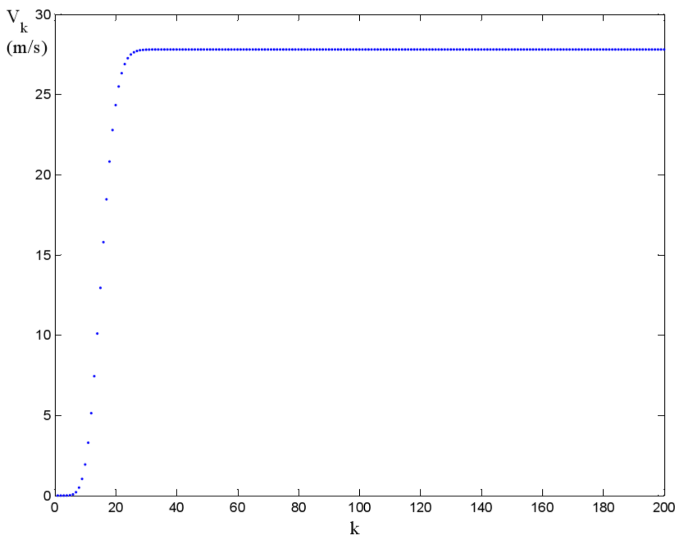



**Fig. 4** Plot of Vk(t) versus k (cars) at time equal to 5s. Each dot indicates a car.

Fig. (6) depicts the velocity-car curve for different values of sampling time:5s, 10s, 15s, 20s and finally 40s. It is similar to a moving wave that propagates to the right side. Obviously, as the wave propagates to the right hand side, its shape changes a bit. This change is recognized by the change in the number of cars that their velocities are between the rated value $V_0$ and zero. This means that the value of $\tau$ calculated in Sec. II. C depends on t.

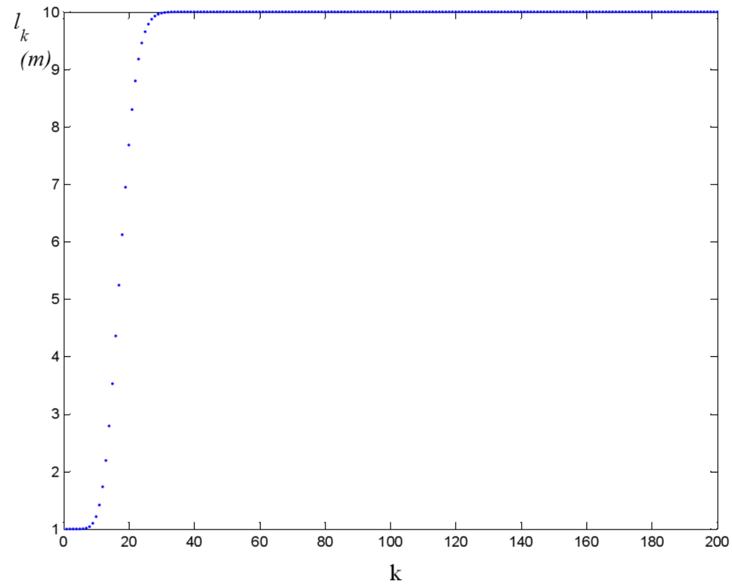



**Fig. 5** Plot of the safe distance function $l_k$(t) for each 200 cars at the fixed time t=5s. Same as Fig. (4), dots represent the cars

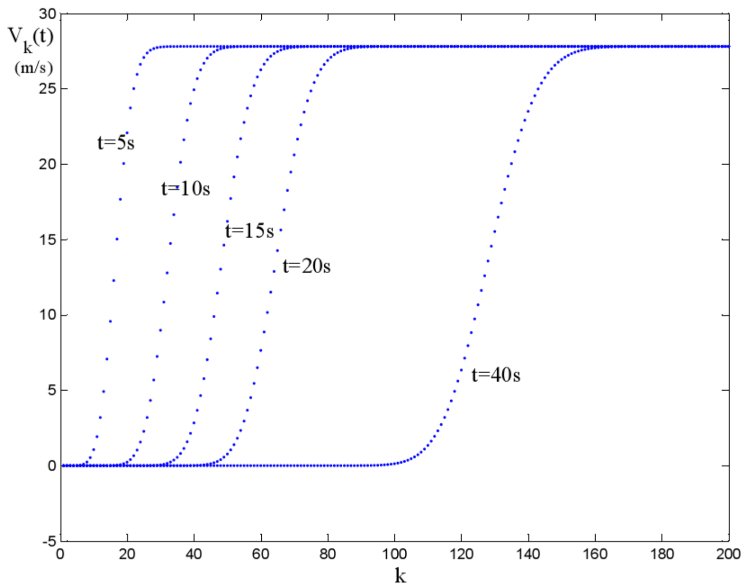



**Fig. 6** Graph of velocity versus k at different sampling times. We call it Brake-wave. Numbers written on the plot are time values.

As you noticed the Fig. (6), this dependency is not too much to be worried about it. The wave function $V_k(t)$ moves with speed $\alpha(\approx 3)$ cars per second approximately. However, the upper side of the wave-shape moves faster than the lower side. That is, the number of cars getting to stop increases as the time passes. We will discuss more about it in Sec. IV.

### 3.2 Running Process

In this section, assume that all cars are in stationary position and the distance between each car is $l$'. Therefore, initial condition of all cars is $V_k(0) = 0$. At t=0 the first car starts to run and it reaches to the rated velocity immediately. In another word, we are interested to find the step response of the car-traffic system. Thus, the input of the system is,

$$V1(t) = V_0.u(t) \tag{10}$$

Where u(t) is the unit step function.
Fig. (7) shows the result of running simulation after passing 5s.
Similar to Fig.(6), we have depicted the moving wave function $V_k(t)$ for different values of time: 5s, 10s, 15s, 20s and 40s. As you notice the figure, the shape of the curve changes as it passes over the cars. That is, the lower side of the curve propagates faster than the upper side.

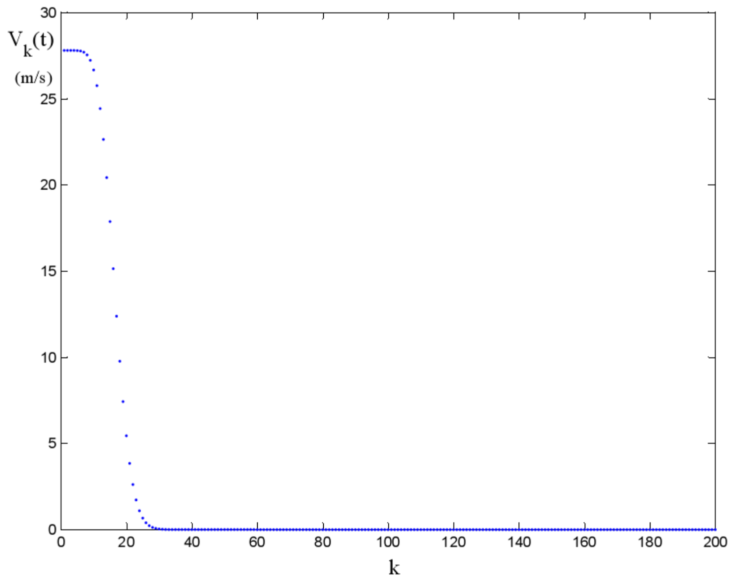



**Fig. 7** Plot of Vk(t) versus k (cars) at time equal to 5s in running process. Each dot indicates a car.

### 3.3 Brake and Run Process: Interruption Phenomena

Now, let us consider the situation discussed in section II. B, i.e. the braking process. But this time, the first car continues its path after a delay time. In this section, we assume that the first car is at rest at $t \leq 0$ and stays the same until t=T and then starts to run. Therefore, the input function to the system would be,

$$V_1(t) = V_0.u(t - T) \tag{11}$$

Same as previous sections, we want to find the step response of the car system. Fig. (9) shows the wave function $V_k(t)$ versus k for T=20s. Fig.(9) is similar to Fig. (6) apart from the Run process. According to Fig. (9), almost 30 cars are still at rest.

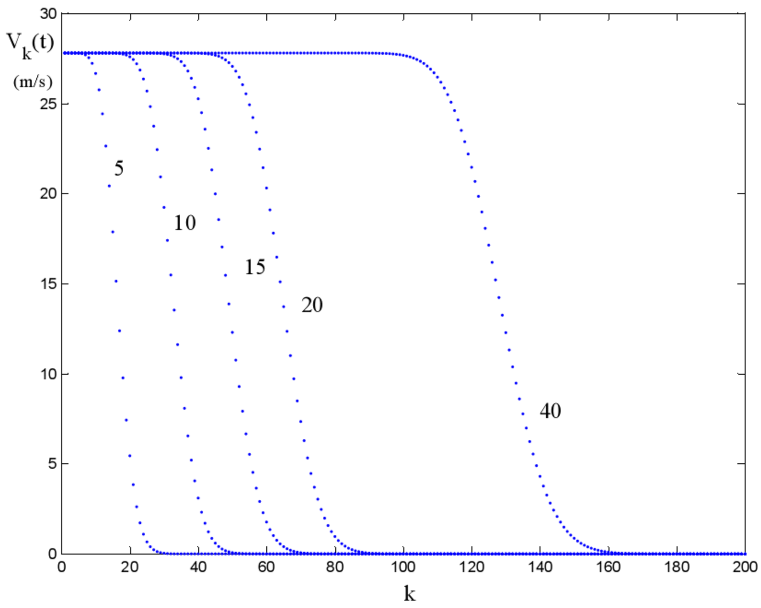



**Fig. 8** Graph of velocity versus k at different sampling times. We call it Run-wave. Numbers written on the plot are time values in sec.

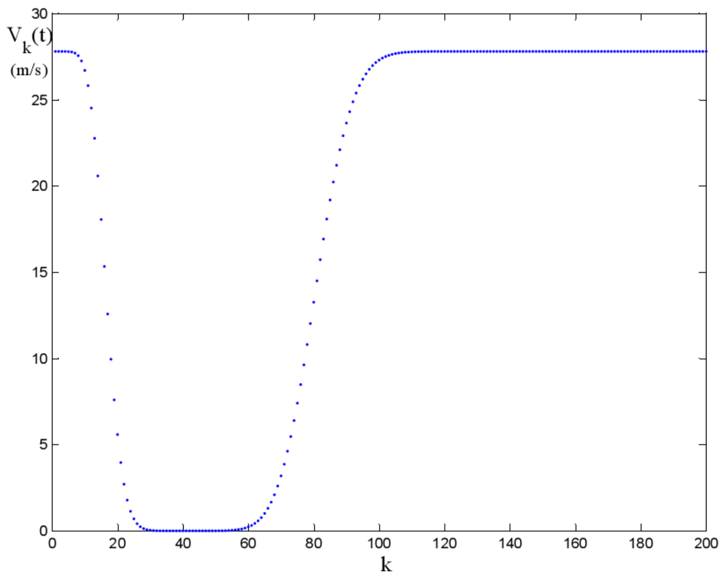



**Fig. 9** Plot of Vk(t) versus k at t=25s and T=20s.

Fig. (10) depicts the same plot at t=45s. As you notice, at this time, nearly 7 or 8 cars are at rest. It seems that the Run-wave travels faster than the Brake wave. This is because we had seen in Figs. (6) and (8) that the lower side of the Run wave travels faster than the upper side and also in Brake wave, the lower side moves more slowly than the upper part of the wave-shape. Since we have symmetry in the problem (the equation of motion is the same) thus, the lower part of the Run wave moves faster than the Brake's lower side. It means that, after a specific time, no car would stop completely, however speed reduction occurs. In the real life experiment, this phenomena happens. That is, if the interruption time is T seconds for the first car, it is shorter for the k-th car and

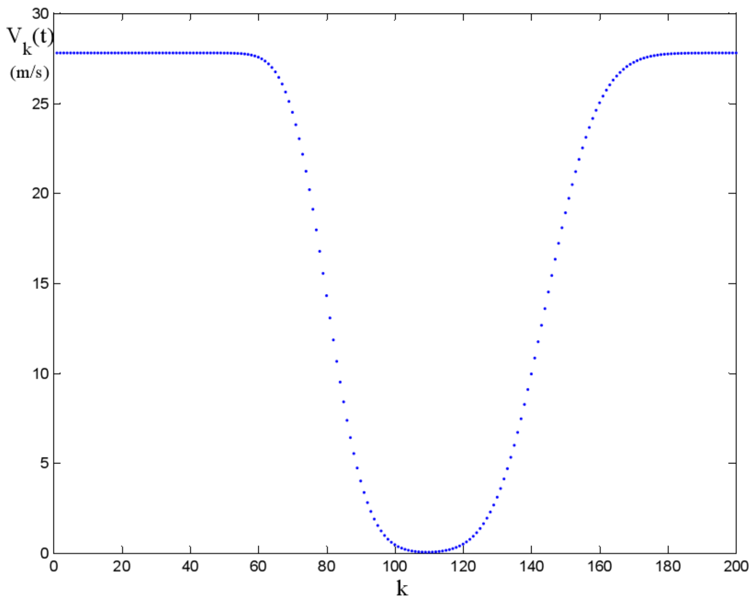



**Fig. 10** Plot of $V_k$ versus k at t=45s and the same t=20sec

maybe it becomes zero for the cars at the end of the tail and that is what we say then "the information has been died during the wave propagation".

## 4. Delayed Step Response

In this case, we just focus on the delayed step response of the system with input in form of Eq. (11), that is,

$$V_1(t) = V_0.u(t-T) \tag{11}$$

First of all, let us go through the Eq.(4) and use Eq. (5) in order to find a dependency of $\tau$(wave period) to the time or k. In general, $\tau$(t,k) is a function of both t and k. In Eqs. (6) and (7) we had approximately determined $\tau$ as a constant value. But now, let us expand the Eq. (6) to the second order of $\tau$. So we have,

$$V_k\left(t+\tau\right) \approx V_k\left(t\right) + \frac{dV_k\left(t\right)}{dt}\tau + \frac{d^2V_k\left(t\right)}{dt^2}\frac{\tau^2}{2} \tag{12}$$

Substituting Eq. (12) in Eq. (4), it changes to

$$\frac{dV_k\left(t\right)}{dt} = \alpha\left(\frac{dV_k\left(t\right)}{dt}\tau \ + \frac{d^2V_k\left(t\right)}{dt^2}\frac{\tau^2}{2}\right) \qquad (13) \tag{13}$$

As we had found the first order of $\tau$, we can expand $\tau$ to its second order as below,

$$\tau \approx \frac{1}{\alpha} + \frac{\beta}{\alpha^2} \tag{14}$$

Simplifying Eq. (13) by using Eq. (14), we get to a new equation for $\tau$.

$$\frac{1}{2}\frac{d^2V_k\left(t\right)}{dt^2} + \beta\frac{dV_k\left(t\right)}{dt} = 0 \tag{15}$$

Therefore,

$$\tau \approx \frac{1}{\alpha} - \frac{1}{2\alpha^2} \frac{\frac{d^2 V_k(t)}{dt^2}}{\frac{dV_k(t)}{dt}} \tag{16}$$

If we use Eq. (4) again, it seems better to understand.

$$\tau \approx \frac{1}{2\alpha} \left( 3 - \frac{\frac{dV_{k-1}(t)}{dt}}{\frac{dV_k(t)}{dt}} \right) \tag{17}$$

If you focus on the lower side of the Brake wave, the magnitude of acceleration of k-th car is more than the k-1-th car. In contrast, at the upper side of the Brake wave, the acceleration magnitude of k-th car is smaller than k-1-th. Thus, according to Eq. (17), the Brake-wave propagates faster at the upper side.
From the symmetry, in Running process the lower side travels faster than the upper side (see Fig. (10). Thus, there would be an interference and that is the two waves (Brake and Run) go through each other at the lower side of the wave shape.

## 5. Flow rate and density function

Let us define q(x , t) as a flow rate of cars, that is, q(x , t) is the number of cars per unit time passing a given point $x$ at time $t$. Therefore,

$$q\,(x, t) = \frac{V_k(t)}{l_k(t)} \tag{18}$$

Using Eq. (1) to substitute Vk(t),

$$q\,(x, t) = \frac{V_0 - \alpha l}{l_k(t)} + \alpha \tag{19}$$

And $\rho$( x, t) is the number of cars per unit length of highway at the same position and time. That is,

$$\rho\,(x, t) = \frac{1}{l_k(t)} \tag{20}$$

Joining Eqs. (19) and (20), we get to equation explaining the relationship between flow rate and the density.

$$q = (V_0 - \alpha l)\rho + \alpha \tag{21}$$

If we assume that the maximum allowable speed is $V_0$ then for $l_k(t) > l$ Eq. (21) changes to

$$q = \rho V0 \tag{22}$$

Fig. (11) displays both Eqs. (21) and (22).

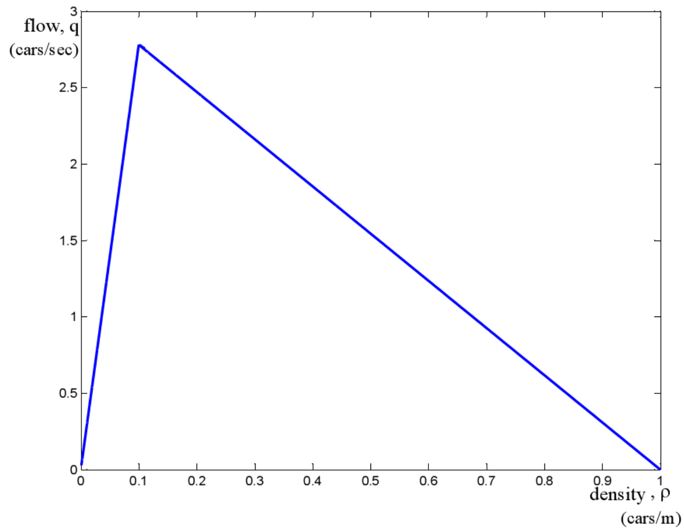



**Fig. 11** Relationship between flow and density on highway.

The behavior of Fig. (11) is similar to what called as a typical relationship for flow and density on highways. That is, the flow increases as the density goes up and it reaches to a maximum value and then starts to decrease with a lower rate [5,6].

## Conclusion

We have presented a microscopic model for traffic flow that shows how macroscopic dynamics emerge from individual car behavior and their interactions. Traffic flow has been modeled from the perspective of the individual driver, making it well suited for simulation.

The traffic wave is investigated and plotted using MatLab§7.6. The group velocity of the wave is determined by using the equation of motion and we have shown that the group velocity differs from side to side of the wave shape.

We had introduce a new approach or briefly a new vision to the traffic model which describes linear control systems using feedback to get to a stable position. In this model, the velocity of each car is a feedback of the distance from the car in front with a constant "set point" $l$.

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