Geometrical Behaviuors of LRS Bianchi Type-I Cosmological Model

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Abstract: By using Einstein's theory of general relativity some properties of spatially homogeneous locally rotationally symmetric (LRS) Bianchi type-I space-time are investigated in empty space. The concept of Riemannian curvature tensor, Ricci tensor and Ricci scalar has been used to discuss the geometrical behavior of the space-time. It is shown that, LRS Bianchi type-I has always flat geometry in empty space. Also we have shown that the vacuum model does not have singularity when time goes to zero.

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1. Introduction

The Bianchi cosmologies play an important role in theoretical cosmology and have been much studied since the 1960s. Bianchi cosmological models are spatially homogenous space-time. More precisely, they are manifolds of the form $M = I \times G$ where $I \subset R$ is an interval and G is a Lie group, endowed with a Lorentzian metric of the form $-dt^2 + g_t$ where $(g_t)_{t \in I}$ is a family of left-invariant Riemannian metrics on G.

The physical content of a space-time M is encoded by a non-linear partial differential equation(PDE) on its Lorentz metric: The so called Einstein equation. For Bianchi

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cosmological models, this PDE (Einstein field equations) reduce to a set of second order ordinary differential equation on the family of metrics $(g_t)_{t\in I}$. However, this is not the case in LRS Bianchi type cosmological models where Einstein field equation leads to a set of non-linear differential equations. A spatially homogenous Bianchi model necessarily has a three-dimensional group, which acts simply transitively on space-like three-dimensional orbits. For simplification and description of the large scale behaviour of the actual universe, locally rotationally symmetric [henceforth referred as LRS]Bianchi type-I space-time have widely studied [1]-[6]. When the Bianchi type-I space-time expands equally in two spatial directions it is called locally rotationally symmetric. Here we confine ourselves to a LRS model of Bianchi type-I. This model is characterized by three metric functions $R_1(t)$, $R_2(t)$ and $R_3(t)$ such that $R_1 \neq R_2 = R_3$.

A study of geometrical aspects of Bertotti-Robinson like space-time [7] has been done by Radojevic [8] and Mohanty et al [9]. Mohanty et al have studied some geometrical aspects of Bianchi type-I in the framwork of general relativity [9]. In this latter, to study the geometrical aspects of the LRS Bianchi type-I space-time, we have calculated Riemannian curvature tensor, Ricci tensor, Ricci scalar. The solution of the Einstein's vacuum field equations also has been derived.

2. Einstein's Vacuum Solution

The Einstein's field equations (in gravitational units $c = 1$, $G = 1$) for the gravitational field (metric) of a matter-energy configuration described by the energy-momentum tensor $T_{\mu\nu}$.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{1}$$

taking the trace of Eq(1) and substituting the result into Eq(1) again one obtains

$$R_{\mu\nu} = (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}) \tag{2}$$

In particular, for the vacuum, $T_{\mu\nu} = 0$, the Einstein equations are simply

$$R_{\mu\nu} = 0 \tag{3}$$

in two and three dimensions, vanishing of the Ricci tensor implies the vanishing of the Riemann tensor. Thus in these cases, the space-times are necessarily flat away from where there is matter, i.e at points at which $T_{\mu\nu} = 0$. Thus there are no true gravitational fields and no gravitational waves.

In four dimensions, however, the situation is completely different. The Ricci tensor has 10 independent components whereas the Riemann tensor has 20. Thus there are 10 components of the Riemann tensor which can curve the vacuum

3. The Metric and Field Equations

We consider the LRS Bianchi type-I metric in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2) \tag{4}$$

where A and B are functions of t only. Einstein's field equations (1) for line-element (4) leads to

$$R_{00} = -(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}) \tag{5}$$

$$R_{11} = A(\ddot{A} + 2\frac{\dot{A}\dot{B}}{B}) \tag{6}$$

$$R_{22} = R_{33} = B(\ddot{B} + \frac{\dot{A}\dot{B}}{A} + \frac{\dot{B}^2}{B}) \tag{7}$$

where an overdot stands for the first and double overdot for the second derivative with respect to t. One can calculate the Ricci scalar; $R = g^{\mu\nu}R_{\mu\nu}$ as

$$R = -2(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}) \tag{8}$$

using the Einstein equation for vacuum dominate we get

$$\ddot{A} + 2\frac{\dot{A}\dot{B}}{B} = 0 \tag{9}$$

$$\ddot{B} + \frac{\dot{A}\dot{B}}{A} + \frac{\dot{B}^2}{B} = 0 \tag{10}$$

$$\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} = 0 \tag{11}$$

solving eqs(9)-(11) one obtains

$$A = (3\alpha_A t + 3\alpha_A)^{\frac{1}{3}}, B = (\frac{3}{2}\alpha_B t + \frac{3}{2}\alpha_B)^{\frac{2}{3}} \tag{12}$$

by assuming $\alpha_{1A} = \frac{\alpha_{1B}}{2}$, $\alpha_{2A} = \frac{\alpha_{2B}}{2}$ and after a suitable coordinate transformations the vacuum model for the metric (4) reduces as

$$ds^2 = dT^2 - T^{2n_1}dx^2 - T^{2n_2}(dy^2 + dz^2) \tag{13}$$

we remark that it is follows from Eq(12) that n_1 and n_2 in Eq(13) can only have the value of $\frac{1}{3}$ and $\frac{2}{3}$ respectively. This may effect the claim made by Mohanty et al. that for Bianchi type-I, n_1, n_2 and n_3 could have any value under the conditions $\sum_{i=1}^{n} n_i = 1$ and $\sum_{i,j=1}^{3} n_i n_j = 0$ if $i \neq j$. we see that this model has the same properties of Bianchi type-I cosmological model. i.e

1- the model is not asymptotically flat at infinite future as in the case of other symmetrical space times and
2- the model collapses at initial epoch.

4. Space of Constant Curvature

A Space is a space of constant curvature if the condition

$$R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{14}$$

holds. Here K shows the curvature of Riemannian manifold. Non vanishing components of Eq(14) for line-element(4) are

$$R_{1010} = -KA^2, \quad R_{2020} = R_{3030} = -KB^2 \tag{15}$$

$$R_{1212} = R_{1313} = KA^2B^2, \quad R_{2323} = KB^4 \tag{16}$$

In other hand one can finds the components of Riemannian curvature tensor via

$$R_{\sigma\rho\mu\nu} = \frac{1}{2}(g_{\sigma\mu,\rho\nu} + g_{\rho\nu,\sigma\mu} - g_{\sigma\nu,\rho\mu} - g_{\rho\mu,\sigma\nu}) + \Gamma_{\lambda\sigma\mu}\Gamma^{\lambda}_{\rho\nu} - \Gamma_{\lambda\sigma\nu}\Gamma^{\lambda}_{\rho\mu} \tag{17}$$

For line-element (4) the above equation reduces as

$$R_{i0i0} = -\frac{1}{2}\frac{\partial^2 g_{ii}}{\partial x_0^2} + g_{ii}(\Gamma^i_{0i})^2, \quad i,j = 1,2,3 \tag{18}$$

and

$$R_{ijij} = -g_{00}\Gamma^0_{ii}\Gamma^0_{jj}, \quad i,j = 1,2,3 \tag{19}$$

So the non vanishing components of Eq(17) for line-element (4) can be calculated as

$$R_{1010} = A\ddot{A}, \quad R_{2020} = R_{3030} = B\ddot{B} \tag{20}$$

$$R_{1212} = R_{1313} = -AB\dot{A}\dot{B}, \quad R_{2323} = -B^2\dot{B}^2 \tag{21}$$

Matching the components of Riemannian curvature tensor from Eqs(15)-(16) and Eqs(20)-(21)

$$K = -\frac{1}{6}(2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}) \tag{22}$$

comparing this result with the result which obtain in Eq(8) we can write

$$K = -\frac{1}{6}R \tag{23}$$

Although the above equation shows that in general, LRS Bianchi type-I is not a space of constant curvature, however, from Eqs(9)-(11) it is easy to show that the only value of curvature constant, K in empty space which can be obtain from Eq(23), is zero. This result is interesting because we see that the behavior of LRS Bianchi type-I in empty space-time is almost similar to the behavior of de Sitter model.

5. Symmetric Space

A space satisfying

$$R_{\mu\nu\rho\sigma;\lambda} = 0 \tag{24}$$

is called a symmetric space. Where the semicolon shows the covariant differentiation. For non vanishing components of Riemannian curvature tensor of metric(4), Eq(24) reads to

$$R_{1212;0} = \frac{\partial}{\partial x^0}(A\dot{A}B\dot{B}) - 2(\frac{\dot{A}}{A} + \frac{\dot{B}}{B})A\dot{A}B\dot{B} = 0 \tag{25}$$

$$R_{1010;0} = \frac{\partial}{\partial x^0}(A\ddot{A}) - 2\dot{A}\ddot{A} = 0 \tag{26}$$

$$R_{2020;0} = \frac{\partial}{\partial x^0}(B\ddot{B}) - 2\dot{B}\ddot{B} = 0 \tag{27}$$

the solution of Eqs(25)-(27) gives

$$A = e^t \tag{28}$$

$$B = e^t \tag{29}$$

This special choice of metric components makes the line-element(4) to be a symmetric space. In view of Eqs(28)-(29) the metric (4) takes the form

$$ds^2 = dt^2 - e^{2t}(dx^2 + dy^2 + dz^2) \tag{30}$$

After a proper choices of coordinates Eq(31) can be transformed to

$$ds^2 = dT^2 - exp[2T(dx^2 + dy^2 + dz^2)] \tag{31}$$

It is interesting to note that the vacuum model (31) has no singularity at $T = 0$.

6. Conclusion

In this paper it is shown that the LRS Bianchi type-I space-time described by the line-element (4) is a space-time of constant curvature in empty space and hence it is an Einstein space. The importance of this result is that although the equality of metric components is the necessary condition for a metric to describe a flat space-time, we found an flat space-time with different time-dependence metric components ($A \neq B$). Also we have shown that with the special choose of metric components i.e. $A = B = exp(t)$, the metric(4) becomes symmetric and hence the vacuum model is free from singularity.

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References

[1] J. Hajj-Boutros and J.Sfeila, Int. J. Theor. Phys. **26** (1987) 98.

[2] Shri Ram, Int. J. Theor. Phys. **28** (1989) 917.

[3] A. Muzumder, Gen. Rel. Grav. **26** (1994) 307.

[4] A. Pradhan and A. K. Vishvakarma, SUJST **XII** Sec. B (2000) 42.

[5] A. Pradhan and A. K. Vishvakarma, Int. J. Mod. Phys. **D 8** (2002) 1195.

[6] A. Pradhan and A. K. Vishvakarma, J. Geom. Phys. **49** (2004) 332.

[7] k. Godel, Rev. Mod. Phys. **21** (1949) 447-450.

[8] D. Radojevic, Teor. Prim. meh **13** (1988) 77-81.

[9] G. Mohanty, R. C. Sahu and P. K. Sahoo, Theor. Appl. Mech **27** (2002) 79-86.