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Underdeterminacy and Redundance in Maxwell's Equations. Origin of Gauge Freedom - Transversality of Free Electromagnetic Waves - Gauge*free* Canonical Treatment *without* Constraints

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Abstract: Maxwell's (1864) original equations are redundant in their description of charge conservation. In the nowadays used, 'rationalized' Maxwell equations, this redundancy is removed through omitting the continuity equation. Alternatively, one can Helmholtz decompose the original set and omit instead the longitudinal part of the flux law. This provides at once a natural description of the transversality of free electromagnetic waves and paves the way to eliminate the gauge freedom. Poynting's inclusion of the longitudinal field components in his theorem represents an additional assumption to the Maxwell equations. Further, exploiting the concept of Newtonian and Laplacian vector fields, the role of the static longitudinal component of the vector potential being *not* determined by Maxwell's equations, but important in quantum mechanics (Aharonov-Bohm effect) is elucidated. Finally, extending Messiah's (1999) description of a gauge invariant canonical momentum, a manifest gauge invariant canonical formulation of Maxwell's theory *without* imposing any contraints or auxiliary conditions will be proposed as input for Dirac's (1949) approach to special-relativistic dynamics.

Keywords: Electromagnetic Waves; Maxwell Equations; Helmholtz Decomposition; Gauge Theory; Poynting's Theorem; Aharonov-Bohm Effect
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1. Introduction

Traditionally, there are two main approaches to classical electromagnetism (CEM), *viz*,

(1) the experimental one going from the phenomena to the rationalized Maxwell equations (*eg*, Maxwell 1873, Mie 1941, Jackson 1999, Feynman, Leighton & Sands 2001);

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(2) the deductive one deriving the phenomena from the rationalized Maxwell equations (*eg*, Hertz 1889, Lorentz 1909, Sommerfeld 2001).

"Rationalized Maxwell equations" (Poynting 1884, Heaviside 1892) means Gauss' laws for the magnetic (1a) and dielectric fields (1c) as well as Faraday's induction (1b) and Ampère-Maxwell flux laws (1d). In SI units,

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{1a}$$
$$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla \times \vec{E}(\vec{r},t) \tag{1b}$$
$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \tag{1c}$$
$$\frac{\partial}{\partial t}\vec{D}(\vec{r},t) = \nabla \times \vec{H}(\vec{r},t) - \vec{j}(\vec{r},t) \tag{1d}$$

For moving charges in vacuo, they can be simplified via $\vec{D}(\vec{r},t) = \varepsilon_0\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t) = \mu_0\vec{H}(\vec{r},t)$ to the microscopic Maxwell equations (Lorentz 1892).

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{2a}$$
$$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\nabla \times \vec{E}(\vec{r},t) \tag{2b}$$
$$\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t) \tag{2c}$$
$$\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0\mu_0}\nabla \times \vec{B}(\vec{r},t) - \frac{1}{\varepsilon_0}\vec{j}(\vec{r},t) \tag{2d}$$

For both sets, two fundamental problems have to be clarified, *viz*,

(1) the origin of the gauge freedom in the potentials, and

(2) the origin of the transversality of free (unbounded) electromagnetic waves.

Stipulated by special relativity, all field variables are usually treated on equal footing. There are quite different types of field variables, however. This comes into play, in particular, when the boundary contains electrodes with fixed potential values, or when the domain under consideration is multiply connected. And Gauss' laws,

$$\nabla\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t) \tag{3}$$
$$\nabla\vec{B}(\vec{r},t) = 0 \tag{4}$$

"are not, properly speaking, equations of motion, but rather constraints imposed on the fields $\mathcal{E}$ $[\vec{E}]$ and $\mathcal{H}$ $[\vec{B}]$. They fix the longitudinal parts of these fields... In order to define the dynamical state of the system, it is therefore sufficient to specify the charge distributions and currents – that is, the positions and the velocities of the particles – on the one hand, and the transverse fields $\mathcal{H}$ $[\vec{B}]$ and $\mathcal{E}_\perp$ $[\vec{E}_T]$ on the other." (Messiah 1999, XXI.22). This suggests to discriminate the transverse and longitudinal components of the field vectors from the very beginning, though this Helmholtz decomposition is not Lorentz *co*variant. However, for being compliant with special relativity, it is sufficient that an equation is Lorentz *in*variant (Barut 1964).

For this, I will – following the recommendation by Boltzmann (2001) – return to Maxwell's (1864) *original* set of equations. Using the Helmholtz (1858) decomposition of

3D vector fields into transverse and longitudinal components, I will show that this set is both underdetermined and redundant (but not inconsistent). Remarkably enough, both deficiencies are related to *longitudinal* vector components.

Thus, to provide the formal basis, Section 2 relates the Helmholtz decomposition to Newtonian, Laplacian and vector fields in multiply connected domains. This will facilitate the understanding of the particular standing of the static longitudinal component of the vector potential, $\vec{A}_L(\vec{r})$, which is not accounted for in any variant of Maxwell equations and which does not enter the Maxwell-Lorentz force.

In Section 3, Maxwell's 1864 set of equations are rewritten in terms of the transverse and longitudinal components of *all* fields. This reveals immediately, that the subset which deals with charge conservation is redundant. In the rationalized Maxwell equations used nowadays (Poynting 1884, Heaviside 1892), this redundance is eliminated through removing the continuity law: $\nabla\vec{j} + \dot{\rho} = 0$, from the basic set of equations. In contrast, I will propose to eliminate this redundance through removing the longitudinal component of Ampère-Maxwell's flux law. A revised set of independent variables being *free* of underdeterminacy and redundance will be proposed.

Moreover, this decomposition will discover the fact, that the incorporation of the longitudinal component of the electrical field strength, $\vec{E}_L$, in Poynting's (1884) theorem represents an *additional* assumption, which, in turn, obscures the transversality of freely propagating electromagnetic waves. Thus, Poynting's theorem separates into *two* theorems: one for the propagating transverse and one for the non-propagating longitudinal field parts, see Section 6.

Section 4 considers the role of $\vec{A}_L(\vec{r}, t)$ for the gauge freedom both in electromagnetism and in Schrödinger wave mechanics, where the latter provides a short-cut to a gauge *invariant* Hamiltonian.

Section 5 treats Mie's approach to the rationalized Maxwell equations in terms of the Helmholtz decomposition. The decomposed Maxwell equations will be used in Section 6 to split Poynting's (1884) theorem into a transverse one for the propagating and a longitudinal one for the non-propagating field momenta, respectively.

There seems to be an astounding difference between the Lagrangian and the Hamiltonian treatments of CEM. In virtually all CEM textbooks, both the Lagrangian and the Hamiltonian are explored for, (i), the motion of charged bodies subject to external electromagnetic fields and, (ii), electromagnetic fields with external charges and currents. In contrast, I'm not aware of a CEM textbook discussing not only the Lagrangian, but also the Hamiltonian for, (iii), closed systems of charges and fields. The latter is needed for the state description, not only in quantum physics (*cf* Dirac 1949), but also in classical physics. Indeed, many textbooks on quantum theory do contain such a Hamiltonian, but usually in momentum space.

As a matter of fact, it is not sufficient simply to insert the canonical momenta in the well-known total energy (84). And in contrast to the Lagrangian, where $L_{tot} = L_{chg} + L_{field} + L_{int}$, the Hamiltonian is *not* additive: $H_{tot} \neq H_{chg} + H_{field} + H_{int}$. Moreover, the fact, that – by virtue of the absence of a time-derivative – Gauss' laws represent

Bergmannian constraints of 1st kind (Dirac 2001, p.8) rather than dynamical equations, prevents a standard treatment of the canonical theory. In Section 7, these difficulties will be overcome for the microscopic theory in common spacetime *without* invoking additional constraints, using Milton & Schwinger's (2006) Lagrangian representation of the microscopic theory and extending it to a Hamiltonian representation.

Section 8 exploits these results for developing a manifest gauge-invariant, *ie*, gauge*free* Lagrangian and Hamiltonian. This includes gaugefree canonical momenta for bodies and fields (following and extending the treatment by Messiah 1999).

Both the microscopic Maxwell equations and the Lagrangian equations of motion are easily written down in terms of Minkowski 4-scalars/vectors/tensors. In contrast, Hamilton's equations of motion distinguish the time coordinate, what prevents a straightforward Lorentz covariant reformulation. Johns (2005) has put space and time variables on equal footing through extending the set of independent variables by an auxiliary parameter. Alternatively, there are proposals for a canonical field momentum density *tensor*, here,

$$\Pi^{\nu}_{\mu} = \frac{\delta L}{\delta(\partial A^{\mu}/\partial x^{\nu})} \tag{5}$$

This remains to be explored. – Anyway, as mentioned above, special-relativistic *in*variance is *not* bound to Lorentz *co*variance (Barut 1964). This is demonstrated in Dirac's (1949) analysis of the possible forms of special-relativistic Hamiltonian dynamics (for a short review of the historical development and recent results, see Stefanovich 2008). Here, moreover, the unity of kinematics and dynamics is guaranteed from the very beginning in that dynamical variables are generators of kinematical transformations; thus, one goal of this contribution consists in providing a *fully interacting* starting Hamiltonian for that approach.

The main results will be summarized and discussed in Section 9.

2. Helmholtz Decomposition of 3D Vector Fields

In order to apply Helmholtz's decomposition theorem appropriately, one has carefully to discriminate between certain types of vector fields, *viz*, Newtonian, Laplacian and vector fields in multiply connected domains.

2.1 Newtonian Vector Fields

Newtonian vector fields are vector fields in unbounded domains with a given distribution of sources and vortices (Schwab 2002). The classical example is Newton's force of gravity. They are the actual subject of

Helmholtz's decomposition theorem: Any sufficiently well-behaving 3D vector field, $\vec{f}(\vec{r})$, can uniquely be decomposed into a transverse or solenoidal, $\vec{f}_T(\vec{r})$, a longitudinal or irrotational, $\vec{f}_L(\vec{r})$, and a constant components (which I will omit in what follows).

$$\vec{f}(\vec{r}) = \iiint_V \vec{f}(\vec{r}')\delta(\vec{r}-\vec{r}')dV'; \quad \vec{r}\in V\backslash\partial V \tag{6}$$

$$= -\frac{1}{4\pi}\iiint_V \vec{f}(\vec{r}')\Delta\frac{1}{|\vec{r}-\vec{r}'|}dV' \tag{7}$$

$$= \frac{1}{4\pi}\nabla\times\nabla\times\iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' - \frac{1}{4\pi}\nabla\nabla\cdot\iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \tag{8}$$

$$= \vec{f}_T(\vec{r}) + \vec{f}_L(\vec{r}) \tag{9}$$

(These notions of longitudinal and transverse fields should not be confused with the notions of longitudinal and transverse waves in waveguides!)

It is thus most useful to introduce scalar, $\phi_f(\vec{r})$, and vector potentials, $\vec{a}_f(\vec{r})$, as

$$\vec{f}_T(\vec{r}) = \nabla\times\vec{a}_{\vec{f}}(\vec{r}); \quad \vec{f}_L(\vec{r}) = -\nabla\phi_{\vec{f}}(\vec{r}) \tag{10}$$

The minus sign is chosen to follow the definitions of the mechanical potential energy and the scalar potential in the electric field strength. $\vec{a}_{\vec{f}}$ is *sourceless*; otherwise, one would increases the number of independent field variables.

As a consequence, each such vector field is uniquely determined by its sources, $\phi_{\vec{f}}$, and source*less* vertices, $\vec{j}_{\vec{f}}$.

$$\nabla\times\vec{f}(\vec{r}) = \nabla\times\vec{f}_T(\vec{r}) = \nabla\times\nabla\times\vec{a}_{\vec{f}}(\vec{r}) = -\Delta\vec{a}_{\vec{f}}(\vec{r}) = \vec{j}_{\vec{f}}(\vec{r}) \tag{11}$$

$$\nabla\cdot\vec{f}(\vec{r}) = \nabla\cdot\vec{f}_L(\vec{r}) = -\Delta\phi_{\vec{f}}(\vec{r}) = \rho_{\vec{f}}(\vec{r}) \tag{12}$$

Including the surface terms (Oughstun 2006, Appendix A), the potentials follow as

$$\phi_{\vec{f}}(\vec{r}) = \frac{1}{4\pi}\nabla\cdot\iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \tag{13}$$

$$= \frac{1}{4\pi}\iiint_V \frac{\rho_{\vec{f}}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' - \frac{1}{4\pi}\oint_{\partial V} \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}\cdot d\vec{\sigma}' \tag{14}$$

$$\vec{a}_{\vec{f}}(\vec{r}) = \frac{1}{4\pi}\nabla\times\iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \tag{15}$$

$$= \frac{1}{4\pi}\iiint_V \frac{\vec{j}_{\vec{f}}(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' + \frac{1}{4\pi}\oint_{\partial V} \frac{\vec{f}(\vec{r}')}{|\vec{r}-\vec{r}'|}\times d\vec{\sigma}' \tag{16}$$

For the balance equations I will also need the

Orthogonality theorem: Integrals over mixed scalar products vanishe,

$$\iiint_V \vec{f}_T\cdot\vec{g}_L dV = -\iiint_V \nabla\times\vec{a}_{\vec{f}}\cdot\nabla\phi_{\vec{g}}dV \tag{17}$$

$$= -\iiint_V \nabla\left(\phi_{\vec{g}}\nabla\times\vec{a}_{\vec{f}}\right)dV = -\iiint_V \nabla\left(\vec{a}_{\vec{f}}\times\nabla\phi_{\vec{g}}\right)dV \tag{18}$$

$$= -\oint_{\partial V}\phi_{\vec{g}}\nabla\times\vec{a}_{\vec{f}}\cdot d\vec{\sigma} = -\oint_{\partial V}\vec{a}_{\vec{f}}\times\nabla\phi_{\vec{g}}\cdot d\vec{\sigma} = 0 \tag{19}$$

if the surface, ∂V, lies infinitely away from the sources of the fields (*cf* Stewart 2008), or if the fields obey appropriate periodic boundary conditions on ∂V (Heitler 1954, I.6.3).

If the Orthogonality theorem holds true, the integrals over the scalar products of two vectors separates as

$$\iiint_V \vec{f}(\vec{r}) \cdot \vec{g}(\vec{r}) dV = \iiint_V \vec{f}_T(\vec{r}) \cdot \vec{g}_T(\vec{r}) dV + \iiint_V \vec{f}_L(\vec{r}) \cdot \vec{g}_L(\vec{r}) dV \tag{20}$$

In particular, both the Joule power and the electric field energy decompose into the contributions of the transverse and longitudinal components of the (di)electric field vectors.

The validity of this theorem will be assumed throughout this series of papers.

Notice that the electromagnetic vector potential, $\vec{A}$, is a vector potential in the sense of Helmholtz's theorem only w.r.t. the magnetic induction, $\vec{B}$, not, however, w.r.t. the electric field strength, $\vec{E}$. As a consequence, both its transverse and longitudinal components are physically significant. The clue is thus to Helmholtz-decompose $\vec{A}$, too.

It should also be noted that the longitudinal and transverse components of a *localized* vector field are *spread over the whole* volume of definition. For instance, for a point-like body of charge q moving along the trajectory $\vec{r}(t)$,

$$\rho(\vec{r}, t) = q\delta(\vec{r} - \vec{r}(t)); \quad \vec{j}(\vec{r}, t) = q\vec{v}(t)\delta(\vec{r} - \vec{r}(t)) \tag{21}$$

one has ($\vec{r}_t \equiv \vec{r}(t)$; here, t is merely a parameter)

$$\rho_{\vec{j}} = \nabla \vec{j} = q\vec{v} \cdot \nabla\delta(\vec{r} - \vec{r}(t)) = -\frac{\partial\rho}{\partial t} \tag{22a}$$

$$\vec{a}_{\vec{j}} = \nabla \times \vec{j} = q\nabla\delta(\vec{r} - \vec{r}_t) \times \vec{v} \tag{22b}$$

$$\phi_{\vec{j}} = \frac{q}{4\pi}\nabla \cdot \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = -\frac{q}{4\pi}\frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3} \tag{23a}$$

$$\vec{a}_{\vec{j}} = \frac{q}{4\pi}\nabla \times \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = -\frac{q}{4\pi}\frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3} \tag{23b}$$

$$\vec{j}_L = \frac{q}{4\pi}\nabla\frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3}; \quad \vec{j}_T = -\frac{q}{4\pi}\nabla \times \frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3} \tag{24}$$

2.2 Laplacean Vector Fields

Laplacean vector fields are vector fields outside any sources and vortices, they (or their potentials) satisfy the Laplace equation and are essentially determined by the (inhomogeneous) boundary conditions (Schwab 2002). A typical example is the electric field between electrodes.

Since both their divergence and curl vanishe identically, Helmholtz's theorem is not really useful for them.

2.3 Vector Fields in Multiply Connected Domains

Vector fields in multiply connected domains assume an 'androgyne' position in that they (or their potentials) satisfy the Laplace equation in certain, bounded domains, but not globally. A well known example is the magnetic field strength, $\vec{H}$, of a constant current, I, through an infinite straight conductor in vacuo. The 'magnetic ring voltage', $\oint \vec{H} \cdot d\vec{s}$, vanishes identically, as long as the path of integration lies *entirely outside* the conductor, so that no current flows through the area bounded by it. But it equals

$$n \iint_\sigma \left(\nabla \times \vec{H}\right) \cdot d\vec{\sigma} = n \iint_\sigma \vec{j} \cdot d\vec{\sigma} = nI \tag{25}$$

if the path *surrounds* the conductor n times (n integer). That means, that *inside* the conductor, $\vec{H}$ is a *vortex* field: $\nabla \times \vec{H} = \vec{j} \neq \vec{0}$, while *outside* the conductor, $\vec{H}$ is a *gradient* field: $\nabla \times \vec{H} = \vec{0}$. Obviously, Helmholtz's theorem is only conditionally applicable, since the integral rather than the differential form of Ampère's flux law is appropriate.

An analogous example is the vector potential in the Aharonov-Bohm (1959) setup. A constant current through an ideal straight infinite coil in vacuo with no spacing between its windings creates a magnetic field strength and induction being constant inside and vanishing outside the coil. However, by virtue of its continuity, the vector potential does *not* vanishe outside the coil, but represents a gradient field there. I will return to this issue in Section 4.

3. Maxwell's (1864) Original Equations Revisited

"He [Maxwell] would not have been so often misunderstood, if one would have started the study not with the treatise, while the specific Maxwellian method occurs much more clearly in his earlier essays." (Boltzmann 2001; *cf* also Sommerfeld 2001, §1) For this, let us return to Maxwell's (1864) original set of "20 equations for the 20 variables" $(F, G, H) = \vec{A}$, $(\alpha, \beta, \gamma) = \vec{H}$, $(P, Q, R) = \vec{E}$, $(p, q, r) = \vec{j}$, $(f, g, h) = \vec{D}$, $(p', q', r') = \vec{J}$, $e = \rho$, $\psi = \Phi$. I will rewrite them in modern notation (the r.h.s. of the foregoing relations), SI units and together with their Helmholtz decomposition. For easier reference, Maxwell's equation numbering is applied. In place of his eqs. (D) for moving conductors his eqs. (35) for conductors at rest is used. The signs in his eqs. (F) and (G) are changed according to the nowadays use.

3.1 Helmholtz Decomposition

A) The total current density, $\vec{J}$, is the sum of electric (conduction, convection) current density, $\vec{j}$, and displacement ('total polarization') current density, $\partial\vec{D}/\partial t$.

$$\vec{J}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \frac{\partial}{\partial t}\vec{D}(\vec{r}, t) \tag{A}$$

This is Maxwell's famous and crucial step to generalize Ampère's flux law to open circuits and to convective currents. The time derivative is a precondition to obtain wave equations for the field variables.

The Helmholtz decomposition of this equation is obvious.

$$\vec{J}_{T,L}(\vec{r},t) = \vec{j}_{T,L}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}_{T,L}(\vec{r},t) \tag{A$_{T,L}$}$$

B) The "magnetic force" (induction, flux density), $\mu\vec{H}$, is the vortex of the vector potential, $\vec{A}$: $\mu\vec{H} = \nabla \times \vec{A}$. Hence, it has got no longitudinal component.

$$(\mu\vec{H})_T(\vec{r},t) = \nabla \times \vec{A}_T(\vec{r},t) \tag{B$_T$}$$

$$(\mu\vec{H})_L(\vec{r},t) \equiv \vec{0} \tag{B$_L$}$$

C) The total current density, $\vec{J}$, is the vortex of the magnetic field strength, $\vec{H}$.

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) \tag{C}$$

Hence, it has got no longitudinal component, too.

$$\vec{J}_T(\vec{r},t) = \nabla \times \vec{H}_T(\vec{r},t) \tag{C$_T$}$$

$$\vec{J}_L(\vec{r},t) = \vec{0} \tag{C$_L$}$$

D) The "electromotive force" (electric field strength) equals

$$\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) - \nabla\Phi(\vec{r},t) \tag{M-35}$$

Therefore,

$$\vec{E}_T(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_T(\vec{r},t) \tag{M-35$_T$}$$

$$\vec{E}_L(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}_L(\vec{r},t) - \nabla\Phi(\vec{r},t) \underset{def}{=} -\nabla\phi_{\vec{E}}(\vec{r},t) \tag{M-35$_L$}$$

The longitudinal component consists of *two* terms, for which, however, there is *no other* equation. This makes the whole set to be *underdetermined* and is the origin of the gauge freedom in the potentials $\vec{A}$ and Φ. Due to the redundancy in some equations below, it is not inconsistent, however.

This underdeterminacy is overcome, if one can work solely with $\phi_E(\vec{r},t)$, the 'total scalar potential of $\vec{E}(\vec{r},t)$', or if one finds an additional equation for $\vec{A}_L$ and Φ, respectively. An example is the boundary conditions in the Aharonov-Bohm (1959) setup, which determine $\vec{A}_L$ outside the coil.

E) Electric field strength and dielectric displacement are related through the "equation of electric elasticity".

$$\vec{E}(\vec{r},t) = \frac{1}{\varepsilon}\vec{D}(\vec{r},t) \tag{E}$$

Thus,

$$\vec{E}_{T,L}(\vec{r},t) = \frac{1}{\varepsilon}\vec{D}_{T,L}(\vec{r},t) \tag{$E_{T,L}$}$$

if ε is a scalar constant.

F) Electric field strength and electric current density are related through the "equation of electric resistance" (σ being the specific conductivity).

$$\vec{E}(\vec{r},t) = \frac{1}{\sigma}\vec{j}(\vec{r},t) \tag{F}$$

Thus,

$$\vec{E}_{T,L}(\vec{r},t) = \frac{1}{\sigma}\vec{j}_{T,L}(\vec{r},t) \tag{$F_{T,L}$}$$

if σ is a scalar constant.

For N point-like charges $\{q_a\}$ in vacuo ($\sigma = 0$), eq. (F) is to be replaced with

$$\sum_{a=1}^{N} q_a\vec{v}_a(t)\delta(\vec{r}-\vec{r}_a(t)) = \vec{j}(\vec{r},t) \tag{26}$$

G) The "free" charge density is related to the dielectric displacement through the "equation of free electricity".

$$\rho(\vec{r},t) - \nabla\vec{D}(\vec{r},t) = 0 \tag{G}$$

Obviously, it concerns the longitudinal component of $\vec{D}$ only.

$$\rho(\vec{r},t) - \nabla\vec{D}_L(\vec{r},t) = 0 \tag{G_L}$$

H) In a conductor, there is – in analogy to hydrodynamics – "another condition", the "equation of continuity".

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla\vec{j}(\vec{r},t) = 0 \tag{H}$$

It concerns the longitudinal component of $\vec{j}$ only.

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla\vec{j}_L(\vec{r},t) = 0 \tag{H_L}$$

At once, by virtue of eq.(C_L), it is merely a consequence of eq.(G_L). Here is the redundancy mentioned above.

With

$$\vec{j} = \vec{j}_T + \vec{j}_L = \nabla\times\vec{a}_{\vec{j}} - \nabla\phi_{\vec{j}} \tag{27}$$

one obtains the continuity equation in the form

$$-\Delta\phi_{\vec{j}}(\vec{r},t) + \frac{\partial}{\partial t}\rho(\vec{r},t) = 0 \tag{28}$$

It has the advantage of being a single equation relating *two* scalar quantities one to another rather than four, as in its usual form (H).

3.2 Elimination of Underdeterminacy and Redundance

The underdeterminacy and redundance in Maxwell's original set can be eliminiated through removing $(\mu\vec{H})_L$, Φ and $\vec{A}_L$ from the set of field variables, but retaining $\phi_{\vec{E}} = -\partial\phi_{\vec{A}}/\partial t + \Phi$. I also remove the total current in view of its merely historical relevance. Then, it remains 18 equations for the 18 variables $(\mu\vec{H})_{(T)} = \vec{B}$, $\vec{H}$, $\vec{A}_T$, $\vec{D}$, $\vec{E}$, $\phi_{\vec{E}}$, $\vec{j}$ and ρ.

B') The magnetic induction (flux density), $\mu\vec{H}$, is solenoidal, since it is the vortex of the transverse component of the vector potential.

$$\mu\vec{H} = (\mu\vec{H})_T = \nabla \times \vec{A}_T \qquad \text{(B')}$$

C') The transverse components of the conduction/convection and displacement current densities build the vortex of the transverse component, $\vec{H}_T$, of the magnetic field strength, $\vec{H}$.

$$\nabla \times \vec{H}_T = \vec{j}_T + \frac{\partial}{\partial t}\vec{D}_T \qquad \text{(C')}$$

D') The electric field strength equals (and Helmholtz decomposes as)

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A}_T - \nabla\phi_{\vec{E}} \qquad \text{(M-35')}$$

E) Electric field strength and dielectric displacement are related through the "equation of electric elasticity" (E).

F) Electric field strength and electric current density are related through the "equation of electric resistance" (F).

G') The "free" charge density is related to the longitudinal component of the dielectric displacement through the "equation of free electricity".

$$\rho(\vec{r},t) - \nabla\vec{D}_L(\vec{r},t) = 0 \qquad \text{(G')}$$

H') The conservation of charge is expressed through the equation of continuity.

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla\vec{j}_L(\vec{r},t) = 0 \qquad \text{(H')}$$

Therefore, the redundance is removed in the flux law rather than eliminating the continuity equation from the set of basic equations. The continuity equation is retained, because it is a direct consequence of the fact, that – within this approach – the charge of a point-like body is a given, invariant property of it (like its mass). This also allows for an immediate explanation of the transversality of free electromagnetic waves (see below).

4. Gauge Freedom and the Role of $\vec{A}_L$

4.1 Classical Gauge Freedom

As mentioned after eq.(M-35$_L$) above, there is only *one* equation for the *two* fields $\partial\vec{A}_L/\partial t$ and Φ. Hence, any change of the scalar and vector potentials such, that the expression

$(-\partial\phi_{\vec{A}}/\partial t+\Phi) = \phi_{\vec{E}}$ remains unchanged, is without any physical effect within Maxwell's theory.

In fact, the Helmholtz components and potentials of vector potential, $\vec{A}$, and electrical field strength, $\vec{E}$,

$$\vec{A} = \vec{A}_T + \vec{A}_L = \nabla \times \vec{a}_{\vec{A}} - \nabla\phi_{\vec{A}} \tag{29a}$$

$$\vec{E} = \vec{E}_T + \vec{E}_L = \nabla \times \vec{a}_{\vec{E}} - \nabla\phi_{\vec{E}} \tag{29b}$$

are known to be interrelated as

$$\vec{E}_T = -\frac{\partial}{\partial t}\vec{A}_T; \quad \vec{a}_{\vec{E}} = -\frac{\partial}{\partial t}\vec{a}_{\vec{A}} \tag{30}$$

$$\vec{E}_L = -\frac{\partial}{\partial t}\vec{A}_L - \nabla\Phi; \quad \phi_{\vec{E}} = -\frac{\partial}{\partial t}\phi_{\vec{A}} + \Phi \tag{31}$$

Hence, the gauge transformation,

$$\vec{A} = \vec{A}' - \nabla\chi; \quad \Phi = \Phi' + \frac{\partial\chi}{\partial t} \tag{32}$$

actually concerns only the scalar potential, $\phi_{\vec{A}}$, of $\vec{A}$ as

$$\phi_{\vec{A}} = \phi'_{\vec{A}} + \chi \tag{33}$$

but not the vector potential, $\vec{a}_{\vec{A}}$, of $\vec{A}$.

In the Lorenz (1867) gauge used in Lorentz covariant formulations of the theory, one has

$$\nabla\vec{A} = -\Delta\phi_{\vec{A}} = -\frac{\partial\Phi}{\partial t} \tag{34}$$

while in the Coulomb (transverse, radiation) gauge being popular in quantum electrodynamics,

$$\nabla\vec{A} = -\Delta\phi_{\vec{A}} = 0 \tag{35}$$

This all suggests to avoid the gauge indeterminacy at all through working solely with $\vec{A}_T$ and $\phi_{\vec{E}}$. If necessary, $\vec{A}_L$ can be determined as boundary value problem.

4.2 Quantum Gauge Freedom (Schrödinger Theory)

Although this series of papers deals with classical electromagnetism, it is enlightening and pedagogically useful to sidestep for looking at gauge freedom within Schrödinger wave mechanics.

In order to be independent of the interpretation of the quantum mechanical formalism, let me proceed a follows (Enders 2006, 2008a,b).

$|\psi|^2$ and $< \psi|\hat{H}|\psi >$ are 'Newtonian state functions' of a non-relativistic quantum system as they are time-independent in stationary states and as their time-dependence is governed by solely the time-dependent part of the Hamiltonian. This suggests to extend Helmholtz's (1847, 1911) explorations about the relationships between forces and energies to the question, which 'external influences' leave $|\psi|^2$ and $< \psi|\hat{H}|\psi >$ unchanged?

Obviously, $|\psi|^2$ is unchanged, if an external influence, w, affects only the phase, φ, of ψ. (Dirac 1931 required the phase to be independent of the state.)

$$\psi_w = \psi_0 e^{i\varphi(w)}; \quad \varphi(0) = 0 \tag{36}$$

Then, if $\psi_0(\vec{r},t)$ obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_0 = \frac{\hat{p}^2}{2m}\psi_0 + V\psi_0 \tag{37}$$

$\psi_w(\vec{r},t)$ obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_w = \hat{H}_w\psi_w = \frac{1}{2m}\left(\hat{p} - \hbar\nabla\varphi\right)^2\psi_w + \left(V - \hbar\frac{\partial\varphi}{\partial t}\right)\psi_0 \tag{38}$$

Consequently, in stationary states, $< \psi_w|\hat{H}_w|\psi_w >$ is independent of w, because $i\hbar\frac{\partial}{\partial t}\psi_w = E\psi_w$, where – by the very definition of w – E is independent of w. This is essentially the gauge invariance of the Schrödinger (Pauli 1926) and Dirac equations (Fock 1929) (see also Weyl 1929, 1931).

For influences caused by external electromagnetic fields, this quite general arguing leads to the following important observation, which will be exploited below when formulating an gaugefree canonical theory.

The common quasi-classical Schrödinger equation for a point-like charge, q, in an electromagnetic field reads

$$i\hbar\frac{\partial}{\partial t}\psi_{\vec{A},\Phi}(\vec{r},t) = \left[\frac{1}{2m}\left(\hat{p} - q\vec{A}(\vec{r},t)\right)^2 + q\Phi(\vec{r},t)\right]\psi_{\vec{A},\Phi}(\vec{r},t) \tag{39}$$

Thus, the wave function

$$\psi_{\vec{A}_T,\phi_{\vec{E}}}(\vec{r},t) = \psi_{\vec{A},\Phi}(\vec{r},t)e^{i\frac{q}{\hbar}\phi_{\vec{A}}(\vec{r},t)} \tag{40}$$

obeys a Schrödinger equation with a *manifest gauge invariant* Hamiltonian.

$$i\hbar\frac{\partial}{\partial t}\psi_{\vec{A}_T,\phi_{\vec{E}}}(\vec{r},t) = \left[\frac{1}{2m}\left(\hat{p} - q\vec{A}_T(\vec{r},t)\right)^2 + q\phi_{\vec{E}}(\vec{r},t)\right]\psi_{\vec{A}_T,\phi_{\vec{E}}}(\vec{r},t) \tag{41}$$

This suggests that manifest gauge invariant theories can be obtained through replacing $\vec{A}$ with $\vec{A}_T$ and Φ with $\phi_{\vec{E}}$.

It's noteworthy that in both Hamiltonians the canonical momentum *operator*: $\hat{p} = -i\hbar\nabla$, is the *same*, while the corresponding *classical* canonical momenta are *different*.

It is noteworthy, that in *multiply* connected domains, notably outside an infinite coil, where the $\vec{B}$-field vanishes, $\phi_{\vec{A}}$ is not globally integrabel. The phase of the wave function can aquire physical significance, as in the Aharonov-Bohm (1959) effect. This underpins the physical significance of the Helmholtz decomposition of the field variables.

Thus, the longitudinal component of a static vector potential, $\vec{A}_L(\vec{r})$, is classically *not* observable, because it does not contribute to the Maxwell-Lorentz force (Maxwell 1864, Lorentz 1892),

$$q\vec{E} + q\vec{v}\times\vec{B} = q\left(-\frac{\partial\vec{A}}{\partial t} - \nabla\Phi + \vec{v}\times\nabla\times\vec{A}\right) \tag{42}$$

This suggests to remove $\vec{A}_L(\vec{r})$ from the classical theory altogether and to consider it to be a 'quantum potential' being proportional to Planck's quantum of action, h. On the other hand, if one requires – for good reasons – $\vec{A}(\vec{r})$ to be continuous, $\vec{A}_L(\vec{r})$ can be finite even in the classical (limit) case.

Eq.(40) suggests to incorporate other non-dynamical fields not entering the Hamiltonian and being determined by Laplacean boundary-value problems, by means of appropriate phase factors, too.

"As emphasized by Yang [1974] the vector potential is an over complete specification of the physics of a gauge theory but the gauge covariant field strength underspecifies the content of a gauge theory. The Bohm-Aharonov [1959] effect is the most striking example of this, wherein there exist physical effects on charged particles in a region where the field strength vanishes. The complete and minimal set of variables necessary to capture all the physics are the non-integrable phase factors." (Gross 1992, II.4) Because there are no such phase factors within classical electromagnetism, their classical limit is rather unclear. The complete and minimal set of classical variables obtained below is only loosely related to those. It is thus hoped that the gauge-free representation presented below will narrow this gap between classical and quantum theory.

5. Helmholtz Decomposition of the "Rationalized" Maxwell's Equations within Mie's Approach

Newton (1999, Definitions) has assumed that the mass is a constant property of a given body. Likewise, it is meaningful to consider the electric charge to be such a property. As a consequence, one has the continuity equation,

$$\nabla\vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\rho(\vec{r},t) = 0 \tag{43}$$

as a precondition of the dynamics of the fields created by the charges. This is in contrast with those approaches which see the continuity as contained in or even as a consequence of the rationalized Maxwell equations, but it is compatible with Maxwell's original equations (see above).

Given that, Mie (1941) argues as follows (after Hehl & Obukhov 2003; see also Bopp 1962, Enders 2008).

1) Mathematically, to each given charge distribution, $\rho(\vec{r},t)$, there is a vector field, $\vec{D}(\vec{r},t)$, such, that

$$\nabla\vec{D}(\vec{r},t) = \rho(\vec{r},t) \tag{44}$$

According to Maxwell (1864, 1873), $\vec{D}(\vec{r},t)$ has got a physical meaning, *viz*, as (di)"electric displacement".

Actually, there are infinitely many such vector fields, because $\vec{D}_T$ is not specified. Moreover, this conclusion is not unique as one can associate to $\rho(\vec{r},t)$ also a scalar field, $\phi(\vec{r},t)$, such, that the latter obeys the Helmholtz equation (Enders 2009)

$$\nabla^2\phi(\vec{r},t) + \kappa\phi(\vec{r},t) = \rho(\vec{r},t) \tag{45}$$

2) By virtue of charge conservation (43),

$$\frac{\partial}{\partial t}\nabla\vec{D}(\vec{r},t) = \frac{\partial}{\partial t}\rho(\vec{r},t) = -\nabla\vec{j}(\vec{r},t) \tag{46}$$

$$0 = \nabla\left(\frac{\partial}{\partial t}\vec{D}(\vec{r},t) + \vec{j}(\vec{r},t)\right) \tag{47}$$

Hence, mathematically, there is a vector field, $\vec{H}(\vec{r},t)$, such, that

$$\nabla\times\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \tag{48}$$

According to Maxwell (1864, 1873), $\vec{H}(\vec{r},t)$ has got a physical meaning, *viz*, as the magnetic field strength.

Actually, there are infinitely many such vector fields, because $\vec{H}_L$ is not specified.

Thus, in the spirit of Helmholtz's (1858) decomposition theorem, this approach can be formulated more precisely as follows.

1') Mathematically, to each given charge distribution, $\rho(\vec{r},t)$, there is a vector field, $\vec{D}(\vec{r},t) = \vec{D}_T(\vec{r},t) + \vec{D}_L(\vec{r},t)$, such, that $\rho(\vec{r},t)$ is the source of its scalar component.

$$\nabla\vec{D}_L(\vec{r},t) = \rho(\vec{r},t) \tag{49}$$

According to Maxwell, $\vec{D}_L$ is the longitudinal component of the (di)"electric displacement", if $\rho(\vec{r},t)$ represents the "free" charges.

2') By virtue of charge conservation,

$$\nabla\left(\frac{\partial}{\partial t}\vec{D}_L(\vec{r},t) + \vec{j}_L(\vec{r},t)\right) = \nabla\vec{J}_L(\vec{r},t) = 0 \tag{50}$$

Hence, the longitudinal component, $\vec{J}_L$, of the total current, $\vec{J} = \frac{\partial}{\partial t}\vec{D} + \vec{j}$, vanishes identically.

$$\vec{J}_L(\vec{r},t) = \vec{j}_L(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}_L(\vec{r},t) \equiv \vec{0} \tag{51}$$

Its transverse component can – as every solenoidal field – be written as the vortex of a vector field.

$$\vec{J}_T(\vec{r},t) = \vec{j}_T(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}_T(\vec{r},t) = \nabla\times\vec{H}_T(\vec{r},t) \tag{52}$$

According to Maxwell, $\vec{H}_T$ is the transverse component of the magnetic field strength.

In other words, the longitudinal part (51) of Ampère-Maxwell's flux law (48) merely duplicates the conservation of charge, whence its transverse part (52) becomes the *effective* flux law.

The two homogeneous rationalized Maxwell equations emerge from his 1864 set through, (i), setting $\mu\vec{H} = \vec{B}$, the magnetic flux density (induction; Maxwell 1873, §604), and, (ii), eliminating the potentials.

$$\nabla\vec{B}(\vec{r},t) = 0 \tag{53}$$

$$\nabla\times\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \tag{54}$$

Here, the latter equation, nowadays called Faraday's induction law, assumes a primary axiomatic position, while it was a secondary, to be derived feature in the original1864 set of equations.

Actually, by virtue of $\nabla\times\vec{E} = \nabla\times\vec{E}_T$, it contains *solely transverse* field components.

$$\vec{B}_L(\vec{r},t) \equiv \vec{0} \tag{55}$$

$$\nabla\times\vec{E}_T(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}_T(\vec{r},t) \tag{56}$$

Thus, the Helmholtz-decomposed "rationalized" Maxwell equations represent a set of 6 equations for the 10 independent components of $\vec{D}_L$, $\vec{D}_T$, $\vec{B}_L$, $\vec{B}_T$, $\vec{E}_T$ and $\vec{H}_T$, where ρ and $\vec{j}_T$ are considered to be external sources, independent variables. As for the full set, it can be closed through material equations; here, $\vec{D}_T = \varepsilon\vec{E}_T$ and $\vec{B}_T = \mu\vec{H}_T$.

$\vec{E}_L$ and $\vec{H}_L$ are needed to write the "rationalized" Maxwell equations in a manifest Lorentz invariant manner; $\vec{E}_L$ is also needed in the Maxwell-Lorentz force.

6. Transverse and Longitudinal Poynting Theorems

In the common derivations of Poynting's (1884) theorem, it is discarded that both the l.h.s. of the flux law (48) and Faraday's induction law (54) contain *solely transverse* field components, see eqs. (56) and (56), respectively. This fact is accounted for in the **Transverse Poynting theorem:**

$$\iiint_V \vec{E}_T\cdot\vec{j}_T dV = -\iiint_V \left(\vec{E}_T\cdot\frac{\partial\vec{D}_T}{\partial t} + \vec{H}_T\frac{\partial\vec{B}_T}{\partial t}\right) dV - \oint_{\partial V}\left(\vec{E}_T\times\vec{H}_T\right)\cdot d\vec{\sigma} \tag{57}$$

Indeed, (i), by virtue of the orthogonality theorem (17) and $\vec{B}\equiv\vec{B}_T$,

$$\iiint \vec{H}\cdot\left(\nabla\times\vec{E}\right) dV = \iiint \vec{H}_T\cdot\left(\nabla\times\vec{E}_T\right) dV \tag{58}$$

$$= -\iiint \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}dV = -\iiint \vec{H}_T\frac{\partial\vec{B}}{\partial t}dV \tag{59}$$

and, (ii), due to $\vec{j}_L + \frac{\partial}{\partial t}\vec{D}_L = \vec{0}$,

$$\iiint \vec{E}\cdot\left(\nabla\times\vec{H}\right) dV = \iiint \vec{E}_T\cdot\left(\nabla\times\vec{H}_T\right) dV \tag{60}$$

$$= \iiint \vec{E}\cdot\vec{j}dV + \iiint \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}dV = \iiint \vec{E}_T\cdot\vec{j}_T dV + \iiint \vec{E}_T\cdot\frac{\partial\vec{D}_T}{\partial t}dV \tag{61}$$

Its interpretation is quite analogous to the standard theorem.

- $\iiint \vec{E}_T\cdot\vec{j}_T dV = \iiint \vec{E}_T\cdot\vec{j}dV$ is the Joule power of $\vec{E}_T$ transfered from the field to the charged bodies.

- $\iiint \left(\vec{E}_T \cdot \frac{\partial}{\partial t}\vec{D}_T + \vec{H}_T \frac{\partial}{\partial t}\vec{B} \right) dV$ is the power of the transverse fields; for $\int \vec{E}_T \cdot d\vec{D}_T$ is the work density done by the transverse electric field to produce the transverse displacement (*cf* the arguing for $\int \vec{E} \cdot d\vec{D}$ and $\frac{1}{2}\iiint \vec{E} \cdot \vec{D} dV$ in Maxwell 1864, §72); analogously, $\int \vec{H} \cdot d\vec{B}$ is the work density done by the magnetic field to produce the (always transverse) magnetic induction (flux density); note, that, $\frac{1}{2}\iiint \vec{H} \cdot \vec{B} dV = \frac{1}{2}\int \vec{J} \cdot \vec{A} dV$, the work done by the (always transverse) total current density to produce the vector potential (*cf* Maxwell 1864, §§ 33, 71).
- $\oint_{\partial V} (\vec{E}_T \times \vec{H}_T) \cdot d\vec{\sigma}$ is the power propagating through the surface ∂V into the exterior of the volume, V, under consideration, where $\vec{E}_T \times \vec{H}_T = \vec{S}^{(T)}$ is the *'propagating part'* (not the transverse component!) of the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$.

The energy balance for the longitudinal components is separately guaranteed through the

Longitudinal Poynting theorem:

$$\iiint_V \vec{E}_L \cdot \vec{j}_L dV = -\iiint_V \vec{E}_L \cdot \frac{\partial}{\partial t}\vec{D}_L dV \tag{62}$$

The Joule power of $\vec{E}_L$ is balanced by the field power of the longitudinal (di)electric field vectors.

The *'non-propagating part'* of the Poynting vector: $\vec{S}^{(L)} = \vec{E}_L \times \vec{H}_T$ (this is not its longitudinal component!), does *not* contribute to the power/energy balance, since $\oint_{\partial V} \vec{S}^{(L)} \cdot d\vec{\sigma} = 0$.

This splitting of Poynting's theorem into its longitudinal and transverse parts supports the view that the Helmholtz decomposition helps to discover the physical content of Maxwell's theory.

It's noteworthy that, energetically, $\vec{E}$ and $\vec{H}$ are a pair in both being intensive, driving quantities, while $\vec{D}$ and $\vec{B}$ are a pair in both being extensive and driven quantities.

7. Standard Canonical Classical Electromagnetism

7.1 Standard Lagrangian

Milton & Schwinger (2006, 1.2) have formulated the microscopic theory in a most elegant manner through accounting explicitly for the fact, that the free-field Lagrangian is expressed much more concisely in terms of the fields $\vec{E}$ and $\vec{B}$ than in terms of the potentials Φ and $\vec{A}$. In SI units and for one body of mass m and charge q moving with velocity $v \ll c$ along the trajectory $\vec{r}_m(t)$, the total Lagrangian reads

$$L(t) = \iiint \mathfrak{L}(\vec{r}, t) dV \tag{63}$$

$$= \iiint \mathfrak{L}_{field}(\vec{r}, t) dV + q\vec{v}(t) \cdot \vec{A}(\vec{r}_m(t), t) - q\Phi(\vec{r}_m(t), t) + \frac{m}{2}\vec{v}(t)^2 \tag{64}$$

with the field's Lagrange density

$$\mathfrak{L}_{field}(\vec{r}, t) = \frac{\varepsilon_0}{2}\vec{E}(\vec{r}, t)^2 - \frac{1}{2\mu_0}\vec{B}(\vec{r}, t)^2 \tag{65}$$

Throughout this paper it will be assumed that, for a point-like body, the discrete and the continuum representations can be freely interchanged, where

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r}_m(t)); \quad \vec{j}(\vec{r},t) = q\vec{v}(t)\delta(\vec{r}-\vec{r}_m(t));$$
$$\rho_m(\vec{r},t) = m\delta(\vec{r}-\vec{r}_m(t)) \quad (66)$$

Thus, the variational derivatives are, (i),

$$\frac{\delta L}{\delta(\partial\vec{A}/\partial t)} = \frac{\partial\mathfrak{L}}{\partial(\partial\vec{A}/\partial t)} - \frac{\partial\mathfrak{L}}{\partial\vec{E}} = -\varepsilon_0\vec{E} = \vec{\Pi}_{\vec{A}} \quad (67)$$

ie, the canonical momentum density of the vector potential; (ii),

$$\frac{\delta L}{\delta\vec{A}} = \frac{\partial\mathfrak{L}}{\partial\vec{A}} + \nabla\times\frac{\partial\mathfrak{L}}{\partial\vec{B}} = \vec{j} - \nabla\times\frac{1}{\mu_0}\vec{B} = -\varepsilon_0\frac{\partial}{\partial t}\vec{E} \quad (68)$$

by virtue of the flux law (being the Lagrangian equation of motion for $\vec{A}$); (iii),

$$\frac{\delta L}{\delta\Phi} = \frac{\partial\mathfrak{L}}{\partial\Phi} + \nabla\cdot\frac{\partial\mathfrak{L}}{\partial\vec{E}} = -\rho + \nabla\varepsilon_0\vec{E} \equiv 0 \quad (69)$$

by virtue of Gauss' law. Together with ($\dot{\Phi} \equiv \partial\Phi/\partial t$)

$$\frac{\delta L}{\delta\dot{\Phi}} = \frac{\partial\mathfrak{L}}{\partial\dot{\Phi}} \equiv 0, \quad (70)$$

this means, that the scalar potential does not exhibit an own dynamics.

In order to obtain a manifest Lorentz covariant formulation of the theory, such a dynamics is usually *created* by means of the Lorenz (1867) gauge

$$\nabla\vec{A} + \dot{\Phi} = 0, \quad (71)$$

providing an equation for $\dot{\Phi}$ and adding the 'nutritious zero' $(-1/2\mu_0)(\nabla\vec{A}+\dot{\Phi})^2$ to the Lagrange density (Heisenberg & Pauli 1929; see also Fock & Podolsky 1932, Dirac, Fock & Podolsky 1932). Within quantum electrodynamics, this leads to the not observable longitudinal and scalar (time-like) photons which are lateron projected out, however.

In contrast, within the gaugefree formulation, the scalar potential, $\phi_{\vec{A}}$, of the vector potential is combined with the common scalar potential, Φ, to the scalar potential of the electric field strength, $\phi_{\vec{E}} = \Phi - \partial\phi_{\vec{A}}/\partial t$. Via the Poisson equation,

$$\Delta\phi_{\vec{E}} = -\frac{\rho}{\varepsilon_0}, \quad (72)$$

the dynamics of $\phi_{\vec{E}}$ is tied to that of the charge density. A self-standing dynamics is exhibited by the propagating transverse field components only. Its canonical theory will be formulated in the next section.

The body's dynamics is dealt with conventionally. (iv),

$$\frac{\partial L}{\partial\vec{v}} = \frac{\partial}{\partial\vec{v}}\left(q\vec{v}\cdot\vec{A} + \frac{m}{2}\vec{v}^2\right) = m\vec{v}(t) + q\vec{A}(\vec{r}_m,t) = p(\vec{r}_m,t), \quad (73)$$

ie, the canonical momentum of the body; (v),

$$\frac{\partial L}{\partial \vec{r}_m} = \frac{\partial}{\partial \vec{r}_m}\left(q\vec{v}\cdot\vec{A}(\vec{r}_m,t) - q\Phi(\vec{r}_m,t)\right) \tag{74}$$

$$= q\left(\vec{v}\cdot\nabla_m\right)\vec{A} + q\vec{v}\times\nabla_m\times\vec{A} - q\nabla_m\Phi \tag{75}$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \vec{v}} \tag{76}$$

$$= \frac{d}{dt}\left(m\vec{v}(t) + q\vec{A}(\vec{r}_m(t),t)\right) \tag{77}$$

$$= m\frac{d\vec{v}}{dt} + q\left(\vec{v}\cdot\nabla_m\right)\vec{A} + q\frac{\partial\vec{A}}{\partial t} \tag{78}$$

This is the Newtonian equation of motion with the Maxwell-Lorentz force (42).

$$m\frac{d\vec{v}}{dt} = -q\frac{\partial\vec{A}}{\partial t} - q\nabla_m\Phi + q\vec{v}\times\nabla_m\times\vec{A} \tag{79}$$

$$= q\vec{E} + q\vec{v}\times\vec{B} \tag{80}$$

7.2 Hamiltonian and Total Energy

By definition, 'external influences' act upon a system without relevant back-reaction. The best known example is, perhaps, forced oscillations. In contrast, without external influences, the change rates are determined solely by the system's own properties, while the *origin* of the system's time scale plays no role. This implies, that the Lagrangian does not explicitly depend on time. Its implicit time-dependence is visible from

$$\frac{dL}{dt} = \iiint\left(\frac{\delta L}{\delta\vec{A}}\cdot\frac{\partial\vec{A}}{\partial t} + \frac{\delta L}{\delta(\partial\vec{A}/\partial t)}\cdot\frac{\partial^2\vec{A}}{\partial t^2}\right)dV + \frac{\partial L}{\partial\vec{r}_m}\cdot\frac{d\vec{r}_m}{dt} + \frac{\partial L}{\partial\vec{v}}\cdot\frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\iiint\left(\frac{\delta L}{\delta(\partial\vec{A}/\partial t)}\cdot\frac{\partial\vec{A}}{\partial t}\right)dV + \frac{d}{dt}\left(\frac{\partial L}{\partial\vec{v}}\cdot\vec{v}\right) \tag{81}$$

(*cf* Milton & Schwinger 2006, 1.3). Hence, the energy function,

$$h = \iiint\left(\frac{\delta L}{\delta(\partial\vec{A}/\partial t)}\cdot\frac{\partial\vec{A}}{\partial t}\right)dV + \frac{\partial L}{\partial\vec{v}}\cdot\vec{v} - L \tag{82}$$

$$= \iiint\left[\frac{\varepsilon_0}{2}\vec{E}^2 + \varepsilon_0\vec{E}\cdot\nabla\Phi + \frac{1}{2\mu_0}\vec{B}^2\right]dV + q\Phi + \frac{m}{2}\vec{v}^2 \tag{83}$$

of closed systems is time-*in*dependent. Actually, if surface terms make no contribution (as assumed throughout this paper, except for the Poynting vector below), it equals the total energy,

$$E = \iiint\left(\frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2\right)dV + \frac{m}{2}\vec{v}^2 = const \tag{84}$$

The total energy is conserved, if the external spacetime is homogeneous in time. And, as it should be, it is gauge invariant.

Replacing in the energy function (82) $\vec{E}$ with $(-\vec{\Pi}_{\vec{A}}/\varepsilon_0)$ and $\vec{v}$ with $(\vec{p}-q\vec{A})/m$, one obtains the Hamiltonian,

$$H = \iiint \left[\frac{1}{2\varepsilon_0}\vec{\Pi}^2_{\vec{A}} - \vec{\Pi}_{\vec{A}}\cdot\nabla\Phi + \frac{1}{2\mu_0}\vec{B}^2\right] dV + q\Phi + \frac{1}{2m}\left(\vec{p}-q\vec{A}\right)^2 \tag{85}$$

$$= \iiint \mathfrak{H} dV \tag{86}$$

with the Hamilton density

$$\mathfrak{H} = \frac{1}{2\varepsilon_0}\vec{\Pi}^2_{\vec{A}} + \frac{1}{2\mu_0}\vec{B}^2 - \vec{\Pi}_{\vec{A}}\cdot\nabla\Phi + \rho\Phi + \frac{1}{2\rho_m}\left(\vec{\pi}-\rho\vec{A}\right)^2; \qquad \vec{\pi} = \rho_m\vec{v} + \rho\vec{A} \tag{87}$$

This Hamiltonian is numerically gauge *in*variant, because the two Φ-dependent terms cancel each another (see the total energy, E, above), and $\vec{p}-q\vec{A}=\vec{v}$ is gauge-*in*dependent, too. Notice, that it is necessary to keep *explicitly* those two terms, $q\Phi$ and $-\iiint \vec{\Pi}_{\vec{A}}\cdot\nabla\Phi dV$, in order to obtain the correct Hamiltonian equations of motion.

For the fields, these are, (i),

$$\frac{\partial\Phi}{\partial t} = \frac{\delta H}{\delta\Pi_\Phi} \equiv 0 \tag{88}$$

This is a formal equation (as $\Pi_\Phi \equiv 0$); it indicates, again, that Φ does not exhibit a dynamics on its own.

(ii),

$$\frac{\partial\vec{A}}{\partial t} = \frac{\delta H}{\delta\vec{\Pi}_{\vec{A}}} = \frac{\partial\mathfrak{H}}{\partial\vec{\Pi}_{\vec{A}}} = \frac{1}{\varepsilon_0}\vec{\Pi}_{\vec{A}} - \nabla\Phi \tag{89}$$

As usual (Goldstein 1950), the equations for the potentials merely reproduce the definitions of the canonical momenta.

(iii),

$$\frac{\partial\Pi_\Phi}{\partial t} = -\frac{\delta H}{\delta\Phi} = -\frac{\partial\mathfrak{H}}{\partial\Phi} + \nabla\cdot\frac{\partial\mathfrak{H}}{\partial(\nabla\Phi)} = -\nabla\vec{\Pi}_{\vec{A}} - \rho \equiv 0 \tag{90}$$

by virtue of Gauss' law. This, again, is not a dynamical equation, but a Bergmannian "primary constraint" (Dirac 2001, p.8). In order to avoid this constraint, Goenner (2004, 5.2.6) has proposed to restrict $\vec{\Pi}_{\vec{A}}$ to this class of values in a rather formal manner. Below, I will show, that such restrictions are not necessary, when working with the Helmholtz-decomposed vector fields.

(iv),

$$\frac{\partial\vec{\Pi}_{\vec{A}}}{\partial t} = -\frac{\delta H}{\delta\vec{A}} = -\frac{\partial\mathfrak{H}}{\partial\vec{A}} - \nabla\times\frac{\partial\mathfrak{H}}{\partial\vec{B}} = -\frac{1}{\mu_0}\nabla\times\vec{B} + \rho\vec{v} \tag{91}$$

This is the microscopic flux law.

8. Gaugefree Canonical Theory

Recall that the Hamiltonian in eq.(41) is *manifest gauge invariant.* This suggests, that manifest gauge invariant entities are obtained from their standard expressions through replacing $\vec{A}$ with $\vec{A}_T$ and Φ with $\phi_{\vec{E}}$. We will see, however, that it is not that simple.

8.1 The Helmholtz Decomposed Microscopic Maxwell Equations

The Helmholtz decomposed microscopic Maxwell equations follow from the Helmholtz decomposed macroscopic Maxwell equations (see above) in the same manner as the not decomposed ones do.

Helmholtz decomposed Gauss' law for the electrical field: Gauss' law for the electric field reduces to a Poisson equation for the scalar potential, $\phi_{\vec{E}}$, of the electric field.

$$\nabla\vec{E} = \nabla\vec{E}_L = -\Delta\phi_{\vec{E}} = \frac{1}{\varepsilon_0}\rho \tag{92}$$

Helmholtz decomposed Gauss' law for the magnetic field: Gauss' law for the magnetic field states that the latter is purely transverse.

$$\nabla\vec{B} = \nabla\vec{B}_L = 0 \tag{93}$$

$$\vec{B}_L \equiv \vec{0}; \quad \vec{B} = \vec{B}_T \tag{94}$$

Helmholtz decomposed Faraday's induction law:

$$\nabla\times\vec{E} = \nabla\times\vec{E}_T = -\frac{\partial}{\partial t}\vec{B}_{(T)} \tag{95}$$

The induction law effectively connects *solely transverse* field components.

Helmholtz decomposed Ampère-Maxwell's flux law: The flux law separates into a transverse and a longitudinal parts.

$$\frac{1}{\mu_0}\nabla\times\vec{B}_{(T)} = \left(\vec{j}_L + \vec{j}_T\right) + \varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}_L + \vec{E}_T\right) \tag{96}$$

Together with Gauss' law (92), the longitudinal part,

$$\vec{0} = \vec{j}_L + \varepsilon_0\frac{\partial}{\partial t}\vec{E}_L \tag{97}$$

is equivalent to the continuity equation and, thus, can be dispensed in favour of that (see above). Consequently, the transverse one,

$$\frac{1}{\mu_0}\nabla\times\vec{B}_{(T)} = \vec{j}_T + \varepsilon_0\frac{\partial}{\partial t}\vec{E}_T \tag{98}$$

represents the *effective* flux law.

In contrast to the scalar potential, $\Phi(\vec{r},t)$, and to the vector potential, $\vec{A}(\vec{r},t)$, the (total) scalar potential of the electric field strength,

$$\phi_{\vec{E}}(\vec{r},t) = \Phi(\vec{r},t) - \frac{\partial}{\partial t}\phi_{\vec{A}} \tag{99}$$

and the transverse component, $\vec{A}_T(\vec{r},t)$, of the vector potential are gauge invariant, if not to say gaugefree. By virtue of the effective flux law (98), the latter one obeys the wave equation

$$-\frac{1}{\mu_0}\Delta\vec{A}_T = \vec{j}_T - \varepsilon_0\frac{\partial^2}{\partial t^2}\vec{A}_T \tag{100}$$

Manifest gauge invariant Lagrangian and Hamiltonian equations of motion have to reproduce this wave equation and $\Delta\phi_{\vec{E}} = -\rho/\varepsilon_0$.

8.2 Gaugefree Canonical Particle Momentum

Obviously,

$$\vec{p}^{(L)}(t) = m\vec{v}(t) + q\vec{A}_T(\vec{r}_m(t),t) \tag{101}$$

is a *gauge invariant* canonical momentum (*cf* Messiah 1999, XXI.23, p.1025, fn.1). More accurately, it is gauge*free*, since $\vec{A}_T$ is not affected by gauge. The body's gaugefree canonical momentum density thus equals

$$\vec{\pi}^{(L)} = \rho_m\vec{v} + \rho\vec{A}_T \tag{102}$$

At first glance, it seems that the field term, $q\vec{A}_T$, represents the influence of the *transverse* field. However, via partial integration and vanishing of all fields outside the volume of integration, one obtains (reversing the calculations in Messiah 1999, XXI.23)

$$\iiint \rho\vec{A}_T dV = -\varepsilon_0\iiint(\Delta\phi_{\vec{E}})\vec{A}_T dV = -\varepsilon_0\iiint \phi_{\vec{E}}\Delta\vec{A}_T dV \tag{103}$$

$$= \varepsilon_0\iiint \phi_{\vec{E}}\nabla\times\vec{B}dV = -\varepsilon_0\iiint \nabla\phi_{\vec{E}}\times\vec{B}dV \tag{104}$$

$$= \varepsilon_0\iiint \vec{E}_L\times\vec{B}dV = \frac{1}{c_0^2}\iiint \vec{S}^{(L)}dV = \vec{p}^{(L)}_{field}(t) \tag{105}$$

Hence, $q\vec{A}_T$ actually contains the contribution of the *longitudinal* electric field component to the field momentum (*cf* above). This is another backing for the view, that, *cum grano salis*, the motion of the longitudinal field, $\vec{E}_L$, is tied to the motion of the charged bodies, while the self-standing motion of the field is realized by solely the transverse fields, $\vec{E}_T$ and $\vec{B}$.

This way, the total momentum of the system charge & fields gains the intuitive expression

$$\vec{p}_{tot} = \vec{p}^{(L)} + \vec{p}_{prop} \tag{106}$$

where

$$\vec{p}_{prop} = \varepsilon_0\iiint \vec{E}_T\times\vec{B}dV \tag{107}$$

is the momentum of the *prop*agating part of the field (see above). Although it is mathematically equivalent to the standard expression,

$$\vec{p}_{tot} = \vec{p}_{kin} + \vec{p}_{field} = m\vec{v} + \varepsilon_0\iiint \vec{E}\times\vec{B}dV \tag{108}$$

it is physically preferable, because, in the latter one, neither the standard canonical momentum, $\vec{p} = m\vec{v} + q\vec{A}$, nor the gauge invariant canonical momentum, $\vec{p}^{(L)}$, have got a self-standing place. The difference

$$\vec{p}_{tot} - \vec{p}_{can} = \vec{p}_{field} - q\vec{A} = \varepsilon_0 \iiint \vec{E}_T \times \vec{B} dV - q\vec{A}_L \tag{109}$$

has not got an own physical meaning.

8.3 Gaugefree Lagrangian

In terms of the Helmholtz decomposed and *g*auge*f*ree field variables, the Lagrangian reads

$$L^{gf}(t) = \iiint \mathfrak{L}^{gf}(\vec{E}_T, \vec{E}_L, \vec{B}, \phi_{\vec{E}}, \vec{A}_T, \vec{v}) dV \tag{110}$$

$$\mathfrak{L}^{gf} = \frac{\varepsilon_0}{2}\vec{E}_T^2 + \frac{\varepsilon_0}{2}\vec{E}_L^2 - \frac{1}{2\mu_0}\vec{B}^2 + \vec{j}\cdot\vec{A}_T - \rho\phi_{\vec{E}} + \frac{1}{2}\rho_m \vec{v}^2 \tag{111}$$

Because there is no $\vec{v}_T$, it is necessary to keep $\vec{j}\cdot\vec{A}_T$, although $\iiint \vec{j}\cdot\vec{A}_T dV = \iiint \vec{j}_T\cdot\vec{A}_T dV$ (see above). The difference to the standard Lagrangian is a total time-derivative.

$$L^{gf}(t) - L(t) = \frac{d}{dt} q\phi_{\vec{A}}(\vec{r}(t), t) \tag{112}$$

For the field, there is a canonical momentum density only for the transverse component of the vector potential, *viz*,

$$\vec{\Pi}_{\vec{A}_T} = \frac{\delta L^{gf}}{\delta(\partial\vec{A}_T/\partial t)} = \frac{\partial\mathfrak{L}^{gf}}{\partial(\partial\vec{A}_T/\partial t)} - \frac{\partial\mathfrak{L}^{gf}}{\partial\vec{E}_T} = -\varepsilon_0\vec{E}_T \tag{113}$$

The corresponding Lagrangian equation of motion is the transverse, effective part of the flux law.

$$\frac{\delta L^{gf}}{\delta\vec{A}_T} = \frac{\partial\mathfrak{L}^{gf}}{\partial\vec{A}_T} + \nabla\times\frac{\partial\mathfrak{L}^{gf}}{\partial\vec{B}} = \vec{j}_T - \nabla\times\frac{1}{\mu_0}\vec{B} = -\varepsilon_0\frac{\partial}{\partial t}\vec{E}_T \tag{114}$$

$\phi_{\vec{E}}$ does not exhibit an own dynamics, because

$$\frac{\delta L^{gf}}{\delta\phi_{\vec{E}}} = \frac{\partial\mathfrak{L}^{gf}}{\partial\phi_{\vec{E}}} + \nabla\cdot\frac{\partial\mathfrak{L}^{gf}}{\partial\vec{E}_L} = -\rho + \nabla\varepsilon_0\vec{E}_L \equiv 0 \tag{115}$$

by virtue of Gauss' law ($\nabla\vec{E} = \nabla\vec{E}_L$) and

$$\frac{\delta L^{gf}}{\delta\dot{\phi}_{\vec{E}}} = \frac{\partial\mathfrak{L}^{gf}}{\partial\dot{\phi}_{\vec{E}}} \equiv 0, \tag{116}$$

(*cf* Messiah 1999, XXI.22). Since $\phi_{\vec{E}}$ is gauge invariant, this cannot be changed (in contrast to Φ).

8.4 Manifest Gauge Invariant Hamiltonian

Accordingly, the *manifest* gauge invariant, or gauge*free* Hamiltonian becomes

$$H^{gf} = \vec{p}^{(L)} \cdot \vec{v} + \iiint \vec{\Pi}_T \cdot \frac{\partial \vec{A}_T}{\partial t} dV - L^{gf} = \iiint \mathfrak{H}^{gf} dV \tag{117}$$

$$\mathfrak{H}^{gf} = \frac{1}{2\rho_m}\left(\vec{\pi}^{(L)} - \rho \vec{A}_T\right)^2 + \rho \phi_{\vec{E}} + \frac{1}{2\varepsilon_0}\vec{\Pi}_T^2 - \frac{\varepsilon_0}{2}\vec{E}_L^2 + \frac{1}{2\mu_0}\vec{B}^2 \tag{118}$$

H^{gf} equals numerically the total energy (84).

$$H^{gf} - E = \iiint \left(\rho \phi_{\vec{E}} - \varepsilon_0 \vec{E}_L^2\right) dV = \iiint \phi_{\vec{E}} \left(\rho - \varepsilon_0 \nabla \vec{E}_L\right) dV = 0 \tag{119}$$

Thus, on comes from the total energy to the Hamiltonian also through accounting for Gauss' law as a constraint.

$$\mathfrak{H}^{gf} = \mathfrak{E}^{gf} + \Lambda \left(\nabla \vec{E}_L - \frac{\rho}{\varepsilon_0}\right) \tag{120}$$

where

$$\mathfrak{E}^{gf} = \frac{\varepsilon_0}{2}\vec{E}_T^2 + \frac{\varepsilon_0}{2}\vec{E}_L^2 + \frac{1}{2\mu_0}\vec{B}^2 + \frac{1}{2}\rho_m \vec{v}^2 \tag{121}$$

is the Helmholtz-decomposed total energy density. The Lagrangian multiplier, Λ, follows from the Hamiltonian equations of motion. The advantages of this approach consists in that one needs not (more or less) guess the (not unique) Lagrangian, but, in this case, one can even derive one (together with Λ), for $\mathfrak{L}^{gf}$ and $\mathfrak{H}^{gf}$ are bilinear in the dynamical variables.

The Hamiltonian equations of motion for the field variables (*cf* Heisenberg & Pauli 1929, Fock & Podolski 1932, Dirac, Fock & Podolsky 1932, Goldstein, eq.(11-56)) reproduce the Helmholtz decomposed microscopic Maxwell equations.

(1)

$$\frac{\partial}{\partial t}\phi_{\vec{E}} = \frac{\partial \mathfrak{H}^{gf}}{\partial \Pi_{\phi_{\vec{E}}}} \equiv 0 \tag{122}$$

The absence of $\Pi_{\phi_{\vec{E}}}$ means, firstly, that the longitudinal electric field component, $\vec{E}_L$, does not have got a dynamics on its own, so that it depends on time only via the positions of the charges creating it. Consequently, not only Φ, but also $\vec{A}_L$ does not represent an independent dynamical variable.

(2)

$$\frac{\partial}{\partial t}\Pi_{\phi_{\vec{E}}} = -\frac{\partial \mathfrak{H}^{gf}}{\partial \phi_{\vec{E}}} + \nabla \cdot \frac{\partial \mathfrak{H}^{gf}}{\partial \vec{E}_L} \tag{123}$$

$$= -\rho - \Delta \phi_{\vec{E}} \equiv 0 \tag{124}$$

The absence of $\Pi_{\phi_{\vec{E}}}$ means, secondly and again, that Gauss' law for the electric field (here in Poisson's form) is a constraint rather than an equation of motion.

(3)

$$\frac{\partial}{\partial t}\vec{A}_T = \frac{\partial \mathfrak{H}^{gf}}{\partial \vec{\Pi}_T} = \frac{1}{\varepsilon_0}\vec{\Pi}_T = -\vec{E}_T \tag{125}$$

As above, this equation merely reproduces the definition of the canonical momentum density (and thus may be considered to be an identity rather than an equation of motion).

(4)

$$\frac{\partial}{\partial t}\Pi_T^j = -\frac{\partial \mathfrak{H}^{gf}}{\partial A_T^j} + \sum_{k=1}^{3}\frac{\partial}{\partial r_k}\frac{\partial \mathfrak{H}^{gf}}{\partial\left(\partial A_T^j/\partial r_k\right)}; \quad j = x, y, z \tag{126}$$

$$= -\frac{\partial \mathfrak{H}^{gf}}{\partial A_T^j} - \nabla \times \frac{\partial \mathfrak{H}^{gf}}{\partial \vec{B}} = \vec{j}_T - \frac{1}{\mu_0}\nabla \times \vec{B} \tag{127}$$

By virtue of $\vec{\Pi}_T = \varepsilon_0 \partial \vec{A}_T/\partial t$, this is equivalent to the wave equation for $\vec{A}_T$ (since $\nabla \vec{A}_T = 0$, we have $\nabla \times \nabla \times \vec{A}_T = -\Delta \vec{A}_T$).

$$\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A}_T = \vec{j}_T + \frac{1}{\mu_0}\Delta \vec{A}_T \tag{128}$$

And this is equivalent to the transverse, effective flux law.

8.5 Time-dependence of the Gaugefree Hamiltonian, H^{gf}: Conservation of Total Energy

The conservation law for the total energy has been shown above to separate into one for the transverse and one for the longitudinal field components. For this, it is sufficient here to consider the time-dependence of the gaugefree Hamiltonian, H^{gf}. Let me rewrite H^{gf} as

$$H^{gf}(t) = H_{body}(t) + H_{non-prop}(t) + H_{prop}(t)$$

Because a magnetic field does not change the kinetic energy of a charged body, the field-independent part, $H_{body}(t)$, is – as in $H(t)$ – effectively built by the (kinetic) energy of the free body.

$$H_{body}(t) = \frac{1}{2m}\left(\vec{p}_T(t) - q\vec{A}_T(\vec{r}(t), t)\right)^2 = \frac{m}{2}v^2(t) \tag{129}$$

By virtue of the Maxwell-Lorentz force, the rate of its change equals the Joule power, P_{Joule}.

$$\frac{d}{dt}H_{body} = m\vec{v} \cdot \frac{d\vec{v}}{dt} = q\vec{v} \cdot \vec{E} \equiv P_{Joule} \tag{130}$$

H_{prop} contains the propagating field energy, *ie*, that of the transverse field components.

$$H_{prop}(t) = \iiint \left(\frac{\varepsilon_0}{2}\vec{E}_T(\vec{r}, t)^2 + \frac{1}{2\mu_0}\vec{B}(\vec{r}, t)^2\right) dV \tag{131}$$

By virtue of the effective flux law (98) and the effective induction law (95), it is diminished by the transverse part of the Joule power (130) and by radiation out of the volume under consideration.

$$\frac{dH_{prop}}{dt} = \iiint \left(\varepsilon_0 \vec{E}_T \cdot \frac{\partial}{\partial t}\vec{E}_T + \frac{1}{\mu_0}\vec{B} \cdot \frac{\partial}{\partial t}\vec{B} \right) dV \tag{132}$$

$$= \iiint \left(\vec{E}_T \cdot \frac{1}{\mu_0}\nabla \times \vec{B} - \vec{E}_T \cdot \vec{j}_T - \frac{1}{\mu_0}\vec{B} \cdot \nabla \times \vec{E}_T \right) dV \tag{133}$$

$$= -q\vec{v} \cdot \vec{E}_T - \frac{1}{\mu_0} \oint_{\partial V} \vec{E}_T \times \vec{B} \cdot d\vec{\sigma} \tag{134}$$

As explained above, the radiation term contains $\vec{E}_T$ rather than $\vec{E}$.

$H_{non-prop}$ contains the field energy of the longitudinal electric field.

$$H_{non-prop}(t) = q\phi_{\vec{E}}(\vec{r}(t)) - \frac{\varepsilon_0}{2} \iiint [\nabla \phi_{\vec{E}}]^2 \, dV = \frac{\varepsilon_0}{2} \iiint \vec{E}_L^2 dV \tag{135}$$

By virtue of the longitudinal part of Ampère-Maxwell's flux law: $\vec{0} = \vec{j}_L + \varepsilon_0 \frac{\partial}{\partial t}\vec{E}_L$, it is diminished by the longitudinal part of the Joule power (130).

$$\frac{dH_{non-prop}}{dt} = \varepsilon_0 \iiint \vec{E}_L \cdot \frac{\partial \vec{E}_L}{\partial t} dV = -q\vec{v} \cdot \vec{E}_L \tag{136}$$

Altogether,

$$\frac{dH^{gf}}{dt} = -\frac{1}{\mu_0} \oint_{\partial V} \vec{E}_T \times \vec{B} \cdot d\vec{S} \tag{137}$$

In closed systems, $H = E = const$, the total energy (84) of the system body & field.

External fields acting on the charged body, are to be added to $\vec{A}_T$ in H_{body} and to $q\phi_{\vec{E}}$ in $H_{non-prop}$ in the usual manner.

Summary and Discussion

The theoretical classical electromagnetism rests essentially on the so-called 'rational(ized)' (1) and microscopic Maxwell equations (2), respectively. Both sets, however, hide the origins of gauge invariance and transversality of free electromagnetic waves. For this I returned to Maxwell's (1864) original set of equations (A)...(H). According to Boltzmann (2001), they express the essence of Maxwell's theory, while those subsequent modifications have led to rather misunderstand it.

As a matter of fact, when deriving the microscopic Maxwell equations (2) from a Helmholtzian analysis of the relationships between forces and energies, a factorization of the forces into geometrical and body-dependent quantities à la Newton and Hertz's interaction principle (Enders 2004, 2006, 2008, 2009), one arrives at the set

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\Phi \tag{138}$$

$$\vec{B} = \nabla \times \vec{A} \tag{139}$$

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \tag{140}$$

$$\nabla \times \vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \tag{141}$$

As it contains the potentials explicitly, it is close to Maxwell's (1864) original set (A)...(H).

In view of the Helmholtz (1858) decomposition of 3D vector fields, this set seems to be not well-defined. The representation (M-35) of the electric field strength in terms of the potentials contains *two* contributions to its longitudinal component, one from the vector potential, $-\partial\vec{A}_L/\partial t$, and one from the scalar potential, $-\nabla\Phi$. For these two, no other equation is established. Consequently, this set is actually *not* "20 equations for 20 variables" (Maxwell 1864, §70), but only 19 equations for 20 variables. It is not inconsistent, however, because the equations being related to charge conservation are redundant.

Both deficiencies can be removed in two different ways. The standard way results in the 'rational(ized)' set (1), where the potentials and the continuity equation have been eliminated. Historically, this has led even to a *principal underestimation* of the potentials (see, for instance, Drude 1906 referring to Heaviside, Hertz and Cohn).

An advantage of this concentration on the field equations is their Lorentz covariance (after re-introducing the potentials). A disadvantage consists in that the experimentally observed transversality of free electromagnetic waves does not naturally emerges out of the theory.

The alternative way proposed here considers the conservation of charge to be a fundamental property of given bodies and, consequently, primary w.r.t. the fields created by such bodies. In other words, the continuity equation represents a *primary, independent* equation against the field equations. At once, the transversality of freely propagating electromagnetic waves is natural consequence of this approach.

Accordingly, both the original and the rationalized sets of Maxwell equations *effectively* contain

- the fact that a charge density creates a longitudinal (di)electric field (Gauss' law);
- the continuity equation expressing the conservation of charge (Gauss' law together with the longitudinal part of Ampère-Maxwell's flux law);
- the propagation of *transverse* electromagnetic waves (Faraday's induction law together with the transverse part of Ampère-Maxwell's flux law).

Consequently, a complete set of independent dynamical field variables contains 4 field components. For instance, 2 dynamically independent components of $\vec{B}$ and of $\vec{E}_T$ each represent such a set; another example is given through 2 dynamically independent components of $\vec{A}_T$ and the corresponding 2 ones of $\vec{\Pi}_T$.

It is perhaps no accident that the history of the electromagnetic potentials is even more curvilineal than that of the field strengths. Maxwell (1861, 1862, 1864) saw the vector potential to represent Faraday's "electrotonic state" and the electromagnetic field momentum, respectively. Later, the potentials were considered to be superfluous or merely mathematical tools for solving the rationalized Maxwell's equations. This mistake lived for a surprisingly long time, in spite of their appearance in the principle of least action (Schwartzschild 1903), in the Hamiltonian (Pauli 1926, Fock 1929) and, last but not least, in the Aharonov-Bohm (1959) effect. The double role of the vector potential, $\vec{A}$, in the electric field strength, $\vec{E}$, where $\partial\vec{A}/\partial t$ contributes to both the transverse and the longitudinal components, has surely hindered the clarification.

The Helmholtz decomposition of the 'rationalized' Maxwell equations also facilitates to understand Poynting's (1884) theorem and the transversality of freely propagating electromagnetic waves. In the common treatments, the propagating field is connected with the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, which, however, contains *both*, transverse *and longitudinal* field components.

As a matter of fact, Poynting's (1884) theorem rests on Faraday's induction law (54) and Ampère-Maxwell's flux law (48). The first one contains solely transverse field components. The same holds true for Ampère-Maxwell's flux law after extraction of those parts which are related to charge conservation rather than to the interaction of electric and magnetic fields, see eq. (52). Consequently, *free propagating* electromagnetic fields contain *solely Helmholtz-transverse* field components. (Notice that the notation for waveguides is slightly different from that). The longitudinal electric field components ($\vec{E}_L$, $\vec{D}_L$) obey a seperate energy balance with the kinetic energy of the charged bodies ('Longitudinal Poynting's theorem').

In other words, the common derivation of Poynting's theorem contains the *additional assumption* that the longitudinal (di)electric field components enter the radiation field, too. The vector identity

$$\vec{E} \cdot \left(\nabla \times \vec{H}\right) - \vec{H} \cdot \left(\nabla \times \vec{E}\right) \equiv \nabla \left(\vec{E} \times \vec{H}\right) \tag{142}$$

serves to interpret the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, as propagating energy flux density, *ie*, as if *all* field components, *both* the longitudinal *and* the transverse ones, would propagate *in the same manner towards infinity.* Though even here, if surface contributions are absent, one has

$$\iiint \nabla \left(\vec{E} \times \vec{H}\right) dV = \iiint \left[\vec{E} \cdot \left(\nabla \times \vec{H}\right) - \vec{H} \cdot \left(\nabla \times \vec{E}\right)\right] dV \tag{143}$$

$$= \iiint \left[\vec{E}_T \cdot \left(\nabla \times \vec{H}_T\right) - \vec{H}_T \cdot \left(\nabla \times \vec{E}_T\right)\right] dV = \iiint \nabla \left(\vec{E}_T \times \vec{H}_T\right) dV \tag{144}$$

That means, again, that, in homogeneous isotropic media, the rationalized Maxwell equations actually contain the propagation of transverse fields only.

Within quantum electrodynamics, this additional hypothesis leads to the appearance of 4 equal photon states, where, actually only the 2 transverse ones are observable, while the longitudinal and the scalar (time-like) ones are not. This seems to speak against that hypothesis. Its only justification consists in that it is necessary for the manifest Lorentz covariant formulation in Minkowski space. However, compatibility with special relativity can also be reached without this formulation (Barut 1964), in particular, by means of Dirac's (1949) approach to relativistic canonical mechanics.

In order to avoid the difficulties just mentioned, I propose to treat the longitudinal and transverse field components from the very beginning as being *physically different*. Such an approach enables one to get *manifest gauge invariant* Lagrangians and Hamiltonians.

By virtue of Gauss' law, the time-dependence of the longitudinal component of the electric field strength, $\vec{E}_L(\vec{r},t)$, follows rigidly that of the charge density, $\rho(\vec{r},t)$; hence, $\vec{E}_L$ is not an independent dynamical variable. This fact is not changed by any gauge. Thus, if one introduces via gauge new dynamical variables, these are finally unphysical (*cf* Pauli 2000, p.72). For instance, the Lorenz gauge allows for a separate wave equation for Φ. This suggest both Φ and $\dot{\Phi}$ to be independent variables – however, $\dot{\Phi}$ is not, because $\dot{\Phi} = -\nabla\vec{A}$.

Littlejohn (2008) has stressed correctly, that the gauge transformation changes only the longitudinal component of the vector potential. His conclusion, however, that this is the "nonphysical" part, while the transverse component is the physical one (Sect. 34.8), overlooks its role in the Aharonov-Bohm effect. Such contradictions have been avoided in this paper through, (i), working with combinations of Φ and $\vec{A}$, in which those "nonphysical parts", if present, cancel each another and, (ii), treating this gauge invariant combination separately from the dynamics of the other field components.

This represents a consequent development of Messiah's (1999) treatment of the radiation field, where, however, the longitudinal field is "eliminated" (loc. cit., XXI.22). In this paper, the longitudinal field is treated on equal footing with, though partly separately from the transverse field. Due to this modification, the results presented here are not bound to the radiation gauge, $\nabla\vec{A} = 0$, used by Messiah, but hold true for *any* gauge.

The approach presented in thispaper benefits from the methodological advantages of the treatments by Newton, Euler and Helmholtz, where the subject under investigation (here, moving charged bodies and the electromagnetic fields created by them and acting back onto them) is defined *before* the mathematical formalism is developed (*cf* Suisky & Enders 2001; Enders & Suisky 2004, 2005; Enders 2006, 2008, 2009). This keeps the latter physically clear.

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