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Dear Colleagues and Friends of EJTP,

We made you wait, but here we are with an issue we think you will find interesting and useful!


Three papers (Bipin Singh Koranga; H. Kleinert and P. Kienle; N. Ivanov, P. Kienle, E. L. Kryshen, M. Pitschmann) deal with the ”hot” question of neutrino mixing; and five papers (Pratima Singh and Pawan Kumar Rai; Ravi Prakash Singh, Lallan Yadav; Hassan Amirhashchi, Hishamuddin Zaimuddin and Hamid Nil Saz Dezfooli; J. P. Singh and P. S. Baghely; M.R. Bordbar and N.Riazi) are centered on cosmological themes. That is the sign of a strong motion in this field promising ”radical” re-reading of the models into play. That is why we have chosen Spiritual Pilgrim, Woodcut, (anonymous German artist, around 1530) as cover.

N. Mebarki, F. Khelili and O. Benabbes & Walaa I. Eshrami and Nasser I. Farahat lead us to the hard and elegant lands of the mathematics of Quantum Field Theory and Standard Model; C. J. Papachristou, B. Kent Harrison delight us with the ”long-standing” problem of Mathematical Physics Constructing a Lax Pair for the Ernst Equation.

Passing through the ever fascinating non-linearity world (Kehui Sun and J. C. Sprott) quickly take us to the works of Robert L. Oldershaw, Mayer Humi and Peter Enders which recall us the never-ending richness and fecundity of classical themes.

In the spirit of Theoretical Physics problem-oriented, not all-the-rage-oriented, enjoy your reading!
The editorial board thanks all its referees for their valuable notes and comments in correcting author(s)’ mistakes. This time it has been my turn to write Foreword, but it goes without saying that the hard job involves the whole Editorial Board. I particularly thank Ammar Sakaji, José Luis Lopez-Bonilla and Leonardo Chiatti.

Ignazio Licata, EJTP Editor
Equivalence Principle and Field Quantization in Curved Spacetime

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Received 20 October 2009, Accepted 22 October 2009, Published 30 October 2009

Abstract: To comply with the equivalence principle, fields in curved spacetime can be quantized only in the neighborhood of each point, where one can construct a freely falling Minkowski frame with zero curvature. In each such frame, the geometric forces of gravity can be replaced by a self-interacting spin-2 field, as proposed by Feynman in 1962. At a fixed distance $R$ from a black hole, the vacuum in each freely falling volume element acts like a thermal bath of all particles with Unruh temperature $T_U = \frac{\hbar G M}{2 \pi c R^2}$. At the horizon $R = \frac{2GM}{c^2}$, the falling vacua show the Hawking temperature $T_H = \frac{\hbar c^3}{8\pi GMk_B}$.

Keywords: Quantum Gravity; Quantum Fields; Curved Spacetime

PACS (2008): 04.62.+v; 04.60.-m

1.) When including Dirac fermions into the theory of gravity, it is important to remember that the Dirac field is initially defined only by its transformation behavior under the Lorentz group. The invariance under general coordinate transformations can be incorporated only with the help of a vierbein field $e^\alpha_\mu(x)$ which couples Einstein indices with Lorentz indices $\alpha$. These serve to define anholonomic coordinate differentials $dx^\alpha$ in curved spacetime $x^\mu$:

$$dx^\alpha = e^\alpha_\mu(x)dx^\mu,$$

\[1\]

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which at any given point have a Minkowski metric:

\[ ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta, \quad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\alpha\beta}. \] (2)

With the help of the vierbein field one can write the action simply as [1]

\[ A = \int d^4x \sqrt{-g} \bar{\psi}(x) \left[ \gamma^a e_a^\mu(x) (i\partial_\mu - \Gamma^\alpha_{\mu\beta} \Sigma_{\alpha\beta}) - m \right] \psi(x), \quad g_{\mu\nu}(x) \equiv \eta_{\alpha\beta} e^a_\mu(x) e^\beta_\nu(x), \] (3)

where \( \Sigma_{\alpha\beta} \) is the spin matrix, which is formed from the commutator of two Dirac matrices as \( i[\gamma_\alpha, \gamma_\beta]/4 \), and \( \Gamma^\mu_{\alpha\beta} \equiv e^\alpha_\nu e^\beta_\lambda \Gamma^\mu_{\nu\lambda} \) is the spin connection, It is constructed from combination of the so-called objects of anholonomity \( \Omega_{\mu\nu\lambda} = \frac{1}{2} [e_\alpha^\lambda \partial_\mu e^\alpha_\nu - (\mu \leftrightarrow \nu)] \), by taking the sum \( \Omega^\mu_{\nu\lambda} - \Omega^\nu_{\mu\lambda} + \Omega^\lambda_{\mu\nu} \) and lowering two indices with the help of the metric \( g_{\mu\nu}(x) \).

The theory of quantum fields in curvilinear spacetime has been set up on the basis of this Lagrangian, or a simpler version for bosons which we do not write down. The classical field equation is solved on the background metric \( g_{\mu\nu}(x) \) in the entire spacetime. The field is expanded into the solutions, and the coefficients are quantized by canonical commutation rules, after which they serve as creation and annihilation operators on some global vacuum of the quantum system.

The purpose of this note is to make this procedure compatible with the equivalence principle.

2.) If one wants to quantize the theory in accordance with the equivalence principle, one must introduce creation and annihilation operators of proper elementary particles. These, however, are defined as irreducible representations of the Poincaré group with a given mass and spin. The symmetry transformations of the Poincaré group can be performed only in a Minkowski spacetime. According to Einstein’s theory, and confirmed by Satellite experiment, we can remove gravitational forces locally at one point. The neighborhood will still be subject to gravitational fields. For the definition of elementary particles we need only a small neighborhood. In it, the geometric forces can be replaced by the forces coming from the spin-2 gauge field theory of gravitation, which was developed by R. Feynman in his 1962 lectures at Caltech [2]. This can be rederived by expanding of the metric in powers its deviations from the flat Minkowski metric. We define a Minkowski frame \( x^a \) around the point of zero gravity, and extend it to an entire finite box without spacetime curvature. Inside this box, particle experiments can be performed and the transformation properties under the Poincaré group can be identified.

Inside the box, the fields are governed by the flat-spacetime action

\[ A = \int d^4x \sqrt{-g} \bar{\psi}(x) \left\{ \gamma^a e_a^b (i\partial_b - \Gamma^c_{ab} \Sigma_{bc}) - m \right\} \psi(x). \] (4)
In this expression, $e^{ab} = \delta^{ab} + \frac{1}{2} h_{ab}(x)$. The metric and the spin connection are defined as above, exchanging the indices $\alpha, \beta, \ldots$ by $a, b, \ldots$. All quantities must be expanded in powers of $h_{ab}$.

Thus we have arrive at a standard local field theory in the freely falling Minkowski laboratory around the point of zero gravity. This action is perfectly Lorentz invariant, and the Dirac field can now be quantized without problems, producing an irreducible representation of the Poincaré group with states of definite momenta and spin orientation $|p, s\rangle$.

The Lagrangian governing the dynamics of the field $h_{ab}(x)$ is well known from Feynman’s lecture [2]. If the laboratory is sufficiently small, we may work with the Newton approximation:

$$A^b = -\frac{1}{8\kappa} \int d^4x \, h_{abc} \epsilon^{cde} \epsilon^{bfg} \partial_d \partial_f h_{eg} + \ldots , \quad \kappa = 8\pi G/c^3, \quad G = \text{Newton constant}, \quad (5)$$

where $\epsilon^{cde}$ is the antisymmetric unit tensor. If the laboratory is larger, for instance, if it contains the orbit of the planet mercury, we must include also the first post-Newtonian corrections.

Thus, although the Feynman spin-2 theory is certainly not a valid replacement of general relativity, it is so in a neighborhood of any freely falling point.

The vacuum of the Dirac field is, of course, not universal. Each point $x^\mu$ has its own vacuum state restricted to the associated freely falling Minkowski frame.

3.) There is an immediate consequence of this quantum theory. If we consider a Dirac field in a black hole, and go to the neighborhood of any point, the quantization has to be performed in the freely falling Minkowski frame with smooth forces. These are incapable of creating pairs. An observer at a fixed distance $R$ from the center, however, sees the vacua of these Minkowski frames pass by with acceleration $a = GM/R^2$, where $G$ is Newton’s constant. At a given $R$, the frequency factor $e^{i\omega t}$ associated with the zero-point oscillations of each scalar particle wave of the world will be Doppler shifted to $e^{i\omega t\epsilon/c a}$, and this wave has frequencies distributed with a probability that behaves like $1/(e^{2\pi\Omega c/a} - 1)$. Indeed, if we Fourier analyze this wave [3]:

$$\int_0^\infty dt \, e^{it\Omega} e^{i\omega t\epsilon/c a} = e^{-\pi c/2a} \Gamma(i\Omega c/a) e^{-\pi \Omega c/a \log(\omega c/a)(c/a)}(c/a). \quad (6)$$

we see that the probability to find the frequency $\Omega$ is $|e^{-\pi c/2a} \Gamma(i\Omega c/a)c/a|^2$, which is equal to $2\pi c/(\Omega a)$ times $1/(e^{2\pi\Omega c/a} - 1)$. The latter is a thermal Bose-Einstein distribution with an Unruh temperature $T_U = \hbar a/2\pi c k_B$, where $k_B$ is the Boltzmann constant. The particles in this heat bath can be detected by suitable particle reactions as described in Ref. [5].

The Hawking temperature $T_H$ is equal to the Unruh temperature of the freely falling Minkowski vacua at the surface of the black hole, which lies at the horizon $R = r_S \equiv 2GM/c^2$. There the Unruh temperature is equal to $T_U|_{a=GM/R^2, R=2GM/c^2} = \hbar c^3/(8\pi GMk_B) = T_H$. 


Acknowledgment:

The author thanks V. Belinski pointing out the many papers on the semiclassical explanation of pair creation in a black hole.

References

[1] Our notation follows H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore 2009, pp. 1–497 (http://www.physik.fu-berlin.de/~kleinert/b11). The only exception is that the vierbein field is here called $e^\alpha_\mu$ rather than $h^\alpha_\mu$ to have the notation $h^a_b$ free for the small deviations of $e^a_b$ from the flat limit $\delta^a_b$.


New Seiberg-Witten Fields Maps Through Weyl Symmetrization and The Pure Geometric Extension of The Standard Model

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Abstract: A unified description of a symmetrized and anti-symmetrized Moyal star product of the noncommutative infinitesimal gauge transformations is presented and the corresponding Seiberg-Witten maps are derived. Moreover, the noncommutative covariant derivative, field strength tensor as well as gauge transformations are shown to be consistently constructed not on the enveloping but on the Lie and/or Poisson algebra. As an application, a pure geometric extension of the standard model is shown explicitly.

Keywords: Gauge Theories; Noncommutative Geometry; Moyal-Weyl Ordering; Seiberg-Witten Maps

PACS (2008): 11.15.-q; 11.10.Nx

1. Introduction

In the last few years, noncommutative geometry becomes the focus of interest in theoretical physics and for models building [1] – [9]. There are several motivations to speculate that the space-time becomes non commutative at very short distances when quantum gravity becomes relevant and even better, if we believe that the extra dimension approach can push the non commutativity scale lower. Moreover, in string theories, the noncommutative gauge theory appears as a certain limit in the presence of a background field[11].

One approach to the non commutative geometry, is the one based on the deformation of the space-time. If fields are assumed to be Lie algebra valued and allow for the closure

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of the Lie algebra valued noncommutative transformation gauge parameters, it turns out only $U(N)$ structure groups are conceivable as well as the corresponding gauge transformations must be in the fundamental representation of this group. This is unsuitable to build realistic models for the electroweak and strong interactions and even the $U(1)$ case, charges are quantized and it will be impossible to describe quarks (the choice of charges introduced in the theory is very restricted to $±1$ or $0$)[7]. The matching of the noncommutative action to the ordinary one, requires first to map the noncommutative space-time coordinates to the ordinary ones by introducing a star product[10], than the noncommutative fields are mapped to commutative ones by means of the Seiberg-Witten maps[1] – [6], [11]. The latter has the remarkable property that ordinary gauge transformations induce noncommutative ones. In this case, the low energy action is local in the sense that there is no UV/IR mixing. However, the basic assumption is that the noncommutative fields are not Lie algebra valued but are in the enveloping algebra and allows to consider $SU(N)$ groups.

In section 2, we define a new deformed Moyal-Weyl ordering product in the sense of a Weyl symmetrization using star product Lie or Poisson brackets with some golden rules to preserve the invariance of the action and the closure of the algebra and construct noncommutative gauge transformations, covariant derivative, field strength tensor and Seiberg-Witten maps fields without recourse to the enveloping algebra approach. As an application, we have constructed in section 3 a pure geometric extension of the standard model. Finally, in section 4, we draw our conclusions.

2. The Formalism

In ordinary quantum mechanics (Q.M), to find out the quantum equivalent of a classical observable $F(x, p)$ depending on the canonical variables $x$ and $p$, one has to go through the symmetrization procedure using the so called Weyl-ordering. In this scheme, and as an example, the Weyl ordering $(x^n p)_w$ of any monomial of the form $x^n p$ is given by:

$$
(x^n p)_w = \frac{1}{n+1} \sum_{l=0}^{n} x^{n-l} p x^l
$$

By analogy to Q.M, one can define a Moyal-Weyl ordering $(f \ast h)_w$ of any two functions $f$ and $h$ on a noncommutative space-time as follows:

$$
(f \ast h)_w \equiv \widehat{f \ast g}
$$

where

$$
\widehat{f \ast g} \equiv 2[f, g]_\eta^*
$$

with
\[ [f, g]_{\eta}^* \equiv \frac{1}{2}(f \ast g + \eta g \ast f) \]  
(4)

where \( \eta = \pm \). It is worth to mention that sometimes we use notations like in eq.(2) (with curly bracket on the top) like in the definition of the ”w” (Moyal-Weyl) ordering etc...But if we want to get more simplified forms (compact) like in eq.(12) etc...it is better to use notations of eq.(3)(with generalized \( \eta - \)commutators). Of course both notations are equivalent.

Now, let us consider the two matrices valued functions \( \Lambda = T_a \Lambda^a \) and \( V_\mu = T_a V^a_\mu \) as elements of a Lie algebra \( \mathcal{G} \) of a Lie group \( G \) (\( SU(N), U(N) \), etc..) and \( \psi \) a matter multiplet in a certain group representation such that it transforms as:

\[
\delta_\Lambda \psi = iT \Lambda^a \psi; \quad \delta_{\overline{\Lambda}} \overline{\psi} = -iT \overline{\Lambda^a \psi}
\]  
(5)

Here \( \Lambda^a \) is the group transformation parameter and \( T_a \) are the group generators.

In the noncommutative space-time, one can generalize the ordinary gauge transformations (eq.(5)) by respecting the Weyl ordering as:

\[
\delta_{\hat{\Lambda}} \hat{\psi} = \left( iT \hat{\Lambda}^a \hat{\psi} \right)_{\eta} w = \frac{1}{2} iT \hat{\Lambda}^a \ast \hat{\psi};
\]

\[
\delta_{\hat{\psi}} \hat{\Lambda} = \left( -iT \hat{\psi} \ast \hat{\Lambda}^a \right)_{\eta} w = -\frac{1}{2} iT \hat{\psi} \ast \hat{\Lambda}^a
\]  
(6)

The \( \hat{\Lambda}^a \) and \( \hat{\psi} \) are the gauge transformation parameter and matter field in the noncommutative space-time respectively. Now, we impose the following golden rules:

\[
(T_a T_b \hat{\Lambda}^a \ast \hat{\Sigma}^b)_{w} = T_a T_b \hat{\Lambda}^a \ast \hat{\Sigma}^b
\]

\[
(T_a T_b \hat{\Sigma}^b \ast \hat{\Lambda}^a)_{w} = T_a T_b \hat{\Sigma}^b \ast \hat{\Lambda}^a
\]  
(7)

and

\[ (T_a T_b \hat{\psi} \ast \hat{\Lambda}^a)_{w} = T_a T_b \hat{\psi} \ast \hat{\Lambda}^a \]

\[
\widehat{A \ast B \ast C} \equiv \left( A \ast B \ast C + \eta C \ast B \ast A \right)
\]  
(8)

Notice that, in eqs.(7) we have to respect the order of the indices a,b,...before the symmetrization procedure take place. Moreover, the generators indices must be in the same
order as those of $\hat{\Sigma}$ and $\hat{\Lambda}$. Now using eq.(6) and the golden rules of eqs.(7), one can show easily that (see Appendix A):

$$\delta_\Sigma (\delta_\Lambda \hat{\psi}) = \left( -T_a T_b \left[ \hat{\Lambda}^a, [\hat{\Sigma}^b, \hat{\psi}]^*_\eta \right]\right)_w$$

$$= \frac{1}{4} \left[ -T_a T_b \hat{\Sigma}^b \hat{\Lambda}^a \hat{\psi} - T_b T_a \eta \hat{\Sigma}^b \hat{\psi} \hat{\Lambda}^a - T_a T_b \eta \hat{\Lambda}^a \hat{\psi} \hat{\Sigma}^b - T_b T_a \eta \hat{\Sigma}^b \hat{\Lambda}^a \hat{\psi} \right]$$

$$- T_a T_b \eta^2 \hat{\psi} \hat{\Sigma}^b \hat{\Lambda}^a$$  \hspace{1cm} (9)

and

$$\delta_\Lambda (\delta_\Sigma \hat{\psi}) = \left( -T_b T_a \left[ \hat{\Sigma}^b, [\hat{\Lambda}^a, \hat{\psi}]^*_\eta \right]\right)_w$$

$$= \frac{1}{4} \left[ -T_b T_a \hat{\Sigma}^b \hat{\Lambda}^a \hat{\psi} - T_a T_b \eta \hat{\Sigma}^b \hat{\psi} \hat{\Lambda}^a - T_b T_a \eta \hat{\Sigma}^b \hat{\Lambda}^a \hat{\psi} \right]$$

$$- T_b T_a \eta^2 \hat{\psi} \hat{\Lambda}^a \hat{\Sigma}^b$$  \hspace{1cm} (10)

Using eqs. (3) and (4) together with the relations:

$$\hat{\Sigma}^b \hat{\Lambda}^a \hat{\psi} = \eta \hat{\Lambda}^a \hat{\Sigma}^b \hat{\psi}$$

$$\hat{\psi} \hat{\Lambda}^a \hat{\Sigma}^b = \eta \hat{\psi} \hat{\Sigma}^b \hat{\Lambda}^a$$

$$\hat{\Sigma}^b \hat{\psi} \hat{\Lambda}^a = \eta \hat{\Lambda}^a \hat{\psi} \hat{\Sigma}^b$$  \hspace{1cm} (11)

and after straightforward simplifications, we obtain:

$$[\delta_\Lambda, \delta_\Sigma] \hat{\psi} = [T_a, T_b] \eta \left[ [\hat{\Lambda}^a, \hat{\Sigma}^b]_*^\eta, \hat{\psi} \right]^*_\eta$$  \hspace{1cm} (12)

Thus, the algebra is closed and the gauge parameters $\hat{\Sigma}$ and $\hat{\Lambda}$ are not elements of the enveloping Lie algebra $G$.

Concerning the covariant derivative $\hat{D}_\mu \hat{\psi}$ of a matter field, its expression can be generalized easily in the noncommutative case as follows:

$$\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} - i T_a \left[ \hat{V}^a_\mu, \hat{\psi} \right]^*_\eta$$  \hspace{1cm} (13)

where, $\hat{V}^a_\mu$ denotes the gauge field in the noncommutative space-time. It is worth to mention that the covariant derivative of eq.(13) transforms as:
On the other hand, a tedious but direct calculation gives (see Appendix B):

\[
\delta_\Lambda (\hat{D}_\mu \hat{\psi}) = i T^a \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right]_\eta^* + \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \left[ \hat{V}^b_\mu, \hat{\psi} \right]_\eta \right] \right)_w \\
= iT^a \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right]_\eta^* + \frac{1}{4} \left( -iT^a iT^b \hat{V}^b_\mu * \hat{\psi} \right) - \eta iT^b iT^a \hat{V}^b_\mu * \hat{\psi} * \hat{\Lambda}^a \\
- \eta i T^a iT^b \hat{V}^b_\mu * \hat{\psi} * \hat{\Lambda}^a - \eta^2 iT^a iT^b \hat{\psi} * \hat{\Lambda}^a - \hat{\psi} * \hat{\Lambda}^a + \hat{\Lambda}^a \right) \\
\tag{14}
\]

Now, from eqs.(13) and (14), it follows the noncommutative gauge boson transformation law (see Appendix B):

\[
\delta_\Lambda \hat{V}_\mu = \partial_\mu \hat{\Lambda} + i [T_a, T_b]_{\eta} \left[ \hat{\Lambda}^a, \hat{V}^b_\mu \right]_\eta^* \\
\tag{16}
\]

(\(\hat{V}_\mu = T_a \hat{V}_\mu^a, \hat{\Lambda} = T_a \hat{\Lambda}^a\)). Now, within the infinitesimal noncommutative gauge transformations of eqs.(6) and (16), one can show the invariance of the action \(I_1\) representing the kinetic term of the noncommutative matter fields \(\hat{\psi}\) and their interaction with the noncommutative vector gauge boson \(\hat{V}^b_\mu\):

\[
I_1 = i \int d^4 x \ \hat{\psi} * \hat{\mathcal{D}} * \hat{\psi} \tag{17}
\]

where

\[
\hat{\mathcal{D}} = \gamma^\mu \hat{D}_\mu
\]

(\(\gamma^\mu\) stands for Dirac matrices). In fact:

\[
\delta_\Lambda I_1 = \int d^4 x \left\{ -iT^a \left[ \hat{\psi}, \hat{\Lambda}^a \right]_\eta^* + i \gamma^\mu (\partial_\mu \hat{\psi} - iT^b \left[ \hat{V}^b_\mu, \hat{\psi} \right]_\eta) \right\}_w \\
+ \int d^4 x \left\{ \hat{\psi} * \left( i \gamma^\mu iT^a \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right]_\eta + \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \left[ \hat{V}^b_\mu, \hat{\psi} \right]_\eta \right] \right) \right) \right\}_w \\
\tag{18}
\]

Direct simplifications give;
\[ \delta \hat{\Lambda} I_1 = -\frac{\eta}{2} \int d^4x (i T^a \hat{\Lambda}^a \ast \hat{\psi} \ast i \gamma^\mu \partial_\mu \hat{\psi} - \hat{\psi} \ast i \gamma^\mu i T^a \partial_\mu \hat{\psi} \ast \hat{\Lambda}^a) \]
\[ + \int d^4x (\frac{-i}{2} \eta T^a \hat{\Lambda}^a \ast \hat{\psi} \ast i \gamma^\mu (-\frac{i}{2} T^b V^b_\mu \ast \hat{\psi} - \eta \frac{i}{2} T^b \hat{\psi} \ast V^b_\mu) \]
\[ + \frac{1}{4} i \hat{\psi} \ast i \gamma^\mu (-i T^a i T^b (\eta \hat{\Lambda}^a \ast \hat{\psi} \ast \hat{\psi} + \eta^2 \hat{\psi} \ast \hat{V}^b_\mu \ast \hat{\Lambda}^a)) \]
\[ (19) \]

Using the following property of the star product:
\[ \int d^4x f \ast (g \ast h) = \int d^4x (f \ast g) \ast h = \int d^4x (g \ast h) \ast f \]
\[ (20) \]
one has:
\[ \int d^4x (i T^a \hat{\Lambda}^a \ast \hat{\psi} \ast i \gamma^\mu \partial_\mu \hat{\psi}) = \int d^4x (\hat{\psi} \ast i \gamma^\mu i T^a \partial_\mu \hat{\psi} \ast \hat{\Lambda}^a) \]
\[ (21) \]
\[ \int d^4x \frac{i}{2} T^a \hat{\Lambda}^a \ast \hat{\psi} \ast i \gamma^\mu (-\frac{i}{2} T^b \hat{\psi} \ast \hat{\Lambda}^a) = \frac{1}{4} \int d^4x (\hat{\psi} \ast i \gamma^\mu (-i T^a i T^b \hat{\psi} \ast \hat{V}^b_\mu) \ast \hat{\Lambda}^a) \]
\[ (22) \]
and
\[ \int d^4x ((\frac{i}{2} T^a \hat{\Lambda}^a \ast \hat{\psi} \ast i \gamma^\mu (\frac{i}{2} T^b \hat{\psi} \ast \hat{V}^b_\mu) = -\frac{1}{4} \int d^4x (\hat{\psi} \ast i \gamma^\mu (-i T^a i T^b \hat{\psi} \ast \hat{V}^b_\mu) \ast \hat{\Lambda}^a) \]
\[ (23) \]
Therefore, we deduce that:
\[ \delta \hat{\Lambda} I_1 = 0 \]
\[ (24) \]
Similarly, one can show that the mass term \( I_2 = \int d^4x \ \hat{\psi} \ast \hat{\psi} \) is gauge invariant. In fact, taking into account the noncommutative gauge transformation laws of eq.( 6), one can write
\[ \delta \hat{\Lambda} I_2 = \int d^4x \left( -i T^a \left[ \hat{\psi}, \hat{\Lambda}^a \right]^{\eta}_\eta \ast \hat{\psi} + \hat{\psi} \ast i T^a \left[ \hat{\Lambda}^a, \hat{\psi} \right]^{\eta}_\eta \right) \]
\[ = \frac{-i \eta}{2} T^a \int d^4x (\hat{\Lambda}^a \ast \hat{\psi} \ast \hat{\psi} - \hat{\psi} \ast \hat{\psi} \ast \hat{\Lambda}^a) \]
\[ (25) \]
Again, using the associativity of the star product and the relation in eq.(20 ) we obtain:
\[ \delta \hat{\Lambda} I_2 = 0 \]
\[ (26) \]
Regarding the noncommutative field strength \( \hat{F}^a_{\mu \nu} \), one can generalize the definition of the curvature tensor such that:
\[
\left[ \hat{D}_\mu, \hat{D}_\nu \right] \hat{\psi} = -iT_a \left[ \hat{F}_{\mu\nu}^a, \hat{\psi} \right]_\eta^* 
\] (27)

Using the expression of the noncommutative covariant derivative given by eq.(13), one can show that:

\[
\hat{D}_\mu \hat{D}_\nu \hat{\psi} = \left( \partial_\mu \hat{D}_\nu \hat{\psi} - iT_a \left[ \hat{V}_\mu^a, \hat{D}_\nu \hat{\psi} \right]_\eta^* \right)_w 
\] (28)

With the help of eqs.(7) as well as eq.(13), eq.(28) can be rewritten as:

\[
\begin{align*}
&\hat{D}_\mu \hat{D}_\nu \hat{\psi} = \partial_\mu \partial_\nu \hat{\psi} - iT_a \left[ \partial_\mu \hat{V}_\nu^a, \hat{\psi} \right]_\eta^* - i T_a \left[ \hat{V}_\nu^a, \partial_\mu \hat{\psi} \right]_\eta^* - iT_a \left[ \hat{V}_\nu^a, \partial_\nu \hat{\psi} \right]_\eta^* \\
&\quad+ \frac{1}{4} \left( -T_a T_b \hat{V}_\nu^a * \hat{V}_\mu^b * \hat{\psi} - \eta T_a T_b \hat{V}_\nu^a * \hat{\psi} * \hat{V}_\mu^b \\
&\quad- \eta T_a T_b \hat{V}_\mu^a * \hat{\psi} * \hat{V}_\nu^b - \eta^2 T_a T_b \hat{\psi} * \hat{V}_\nu^a * \hat{V}_\mu^b \right) 
\end{align*}
\] (29)

Using the relations of eqs.(11), we deduce that:

\[
\left[ \hat{D}_\mu, \hat{D}_\nu \right] \hat{\psi} \equiv -iT_a \left[ \hat{F}_{\mu\nu}^a, \hat{\psi} \right]_\eta^* 
\] (30)

\[
\begin{align*}
&\quad= -iT_a \left[ \partial_\mu \hat{V}_\nu^a - \partial_\nu \hat{V}_\mu^a, \hat{\psi} \right]_\eta^* - \frac{1}{4} \left[ T_a, T_b \right]_{-\eta} \hat{V}_\mu^a * \hat{V}_\nu^b * \hat{\psi} \\
&\quad+ \frac{1}{4} \left[ T_a, T_b \right]_{-\eta} \hat{\psi} * \hat{V}_\mu^a * \hat{V}_\nu^b 
\end{align*}
\]

Finally,

\[
\left[ \hat{F}_{\mu\nu}^a, \hat{\psi} \right]_\eta^* = \left[ \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu, \hat{\psi} \right]_{-\eta}^* - i \left[ T^a, T^b \right]_{-\eta} \left[ \hat{V}_\mu^a, \hat{V}_\nu^b \right]_{-\eta}^* \hat{\psi} 
\] (31)

or

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i \left[ T^a, T^b \right]_{-\eta} \left[ \hat{V}_\mu^a, \hat{V}_\nu^b \right]_{-\eta}^* 
\] (32)

Notice that one can rewrite eq.(32) as:

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - \frac{i}{2} \left( T^a T^b \hat{V}_\mu^a * \hat{V}_\nu^b + \eta T^a T^b \hat{V}_\mu^a * \hat{V}_\nu^b \right) 
\] (33)

or in a more compact form as:
\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu - i \left[ \hat{V}_\mu, \hat{V}_\nu \right]^* - i\eta \left[ \hat{V}_\mu, \hat{V}_\nu \right] \]  

(34)

where

\[ f * g = g * f \]  

(35)

with

\[ f * g = G(f, g, \theta^{\mu\nu}) \]  

(36)

and

\[ f \tilde{*} g = G(f, g, -\theta^{\mu\nu}) \]  

(37)

Here \( G(f, g, \theta^{\mu\nu}) \) is a function of \( f, g \) and \( \theta^{\mu\nu} \). Similarly one can show that:

\[ \delta_\Lambda \hat{V}_\mu = \partial_\mu \hat{\Lambda} + \frac{i}{2} \left( \hat{\Lambda} \ast \hat{V}_\mu + \eta \hat{\Lambda} \ast \hat{V}_\mu - \eta \hat{V}_\mu \ast \hat{\Lambda} - \eta^2 \hat{V}_\mu \ast \hat{\Lambda} \right) \]  

(38)

Now, from eqs. (34) and (38), one has:

\[ \delta_\Lambda \hat{F}_{\mu\nu} = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 \]  

(39)

where

\[ \Omega_1 = i \left[ \hat{\Lambda}, \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu \right]^* + i\eta \left[ \hat{\Lambda}, \partial_\mu \hat{V}_\nu - \partial_\nu \hat{V}_\mu \right]^* \]  

(40)

\[ \Omega_2 = \left[ \hat{\Lambda}, \left[ \hat{V}_\mu, \hat{V}_\nu \right]^* \right]^* \]  

(41)

\[ \Omega_3 = \eta \left[ \hat{\Lambda}, \left[ \hat{V}_\mu, \hat{V}_\nu \right]^* \right]^* \]  

(42)

\[ \Omega_4 = \eta \left[ \hat{\Lambda}, \left[ \hat{V}_\mu, \hat{V}_\nu \right]^* \right]^* \]  

(43)

and

\[ \Omega_5 = \left[ \hat{\Lambda}, \left[ \hat{V}_\mu, \hat{V}_\nu \right]^* \right]^* \]  

(44)

straightforward simplifications give:

\[ \delta_\Lambda \hat{F}_{\mu\nu} = i \left[ \hat{\Lambda}, \hat{F}_{\mu\nu} \right]^* + i\eta \left[ \hat{\Lambda}, \hat{F}_{\mu\nu} \right]^* \]  

(45)

since
\[
\left[ \hat{\Lambda}, \hat{F}_{\mu\nu} \right]^* + \eta \left[ \hat{\Lambda}, \hat{F}_{\mu\nu} \right] = T^a T^b \hat{\Lambda}^a \hat{F}_{\mu\nu}^b - T^b T^a \hat{F}_{\mu\nu}^b \hat{\Lambda}^a + \eta T^a T^b \hat{\Lambda}^a \hat{F}_{\mu\nu}^b - \eta T^b T^a \hat{F}_{\mu\nu}^b \hat{\Lambda}^a
\]
\[
= T^a T^b \hat{\Lambda}^a \hat{F}_{\mu\nu}^b - T^b T^a \hat{F}_{\mu\nu}^b \hat{\Lambda}^a + \eta T^a T^b \hat{\Lambda}^a \hat{F}_{\mu\nu}^b - \eta T^b T^a \hat{\Lambda}^a \hat{F}_{\mu\nu}^b
\]
\[
= [T^a, T^b] - \eta \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \eta (46)
\]

therefore
\[
\delta \hat{\Lambda} \hat{F}_{\mu\nu} = i [T^a, T^b] - \eta \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \eta (47)
\]

Regarding the gauge invariance of the noncommutative Yang-Mills action \( I_{YM} \) defined as:
\[
I_{YM} = \int d^4x \text{Tr} \left( \hat{F}_{\mu\nu} \hat{F}_{\mu\nu}^* \right) (48)
\]
and with the help of the transformation law of eq. (47), one has:
\[
\delta \hat{\Lambda} I_{YM} = i \text{Tr} \left( [T^a, T^b] - \eta \right) \int d^4x \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \hat{F}_{\mu\nu}^c
\]
\[
- i \text{Tr} \left( T^c \left[ T^a, T^b \right] - \eta \right) \int d^4x \hat{F}_{\mu\nu}^c \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \eta (49)
\]

Using the fact that:
\[
\text{Tr}(ABC) = \text{Tr}(CBA) \quad (50)
\]
(A, B and C are matrices) and the star product property in eq.(20), then
\[
\text{Tr} \left( [T^a, T^b] - \eta \right) = \text{Tr} \left( T^c \left[ T^a, T^b \right] - \eta \right) (51)
\]
and
\[
\int d^4x \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \hat{F}_{\mu\nu}^c = \int d^4x \hat{F}_{\mu\nu}^c \left[ \hat{\Lambda}^a, \hat{F}_{\mu\nu}^b \right]^* \eta (52)
\]

Consequently
\[
\delta \hat{\Lambda} I_{YM} = 0 (53)
\]

Regarding the Seiberg-Witten maps, if one sets:
\[
\hat{\psi} = \hat{\psi} \left[ V, \psi \right] = V + \tilde{\psi} \quad (54)
\]
\[
\hat{\psi} = \hat{\psi} \left[ V, \psi \right] = \psi + \tilde{\psi} \quad (55)
\]
and
\[ \hat{\Lambda} = \hat{\Lambda} [V, \Lambda] = \Lambda + \tilde{\Lambda} \] (56)

where

\[ \delta V_{\mu} = \partial_{\mu} \Lambda + i [\Lambda, V_{\mu}] \] (57)

and uses the transformation laws in eqs.(6) and (16), we obtain:

\[ \tilde{V}_{\mu} = \frac{1}{4} \theta^{\alpha\beta} \left\{ \delta_{\eta,-} \left( [F_{\alpha\mu}, V_{\beta}]_{\eta} + [V_{\beta}, \partial_{\alpha} V_{\mu}]_{\eta} \right) + 4 \delta_{\eta,+} \left( [V_{\beta}, \partial_{\alpha} V_{\mu}]_{\eta} - \frac{1}{2} [V_{\beta}, \partial_{\mu} V_{\alpha}]_{\eta} \right) \right\} + O(\theta^2) \] (59)

\[ \tilde{\psi} = -\frac{i}{8} \theta^{\alpha\beta} \left\{ \delta_{\eta,-} \left[ V_{\alpha}, V_{\beta} \right]_{\eta} \psi + 8i \delta_{\eta,+} \left[ V_{\beta} \partial_{\alpha} \psi + F_{\alpha\beta} \psi \right] \right\} + O(\theta^2) \] (60)

and

\[ \tilde{\Lambda} = \frac{1}{4} \theta^{\alpha\beta} \left\{ \delta_{\eta,-} \left[ V_{\beta}, \partial_{\alpha} \Lambda \right]_{\eta} + 2 \delta_{\eta,+} \left[ V_{\beta}, \partial_{\alpha} \Lambda \right]_{\eta} \right\} + O(\theta^2) \] (61)

To give further clarifications, it is worth to mention that the paper may give the impression that we have defined in our approach a new ”Moyal-Weyl ordering” related to the noncommutative space-time which makes a confusion. In reality, we have just defined new gauge transformations , covariant derivatives, Seiberg-Witten maps through what we have called ”Moyal-Weyl ordering”. It is just a breach of trust. We mean Weyl symmetrization through Poisson or Lie brackets in the expressions of gauge transformations, definitions of covariant derivative etc., by respecting certain golden rules which are necessary for the invariance of the action and the closure of the algebra and which involve Moyal star product. So, in this paper we did not change the Moyal-Weyl product initially introduced in the noncommutative space-time mathematical formalism and the isomorphism between the classical functions and the corresponding operators still holds. In fact, we can always associate to any function \( f(x) \) of the classical commutative space-time an operator denoted by a \( W(f) \) and defined by:

\[ W(f) = (2\pi)^{-\frac{3}{2}} \int d^4 k \ e^{-ik\hat{x}} \tilde{f}(k) \] (62)

where \( \tilde{f}(k) \) is its Fourier transform and \( \hat{x}_\mu \) the noncommutative variable.

\[ \tilde{f}(k) = (2\pi)^{-\frac{3}{2}} \int d^4 x \ e^{ikx} f(x) \] (63)

These operators \( W(f), W(g) \), etc., can be multiplied to give other operators. The product operator \( W(f)W(g) \) is itself associated to a classical function \( h(x) = (f*g)(x) \) such that:
\[ W(f)W(g) = W(f * g) \]  
(64)

where \((f * g)(x)\) is a function of the classical variable \(x_\mu\)

\[(f * g)(x) = \left[ e^{\frac{i}{2} \theta^{\mu \nu} \partial_\mu \partial_\nu (f(x)g(y))} \right]_{x=y} \]  
(65)

Notice that at the level of the infinitesimal gauge transformations introducing what we have called Moyal-Weyl ordering and meaning the symmetrization by using Poisson or Lie brackets, the operator counterpart in the lagrangian does not change. However for the definition of the covariant derivative it is exactly equivalent to have this symmetrization procedure at the level of the operators counterparts. for example

\[ \hat{\Psi}(x) \gamma^\mu \hat{D}_\mu \hat{\Psi}(x) = \hat{\Psi}(x) \gamma^\mu \partial_\mu \hat{\Psi}(x) - iT_a \hat{\Psi}(x) \gamma^\mu \left[ \hat{A}_a(x), \hat{\Psi}(x) \right]_\eta \]  
(66)

This is always possible because we are dealing with operators and thus, always the ordering ambiguities arise. Now, the choice of a one symmetrization from another depends on what we want to achieve (of course we have to respect certain golden rules if we are dealing also with the Lie generators algebra \(T^a\)). Essentially, which is important is the closure of the algebra. where the noncommutative gauge transformations parameters \(\hat{\Lambda}\) are elements of the Lie or Poisson (not the enveloping) algebra of the local gauge group \(\mathcal{G}\).

Moreover, one may ask about the necessity and a possible use of the antisymmetrization through star product commutators. If we take for example a \(U(1)\) gauge theory, where a singlet matter field \(\Phi\) transforms trivially under the symmetry group as:

\[ \delta_\Lambda \Phi(x) = 0 \]  
(68)

The noncommutative equivalent of this transformation could be generalized to:

\[ \hat{\delta}_\Lambda \hat{\Phi}(x) = i \left[ \hat{\Lambda}, \hat{\Phi} \right]_\star \equiv i \left[ \hat{\Lambda} \star \hat{\Phi} \right] \]  
(69)

where of course in the limit \(\theta \to 0\) we find the classical commutative case. Another important possible application of the antisymmetrisation in defining Moyal-Weyl ordering is the creation of new interactions invariant under new symmetries of a noncommutative gauge group of the \(O(\theta)\) or higher and such that at the limit \(\theta \to 0\), these interactions will
be switched off and disappear. For example, if we take in the commutative space-time the kinetic term of the action $I$ of a Dirac spinor field $\psi$ such that:

$$I = i \int d^4x \bar{\psi} \gamma^\mu \partial_\mu \psi$$

(70)

This action is invariant under infinitesimal local gauge transformations of a certain Lie group $G$ with generators $T^a$ if and only if $\psi$ transforms as a singlet. That is:

$$\delta_A \psi = \delta_A \bar{\psi} = 0$$

(71)

Now, in the noncommutative space-time, the action $I$ becomes $I^*$ such that:

$$I^* = i \int d^4x \bar{\psi} \gamma^\mu \partial_\mu \psi$$

(72)

and the transformation laws of eq.(6) (within the antisymmetrization idea and generalization of a singlet state) becomes:

$$\delta_A \bar{\psi} = \frac{i}{2} \left( \Lambda \bar{\psi} - \psi \Lambda \right)$$

$$= -\frac{i}{2} \theta^{\mu\nu} \partial_\mu \Lambda \partial_\nu \bar{\psi} + O(\theta^2)$$

(73)

Now, the action $I^*$ is no more invariant under these noncommutative transformations of the matter field. To do so, we transform the ordinary derivative $\partial_\mu$ into a covariant one $\hat{D}_\mu$ such that:

$$\hat{D}_\mu \psi \equiv (\partial_\mu \psi - iT^a \left[ \hat{V}_\mu^a, \psi \right])$$

(74)

and

$$\delta_A \hat{V}_\mu = \partial_\mu \Lambda + i \left\{ T^a, \hat{V}^b \right\}$$

(75)

where $\hat{V}_\mu$ is the noncommutative gauge boson. Then, the action $I^*$ becomes $I^{*'}$ such that:

$$I^{*'} = i \int d^4x \bar{\psi} \gamma^\mu \hat{D}_\mu \psi$$

(76)

Notice that at the limit $\theta \to 0$, and since $\left[ \hat{V}_\mu^a, \psi \right] \to 0$, one has $\hat{D}_\mu \psi \to \partial_\mu \psi$ and we get back the action $I$. This means that although the interaction (force) between the matter and gauge field is absent in the commutative space-time, it is not (thanks to the antisymmetric transformations and the noncommutative generalization of the singlet state) in the noncommutative space-time. This is the way to generate new interactions within this
approach. The most important point is that the order noncommutative parameter $\theta$ becomes like a scale for which new physics (interactions) becomes relevant. As a conclusion, if we want to extend any gauge theory and generate models beyond with a pure geometric noncommutative scale $\theta$, we need to consider the antisymmetric noncommutative gauge transformations with the corresponding Seiberg-Witten maps.

3. Applications

Using the previous formalism, we construct a non abelian non commutative gauge theory invariant under the infinitesimal transformations of the gauge Lie group $SU(2)_L \times SU(2)_R \times U(1)_Y$ (left-right model). To keep our idea transparent, we will not consider the Higgs and Yukawa sectors. The matter physical states are the doublets:

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$$

(77)

The isospin generators of the tensor group $SU(2)_L \times SU(2)_R$, are given by $T_a^{(L)} = T_a^{(R)} = \frac{\tau_a}{2}$ ($\tau_a$ are the Pauli matrices) and $Y$ is the $U(1)$ hypercharge generator such that:

$$Y_L = -\frac{1}{2} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

$$Y_R = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$$

(78)

and

$$Y^{(L)} - Y^{(R)} = \frac{1}{2} \tau_3$$

If we denote by $W_a^\mu, B^a_\mu$ and $A_\mu$ the gauge potentials of the groups $SU(2)_L, SU(2)_R$, and $U(1)_Y$, respectively, and using the previous formalism of section 2 the noncommutative matter fields covariant derivatives takes the following forms:

$$\hat{D}_\mu L = \partial_\mu L - ig_L T_a^{(L)} \left\{ \tilde{W}_{\mu a}^\nu \hat{L}^\nu \right\} - ig' Y^{(L)} \left\{ \hat{A}_{\mu a}^\nu \hat{L}^\nu \right\}$$

(79)

and

$$\hat{D}_\mu R = \partial_\mu R - ig_R T_a^{(R)} \left[ \tilde{B}_{\mu a} \hat{R} \right] - ig' Y^{(R)} \left\{ \hat{A}_{\mu a} \hat{R} \right\}$$

(80)
(here \( g', g_L \) and \( g_R \) denote the \( U(1)_Y, SU(2)_L \) and \( SU(2)_R \) coupling constants respectively). The left and right symmetrized noncommutative states \( \hat{L} \) and \( \hat{R} \) transform as follows:

\[
\delta_{\hat{\lambda}_L} \hat{L} = ig_L T_a^{(L)} \left\{ \hat{\lambda}_L \hat{L} \right\} + ig' Y^{(L)} \left\{ \hat{\alpha} \hat{L} \right\} \quad (81)
\]

and

\[
\delta_{\hat{\lambda}_R} \hat{R} = ig_R T_a^{(R)} \left[ \hat{\lambda}_R \hat{R} \right] + ig' Y^{(R)} \left\{ \hat{\alpha} \hat{R} \right\} \quad (82)
\]

(\( \hat{\lambda}_L, \hat{\lambda}_R \) and \( \hat{\alpha} \) are infinitesimal noncommutative gauge parameters). Thus, the noncommutative Lagrangian density \( \mathcal{L}_{NC} \) is given by:

\[
\mathcal{L}_{NC} = \bar{\hat{L}} \hat{L} \delta^{\alpha}_{\hat{\lambda}_L} - \bar{\hat{R}} \hat{R} \delta^{\alpha}_{\hat{\lambda}_R} - \frac{1}{2g_L^2} T_1 \hat{F}^{\mu\nu} \hat{F}_{\mu\nu} - \frac{1}{2g_R^2} T_2 \hat{G}^{\mu\nu} \hat{G}_{\mu\nu} - \frac{1}{4g^2} \hat{f}^{\mu\nu} \hat{f}_{\mu\nu} \quad (83)
\]

(\( T_1 \) and \( T_2 \) represent the trace over the vector space of the fields) where

\[
\hat{F}^{\mu\nu} = \partial_\mu \hat{W}_\nu - \partial_\nu \hat{W}_\mu - ig_L \left[ T_a^{(L)}, T_b^{(L)} \right] \left\{ \hat{W}_a^{\mu}, \hat{W}_b^{\nu} \right\}
\]

\[
\hat{G}^{\mu\nu} = \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu - ig_R \left[ T_a^{(R)}, T_b^{(R)} \right] \left\{ \hat{B}_a^{\mu}, \hat{B}_b^{\nu} \right\}
\]

and

\[
\hat{f}^{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu \quad (85)
\]

The noncommutative fields \( \hat{L}(x), \hat{R}(x), \hat{W}_\mu^a, \hat{B}_\mu^a, \hat{A}_\mu \) are the Seiberg-Witten maps of the classical fields \( L(x), R(x), W_\mu^a, B_\mu^a, A_\mu \) such that:

\[
\hat{L} = L - i \theta^{\mu\nu} [W_\mu, W_\nu] L + O(\theta^2)
\]

\[
\hat{R} = R - \theta^{\mu\nu} B_\mu \partial_\nu R + \theta^{\mu\nu} G_{\mu\nu} R + O(\theta^2)
\]

\[
\hat{W}_\mu = W_\mu + \frac{1}{4} \theta^{\mu\lambda} [W_\lambda, \partial_\nu W_\mu - F_{\nu\mu}] + O(\theta^2)
\]

\[
\hat{B}_\mu = B_\mu - \theta^{\lambda\nu} [B_\lambda, \partial_\nu B_\mu] + \frac{1}{2} \theta^{\lambda\nu} [B_\lambda, \partial_\mu B_\nu] + O(\theta^2)
\]

\[
\hat{A}_\mu = A_\mu + O(\theta^2)
\]

Straightforward simplifications give:

\[
\mathcal{L}_{NC} = \mathcal{L} + \mathcal{L}_{L}^{(1)} + \mathcal{L}_{R}^{(1)} + \mathcal{L}_{g}^{(1)} \quad (87)
\]

where \( \mathcal{L} \) is the classical Lagrangian given by:

\[
\mathcal{L} = \bar{\hat{L}} \hat{L} + \bar{\hat{R}} \hat{R} - \frac{1}{2g_L^2} T_1 \hat{F}^{\mu\nu} \hat{F}_{\mu\nu} - \frac{1}{2g_R^2} T_2 \hat{G}^{\mu\nu} \hat{G}_{\mu\nu} - \frac{1}{4g^2} \hat{f}^{\mu\nu} \hat{f}_{\mu\nu} \quad (88)
\]
and $\mathcal{L}_L^{(1)}$, $\mathcal{L}_R^{(1)}$, $\mathcal{L}_g^{(1)}$, represent the contribution of the space-time noncommutativity and have the following expressions:

$$\mathcal{L}_L^{(1)} = \frac{1}{4} \theta^{\mu\nu} \mathcal{L} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{P} L + \frac{1}{4} \theta^{\mu\nu} \mathcal{L} \mathcal{W}_\mu \mathcal{W}_\nu L + \mathcal{L}_2^{(1)}\mathcal{W}_\mu (1)$$

(89)

and

$$\mathcal{L}_R^{(1)} = -i \theta^{\mu\nu} \partial_\nu (\mathcal{R}) \mathcal{B}_\mu \mathcal{P} R - i \theta^{\mu\nu} \mathcal{L} \mathcal{B}_\mu \partial_\nu R + g_R \theta^{\mu\nu} \mathcal{L} \mathcal{B}_\mu \partial_\nu R + \theta^{\mu\nu} \mathcal{G}_{\mu\nu} \mathcal{P} R + \theta^{\mu\nu} \mathcal{G}_{\mu\nu} R$$

(90)

where

$$D_\mu L = \left( \partial_\mu - i g_L \mathcal{W}_\mu \frac{\tau_a}{2} + \frac{i}{2} g' A_\mu \right) L$$

(91)

$$D_\mu R = \left( \partial_\mu - i g' A_\mu (R) \right) R$$

(92)

and

$$\mathcal{W}_\mu^{(1)} = \frac{1}{4} \theta^{\alpha\beta} \left[ F_{\alpha\beta}, \mathcal{W}_\nu \right] + \frac{1}{4} \theta^{\alpha\beta} \left[ \mathcal{W}_\beta, \partial_\mu \mathcal{W}_\mu \right]$$

(93)

Notice here that the right currents are of the order $\theta$, thus, they vanish in the classical limits when $\theta \to 0$. Finally, $\mathcal{L}_g^{(1)}$ has the form:

$$\mathcal{L}_g^{(1)} = -\frac{1}{g_L^2} T_{r_1} \mathcal{F}_{\mu
u}^{(1)} \mathcal{F}_{\mu
u}^{(1)} - \frac{1}{g_R^2} T_{r_2} \mathcal{G}_{\mu\nu}^{(1)} \mathcal{G}_{\mu\nu}^{(1)} - \frac{1}{2g'^2} \mathcal{F}_{\mu\nu}^{(1)} \mathcal{F}_{\mu\nu}^{(1)}$$

(94)

with

$$\mathcal{F}_{\mu\nu}^{(1)} = -\frac{i}{8} \theta^{\alpha\beta} \left[ \mathcal{F}_{\mu\nu}, [\mathcal{W}_\beta, \mathcal{W}_\alpha] \right] + \frac{i}{8} \theta^{\alpha\beta} \left( \partial_{\mu} [\mathcal{W}_\nu, [\mathcal{W}_\beta, \mathcal{W}_\alpha]] - \partial_{\nu} [\mathcal{W}_\mu, [\mathcal{W}_\beta, \mathcal{W}_\alpha]] \right)$$

(95)

and

$$\mathcal{G}_{\mu\nu}^{(1)} = \partial_\mu B_\nu^{(1)} - \partial_\nu B_\mu^{(1)} - i \left[ B_\mu^{(1)}, B_\nu \right] - i \left[ B_\mu, B_\nu^{(1)} \right]$$

(96)

and

$$\mathcal{F}_{\mu\nu}^{(1)} = 0$$

(97)

with

$$B_\mu^{(1)} = -\theta^{\lambda\nu} \{ B_\lambda, \partial_\nu B_\mu \} + \frac{1}{2} \theta^{\lambda\nu} \{ B_\lambda, \partial_\mu B_\nu \}$$

(98)

The electroweak currents $\mathcal{L}_{(currents)}^{NC}$ can be deduced directly from the previous Lagrangian to get:

$$\mathcal{L}_{(currents)}^{NC} = \mathcal{L}_{(currents)}^{NC(L)} + \mathcal{L}_{(currents)}^{NC(R)}$$

(99)
where

$$\mathcal{L}^{NC(L)}_{\text{(currents)}} = \overline{L}i\gamma^\mu \left( -ig_L W_\mu + \frac{i}{2} g' A_\mu \right) L + \overline{L} \gamma^\mu W^{(1)}_\mu L +$$

$$\frac{1}{4} g^{\alpha\beta} \overline{W}_\alpha W_\beta \gamma^\mu \left( \partial_\mu - ig_L W_\mu + \frac{i}{2} g' A_\mu \right) L + \frac{1}{4} g^{\alpha\beta} \overline{W}_\gamma \gamma^\mu \left( \partial_\mu - ig_L W_\mu + \frac{i}{2} g' A_\mu \right) W_\alpha W_\beta L$$

(100)

and

$$\mathcal{L}^{NC(R)}_{\text{(currents)}} = \overline{R}i\gamma^\mu \left( -ig' A_\mu Y^{(R)} \right) R - i\theta^{\alpha\beta} \overline{R} \gamma^\mu \left( \partial_\mu - ig' A_\mu Y^{(R)} \right) B_\alpha \partial_\beta R + g_R \theta^{\mu\nu} \overline{R} i\partial_\mu B \partial_\nu R - i\theta^{\mu\nu} \partial_\nu \left( \overline{R} \right) B_\mu \gamma^\mu \left( \partial_\mu - ig' A_\mu Y^{(R)} \right) R.$$  

(101)

\((L \text{ and } R \text{ stand for left and right). From the above expressions, one can deduce the neutral and charged currents. Regarding the neutral electroweak currents, the Lagrangian } \mathcal{L}^{NC}_{(n,c)} \text{ has as expression:}

$$\mathcal{L}^{NC}_{(n,c)} = \mathcal{L}_{(n,c)} + \mathcal{L}^{(1)}_{(n,c)}$$  

(102)

where \(\mathcal{L}_{(n,c)}\) is the classical electroweak neutral current given by

$$\mathcal{L}_{(n,c)} = g_L J^\mu \overline{W}^\mu + \frac{1}{2} g' J^m_\mu A_\mu = e J^{e.m}_\mu A_\mu + \frac{g_L}{\cos \theta_W} J^0_\mu Z^\mu$$

(103)

with

$$J^Y = - \left( \overline{\nu}_L \gamma^\mu \nu_L + \overline{e}_L \gamma^\mu e_L + 2 \overline{\tau}_R \gamma^\mu e_R \right)$$

$$J^{e.m}_\mu = \overline{\nu}_L \gamma^\mu e_L + \overline{\tau}_R \gamma^\mu e_R = \overline{\tau}_R \gamma^\mu e$$

$$J^3 = \frac{1}{2} \left( \overline{\nu}_L \gamma^\mu \nu_L - \overline{\tau}_L \gamma^\mu e_L \right)$$

$$J^L = - \left( \overline{\nu}_L \gamma^\mu \nu_L + \overline{\tau}_L \gamma^\mu e_L \right)$$

(104)

and

$$J^0_\mu = J^3 + \sin^2 \theta_W J^{e.m}_\mu$$

The fields \(Z^\mu, A^\mu\), are defined through the Weinberg angle \(\theta_W\) rotation as follows:

$$B^\mu = \cos \theta_W Z^\mu + \sin \theta_W A^\mu$$

(105)

and

$$A^\mu = - \sin \theta_W Z^\mu + \cos \theta_W A^\mu$$

The term \(\mathcal{L}^{(1)}_{(n,c)}\) is the pure noncommutative neutral electroweak current and has the form:
\[ L^{(1)}_{(n,c)} = L^{(1)L}_{(n,c)} + L^{(1)R}_{(n,c)} \]  
\[ \text{with} \]
\[ L^{(1)L}_{(n,c)} = \frac{i}{8} (g_L)^3 \theta^{\alpha\beta} J^L_{\mu} W^3_{\alpha} \left( W^+_\beta W^-_{\mu} - W^-_{\beta} W^+_{\mu} \right) \]
\[ \text{and} \]
\[ L^{(1)R}_{(n,c)} = \frac{i}{2} g_R \theta^{\alpha\beta} \left[ (\bar{\nu}_R \gamma^\mu \bar{\nu}_R - \bar{e}_R \gamma^\mu \bar{e}_R) G^3_{\alpha\mu} - \frac{1}{2} (\bar{\nu}_R \partial \bar{\nu}_R - \bar{e}_R \partial \bar{e}_R) G^3_{\alpha\beta} \right] \]

Notice here that the right neutral currents are of the order \( \theta \), and vanish in the commutative classical limit when \( \theta \to 0 \).

Regarding the left charged electroweak currents, the Lagrangian \( L^{NCL}_{(c,c)} \) takes the form:
\[ L^{NCL}_{(c,c)} = L^L_{(c,c)} + L^{(1)L}_{(c,c)} \]
where \( L^L_{(c,c)} \) is the classical charged electroweak current which has the following expression:
\[ L^L_{(c,c)} = \frac{g_L}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu \nu_L W^+_{\mu} - \bar{e}_L \gamma^\mu \nu_L W^-_{\mu}) = \frac{g_L}{\sqrt{2}} (J^+_\mu W^{+\mu} - J^-_{\mu} W^{-\mu}) \]
with
\[ J^+_\mu = (J^-_{\mu})^+ = \bar{\nu}_L \gamma^\mu e_L \]
and \( L^{(1)L}_{(c,c)} \) is the left charged electroweak current given by:
\[ L^{(1)L}_{(c,c)} = \frac{i}{4} \left( \frac{g_L}{\sqrt{2}} \right)^2 \theta^{\alpha\beta} A^\mu W^3_{\alpha} \left( J^+_\mu W^+_{\beta} - J^-_{\mu} W^-_{\beta} \right) + \frac{1}{2} \left( \frac{g_L}{\sqrt{2}} \right)^2 \theta^{\alpha\beta} \left[ \tilde{J}^+_\mu W^+_{\beta} - \tilde{J}^-_{\mu} W^-_{\beta} \right] W^3_{\alpha} \]
with
\[ \tilde{J}_+ = \left( \tilde{J}_- \right)^+ = \bar{\nu}_L \partial \bar{e}_L \]
and
\[ W^\pm_{\mu} = \frac{1}{\sqrt{2}} \left( W^1_{\mu} \pm i W^2_{\mu} \right) \]

For the right charged currents, the lagrangian \( L^{NCR}_{(c,c)} \) is given by:
\[ L^{NCR}_{(c,c)} = L^R_{(c,c)} + L^{(1)R}_{(c,c)} \]
with
\[ L^R_{(c,c)} = 0 \ldots \]
\[ L^{(1)}_{\text{R(c)}} = \frac{g_R}{\sqrt{2}} i^R \theta^\alpha \gamma^\mu (\partial_\alpha W_\beta^+ A_\mu + W_\beta^+ \partial_\alpha A_\mu + \partial_\alpha W_\beta^- A_\mu + W_\beta^- \partial_\alpha A_\mu) e_R \]
\[ + \frac{g_R}{\sqrt{2}} i^R \theta^\alpha \gamma^\mu (\partial_\alpha e_R W_\beta^+ A_\mu + \partial_\alpha (\bar{e}_R) \gamma^\mu \nu_R W_\beta^- A_\mu) \]
\[ + \frac{g_R}{\sqrt{2}} i^R (\bar{e}_R \gamma^\mu \partial_\beta \nu_R W_{\alpha \mu}^- + \bar{e}_R \gamma^\mu \partial_\nu \bar{e}_R W_{\alpha \mu}^+) - \frac{g_R}{\sqrt{2}} i^R \theta^\alpha \gamma^\mu (\partial_\alpha e_R W_{\alpha \beta}^- + \bar{e}_R \partial_\nu \bar{e}_R W_{\alpha \beta}^+) \]
where
\[ W_{\alpha \beta}^\pm = \partial_\alpha W_{\beta}^\pm - \partial_\beta W_{\alpha}^\pm. \]

Again, notice that the charged right currents are of the order \( \theta \), and vanish in the classical limit when \( \theta \to 0 \).

**Conclusions**

Through this work, in a unified description and in order to avoid gauge fields transformations that are not Lie or Poisson algebra valued, we have defined a new Weyl ordering using Moyal star product (symmetrization through star Poisson or Lie brackets) together with some golden rules. Based on this new approach, the noncommutative covariant derivative, curvature tensor, Seiberg-Witten maps fields and the corresponding gauge transformations as well as an invariant action are constructed.

The most important idea of this approach is the generalization of the singlet notion under noncommutative gauge transformations and the introduction of a geometric way to create new interactions, extend and enlarge a gauge theory namely the standard model. The physical application was done for the left-right extension of the standard model and the corresponding charged and neutral currents were derived. The right sector is shown to have a pure noncommutative space-time origin.

**Appendix A**

The matter field noncommutative gauge transformation is given by:
\[ \delta_{\tilde{\Lambda}} \hat{\psi} = \left( i T^a \left[ \tilde{\Lambda}^a \hat{\psi} \right] \right) \eta_w = i T^a \left[ \tilde{\Lambda}^a \hat{\psi} \right] \eta_w \]
thus
\[ \delta_{\Sigma} \left( \delta_{\tilde{\Lambda}} \hat{\psi} \right) = \left( i T^a \left[ \tilde{\Lambda}^a \delta_{\Sigma} \hat{\psi} \right] \right) \eta_w = \left( i T^a \left[ \tilde{\Lambda}^a \delta_{\Sigma} \hat{\psi} \right] \right) \eta_w \]
\[ = \left( i T^a i T^b \left[ \tilde{\Lambda}^a \left[ \tilde{\Sigma}^b \hat{\psi} \right] \right] \right) \eta_w = \frac{1}{2} \left( i T^a i T^b \left[ \tilde{\Lambda}^a \tilde{\Sigma}^b \hat{\psi} + \eta \hat{\psi} * \tilde{\Sigma}^b \right] \right) \eta_w \]
\[ = \frac{1}{4} \left( i T^a i T^b \left[ \tilde{\Lambda}^a \tilde{\Sigma}^b * \hat{\psi} + \eta \tilde{\Sigma}^b * \hat{\psi} * \tilde{\Lambda}^a + \eta \tilde{\Lambda}^a * \hat{\psi} * \tilde{\Sigma}^b + \eta^2 \psi * \tilde{\Sigma}^b * \tilde{\Lambda}^a \right] \right) \eta_w \]
In what follows, we denote by:

$$
\hat{A} \ast \hat{B} \equiv 2 [A, B]_\eta = A \ast B + \eta B \ast A
$$

(A3)

Using the following golden rules:

$$
iT_a iT_b \hat{\Lambda}^a \ast \hat{\psi} \ast \hat{\Sigma}^b \rightarrow \left( iT_a iT_b \hat{\Lambda}^a \ast \hat{\psi} \ast \hat{\Sigma}^b \right) \rightarrow iT_a iT_b \left( \hat{\Lambda}^a \ast \hat{\psi} + \eta \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a \right)
$$

(A4)

$$
iT_a iT_b \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a \rightarrow \left( iT_a iT_b \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a \right) \rightarrow iT_a iT_b \left( \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a + \eta \hat{\Lambda}^a \ast \hat{\psi} \ast \hat{\Sigma}^b \right)
$$

(A5)

$$
iT_a iT_b \hat{\Lambda}^a \ast \hat{\Sigma}^b \ast \hat{\psi} \rightarrow \left( iT_a iT_b \hat{\Lambda}^a \ast \hat{\Sigma}^b \ast \hat{\psi} \right) \rightarrow iT_a iT_b \left( \hat{\Lambda}^a \ast \hat{\Sigma}^b \ast \hat{\psi} + \eta \hat{\Lambda}^a \ast \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a \right)
$$

(A6)

$$
iT_a iT_b \hat{\Sigma}^b \ast \hat{\Lambda}^a \ast \hat{\psi} \rightarrow \left( iT_a iT_b \hat{\Sigma}^b \ast \hat{\Lambda}^a \ast \hat{\psi} \right) \rightarrow iT_a iT_b \left( \hat{\Sigma}^b \ast \hat{\Lambda}^a \ast \hat{\psi} + \eta \hat{\Lambda}^a \ast \hat{\Sigma}^b \ast \hat{\psi} \ast \hat{\Lambda}^a \right)
$$

(A7)

$$
iT_a iT_b \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{\Sigma}^b \rightarrow \left( iT_a iT_b \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{\Sigma}^b \right) \rightarrow iT_a iT_b \left( \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{\Sigma}^b \right)
$$

(A8)

$$
iT_a iT_b \hat{\Sigma}^b \ast \hat{\Lambda}^a \ast \hat{\psi} \rightarrow \left( iT_a iT_b \hat{\Sigma}^b \ast \hat{\Lambda}^a \ast \hat{\psi} \right) \rightarrow iT_a iT_b \left( \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{\Sigma}^b \right)
$$

(A9)
we obtain:

\[
\delta_\Sigma \left( \delta_\Lambda \tilde{\psi} \right) = \frac{1}{4} \left( i T^a i T^b \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi} + \eta i T^b i T^a \hat{\Sigma}^b * \tilde{\psi} * \hat{\Lambda}^a \right)
\]

\[
\delta_\Lambda \left( \delta_\Sigma \tilde{\psi} \right) = \frac{1}{4} \left( i T^b i T^a \hat{\Sigma}^b * \hat{\Lambda}^a * \tilde{\psi} + \eta i T^a i T^b \hat{\Lambda}^a * \tilde{\psi} * \hat{\Sigma}^b \right)
\]

Similarly:

\[
\delta_\Lambda \left( \delta_\Sigma \tilde{\psi} \right) = \frac{1}{4} \left( i T^b i T^a \hat{\Sigma}^b * \hat{\Lambda}^a * \tilde{\psi} + \eta i T^a i T^b \hat{\Lambda}^a * \tilde{\psi} * \hat{\Sigma}^b \right)
\]

we deduce that:

\[
\delta_\Lambda \left( \delta_\Sigma \tilde{\psi} \right) - \delta_\Sigma \left( \delta_\Lambda \tilde{\psi} \right) = \left[ \delta_\Lambda, \delta_\Sigma \right] \tilde{\psi}
\]

\[
= \frac{1}{4} \left( i T^b i T^a \hat{\Sigma}^b * \hat{\Lambda}^a * \tilde{\psi} - i T^a i T^b \hat{\Lambda}^a * \tilde{\psi} * \hat{\Sigma}^b \right) - \eta^2 i T^a i T^b \tilde{\psi} * \hat{\Lambda}^a * \hat{\Sigma}^b
\]

using the relations

\[
\hat{\Sigma}^b * \hat{\Lambda}^a * \tilde{\psi} = \eta \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi}
\]

\[
\tilde{\psi} * \hat{\Lambda}^a * \hat{\Sigma}^b = \eta \tilde{\psi} * \hat{\Sigma}^b * \hat{\Lambda}^a
\]

\[
\hat{\Sigma}^b * \tilde{\psi} * \hat{\Lambda}^a = \eta \hat{\Lambda}^a * \tilde{\psi} * \hat{\Sigma}^b
\]

and \( \eta^2 = 1 \), we obtain:

\[
\left[ \delta_\Lambda, \delta_\Sigma \right] \tilde{\psi} = \frac{1}{4} \left( \eta i T^b i T^a \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi} - i T^a i T^b \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi} \right)
\]

\[
= \frac{1}{4} \left( \left[ T^a, T^b \right] - \eta \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi} + \eta \left( T^a T^b - \eta T^b T^a \right) \tilde{\psi} * \hat{\Lambda}^a * \hat{\Sigma}^b \right)
\]

\[
= \frac{1}{4} \left[ \left[ T^a, T^b \right] - \eta \hat{\Lambda}^a * \hat{\Sigma}^b * \tilde{\psi} + \eta \tilde{\psi} * \hat{\Lambda}^a * \hat{\Sigma}^b \right]
\]

\[
= \frac{1}{2} \left[ T^a, T^b \right] - \eta \left[ \hat{\Lambda}^a * \hat{\Sigma}^b, \tilde{\psi} \right] \hat{\Lambda}^a * \hat{\Sigma}^b, \tilde{\psi} \right] \eta
\]
Appendix B

We define the noncommutative covariant derivative as

\[
\hat{D}_\mu \hat{\psi} \equiv \left( \partial_\mu \hat{\psi} - iT^a \left[ V^a_\mu, \hat{\psi} \right] \right) = \partial_\mu \hat{\psi} - iT^a \left[ \hat{V}^a_\mu, \hat{\psi} \right]
\]

(B1)

It is a covariant derivative in the sense:

\[
\delta_\Lambda \left( \hat{D}_\mu \hat{\psi} \right) = \left( iT^a \left[ \Lambda^a, \partial_\mu \hat{\psi} \right] \right) + \left( iT^a \left[ \hat{\Lambda}^a, -iT^b \left[ V^b_\mu, \hat{\psi} \right] \right] \right)
\]

(B2)

Using the golden rules eqs.(A4)-(A9) we get:

\[
\delta_\Lambda \left( \hat{D}_\mu \hat{\psi} \right) = iT^a \left[ \Lambda^a, \partial_\mu \hat{\psi} \right] + \frac{1}{4} \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \hat{\psi} + \eta \hat{V}^b_\mu \hat{\psi} \right] \right) + \frac{1}{4} \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \hat{\psi} + \eta \hat{V}^b_\mu \hat{\psi} \right] \right) \]

(B3)

Moreover, a direct calculation and using the fact that:

\[
\left[ \delta_\Lambda, \partial_\mu \right] = 0
\]

(B4)

gives:

\[
\delta_\Lambda \left( \hat{D}_\mu \hat{\psi} \right) = \partial_\mu \left( iT^a \left[ \Lambda^a, \hat{\psi} \right] \right) + \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right] \right) + \left( -iT^a iT^b \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right] \right)
\]

(B5)
again, with the help of the golden rules of eqs.(A4)-(A9) we deduce that:

\[
\delta_{\Lambda} \left( \hat{D}_\mu \hat{\psi} \right) = iT^a \left[ \hat{\Lambda}^a, \partial_\mu \hat{\psi} \right]_{\eta}^* + iT^a \left[ \partial_\mu \hat{\Lambda}^a, \hat{\psi} \right]_{\eta}^* - iT^b \left[ \delta_{\Lambda} \hat{V}^b_\mu, \hat{\psi} \right]_{\eta}^* \\
+ \frac{1}{4} \left( -iT^b iT^a \hat{V}^b_\mu \ast \hat{\Lambda}^a \ast \hat{\psi} - \eta iT^a iT^b \hat{\Lambda}^a \ast \hat{\psi} \ast \hat{V}^b_\mu \right) - \eta iT^b iT^a \hat{V}^b_\mu \ast \hat{\Lambda}^a - \eta^2 iT^b iT^a \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{V}^b_\mu \right) 
\]

using relations (A13) we obtain:

\[
-i T^b \left[ \delta_{\Lambda} \hat{V}^b_\mu, \hat{\psi} \right]_{\eta}^* = -iT^a \left[ \partial_\mu \hat{\Lambda}^a, \hat{\psi} \right]_{\eta}^* + \frac{1}{4} \left( [T^a, T^b]_{\eta} \hat{\Lambda}^a \ast \hat{V}^b_\mu \ast \hat{\psi} + \eta [T^a, T^b]_{\eta} \hat{\psi} \ast \hat{\Lambda}^a \ast \hat{V}^b_\mu \right) \\
= -iT^a \left[ \partial_\mu \hat{\Lambda}^a, \hat{\psi} \right]_{\eta}^* + \frac{1}{2} \left( [T^a, T^b]_{\eta} \hat{\Lambda}^a \ast \hat{V}^b_\mu \ast \hat{\psi} \right) 
\]

thus

\[
-i T^b \delta_{\Lambda} \hat{V}^b_\mu = -iT^a \partial_\mu \hat{\Lambda}^a + \frac{1}{2} [T^a, T^b]_{\eta} \hat{\Lambda}^a \ast \hat{V}^b_\mu 
\]

or

\[
\delta_{\Lambda} \hat{V}^b_\mu = \partial_\mu \hat{\Lambda}^a + \frac{i}{2} [T^a, T^b]_{\eta} \hat{\Lambda}^a \ast \hat{V}^b_\mu = \partial_\mu \Lambda + i [T^a, T^b]_{\eta} \left[ \hat{\Lambda}^a, \hat{V}^b_\mu \right]_{\eta}^* 
\]

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**References**

A Method for Constructing a Lax Pair for the Ernst Equation

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Abstract: A systematic construction of a Lax pair and an infinite set of conservation laws for the Ernst equation is described. The matrix form of this equation is rewritten as a differential ideal of $gl(2,R)$-valued differential forms, and its symmetry condition is expressed as an exterior equation which is linear in the symmetry characteristic and has the form of a conservation law. By means of a recursive process, an infinite collection of such laws is then obtained, and the conserved “charges” are used to derive a linear exterior equation whose components constitute a Lax pair.

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1. Introduction

The search for the connections between symmetry and integrability has always been a central problem in the study of nonlinear partial differential equations (PDEs). For those PDEs having an underlying variational structure, the work of E. Noether and its extensions (see, e.g., [1,2]) provide an important link between variational symmetries and conservation laws. Non-variational connections between symmetry and integrability, however, also exist. They are often related to the possibility of “linearizing” a nonlinear PDE by use of a Lax pair, i.e., a pair of coupled PDEs linear in an auxiliary function $\psi$ and integrable for $\psi$ on the condition that the original (nonlinear) PDE is satisfied. Linearity is an important issue here, since the symmetry condition (characteristic equation) of a

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PDE is itself a linear PDE for the symmetry characteristic [1,2].

A given nonlinear PDE may often be linearized in more than one way by different choices of a Lax pair. A particularly useful choice is the one in which the Lax pair plays the role of a Bäcklund transformation connecting the PDE with its symmetry condition [3], so that the solution $\psi$ of the pair is a symmetry characteristic for the PDE (or, more generally, is linearly dependent on a symmetry characteristic). Hence, in a sense, the symmetry condition is “built” into the Lax pair. In this way, one obtains a symmetry of the PDE by integrating the associated linear system.

A well-known example where these ideas find wide applications is the self-dual Yang-Mills equation [4,5]. Interestingly, this has been shown to be a sort of prototype equation from which several other known PDEs are derived by reduction [6,7]. One such PDE is the Ernst equation of General Relativity describing stationary, axially symmetric gravitational fields. In a previous paper [8] the authors proposed a new Lax pair for this equation (an older one was found by Belinski and Zakharov [9]) and showed that the solution $\psi$ of this pair is indeed linearly related to a symmetry characteristic. In addition to giving new “hidden” symmetries, the Lax pair also leads to the construction of infinite collections of conservation laws for the Ernst equation.

Admittedly, finding a Lax pair with specific properties almost always requires a certain amount of guessing, as well as a lot of patience in a long trial-and-error process. We now ask the question: Can a linear system such as that of [8] be derived in a systematic way? This article answers this question in the affirmative. As we show, the symmetry condition alone leads one straightforwardly to the discovery of infinite sets of conservation laws, as well as a Lax pair having the desired properties. Our formalism is expressed in the language of exterior differential forms which is both elegant and economical. Hence, for example, differential equations expressing conservation laws, as well as systems of PDEs constituting differential recursion relations or Lax pairs, will now be represented by single exterior equations. In this regard, it would be more appropriate to speak of an **exterior linearization equation**, rather than of a Lax pair in the ordinary sense of this term.

In short, the process is as follows: First, we rewrite the Ernst equation as a differential ideal of matrix-valued differential forms and express its symmetry condition as an exterior equation which is linear in the symmetry characteristic. This latter equation is in conservation-law form, and this fact allows us to introduce a first “conserved charge” or “potential”. A second conservation law is then found, with a new potential, and the process continues indefinitely, yielding a double infinity of conserved charges. These charges are related to each other via a certain recursion relation and are used as Laurent coefficients in a series whose terms involve powers (both positive and negative) of a complex “spectral” parameter. This series (assuming it converges) represents some complex function $\Psi$, which is shown to satisfy an exterior linearization equation equivalent to a Lax pair.
2. Mathematical Preliminaries

The variables \( x^\mu \equiv \rho, z (\mu = 1, 2, \text{respectively}) \) will be regarded as local orthogonal coordinates in a 2-dimensional Euclidean space with metric \( \delta_{\mu \nu} \). Geometrical objects defined in this space (such as functions or differential forms) are assumed matrix-valued, with values generally in \( gl(2, C) \) (with appropriate restrictions, such as real-valuedness, etc., in accordance with physical requirements).

The volume 2-form in our space is

\[
\tau = \frac{1}{2} \varepsilon_{\mu \nu} dx^\mu dx^\nu = d\rho dz
\]

(the usual summation convention is assumed). For any 1-form

\[
\sigma = \sigma_\mu dx^\mu = \sigma_1 d\rho + \sigma_2 dz,
\]

the dual of \( \sigma \) with respect to \( \tau \) is defined as the 1-form \( *\sigma \) with components

\[
(\ast \sigma)_\nu = \tau_{\mu \nu} \sigma^\mu = \varepsilon_{\mu \nu} \delta^{\mu \lambda} \sigma_\lambda,
\]

so that

\[
\ast \sigma = (\ast \sigma)_\mu dx^\mu = -\sigma_2 d\rho + \sigma_1 dz.
\]

In particular, \( \ast d\rho = dz, \ast dz = -d\rho \). Also,

\[
\ast(\ast \sigma) = -\sigma \tag{1}
\]

For 1-forms \( \sigma_1 \) and \( \sigma_2 \), we have that

\[
\ast \sigma_1 \wedge \ast \sigma_2 = \sigma_1 \wedge \sigma_2, \quad \sigma_1 \wedge \ast \sigma_2 = - (\ast \sigma_1) \wedge \sigma_2 \tag{2}
\]

We note that the \( * \) operation is linear, so that

\[
\ast(\alpha \sigma_1 + \beta \sigma_2) = \alpha \ast \sigma_1 + \beta \ast \sigma_2 \tag{3}
\]

where \( \alpha \) and \( \beta \) are 0-forms.

Given any differential forms \( \zeta \) and \( \xi \), we define the commutator

\[
[\zeta, \xi] \equiv \zeta \wedge \xi - \xi \wedge \zeta.
\]

In particular, if \( \sigma \) is a 1-form and \( \psi \) is a 0-form, then \( [\sigma, \psi] = \sigma \psi - \psi \sigma \) and, by the antiderivation property of the exterior derivative,

\[
d[\sigma, \psi] = [d\sigma, \psi] - \{\sigma, d\psi\} \tag{4}
\]

where, in general, curly brackets denote anticommutators:

\[
\{\sigma_1, \sigma_2\} \equiv \sigma_1 \wedge \sigma_2 + \sigma_2 \wedge \sigma_1.
\]

We note that, to simplify our notation, we will often omit the symbol \( \wedge \) of the exterior product. It should be kept in mind, however, that exterior multiplication of differential forms will always be assumed. Thus, an expression like \( \sigma_1 \sigma_2 \) should be understood as \( \sigma_1 \wedge \sigma_2 \).
3. Ernst Equation: Geometrical Formulation and Symmetry

We adopt the following matrix form of the Ernst equation [6,7]:

\[(\rho g^{-1}g_\rho) + (\rho g^{-1}g_z)_z = 0\] (5)

where subscripts denote partial derivatives with respect to the variables \(\rho, z\), collectively denoted \(x^\mu (\mu=1,2,\text{ respectively})\). The matrix function \(g\) is assumed to be \(SL(2,R)\)-valued and symmetric. With the parametrization

\[g = \frac{1}{f} \begin{bmatrix} 1 & \omega \\ \omega & f^2 + \omega^2 \end{bmatrix}\]

and by setting \(E = f+\imath \omega\), we recover the Ernst equation in the usual form,

\[(ReE) \nabla^2 E = (\nabla E)^2.\]

With the substitutions

\[A = g^{-1}g_\rho, \quad B = g^{-1}g_z,\]

equation (5) becomes equivalent to the system of PDEs

\[A + \rho(A_\rho + B_z) = 0 \quad (6)\]

\[B_\rho - A_z + [A, B] = 0 \quad (7)\]

The second equation is just the integrability condition in order that \(g\) may be reconstructed from \(A\) and \(B\).

We introduce the matrix-valued “connection” 1-form

\[\gamma = g^{-1}dg = Ad\rho + Bdz\] (8)

The integrability condition \(d(dg)=0\) in order that \(g\) may be recovered from \(\gamma\), together with the obvious requirement that \(g\) be nonsingular, yield the Mauer-Cartan equation \(\omega=0\), where \(\omega\) is the 2-form

\[\omega = d\gamma + \gamma \wedge \gamma = dBdz - d\rho dA + [A, B] d\rho dz\] (9)

We also construct the 2-form

\[d(\rho \ast \gamma) = A d\rho dz + \rho(dA dz + d\rho dB)\] (10)

where \(\ast \gamma = -Bd\rho + Adz\).

We now observe that Eqs.(6) and (7) correspond to the system of exterior equations

\[d(\rho \ast \gamma) = 0, \quad \omega = 0\] (11)
Indeed, one may consider $d(\rho^*\gamma)$ and $\omega$ as 2-forms in a jet-like space of four variables: the scalar variables $x^\mu = \rho, z$ and the $gl(2,R)$ variables $A$ and $B$. Equations (6) and (7) are recovered by projecting Eqs.(11) onto the base space of the $x^\mu$.

Let $I\{d(\rho^*\gamma),\, \omega\}$ be the ideal of forms [10-12] generated by the 2-forms $d(\rho^*\gamma)$ and $\omega$. The first form is exact, thus its exterior derivative is trivially a member of the ideal, while, as we can easily show, $d\omega = \omega \wedge \gamma - \gamma \wedge \omega$, which also belongs to $I$. We thus conclude that $I$ is a differential (closed) ideal.

The first of Eqs.(11) implies the existence of a matrix potential $X$ such that $\rho^*\gamma = dX$ (that is, $\rho A = X_z, \rho B = -X_\rho$). Then, $^*dX = -\rho \gamma$, and, by the Mauer-Cartan equation $\omega = 0$, we get

$$d\rho \ast dX - \rho d \ast dX + dX dX = 0 \quad (12)$$

[where use has been made of the first of Eqs.(2)]. In component form,

$$X_\rho - \rho (X_{\rho\rho} + X_{zz}) + [X_\rho, X_z] = 0 \quad (13)$$

We introduce the covariant derivatives

$$D_\rho = \partial_\rho + [A, \ ] , \quad D_z = \partial_z + [B, \ ] \quad (14)$$

(where $\partial_\rho = \partial/\partial \rho$ and $\partial_z = \partial/\partial z$) which are seen to be derivations on the Lie algebra of $gl(2,C)$-valued functions. We also define an exterior covariant derivative $D$ which acts on $gl(2,C)$ functions $\Phi$ as follows:

$$D \Phi = d\Phi + [\gamma, \Phi] = (D_\rho \Phi) d\rho + (D_z \Phi) dz \quad (15)$$

We now look at the symmetry problem for system (11). We first note that all symmetries of a system of PDEs can be expressed as infinitesimal transformations of the dependent variables alone [1,2]. Thus, all symmetries may be represented by “vertical” vector fields, i.e., vectors with vanishing projections on the base space of the $x^\mu$. Let $\delta g = \alpha Q[g]$ be an infinitesimal symmetry transformation of Eq.(5), where $\alpha$ is an infinitesimal parameter and $Q$ is a matrix-valued function which may depend locally or nonlocally on $g$. It is convenient to set $Q = g^n \Phi$, where $\Phi$ is another matrix 0-form. The infinitesimal symmetry of Eq.(5) is then written as

$$\delta g = \alpha g \Phi \quad (16)$$

(with appropriate restrictions on $\Phi$ in order that the transformation preserve the symmetric $SL(2,R)$ character of $g$). This induces the symmetry transformations $\delta A = \alpha D_\rho \Phi$, $\delta B = \alpha D_z \Phi$ of system (6)-(7). These are summarized by the formal vector field

$$V = D_\rho \Phi \frac{\partial}{\partial A} + D_z \Phi \frac{\partial}{\partial B} \quad (17)$$

The symmetry condition on the ideal $I$ of the 2-forms $d(\rho^*\gamma)$ and $\omega$ is that the Lie derivative with respect to $V$ should leave this ideal invariant [10-12]:

$$LV I \subset I$$
This is satisfied by requiring that
\[ L_V d(\rho \ast \gamma) = L_V \omega = 0 \mod I \{ d(\rho \ast \gamma), \omega \} \] (18)
By using Eq.(9) for \( \omega \), taking into account that the Lie derivative commutes with the exterior derivative and satisfies the Leibniz rule, and by noting that
\[ L_V \gamma = L_V (A d\rho + B dz) = (D_\rho \Phi) d\rho + (D_z \Phi) dz = D\Phi = d\Phi + [\gamma, \Phi], \]
we find that
\[ L_V \omega = \omega \Phi - \Phi \omega \equiv [\omega, \Phi], \]
which is automatically a member of the ideal \( I \), hence satisfies the condition for \( \omega \) in Eq.(18). On the other hand, by noting that
\[ L_V \ast \gamma = L_V (-B d\rho + A dz) = * D\Phi, \]
we find that the condition for \( d(\rho \ast \gamma) \) is expressed as an exterior equation which is linear in \( \Phi \):
\[ d(\rho \ast D\Phi) = 0 \text{ on solutions} \] (19)
(where “on solutions” means: when Eqs.(11) are satisfied). In component form,
\[ (\rho D_\rho \Phi)_\rho + (\rho D_z \Phi)_z = 0 \text{ on solutions} \] (20)
The reader is invited to derive the symmetry condition (20) directly from the Ernst equation (5) by assuming a symmetry characteristic \( Q=g\Phi \) and by applying the abstract formalism described in [3]. (Note, however, that our present notation is different from that of [3]. Specifically, the symbols \( D_\rho \) and \( D_z \), which here denote covariant derivatives, have the meaning of total derivatives in [3].)

4. Conservation Laws and Exterior Linearization Equation

We now turn to integrability characteristics of the Ernst equation. As is well known, the hallmark of integrability is the existence of a linear system or Lax pair. This system may be compactified into a single exterior equation involving 1-forms, which will be referred to as an exterior linearization equation. The purpose of this section is to describe a systematic construction of such a linearization equation for the Ernst equation, or equivalently, for the exterior system (11).

We begin with the symmetry condition (19):
\[ d(\rho \ast D\Phi) = 0 \] (21)
The corresponding infinitesimal symmetry transformation is \( g' = g + \alpha g\Phi \), according to Eq.(16). This means that \( g' \) is a solution of the general PDE (5) when \( g \) is a solution. However, we will not require here that the new solution \( g' \) conform to the extra
physical restrictions imposed on the original solution \( g \), namely, of being symmetric and having unit determinant. Thus, all real solutions \( \Phi \) of the exterior equation (21) will be admissible (e.g., \( \Phi = g^{-1} g_z = B \)).

As its component form (20) suggests, the exterior equation (21) expresses a conservation law valid for solutions of the Ernst equation. Equation (21) also implies the existence of a “conserved charge” or “potential” \( \Phi' \), such that
\[
d\Phi' = \rho \ast D \Phi = \rho (\ast d\Phi + \ast [\gamma, \Phi])
\]
[where use has been made of the linearity property (3) of the star operation]. Starring this equation, solving for \( d\Phi \), and requiring that \( d(d\Phi) = 0 \), we find another conservation law:
\[
d (\rho \ast D \Phi' - 2\Phi' dz) = 0
\]
by which we introduce a new potential \( \Phi'' \) such that
\[
d\Phi'' = \rho \ast D \Phi' - 2\Phi' dz = \rho (\ast d\Phi' + \ast [\gamma, \Phi']) - 2\Phi' dz
\]
Starring this and applying \( d(d\Phi') = 0 \), we obtain yet another conservation law:
\[
d (\rho \ast D \Phi'' - 4\Phi'' dz) = 0
\]
This process suggests that we consider the following exterior recursion relation:
\[
d\Phi^{(n+1)} = \rho \ast D \Phi^{(n)} - 2n \Phi^{(n)} dz
\]
with \( \Phi^{(0)} = \Phi \) representing a symmetry characteristic of the Ernst equation in its general form (5) [i.e., a solution of Eq. (21)].

In order that the exterior equation (22) be integrable for \( \Phi^{(n+1)} \) for an already known \( \Phi^{(n)} \), the integrability condition \( d(d\Phi^{(n+1)}) = 0 \) must be satisfied. This yields
\[
d (\rho \ast D \Phi^{(n)} - 2n \Phi^{(n)} dz) = 0
\]
We will now show that Eq. (23) is a conservation law valid for solutions of the Ernst equation. The left-hand side of (23) is written as
\[
l.h.s. (23) = d (\rho \ast d\Phi^{(n)} + [\rho \ast \gamma, \Phi^{(n)}] - 2n \Phi^{(n)} dz)
\]
By using property (4) and the second property (2), we have:
\[
d [\rho \ast \gamma, \Phi^{(n)}] = [d(\rho \ast \gamma), \Phi^{(n)}] - \rho \ast \gamma d\Phi^{(n)} - \rho d\Phi^{(n)} \ast \gamma
\]
\[
= [d(\rho \ast \gamma), \Phi^{(n)}] + \rho \gamma \ast d\Phi^{(n)} + \rho \ast d\Phi^{(n)} \gamma,
\]
\[
d\Phi^{(n)} dz = d\Phi^{(n)} \ast d\rho = d\rho \ast d\Phi^{(n)}.
\]
Therefore,
\[
l.h.s. \ (23) = (1-2n) \ d\rho \ast d\Phi^{(n)} + \rho \ d \ast d\Phi^{(n)} + [d(\rho \ast \gamma), \ \Phi^{(n)}] + \rho \gamma \ast d\Phi^{(n)} + \rho \ast d\Phi^{(n)} \gamma.
\]

Now, by rewriting the recursion relation (22) with \((n-1)\) in place of \(n\), we can express \(d\Phi^{(n)}\), thus also \(\ast d\Phi^{(n)}\), in terms of \(\Phi^{(n-1)}\). Substituting for \(\ast d\Phi^{(n)}\) into the expression for the l.h.s. of (23), and taking into account that \(d\gamma + \gamma \gamma = \omega\), we finally find:
\[
l.h.s. \ (23) = [d(\rho \ast \gamma), \ \Phi^{(n)}] - \rho^2 [\omega, \ \Phi^{(n-1)}].
\]

We note that this expression vanishes when \(d(\rho \ast \gamma)=0\) and \(\omega=0\), i.e., for solutions of the Ernst equation. This proves the conservation-law property of Eq.(23).

As we have just shown, the conservation law (23) is the necessary condition for \(\Phi^{(n)}\) in order that the exterior equation (22) be integrable for \(\Phi^{(n+1)}\). For \(n=0\), Eq.(23) is just the symmetry condition (21), which is indeed satisfied by \(\Phi^{(0)}\) since the latter is, by assumption, a symmetry characteristic. Now, we must show that the solution \(\Phi^{(n+1)}\) of Eq.(22) also conforms to condition (23) with \((n-1)\). This will ensure that the recursive process may continue indefinitely for all values of \(n\), yielding an infinite number of conservation laws from any given symmetry characteristic \(\Phi^{(0)}\). This time we need to eliminate \(\Phi^{(n)}\) from Eq.(22) in favor of \(\Phi^{(n+1)}\). By this process we will actually derive the necessary condition for \(\Phi^{(n+1)}\) in order that the exterior equation (22) be integrable for \(\Phi^{(n)}\) when \(\Phi^{(n+1)}\) is already known. This will allow us to use the recursion relation (22) “backwards” to obtain potentials \(\Phi^{(n)}\) and corresponding conservation laws (23) for negative values of \(n\) also. Thus, the validity of Eqs.(22) and (23) will be extended to all integral values \(n=0, \pm 1, \pm 2, \ldots\)

Starring Eq.(22) and solving for \(d\Phi^{(n)}\), we get:
\[
d\Phi^{(n)} = -\frac{1}{\rho} \ast d\Phi^{(n+1)} - [\gamma, \ \Phi^{(n)}] + \frac{2n}{\rho} \Phi^{(n)} d\rho \tag{24}
\]

We apply the integrability condition \(d(\Phi^{(n)})=0\), and use Eq.(24) again to replace \(d\Phi^{(n)}\) where it appears. Then, a lengthy but relatively straightforward calculation, performed with the aid of properties (2) and (4), shows that
\[
d (\rho \ast D \Phi^{(n+1)} - 2 (n+1) \Phi^{(n+1)} dz) = [d(\rho \ast \gamma), \ \Phi^{(n+1)}] - \rho^2 [\omega, \ \Phi^{(n)}].
\]

So, the left-hand side of the above equation vanishes for solutions of the Ernst equation, as it should.

In conclusion, starting with any symmetry characteristic \(\Phi^{(0)}\), we can use the recursion relation (22) to find a double infinity of conserved charges (potentials) \(\Phi^{(n)}\) for \(n = \pm 1, \pm 2, \ldots\). These charges are increasingly nonlocal in \(g\), since they involve integrals of increasing order of expressions containing the function \(g\).

With these charges in hand, we now introduce a complex variable \(\lambda\) (to be identified with a spectral parameter) and construct a function \(\Psi(x^\mu, \lambda)\) having the following series representation for \(\lambda \neq 0\):
\[
\Psi(x^\mu, \lambda) = \sum_{n=-\infty}^{+\infty} \lambda^n \Phi^{(n)}(x^\mu) \tag{25}
\]
We assume that the series (25) converges to the function \( \Psi \) which is single-valued and analytic (as a function of \( \lambda \)) in some annular region centered at the origin of the \( \lambda \)-plane. Hence, Eq. (25) represents a Laurent expansion of \( \Psi \) in this region.

Multiplying the recursion relation (22) by \( \lambda^n \), summing over all integral values of \( n \), and using Eq. (25), we find an exterior equation linear in \( \Psi \):

\[
\rho^* D \Psi - 2 \lambda \Psi_{\lambda} dz = \frac{1}{\lambda} d \Psi
\]

or explicitly,

\[
\rho^* d \Psi + [\rho^* \gamma, \Psi] - 2 \lambda \Psi_{\lambda} dz = \frac{1}{\lambda} d \Psi
\]

Relation (26) is an exterior linearization equation for the Ernst equation, equivalent to a Lax pair. Specifically, the exterior equation (26), linear with respect to \( \Psi \), is integrable for \( \Psi \) when the exterior equations (11) are satisfied.

The proof of this statement is outlined as follows: The integrability condition for solution of Eq. (26) is \( d(d \Psi) = 0 \). So, the exterior derivative of the left-hand side of this equation must vanish. By using algebraic manipulations which are by now familiar to the reader (such as, for example, \( \{\gamma^*, d \Psi\} = -\{\gamma, d \Psi^*\}, d \Psi_{\lambda} dz = d \rho^* d \Psi_{\lambda}, \text{etc.} \)), the above requirement leads to the following exterior equation:

\[
d\rho^* d \Psi + \rho d^* d \Psi + [d(\rho^* \gamma), \Psi] + \rho \{\gamma^*, d \Psi\} - 2 \lambda d \rho^* d \Psi_{\lambda} = 0 \quad (28)
\]

By starring the linear system (27), we find an expression for \( \gamma^* d \Psi \):

\[
\gamma^* d \Psi = -\lambda \rho (d \Psi + [\gamma, \Psi]) + 2 \lambda^2 \Psi_{\lambda} d \rho \quad (29)
\]

Differentiating this with respect to \( \lambda \), we have:

\[
\gamma^* d \Psi_{\lambda} = -\rho (d \Psi + [\gamma, \Psi]) - \lambda \rho (d \Psi_{\lambda} + [\gamma, \Psi_{\lambda}]) + 4 \lambda \Psi_{\lambda} d \rho + 2 \lambda^2 \Psi_{\lambda \lambda} d \rho
\]

Substituting this equation and Eqs. (29) into the integrability condition (28), we finally get:

\[
[d(\rho^* \gamma) - \lambda \rho^2 \omega, \Psi] = 0
\]

where \( \omega = d \gamma + \gamma \gamma \). The above relation is valid independently of \( \Psi \) and \( \lambda \) if \( d(\rho^* \gamma) = 0 \) and \( \omega = 0 \), i.e., for solutions of the Ernst equation. This proves that the integrability of the exterior equation (26) for \( \Psi \) is indeed dependent upon the satisfaction of the Ernst equation.

In component form, Eq. (26) is written as a pair of linear first-order PDEs for \( \Psi \):

\[
\rho D_{\rho} \Psi - 2 \lambda \Psi_{\lambda} = \frac{1}{\lambda} \Psi_{z}
\]

\[
\rho D_{\rho} \Psi = -\frac{1}{\lambda} \Psi_{\rho}
\]

The reader is invited to show that the integrability of system (30) for \( \Psi \) requires that equation (5) is satisfied (see also [8]). Thus, (30) represents a Lax pair for the Ernst equation. In fact, this pair is equivalent to that found by different means in [8]. What we have shown is that this system may actually be constructed by a remarkably straightforward process, by starting with the symmetry condition of the field equation.
5. Connection to Other Linear Systems

It can be shown (see [8,3]) that, by solving the linear system (30) for $\Psi$, for a given solution $g$ of the Ernst equation, one simultaneously obtains an infinitesimal "hidden" symmetry of this equation, given by the expression

$$\delta g = \frac{\alpha}{2\pi i} \int_C \frac{d\lambda}{\lambda} \left( g \Psi(x^\mu, \lambda) + \Psi^T(x^\mu, \lambda) g \right)$$

(31)

where $\alpha$ is an infinitesimal parameter, $C$ is a positively oriented, closed contour around the origin of the $\lambda$-plane, and $\Psi^T$ denotes the transpose of the matrix $\Psi$. (Here, $g$ is assumed to conform to the physical restrictions of being real, symmetric, and of unit determinant. Moreover, $\Psi$ is required to be traceless and to assume real values when $\lambda$ is confined to the real axis. Then, the new solution $g' = g + \delta g$ obeys the same physical restrictions as $g$.) Since solutions of the system (30) [or equivalently, the exterior linearization equation (26)] are of importance in this regard, any mechanism for producing as many solutions as possible would be useful. We now exhibit a simple transformation which maps solutions of (a form of) the Belinski-Zakharov (B-Z) linear system [9] into solutions of our linearization equation (26).

We recall the exterior linearization equation (27):

$$\rho (\ast d\Psi + [\ast\gamma, \Psi]) - 2\lambda\Psi_\lambda dz = \frac{1}{\lambda} d\Psi$$

(32)

where $\Psi$ conforms to the physical conditions mentioned in the previous paragraph; namely, $\text{tr}\Psi=0$ and $\Psi(x^\mu,\lambda^*)=\Psi^*(x^\mu,\lambda)$ (the asterisk here denotes complex conjugation). On the other hand, a variant form of the B-Z linear system, adapted to the particular form of our equations, is the following:

$$\rho (\ast d\Phi + \ast\gamma \Phi) - 2\lambda\Phi_\lambda dz = \frac{1}{\lambda} d\Phi$$

(33)

Let $\Phi(g;\lambda)$ be a non-singular solution of the exterior equation (33) for some solution $g$ of the Ernst equation. We assume that $\Phi$ becomes real for real values of $\lambda$. Consider now the function $\Psi(g;\lambda)$ given by

$$\Psi = \Phi T\Phi^{-1}$$

(34)

where $T$ is an arbitrary traceless matrix function of the form

$$T = F \left( z - \frac{\lambda^2}{2} + \frac{1}{2\lambda} \right)$$

(35)

subject to the condition that $F$ be real-valued for real values of $\lambda$. It may then be proven that $\Psi(g;\lambda)$ is a solution of the linearization equation (32).

Although only a subset of the entirety of solutions of Eq.(32) can be produced in this fashion, the transformation (34)-(35) is an effective way of taking advantage of our knowledge regarding the B-Z formulation for the purpose of finding hidden symmetries of the Ernst equation.
Our method for finding a linear system and an infinite number of nonlocal conserved currents for the Ernst equation is closely related to that of Nakamura [13]. In the latter case, the Lax pair does not contain derivative terms with respect to the spectral parameter. Moreover, the infinite set of conservation laws is accompanied by a corresponding infinite set of nonlocal symmetries, which is not the case with our method for the Ernst equation but which is the case with regard to another familiar nonlinear system, the self-dual Yang-Mills (SDYM) equation. To achieve these extra characteristics, however, one has to perform an analytic continuation of \( g(\rho, z) \) into complex space and introduce more independent variables. In this way the Ernst equation transforms into a reduced form of the SDYM equation, and the mathematical treatments of these two systems become quite similar.

Summary

In this article we have pursued our study of the relation between symmetry and integrability characteristics of the Ernst equation. Taking advantage of the conservation-law form of the symmetry condition, we have inductively produced a double infinity of nonlocal conserved charges by means of a recursion relation. These charges were then used as Laurent coefficients in an infinite series whose terms involve powers (both positive and negative) of a complex “spectral” parameter. Within its domain of convergence, this series represents a function \( \Psi \) which is seen to satisfy a certain linear system, the integrability of which for \( \Psi \) is possible in view of the Ernst equation. Finally, we have presented a simple transformation which maps all solutions of the Belinski-Zakharov Lax pair [9] into solutions of our linear system, and we have compared our results to those of Nakamura [13]. Our formalism was developed in the language of differential forms and exterior calculus, which allowed us to present our equations in a more compact, as well as a more elegant form.

It is remarkable that integrability properties of the Ernst equation, such as the existence of Lax pairs and an infinite number of conservation laws, can be derived in a straightforward way by performing rather natural manipulations on the symmetry condition. This characteristic, which is also observed in the case of the SDYM equation, reveals a profound, non-Noetherian connection between symmetry and integrability. It will be further explored in future publications.

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Plane Symmetric Viscous Fluid Universe in Lyra Geometry

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Abstract: A new class of plane-symmetric homogeneous cosmological models for viscous fluid distribution is obtained in the context of Lyra’s geometry. We have obtained two types of solutions by considering the uniform as well as time dependent displacement field. To get the deterministic solutions of Einstein’s modified field equations, the free gravitational field is assumed to be of type D which is of the next order in the hierarchy of Petrov classification. It has been found that the displacement vector $\beta$ behaves like cosmological term $\Lambda$ in the normal gauge treatment and the solutions are consistent with the observations. The displacement vector $\beta(t)$ affects entropy. Some physical and geometric properties of the models are discussed.

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1. Introduction and Motivations

In 1917 Einstein introduced the cosmological constant into his field equations of general relativity in order to obtain a static cosmological model since, as is well known, without the cosmological term his field equations admit only non-static solutions. After the discovery of the red-shift of galaxies and explanation thereof, Einstein regretted for the introduction of the cosmological constant. Recently, there has been much interest in the cosmological term in context of quantum field theories, quantum gravity, supergravity theories, Kaluza-Klein theories and the inflationary-universe scenario. Shortly after Einstein’s general theory of relativity, Weyl [1] suggested the first so-called unified field theory based on a generalization of Riemannian geometry. With its backdrop, it would seem more appropriate to call Weyl’s theory a geometrized theory of gravitation.
and electromagnetism (just as the general theory was a geometrized theory of gravitation only), instead a unified field theory. It is not clear as to what extent the two fields have been unified, even though they acquire (different) geometrical significance in the same geometry. The theory was never taken seriously in as much as it was based on the concept of non-integrability of length transfer; and, as pointed out by Einstein, this implies that spectral frequencies of atoms depend on their past histories and therefore have no absolute significance. Nevertheless, Weyl's geometry provides an interesting example of non-Riemannian connections, and Folland [2] has given a global formulation of Weyl manifolds clarifying considerably many of Weyl's basic ideas thereby.

In 1951, Lyra [3] proposed a modification of Riemannian geometry by introducing a gauge function into the structure-less manifold, as a result of which the cosmological constant arises naturally from the geometry. This bears a remarkable resemblance to Weyl's geometry. But in Lyra's geometry, unlike that of Weyl, the connection is metric preserving as in Riemannian; in other words, length transfers are integrable. Lyra also introduced the notion of a gauge and in the “normal” gauge the curvature scalar is identical to that of Weyl. In consecutive investigations Sen [4], Sen and Dunn [5] proposed a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein field equations based on Lyra's geometry. It is, thus, possible [4] to construct a geometrized theory of gravitation and electromagnetism much along the lines of Weyl’s “unified” field theory, however, without the inconvenience of non-integrability length transfer.

Halford [6] has pointed out that the constant vector displacement field \( \phi_i \) in Lyra’s geometry plays the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. It is shown by Halford [7] that the scalar-tensor treatment based on Lyra's geometry predicts the same effects within observational limits as the Einstein’s general theory. Several authors Sen and Vanstone [8], Bhamra [9], Karade and Borikar [10], Kalyanshetti and Wagmode [11], Reddy and Innaiah [12], Beesham [13], Reddy and Venkateswarlu [14], Soleng [15], have studied cosmological models based on Lyra’s manifold with a constant displacement field vector. However, this restriction of the displacement field to be constant is merely one for convenience and there is no a priori reason for it. Beesham [16] considered FRW models with time dependent displacement field. He has shown that by assuming the energy density of the universe to be equal to its critical value, the models have the \( k = -1 \) geometry. Singh and Singh [17]– [20], Singh and Desikan [21] have studied Bianchi-type I, III, Kantowski-Sachs and a new class of cosmological models with time dependent displacement field and have made a comparative study of Robertson-Walker models with constant deceleration parameter in Einstein’s theory with cosmological term and in the cosmological theory based on Lyra’s geometry. Soleng [22] has pointed out that the cosmologies based on Lyra’s manifold with constant gauge vector \( \phi \) will either include a creation field and be equal to Hoyle’s creation field cosmology [20]– [24] or contain a special vacuum field, which together with the gauge vector term, may be considered as a cosmological term. In the latter case the solutions are equal to
the general relativistic cosmologies with a cosmological term.

Recently, Pradhan et al. [25], Casama et al. [26], Rahaman et al. [27], Bali and Chandnani [28], Kumar and Singh [29], Singh [30] and Rao, Vinutha and Santhi [31] have studied cosmological models based on Lyra’s geometry in various contexts. With these motivations, in this paper, we have obtained exact solutions of Einstein’s field equations for viscous fluid distribution in plane symmetric homogeneous space-time within the frame work of Lyra’s geometry for uniform and time varying displacement vector. This paper is organized as follows. In Section 1 the motivation for the present work is discussed. The metric and the field equations are presented in Section 2, in Section 3 the solution of field equations. The Section 4 describes the solution of the first model. The Subsections 4.1 and 4.2 deal with the solutions for uniform displacement field (\( \beta = \beta_0 \), constant) and time varying displacement field (\( \beta = \beta(t) \)). Subsections 4.2.1, 4.2.2 and 4.2.3 describe the solutions of Empty Universe, Zeldovich Universe and Radiating Universe respectively whereas Subsection 4.3 deals with the physical and geometric aspects of the first model. The Section 5 describes the solution of the second model. The Subsections 4.1 and 4.2 deal with the solutions for uniform displacement field (\( \beta = \beta_0 \), constant) and time varying displacement field (\( \beta = \beta(t) \)) of the second model. Subsections 5.2.1, 5.2.2 and 5.2.3 describe the solutions of Empty Universe, Zeldovich Universe and Radiating Universe respectively whereas Subsection 5.3 deals with the physical and geometric aspects of the second model. Finally, in Section 6 discussion and concluding remarks are given.

2. The metric and field equations

We consider the metric in the form of Marder [32]

\[
ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2,
\]

where the metric potentials \( A, B \) and \( C \) are functions of \( t \) alone. This ensures the model to be spatially homogeneous. This is a transform form of the metric of Bianchi type I spacetime in comoving coordinates which has been studied by a number of authors e.g. (Heckmann and Schucking [33], Thorne [34] and Roy and Prakash [35]).

The energy-momentum tensor for a viscous fluid distribution is given by Landau and Lifshitz [36]

\[
T^i_j = (\rho + p)v_i v^j + pg^j_i - \eta(v_i^j v^j_i + v^i v^j v_i^j v^j_i + v_i v^j v^j_i) - \left( \xi - \frac{2}{3}\eta \right) v^j_i (g^i_j + v_i v^j).
\]

Here \( \rho, p, \eta \) and \( \xi \) are energy density, isotropic pressure, the coefficient of shear viscosity and bulk viscous coefficient respectively and \( v^i \) is the flow vector satisfying the relation

\[
g_{ij}v^iv^j = -1.
\]
The semicolon (;) indicates covariant differentiation. We choose the coordinates to be comoving, so that \( v^1 = v^2 = v^3 = 0 \) and \( v^4 = \frac{1}{A} \).

The field equations, in normal gauge for Lyra’s manifold, obtained by Sen [4] as

\[
R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -8\pi G T_{ij},
\]

where \( \phi_i \) is the displacement field vector defined as

\[
\phi_i = (0, 0, 0, \beta),
\]

where \( \beta \) is either a constant or a function of \( t \). The other symbols have their usual meaning as in Riemannian geometry.

For the line-element (1), the field Eq. (4) with Eqs. (2) and (5) lead to the following system of equations

\[
\frac{1}{A^2} \left[ \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A} \dot{C}}{AC} - \frac{\dot{B} \dot{C}}{BC} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right] - \frac{3}{4} \beta^2 = 8\pi \left[ p - 2\eta \frac{\dot{A}}{A^2} - \left( \xi - \frac{2}{3} \eta \right) v^l \right],
\]

\[
\frac{1}{A^2} \left[ \frac{\dot{A}^2}{A} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right] - \frac{3}{4} \beta^2 = 8\pi \left[ p - 2\eta \frac{\dot{B}}{AB} - \left( \xi - \frac{2}{3} \eta \right) v^l \right],
\]

\[
\frac{1}{A^2} \left[ \frac{\dot{A}^2}{A} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right] - \frac{3}{4} \beta^2 = 8\pi \left[ p - 2\eta \frac{\dot{C}}{AC} - \left( \xi - \frac{2}{3} \eta \right) v^l \right],
\]

\[
\frac{1}{A^2} \left[ \frac{\dot{A} B}{AB} + \frac{\dot{A} C}{AC} + \frac{\dot{B} \dot{C}}{BC} \right] + \frac{3}{4} \beta^2 = 8\pi \rho.
\]

Here, and also in the following expressions a dot indicates ordinary differentiation with respect to \( t \).

3. Solutions of the Field Equations

Equations (6)-(9) are four equations in six unknowns \( A, B, C, p, \rho, \) and \( \beta \). The coefficients of viscosity \( \eta \) and \( \xi \) are taken as constants. Equations (6)-(9) are not independent, but are related by the contracted Bianchi identities. In the present case they lead to the single condition

\[
\frac{dp}{dt} + (p + \rho) \ln(ABC) - \left( \rho - \frac{2}{3} \eta \right) \frac{1}{A} \left( \frac{d}{dt} \ln(ABC) \right)^2 - 2\eta \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} = 0.
\]
For complete solutions of equations (6)-(9), we need two extra conditions. The research on exact solutions is based on some physically reasonable restrictions used to simplify the Einstein equations. However we proceed from a different consideration. Although the distribution of matter at each point determines the nature of expansion in the model, the latter is also affected by the free gravitational field through its effect on the expansion, vorticity and shear in the fluid flow. A prescription of such a field may therefore be made on an \textit{a priori} basis. The cosmological models of Friedman Robertson Walker, as well as the universe of Einstein and de Sitter, have vanishing free gravitational fields. Here we choose the free gravitational field to be of type D which is of the next order in the hierarchy of Petrov classification. This requires that either

\[(a) \quad C^{12}_{12} = C^{13}_{13},\]

or

\[(b) \quad C^{12}_{12} = C^{23}_{23}.\]

Conditions \((a)\) and \((b)\) are identically satisfied if \(B = C\) and \(A = C\) respectively. However, we shall assume \(A, B, C\) to be unequal on account of the supposed anisotropy.

From equations (6) and (7) we obtain

\[\frac{d}{dt} \left( \frac{\dot{A}}{A} \right) + \frac{\dot{A}}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC} = 16\pi\eta A \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right).\]

(11)

Also from equations (7) and (8) we obtain

\[\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 16\pi\eta A \left( \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right).\]

(12)

In the following Sections 4 and 5, we have derived two models of the universe based on the above two conditions \((a)\) and \((b)\) respectively.

### 4. The First Model

The condition

\[C^{12}_{12} = C^{13}_{13}\]

leads to

\[\frac{\dot{B}}{B} - \frac{\dot{C}}{C} + 2\frac{\dot{A}}{A} \left( \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right) = 0.\]

(14)

Equations (12) and (14) lead to

\[A = \frac{1}{8\pi\eta t + a},\]

(15)

where \(a\) is a constant of integration. From equations (14) and (15) we obtain

\[\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = -\frac{16\pi\eta}{(8\pi\eta t + a)} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right).\]

(16)
which on integration gives
\[ \dot{B}C - B\dot{C} = \frac{b}{(8\pi \eta t + a)^2}, \]
(17)
where \( b \) is an integrating constant. From equations (11) and (15) we get
\[ \left( \frac{8\pi \eta}{8\pi \eta t + a} \right)^2 + \frac{8\pi \eta}{(8\pi \eta t + a)} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{16\pi \eta}{(8\pi \eta t + a)} \frac{\dot{B}}{B} + \frac{\dot{B}}{B} + \frac{\dot{B}C}{BC} = 0. \]
(18)
From equations (17) and (18) we obtain
\[ B = \frac{K(Lt + M)}{(8\pi \eta t + a)^{\frac{1}{2} + \frac{\xi}{2}}} \]
(19)
and
\[ C = \frac{(Lt + M)^{\frac{1}{2} - \frac{\xi}{2}}}{K(8\pi \eta t + a)}, \]
(20)
where \( K, L \) and \( M \) are constants of integration.
Hence, the geometry of the space time (1) takes the form
\[ ds^2 = \frac{1}{(8\pi \eta t + a)^2} (dx^2 - dt^2) + \frac{K^2(Lt + M)^{\frac{1}{2} + \frac{\xi}{2}}}{(8\pi \eta t + a)^2} dy^2 + \frac{(Lt + M)^{\frac{1}{2} - \frac{\xi}{2}}}{K^2(8\pi \eta t + a)^2} dz^2. \]
(21)
For the specification of displacement vector \( \beta \) within the framework of Lyra geometry and for realistic models of physical importance, we consider following two cases:

4.1 When \( \beta \) is a constant i.e. \( \beta = \beta_0 \) (constant)

Using Eqs. (15), (19) and (20) in Eqs. (6) - (9), the expressions for pressure \( p \) and density \( \rho \) for the model (21) are given by
\[ 8\pi p = -192\pi^2 \eta^2 + \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{4(Lt + M)^2} + \frac{32\pi \eta L(8\pi \eta t + a)}{3(Lt + M)} \]
\[ -8\pi \xi \left[ 24\pi \eta - \frac{(8\pi \eta t + a)L}{(Lt + M)} \right] - \frac{3}{4} \beta_0^2, \]
(22)
\[ 8\pi \rho = 192\pi^2 \eta^2 - \frac{16\pi \eta L(8\pi \eta t + a)}{(Lt + M)} + \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{4(Lt + M)^2} + \frac{3}{4} \beta_0^2. \]
(23)
From Eq. (23) it is observed that for \( t > \frac{(a-m)}{(L-8\pi \eta)} \), the energy density decreases with time and is always positive.

The reality conditions (Ellis [37])
\[ (i)\rho + p > 0, \ (ii)\rho + 3p > 0, \]
lead to
\[ \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta t + a)}{3(Lt + m)} > 192\pi^2 \xi \eta, \]
(24)
and
\[
\frac{(L^2 - b^2)(8\pi \eta t + a)^2}{(Lt + M)^2} + \frac{8\pi L(3\xi + 2\eta)(8\pi \eta t + a)}{(Lt + m)} > 192\pi^2 \eta(3\xi + 2\eta) + \frac{3}{2}\beta_0^2,
\]
respectively.

The dominant energy conditions (Hawking and Ellis [38])
\[(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,\]
lead to
\[
24\pi\eta(2\eta + \xi) + \frac{3}{16\pi}\beta_0^2 \geq \frac{(\xi + 6\eta)(8\pi \eta t + a)L}{(Lt + M)},
\]
and
\[
\frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta t + a)}{3(Lt + m)} \geq 192\pi^2 \xi \eta,
\]
respectively. The conditions (25) and (26) impose a restriction on \(\beta_0\).

4.2 When \(\beta\) is a function of \(t\) i.e. \(\beta = \beta(t)\)

In this case to find the explicit value of displacement field \(\beta(t)\), we assume that the fluid obeys an equation of state of the form
\[p = \gamma \rho,\]
where \(\gamma(0 \leq \gamma \leq 1)\) is a constant.

Using Eqs. (15), (19), (20) and (28) in Eqs. (6) - (9), we obtain the expressions for energy density \(\rho\) and displacement vector \(\beta(t)\) for the model (21) as
\[
8\pi(1 + \gamma)\rho = \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta t + a)}{3(Lt + M)} - 192\pi^2 \xi \eta,
\]
\[
(1 + \gamma)\beta_0^2(t) = \frac{(L^2 - b^2)(8\pi \eta t + a)^2(1 - \gamma)}{3(Lt + M)^2} + \frac{32\pi L(8\pi \eta t + a)(3\xi + 2\eta(3\gamma + 2))}{9(Lt + M)} - 256\pi^2 \eta(\xi + \eta(1 + \gamma)).
\]
From Eqs. (29) and (30), we observe that for \(t > \frac{(a-M)}{(2\pi \eta)}\), the energy density \(\rho(t)\) and displacement vector \(\beta(t)\) are decreasing function of time and are always positive.
4.2.1 Empty Universe

Let us consider $\gamma = 0$ in Eq. (28) which leads $p = 0$. Thus, from Eqs. (6) - (8) we obtain

$$\beta^2(t) = \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{3(Lt + M)^2} + \frac{32\pi L(8\pi \eta t + a)(9\xi + 4\eta)}{3(Lt + M)} - 256\pi^2 \eta^2 (\xi + \eta).$$  \hspace{1cm} (31)

Halford [6] has pointed out that the constant vector displacement field $\phi_i$ in Lyra’s geometry plays the role of cosmological constant $\Lambda$ in the normal general relativistic treatment. From Eq. (31), it is observed that for $t > \frac{(a-M)}{(L-8\pi \eta)}$, the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford as well as with the recent observations [39, 40] leading to the conclusion that $\Lambda(t)$ is a decreasing function of $t$.

4.2.2 Zeldovich Universe

Let us consider $\gamma = 1$ in Eq. (28) which yields $p = \rho$. Therefore, in this case, the expressions for $p$, $\rho$ and $\beta(t)$ are given by

$$16\pi p = 16\pi \rho = \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(8\pi \eta t + a)(3\xi - 2\eta)}{3(Lt + M)} - 192\pi^2 \xi \eta,$$  \hspace{1cm} (32)

$$\beta^2(t) = \frac{16\pi L(8\pi \eta t + a)(3\xi + 10\eta)}{9(Lt + M)} - 128\pi^2 \eta (\xi + \eta).$$  \hspace{1cm} (33)

From Eqs. (32) and (33), we observe that for $t > \frac{(a-M)}{(L-8\pi \eta)}$, the energy density $\rho(t)$ and displacement vector $\beta(t)$ are decreasing function of time and are always positive. The reality condition (Ellis [37])

$$(i) \rho + p > 0, \quad (ii) \rho + 3p > 0,$$

lead to

$$\frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(8\pi \eta t + a)(3\xi - 2\eta)}{3(Lt + M)} > 192\pi^2 \eta^2 \xi \eta.$$  \hspace{1cm} (34)

4.2.3 Radiating Universe

Let us consider $\gamma = \frac{1}{3}$ in Eq. (28) which gives $\rho = 3p$. Hence, in this case, the expressions for $p$, $\rho$ and $\beta(t)$ are obtained as

$$8\pi \rho = 24\pi p = \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{8(Lt + M)^2} + \frac{2\pi L(3\xi - 2\eta)(8\pi \eta t + a)}{9(Lt + M)} - 48\pi^2 \xi \eta,$$  \hspace{1cm} (35)

$$\beta^2(t) = \frac{(L^2 - b^2)(8\pi \eta t + a)^2}{6(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta t + a)}{(Lt + M)} - 64\pi^2 \eta (3\xi + 2\eta).$$  \hspace{1cm} (36)

From Eqs. (35) and (36), it is observed that for $t > \frac{(a-M)}{(L-8\pi \eta)}$, the energy density $\rho(t)$ and the displacement vector $\beta(t)$ is decreasing function of time and always positive. Thus we see that $\beta(t)$ behaves like cosmological term $\Lambda$. 
The reality conditions (Ellis \[37\])

\[(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,\]

and the dominant energy conditions (Hawking and Ellis \[38\])

\[(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,\]

lead to

\[
\frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta + a)}{9(Lt + M)} > 192\pi^2 \xi \eta, \tag{37}
\]

and

\[
\frac{(L^2 - b^2)(8\pi \eta t + a)^2}{2(Lt + M)^2} + \frac{8\pi L(3\xi - 2\eta)(8\pi \eta + a)}{9(Lt + M)} \geq 192\pi^2 \xi \eta, \tag{38}
\]

respectively.

### 4.3 Some Geometric Properties of First Model

We shall now give the expressions for kinematic quantities and components of conformal curvature tensor. With regard to the kinematical properties of the velocity vector \(v^i\) in the metric (21), a straightforward calculation leads to the expressions for expansion (\(\theta\)), deceleration parameter \(q\), proper volume \(V^3\) and shear \((\sigma_{ij})\) of the fluid:

\[
\theta = \frac{(8\pi \eta t + a)L}{(Lt + M)} - 24\pi \eta, \tag{39}
\]

\[
q = -1 - \frac{\frac{256\pi^2 \eta^2}{(8\pi \eta t + a)^2} - \frac{L^2}{(Lt + M)^2}}{\frac{(32\pi \eta)^2}{(8\pi \eta t + a)^2} - \frac{64\pi \eta L}{(8\pi \eta t + a)(Lt + M)}}, \tag{40}
\]

\[
V^3 = \sqrt{-g} = \frac{(Lt + M)}{(8\pi \eta t + a)^{\frac{1}{3}}}, \tag{41}
\]

\[
\sigma_{11} = -\frac{L(Lt + M)^{-1}}{3(8\pi \eta t + a)}, \tag{42}
\]

\[
\sigma_{22} = \frac{K^2(L + 3b)(Lt + M)^{\frac{5}{2}}}{6(8\pi \eta t + a)}, \tag{43}
\]

\[
\sigma_{33} = \frac{(L - 3b)(Lt + M)^{-\frac{1}{2}}}{6K^2(8\pi \eta t + a)}, \tag{44}
\]

and other components of the shear tensor \((\sigma_{ij})\) being zero. Hence

\[
\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \left(\frac{(L^2 + 3b^2)}{12}\right) \left(\frac{8\pi \eta t + a}{Lt + M}\right)^2. \tag{45}
\]

From Eqs. (39) and (45) we obtain

\[
\frac{\sigma}{\bar{\theta}} = \frac{\sqrt{(L^2 + 3b^2)}(8\pi \eta t + a)}{(8\pi \eta t + a)L - 24\pi \eta (Lt + M)}. \tag{46}
\]
The non-vanishing components of the conformal curvature tensor are

\[ C^{12}_{12} = C^{13}_{13} = -\frac{1}{2} C^{23}_{23} = \left( \frac{L^2 - b^2}{12} \right) \left( \frac{8\pi\eta t + a}{Lt + M} \right)^2. \]  

(47)

For large \( t \), we find

\[ C^{23}_{23} = -\frac{32}{3} \pi^2 \eta^2 \left( 1 - \frac{b^2}{L^2} \right), \]  

(48)

and

\[ \sigma^2 = \frac{16}{3} \pi^2 \eta^2 \left( 1 + \frac{3b^2}{L^2} \right). \]  

(49)

Here we find

\[ C^{12}_{12} + C^{13}_{13} + C^{23}_{23} = 0. \]  

(50)

The rotation \( \omega \) is identically zero. The rate of expansion \( H_i \) in the directions of \( x, y \) and \( z \) are given by

\[ H_x = \frac{\dot{A}}{A} = -\frac{8\pi\eta}{(8\pi\eta t + a)}, \]  

(51)

\[ H_y = \frac{\dot{B}}{B} = \left( \frac{1}{2} + \frac{b}{2L} \right) \frac{L}{(Lt + M)} - \frac{8\pi\eta}{(8\pi\eta t + a)}, \]  

(52)

\[ H_z = \frac{\dot{C}}{C} = \left( \frac{1}{2} - \frac{b}{2L} \right) \frac{L}{(Lt + M)} - \frac{8\pi\eta}{(8\pi\eta t + a)}. \]  

(53)

Now since

\[ \int_{t_0}^{t} \frac{dt}{V(t)} = \int_{t_0}^{t} \left( \frac{8\pi\eta t + a}{Lt + M} \right)^\frac{2}{3} dt. \]  

(54)

This is convergent integral hence particle horizon exists.

The models represent shearing, non-rotating and Petrov Type D universe in general, in which the flow is geodetic. It is also observed that the viscosity prevents the free gravitational field as well as the shear from withering away. It is also obvious from (39) that the effect of viscosity is to retard expansion of the model. Since \( \lim_{t \to \infty} \frac{\sigma}{\dot{\varphi}} \neq 0 \), the models do not approach isotropy for large values of \( t \). It is observed from Eq. (40) which implies an accelerating model of the universe. Recent observations of type Ia supernovae \cite{39, 40} reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range \(-1 < q < 0\). It follows that our models of the universe are consistent with recent observations. For

\[ 256\pi^2 \eta^2 (Lt + M)^2 = L^2 (8\pi\eta t + a)^2 \]  

(55)

the deceleration parameter \( q \) approaches the value \((-1)\) as in the case of de-Sitter universe.
5. The Second Model

The condition
\[ C_{12}^{12} = C_{23}^{23}, \]  
leads to
\[ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \frac{\ddot{C}}{C} - \frac{B\dot{C}}{BC} + \frac{\dot{A}}{A} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right). \]  
Equations (11), (12) and (57) lead to
\[ \frac{\dot{B}}{B} = -8\pi\eta A. \]  
From equations (12) and (58) we obtain
\[ \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} = 2 \frac{\dot{B}}{B} (\frac{\dot{B}}{B} - \frac{\dot{C}}{C}), \]  
which on integration leads
\[ C = B(k_1 - kt), \]  
where \( k \) and \( k_1 \) are constants of integration. From equations (57) and (60) we get
\[ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \left( \frac{k}{k_1 - kt} \right), \]  
which on integration gives
\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_2}{k_1 - kt}, \]  
where \( k_2 \) is an integrating constant. From equations (58) and (62) we obtain
\[ A = \left[ \frac{8\pi\eta(k_1 - kt)}{k_2 - k} + k_3(k_1 - kt)^{\frac{k_2}{\eta} - 1} \right]^{-1}, \]  
\( k_3 \) being a constant of integration. From equations (58) and (63) we obtain
\[ B = k_4 \left[ \frac{(k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{\eta} - 1}}{8\pi\eta + (k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{\eta} - 1}} \right], \]  
where \( k_4 \) is a constant of integration. Also, from equations (60) and (64) we obtain
\[ C = k_4 \left[ \frac{(k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{\eta}}}{8\pi\eta + (k_2 - k)k_3(k_1 - kt)^{\frac{k_2}{\eta}} - 1} \right]. \]  
By a suitable transformation of coordinates the metric of this model can be put into the form
\[ ds^2 = \left( \frac{8\pi\eta}{\alpha - 1} T + \ell T^\alpha \right)^{-2} \left[ dX^2 - dT^2 + T^{2\alpha} dY^2 + T^{2(\alpha + 1)} dZ^2 \right], \]  
where \( \alpha \) and \( \ell \) are arbitrary constants.

For the specification of displacement vector \( \beta \) within the framework of Lyra geometry and for realistic models of physical importance, we consider following two cases:
5.1 When β is a constant i.e. β = β₀ (constant)

Using Eqs. (63), (64) and (65) in Eqs. (6) and (9) the expressions for pressure p and density ρ for the model (66) are given by

\[ 8\pi p = 64\pi^2 \eta^2 \left( \frac{2 - \alpha}{\alpha - 1} \right) + 16\pi\eta\ell T^{\alpha-1} + \alpha(\alpha - 1)\ell^2 T^{2(\alpha - 1)} \]

\[-\left( \frac{8\pi\eta}{\alpha - 1} + \alpha\ell T^{\alpha-1} \right)^2 - \frac{16}{3}\pi\eta(\alpha + 2) \left( \frac{8\pi\eta}{\alpha - 1} + \alpha\ell T^{\alpha-1} \right) \]

\[ + 8\pi\xi[(\alpha - 1)\ell T^{\alpha-1} - 16\pi\eta] - \frac{3}{4}\beta^2_0, \quad (67) \]

\[ 8\pi\rho = 64\pi^2 \eta^2 \left[ 1 - \frac{\alpha}{(\alpha - 1)^2} \right] + 16\pi\eta\ell \left( 1 - \frac{\alpha}{\alpha - 1} \right) T^{\alpha-1} \]

\[-\alpha\ell^2 T^{2(\alpha - 1)} + \frac{3}{4}\beta^2_0. \quad (68) \]

It is observed from Eq. (68) that for α < 0, the energy density ρ is a decreasing function of t and is always positive.

The reality conditions (Ellis [37])

\[ (i)\rho + p > 0, \quad (ii)\rho + 3p > 0, \]

lead to

\[ \left[ \xi\ell(\alpha - 1) + \frac{2\eta\alpha(\alpha - 2)}{(\alpha - 1)} \right] T^{\alpha-1} - \frac{\alpha\ell^2}{4\pi} T^{2(\alpha - 1)} \]

\[ > 16\pi\xi\eta + \frac{8\pi\eta^2(2\alpha^2 + 5\alpha - 4)}{3(\alpha - 1)^2} \quad (69) \]

and

\[ \left[ 36\xi\ell(\alpha - 2) - \frac{2\eta(\alpha^2 + 4\alpha - 2)}{(\alpha - 1)^2} \right] T^{\alpha-1} - \frac{3\alpha^2\ell^2}{8\pi} T^{2(\alpha - 1)} \]

\[ > 48\pi\xi\eta - \frac{32\pi\eta^2(2\alpha - 3)}{(\alpha - 1)^2} - \frac{3}{16\pi} \beta^2_0, \quad (70) \]

respectively.

The dominant energy conditions (Hawking and Ellis [38])

\[ (i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0, \]

lead to

\[ \left[ \frac{10\eta(5\alpha^2 - 4\alpha + 2)}{(\alpha - 1)} - \xi(\alpha - 1) \right] \geq -\frac{64}{3}\pi\eta^2 - 16\pi\xi\eta - \frac{3}{16\pi} \beta^2_0, \quad (71) \]

and

\[ \left[ \xi(\alpha - 1) + \frac{2\eta\alpha(\alpha - 2)}{(\alpha - 1)} \right] T^{\alpha-1} - \frac{\alpha\ell^2}{4\pi} T^{2(\alpha - 1)} \]

\[ \geq 16\pi\xi\eta + \frac{8\pi\eta^2(2\alpha^2 + 5\alpha - 4)}{3(\alpha - 1)^2}, \quad (72) \]

respectively. The conditions (70) and (71) impose a restriction on β₀.
5.2 When $\beta$ is a function of $T$ i.e. $\beta = \beta(T)$

In this case to find the explicit value of displacement field $\beta(t)$, we assume that the fluid obeys an equation of state given by (28). Using Eqs. (63) - 65) and (28) in Eqs. (6) - (9), we obtain $\rho$ and $\beta$ for the model (66)

$$8\pi(1 + \gamma)\rho = -2\alpha \ell^2 T^{2(\alpha - 1)} - \frac{8\pi\ell}{3(\alpha - 1)} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha - 1}$$

$$- \frac{128\pi^2 \eta^2 (\alpha^2 + \alpha + 1)}{3(\alpha - 1)^2} - 128\pi^2 \xi \eta, \quad (73)$$

and

$$\frac{3}{4}(1 + \gamma)\beta^2 = 2\alpha \ell^2 [4\pi \xi (1 + \gamma) - 1] T^{2(\alpha - 1)} + \frac{8\pi\ell}{(\alpha - 1)} \left[ \xi(\alpha - 1)^2 - 2\eta(\alpha + 1) \right] T^{\alpha - 1} - \frac{64\pi^2 \eta^2 (2\alpha^2 - \alpha + 2)}{3(\alpha - 1)^2} - 64\pi^2 \eta \{ \eta(1 + \gamma) + 2\xi \}. \quad (74)$$

We consider three following cases of physical interest.

5.2.1 Empty Universe

In this case $p = 0$. Hence, we obtain the expression for $\beta$ as

$$\frac{3}{4}\beta^2 = 2\alpha \ell^2 (4\pi \xi - 1) T^{2(\alpha - 1)} + \frac{8\pi\ell}{(\alpha - 1)} \left[ \xi(\alpha - 1)^2 - 2\eta(\alpha + 1) \right] T^{\alpha - 1} - \frac{64\pi^2 \eta^2 (2\alpha^2 - \alpha + 2)}{3(\alpha - 1)^2} - 64\pi^2 \eta \{ \eta(1 + \gamma) + 2\xi \}. \quad (75)$$

From above equation it is observed that the displacement vector $\beta$ is a decreasing function of time for $\alpha < 0$.

5.2.2 Zeldovice Universe

In this case we have $\rho = p$. Therefore, in this case, the expressions for $\beta(t)$ is given by

$$\frac{3}{2}\beta^2 = 2\alpha \ell (8\pi \xi - 1) T^{2(\alpha - 1)} + \frac{8\pi\ell}{(\alpha - 1)} \left[ \xi(\alpha - 1)^2 - 2\eta(\alpha + 1) \right] T^{\alpha - 1}$$

$$- \frac{64\pi^2 \eta^2 (2\alpha^2 - \alpha + 2)}{3(\alpha - 1)^2} - 128\pi^2 \eta \{ \eta + \xi \}. \quad (76)$$

From Eq. (76), it is observed that displacement vector $\beta$ is decreasing function of time for $\alpha < 1$. The expressions for pressure $p$ and energy density $\rho$ are given by

$$8\pi p = 8\pi \rho = -2\alpha \ell^2 T^{2(\alpha - 1)} - \frac{8\pi\ell}{3(\alpha - 1)} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha - 1}$$

$$- \frac{128\pi^2 \eta^2 (\alpha^2 + \alpha + 1)}{3(\alpha - 1)^2} - 128\pi^2 \xi \eta. \quad (77)$$
The reality conditions (Ellis [37])

\[(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,\]

lead to

\[-\alpha\ell^2 T^{2(\alpha-1)} - \frac{4\pi\ell}{3(\alpha - 1)} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha-1} > \frac{64\pi^2\eta^2(\alpha^2 + \alpha + 1)}{3(\alpha - 1)^2} + 64\pi^2\xi\eta. \quad (78)\]

5.2.3 Radiating Universe

In this case we have \(\rho = 3p\). From Eqs. (6) - (9), the expressions for \(\rho\), \(p\) and \(\beta(t)\) are obtained as

\[
\rho = 3p = -\frac{3\alpha\ell^2}{16\pi} T^{2(\alpha-1)} - \frac{\ell}{4(\alpha - 1)^2} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha-1} - \frac{4\pi\eta^2(\alpha^2 + \alpha + 1)}{4(\alpha - 1)^2} - 12\pi\xi\eta, \quad (79)
\]

\[
\beta^2 = \frac{2\alpha\ell^2}{3}(16\pi\xi - 3)T^{2(\alpha-1)} + \frac{8\pi\ell}{(\alpha - 1)} \left[ \xi(\alpha - 1)^2 - 2\eta(\alpha + 1) \right] T^{\alpha-1} - \frac{64\pi^2\eta^2(2\alpha^2 - \alpha + 2)}{3(\alpha - 1)} - \frac{128\pi^2\eta^2(2\eta + 3\xi)}{3}. \quad (80)
\]

From Eq. (80), it is observed that displacement vector \(\beta\) is decreasing function of time for \(\alpha < 1\). For \(\alpha < 0\) and \(\ell < 0\), the energy density decreases with time and is always positive.

The reality conditions (Ellis [37])

\[(i)\rho + p > 0, \quad (ii)\rho + 3p > 0,\]

and the dominant energy conditions (Hawking and Ellis [38])

\[(i)\rho - p \geq 0, \quad (ii)\rho + p \geq 0,\]

lead to

\[-\frac{3\alpha\ell^2}{4\pi} T^{2(\alpha-1)} - \frac{\ell}{(\alpha - 1)^2} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha-1} > \frac{16\pi^2\eta^2(\alpha^2 + \alpha + 1)}{(\alpha - 1)^2} + 48\pi\xi\eta, \quad (81)\]

and

\[-\frac{3\alpha\ell^2}{4\pi} T^{2(\alpha-1)} - \frac{\ell\ell}{(\alpha - 1)^2} \left[ 2n(\alpha^2 + 4\alpha + 1) + 3\xi(\alpha - 1)^2 \right] T^{\alpha-1} \geq \frac{16\pi^2\eta^2(\alpha^2 + \alpha + 1)}{(\alpha - 1)^2} + 48\pi\xi\eta, \quad (82)\]

respectively.
5.3 Some Geometric Properties of Second Model

The expressions for the expansion \( \theta \), Hubble parameter \( H \), the magnitude of shear \( \sigma^2 \), deceleration parameter \( q \) and proper volume \( V^3 \) for the model (66) are given by

\[
\theta = 3H = \ell(\alpha - 1)T^{\alpha - 1} - 16\pi\eta, \tag{83}
\]

\[
\sigma^2 = \frac{1}{3}(\alpha^2 + \alpha + 1) \left[ \frac{8\pi\eta}{\alpha - 1} + \ell T^{\alpha - 1} \right]^2, \tag{84}
\]

\[
q = -1 - \frac{1}{\left[ \frac{(2\alpha + 1)}{3T} - \frac{2\ell(\alpha - 1)T^{\alpha - 1}}{3(8\pi\eta + \ell(\alpha - 1)T^{\alpha - 1})} \right]^2 \times \]

\[
\left[ \frac{(2\ell + 1)}{3T^2} + \frac{2\ell(\alpha - 1)^2(8\pi\eta T^{\alpha - 2} + \ell\alpha T^{\alpha - 1})}{3 \left( 8\pi\eta T^{\alpha - 1} + \ell(\alpha - 1)T^{\alpha - 1} \right)^2} \right], \tag{85}
\]

\[
V^3 = \sqrt{-g} = \left[ \frac{8\pi\eta}{(\alpha - 1)} T + \ell T^{\alpha} \right]^{-2} T^{2\alpha + 1}. \tag{86}
\]

The non-vanishing components of the conformal curvature tensor are

\[
C_{12}^{12} = C_{23}^{23} = -\frac{1}{2} C_{13}^{13} = -\frac{1}{3\alpha} \left[ \frac{8\pi\eta}{\alpha - 1} + \ell T^{\alpha - 1} \right]. \tag{87}
\]

Here we also find

\[
C_{12}^{12} + C_{13}^{13} + C_{23}^{23} = 0. \tag{88}
\]

The rotation \( \omega \) is identically zero.

The rate of expansion \( H \) in the directions of x, y and z are given by

\[
H_x = \frac{\dot{A}}{A} = -\left[ \frac{8\pi\eta + \alpha\ell(\alpha - 1)T^{\alpha - 1}}{8\pi\eta T + \ell(\alpha - 1)T^{\alpha}} \right], \tag{89}
\]

\[
H_y = \frac{\dot{B}}{B} = \alpha T \left[ \frac{8\pi\eta + \alpha\ell(\alpha - 1)T^{\alpha - 1}}{8\pi\eta T + \ell(\alpha - 1)T^{\alpha}} \right], \tag{90}
\]

\[
H_z = \frac{\dot{C}}{C} = \frac{(\alpha + 1)}{T} \left[ \frac{8\pi\eta + \alpha\ell(\alpha - 1)T^{\alpha - 1}}{8\pi\eta T + \ell(\alpha - 1)T^{\alpha}} \right]. \tag{91}
\]

The models represent shearing, non-rotating and Petrov Type D universe in general, in which the flow is geodetic. For this model too, it is observed that the effect of viscosity prevents the shear and the free gravitational field from withering away for large value of \( T \). It also retards expansion of the model. Since \( \lim_{T \to \infty} \frac{\sigma}{\theta} \neq 0 \), the models do not approach isotropy for large values of \( T \). It is observed from Eq. (85) which implies an accelerating model of the universe. It follows that our models of the universe are consistent with recent observations [39, 40]. For the critical time \( T_c \) given by

\[
\frac{(2\ell + 1)}{3T_c^2} = \frac{2\ell(\alpha - 1)^2(8\pi\eta T_c^{\alpha - 2} + \ell\alpha T_c^{\alpha - 1})}{3 \left( 8\pi\eta T_c^{\alpha - 1} + \ell(\alpha - 1)T_c^{\alpha - 1} \right)^2}, \tag{92}
\]
the deceleration parameter \( q \) approaches the value \((-1)\) as in the case of de-Sitter universe.

We also find

\[
\int_{t_0}^{t} \frac{dt}{V(t)} = \int_{t_0}^{t} \left[ \frac{8\pi \eta}{(\alpha-1)} T + \frac{\ell T^\alpha}{T^{2\alpha+1}} \right]^{\frac{2}{3}} dt.
\]

(93)

It is the convergent integral hence particle horizon exits.

The metric (66) is conformal to the metric

\[
ds^2 = dX^2 - dT^2 + T^{2\alpha} dY^2 + T^{2(\alpha+1)} dZ^2.
\]

(94)

The universe (94) represents a viscous fluid cosmological model in which kinematic viscosity \( \eta_0 \) is \(-\frac{\alpha}{8\pi T}\) and the pressure \( p_0 \) and the density \( \rho_0 \) are given by

\[
8\pi p_0 = 8\pi \xi \left( \frac{2\alpha + 1}{T} \right) - \left[ \frac{\alpha(5\alpha + 1)}{3T^2} \right] - \Lambda,
\]

(95)

\[
8\pi \rho_0 = \frac{\alpha(\alpha + 1)}{T^2} + \Lambda.
\]

(96)

It is also remarkable that the space-time (94) becomes flat when \( \alpha \) is zero. The corresponding model

\[
ds^2 = (\beta - 8\pi \eta T)^{-2}(dX^2 - dT^2 + dY^2 + T^2 dZ^2)
\]

(97)

represents a conformally flat viscous fluid cosmological model.

**Discussion and Concluding Remarks**

In this paper, we have presented a new class of exact solutions of Einstein’s field equations for plane-symmetric space-time with bulk viscous fluid distribution within the framework of Lyra’s geometry both for uniform and time dependent displacement field. Generally, the models represent shearing, non-rotating and Petrov type D universe in which the flow vector is geodetic. In all these models, we observe that they do not approach isotropy for large values of time.

It is possible to discuss entropy in our universe. In thermodynamics the expression for entropy is given by

\[
TdS = d(\rho V^3) + \bar{p}(dV^3),
\]

(98)

where \( V^3 = A^2 BC \) is the proper volume in our case and \( \bar{p} \) is the effective pressure given by

\[
\bar{p} = p - (\xi - \frac{2}{3}\eta) v_i^i.
\]

(99)

To solve the entropy problem of the standard model, it is necessary to treat \( dS > 0 \) for at least a part of evolution of the universe. Hence Eq. (98) reduces to

\[
TdS = \dot{\rho} + (\rho + \bar{p}) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) > 0.
\]

(100)
The conservation equation $T^j_{ij} = 0$ for (1) leads to

$$\dot{\rho} + (\rho + p) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{3}{2} \beta \dot{\beta} + \frac{3}{2} \beta^2 \left( 2 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0. \tag{101}$$

Therefore, Eqs. (100) and (101) lead to

$$\frac{3}{2} \beta \dot{\beta} + \frac{3}{2} \beta^2 \left( 2 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) < 0. \tag{102}$$

which gives to $\beta < 0$. Thus, the displacement vector $\beta(t)$ affects entropy because for entropy $dS > 0$ leads to $\beta(t) < 0$.

It is observed that the displacement vector $\beta(t)$ coincides with the nature of the cosmological constant $\Lambda$ which has been supported by the work of several authors as discussed in the physical behaviour of the model in Sections 4 and 5. In recent time $\Lambda$-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a $\Lambda$-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [41] and [42]). In recent past there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [43, 44, 45]. Therefore the study of cosmological models in Lyra’s geometry may be relevant for inflationary models. There seems a good possibility of Lyra’s geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. However, the importance of Lyra’s geometry for astrophysical bodies is still an open question.

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References


Some Bianchi Type I Cosmological Models of the Universe for Viscous Fluid Distribution in Lyra Geometry

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Abstract: Some Bianchi type I cosmological models of the universe with time dependent gauge function $\beta$ for viscous fluid distribution within the framework of Lyra geometry are investigated in which the expansion is considered only in two dimensions i.e. one of the Hubble parameter ($H_1 = \frac{\dot{A}}{A}$) is zero. To get the deterministic solutions of Einstein’s modified field equations, the viscosity coefficient of bulk viscous fluid is assumed to be a power function of mass density and the coefficient of shear viscosity is considered as constant in first case whereas in other case it is taken as proportional to scale of expansion in the model. It has been found that the displacement vector $\beta(t)$ behaves like cosmological term $\Lambda$ in the normal gauge treatment and the solutions are consistent with the observations. Solution in absence of shear viscosity is also obtained. The displacement vector $\beta(t)$ affects entropy. Some physical and geometrical properties of the models are discussed.

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1. Introduction and Motivations

In Einstein’s general relativity, the curvature of the space-time is influenced by matter, and it provides the geometrical description of matter. Einstein succeeded in geometrizing gravitation by expressing gravitational potential in terms of metric tensor. In general relativity, spatially homogeneous space-times either belong to Bianchi type or to Kantowaski-Sachs models and interpreted as cosmological models [1]. Spatially homoge-
neous and isotropic universes can be well described by Friedmann-Robertson-Walker [2, 3] (FRW) model. However, the FRW model has the disadvantage of being unstable near the singularity [4] and it fails to describe the early universe. Therefore spatially homogeneous and anisotropic Bianchi type-I models are undertaken to understand the universe at its early stage of evolution. The idea of geometrizing gravitation by Einstein in 1917 inspired Weyl [5] to develop a theory to geometrize gravitation and electromagnetism. Weyl [5] suggested the first so-called unified field theory based on a generalization of Riemannian geometry. With its backdrop, it would seem more appropriate to call Weyl’s theory a geometrized theory of gravitation and electromagnetism, instead a unified field theory. It is not clear as to what extent the two fields have been unified, even though they acquire (different) geometrical significance in the same geometry. The theory was never taken seriously in as much as it was based on the concept of non-integrability of length transfer; and, as pointed out by Einstein, this implies that spectral frequencies of atoms depend on their past histories and therefore have no absolute significance. Nevertheless, Weyl’s geometry provides an interesting example of non-Riemannian connections, and Folland [6] has given a global formulation of Weyl manifolds clarifying considerably many of Weyl’s basic ideas thereby.

In 1951, Lyra [7] proposed a modification of Riemannian geometry by introducing a gauge function into the structure-less manifold, as a result of which the cosmological constant arises naturally from the geometry. This bears a remarkable resemblance to Weyl’s geometry. But in Lyra’s geometry, unlike that of Weyl, the connection is metric preserving as in Riemannian; in other words, length transfers are integrable. Lyra also introduced the notion of a gauge and in the “normal” gauge the curvature scalar is identical to that of Weyl. In consecutive investigations Sen and co-worker [8, 9] proposed a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein field equations based on Lyra’s geometry. It is, thus, possible [8] to construct a geometrized theory of gravitation and electromagnetism much along the lines of Weyl’s “unified” field theory, however, without the inconvenience of non-integrability length transfer. Halford [10] has pointed out that the constant vector displacement field $\phi_i$ in Lyra’s geometry plays the role of cosmological constant $\Lambda$ in the normal general relativistic treatment. It is shown by Halford [11] that the scalar-tensor treatment based on Lyra’s geometry predicts the same effects within observational limits as the Einstein’s general theory of relativity. Several authors Sen and Vanstone [12], Bhamra [13], Karade and Borikar [14], Kalyanshetti and Wagmode [15], Reddy and Innaiah [16], Beesham [17], Reddy and Venkateswarlu [18], Soleng [19], studied cosmological models based on Lyra’s manifold with a constant displacement field vector. However, this restriction of the displacement field to be constant is merely one for convenience and there is no a priori reason for it. Beesham [20] considered Friedmann-Robertson-Walker (FRW) models with time dependent displacement field. Singh and co-workers [21]– [25] studied Bianchi-type I, III, Kantowski-Sachs and a new class of cosmological models with time dependent displacement field and have made a comparative study of Robertson-Walker models with constant
deceleration parameter in Einstein's theory with cosmological term and in the cosmological theory based on Lyra's geometry. Soleng [19] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector \( \phi \) will either include a creation field and be equal to Hoyle's creation field cosmology [26]–[28] or contain a special vacuum field, which together with the gauge vector term, may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

Most studies in cosmology involve a perfect fluid. Large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation, suggest that we should analyze dissipative effects in cosmology. Further, there are several processes which are expected to give rise to viscous effect. These are the decoupling of neutrinos during the radiation era and the recombination era [29], decay of massive super string modes into massless modes [30], gravitational string production [31, 32] and particle creation effect in grand unification era [33]. It is known that the introduction of bulk viscosity can avoid the big bang singularity. Thus, we should consider the presence of a material distribution other than a perfect fluid to have realistic cosmological models (see Grøn [34] for a review on cosmological models with bulk viscosity). A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy [35]. The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past.

Recently, Pradhan et al. [36], Casama et al. [37], Rahaman et al. [38], Bali and Chandnani [39], Kumar and Singh [40], Singh [41], Rao, Vinutha and Santhi [42] and Pradhan [43] have studied cosmological models based on Lyra's geometry in various contexts. With these motivations, in this paper, we have obtained exact solutions of Einstein's modified field equations for viscous fluid distribution in Bianchi type-I homogeneous space-time within the frame work of Lyra's geometry for time varying displacement vector \( \beta(t) \). This paper is organized as follows. In Section 1 the motivation for the present work is discussed. The metric and the field equations are presented in Section 2, in Section 3 are the solutions of field equations. The Subsection 3.1 describes the solution of Case I where \( \eta = \text{constant} \) and also deals with some physical and geometrical properties of the model. The Subsection 3.2 deals with the solution of Case II where \( \eta = b\theta \), where \( b \) is an arbitrary constant and the physical and geometrical aspects of the model are also described. In Section 4 the solution of field equations in absence of shear viscosity is described. Finally, discussion and concluding remarks are given in Section 5.

2. The Metric and Field Equations

We consider the Bianchi type-I metric in the form

\[
ds^2 = -dt^2 + dx^2 + B^2dy^2 + C^2dz^2,
\]
where the metric potentials \( B \) and \( C \) are functions of \( t \) alone. This metric depicts the case when one of the Hubble parameters (here \( H_1 = \frac{\dot{A}}{A} \)) is zero, i.e. the expansion is only in two directions. The kinematic parameters are then related as \( \theta = -3\sigma_1^1 \). This condition leads to the metric (1).

The energy-momentum tensor for a viscous fluid distribution is given by Landau and Lifshitz [44]

\[
T^j_i = (\rho + p)v_i^j + p g^j_i - \eta (v_i^j + v^j_i + v^j v_i^l + v_i v^j_l) - \left( \xi - \frac{2}{3} \eta \right) v^l_i (g^j_l + v_i v^j) .
\]

(2)

Here \( \rho, p, \eta \) and \( \xi \) are energy density, isotropic pressure, the coefficient of shear viscosity and bulk viscous coefficient respectively and \( v^j \) is the flow vector satisfying the relation

\[
g_{ij} v^i v^j = -1.
\]

(3)

The semicolon (;) indicates covariant differentiation. We choose the coordinates to be comoving, so that \( v^1 = v^2 = v^3 = 0 \) and \( v^4 = \frac{1}{A} \).

The field equations, in normal gauge for Lyra’s manifold, obtained by Sen [8] as

\[
R^j_i - \frac{1}{2} g^j_i \phi + \frac{3}{2} \phi^i \phi^j - \frac{3}{4} g^j_i \phi^k \phi^k = -8\pi T^j_i, \tag{4}
\]

where \( \phi_i \) is the displacement field vector defined as

\[
\phi_i = (0, 0, 0, \beta(t)), \tag{5}
\]

where other symbols have their usual meaning as in Riemannian geometry.

For the line-element (1), the field Eq. (4) with Eqs. (2) and (5) lead to the following system of equations

\[
\frac{\dot{B} \dot{C}}{BC} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{3}{4} \beta^2 = -8\pi \left[ p - \left( \xi - \frac{2}{3} \eta \right) v^l_i \right], \tag{6}
\]

\[
\frac{\dot{C}}{C} + \frac{3}{4} \beta^2 = -8\pi \left[ p - 2\eta \frac{\dot{B}}{B} - \left( \xi - \frac{2}{3} \eta \right) v^l_i \right], \tag{7}
\]

\[
\frac{\dot{B}}{B} + \frac{3}{4} \beta^2 = -8\pi \left[ p - 2\eta \frac{\dot{C}}{AC} - \left( \xi - \frac{2}{3} \eta \right) v^l_i \right], \tag{8}
\]

\[
\frac{\dot{B} \dot{C}}{BC} + \frac{3}{4} \beta^2 = 8\pi \rho. \tag{9}
\]

Here, and also in the following expressions a dot indicates ordinary differentiation with respect to \( t \).
The energy conservation equation \( T_{ij}^a = 0 \) leads to
\[
\dot{\rho} + (\rho + \bar{p}) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0,
\]
(10)
where \( \bar{p} \) is the effective pressure given by
\[
\bar{p} = p - \left( \xi - \frac{2}{3} \eta \right) v_{;i}^i.
\]
(11)
and
\[
(R^j_i - \frac{1}{2} g^j_i R)_{;j} + \frac{3}{2} (\phi_i \phi^j)_{;j} - \frac{3}{4} (g^j_k \phi_k \phi^k)_{;j} = 0.
\]
(12)
Equation (12) leads to
\[
\frac{3}{2} \phi_i \left[ \frac{\partial \phi^j}{\partial x^j} + \phi^j \Gamma^i_{;j} \right] + \frac{3}{2} \phi^j \left[ \frac{\partial \phi_i}{\partial x^j} - \phi_i \Gamma^j_{;i} \right] - \frac{3}{4} g^j_k \phi_k \left[ \frac{\partial \phi^k}{\partial x^j} + \phi^k \Gamma^j_{;i} \right] - \frac{3}{4} g^j_i \phi^k \left[ \frac{\partial \phi^i}{\partial x^j} - \phi^i \Gamma^j_{;k} \right] = 0.
\]
(13)
Equation (13) is identically satisfied for \( i = 1, 2, 3 \). For \( i = 4 \), Eq. (13) reduces to
\[
\frac{3}{2} \beta \ddot{\beta} + \frac{3}{2} \beta^2 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0.
\]
(14)

3. Solutions of the Field Equations

Equations (6)-(9) are four independent equations in seven unknowns \( B, C, p, \rho, \eta, \xi \) and \( \beta \). For complete solutions of equations (6)-(9), we need three extra conditions. The research on exact solutions is based on some physically reasonable restrictions used to simplify the Einstein equations.

From Eqs. (6) - (8), we obtain
\[
\frac{\dot{B}}{B} + \frac{\dot{B} C}{B C} = -16\pi \eta \frac{\dot{B}}{B},
\]
(15)
and
\[
\frac{\dot{B}}{B} - \frac{\ddot{C}}{C} = -16\pi \eta \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right).
\]
(16)
Eq. (16) on integration leads to
\[
C^2 \frac{d}{dt} \left( \frac{B}{C} \right) = L e^{-16\pi \eta t},
\]
(17)
where \( L \) is an integrating constant. Setting \( BC = \mu \) and \( \frac{B}{C} = \nu \) in Eqs. (15) - (17) lead to
\[
\ddot{\mu} + 16\pi \eta \dot{\mu} = 0,
\]
(18)
and

\[ \dot{\nu} = \frac{L}{\mu} e^{-16\pi \eta t}. \]  

(19)

Eq. (18) on integration leads to

\[ \dot{\mu} = M e^{-16\pi \eta}, \]  

(20)

where \( M \) is an integrating constant.

Here we consider two cases:

3.1 Case I: Let \( \eta = \text{constant} = a \) (say)

In this case Eq. (20) on integration gives

\[ \mu = N - \frac{M}{16\pi a} e^{-16\pi at}, \]  

(21)

where \( N \) is a constant of integration.

Equation (19) on integration leads to

\[ \nu = \alpha (16\pi a N - M e^{-16\pi at})^{L/M}, \]  

(22)

where \( \alpha \) is an integrating constant.

Now we set

\[ e^{-16\pi at} = \cos 2\sqrt{16\pi} \tau, \]  

(23)

\[ \alpha = \frac{1}{(16\pi a)^{L/M}}, \]  

(24)

and

\[ N = \frac{M}{16\pi a}. \]  

(25)

Using the above Eqs. (23) - (25) in Eqs. (21) and (22), we obtain

\[ \mu = 2M \left( \sin \frac{16\pi a \tau}{\sqrt{16\pi a}} \right)^{2}, \]  

(26)

and

\[ \nu = (2M)^{L/M} \left( \sin \frac{16\pi a \tau}{\sqrt{16\pi a}} \right)^{2L/M}. \]  

(27)

From Eqs. (26) and (27), we obtain

\[ B^2 = \mu \nu = (2M)^{1+(L/M)} \left( \sin \frac{16\pi a \tau}{\sqrt{16\pi a}} \right)^{2+(2L/M)}, \]  

(28)

\[ C^2 = \frac{\mu}{\nu} = (2M)^{1-(L/M)} \left( \sin \frac{16\pi a \tau}{\sqrt{16\pi a}} \right)^{2-(2L/M)}. \]  

(29)
Hence the metric (1) reduces to the form
\[
\begin{align*}
&ds^2 = -4 \left( \frac{\tan 2\sqrt{16\pi a\tau}}{\sqrt{16\pi a}} \right) d\tau^2 + dx^2 \\
&+ (2M)^{1+(L/M)} \left( \frac{\sin \sqrt{16\pi a\tau}}{\sqrt{16\pi a}} \right)^{2+(2L/M)} dy^2 \\
&+ (2M)^{1-(L/M)} \left( \frac{\sin \sqrt{16\pi a\tau}}{\sqrt{16\pi a}} \right)^{2-(2L/M)} dz^2.
\end{align*}
\] (30)

After suitable transformation of coordinates metric (30) reduces to the form
\[
\begin{align*}
&ds^2 = -4 \left( \frac{\tan 2kT}{k} \right) dT^2 + dX^2 \\
&+ \left( \frac{\sin kT}{k} \right)^{2+(2L/M)} dY^2 + \left( \frac{\sin kT}{k} \right)^{2-(2L/M)} dZ^2,
\end{align*}
\] (31)

where \( k = \sqrt{16\pi a} \).

The pressure and density for the model (31) are given by
\[
\begin{align*}
8\pi p &= \frac{1}{4M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ M(M - L) \sin^2 kT \cos 2kT \\
&- \frac{(L + M)}{4} \cos 2kT \{(L + M) \cos 2kT - 2M\} \\
&+ 16\pi M^2 \left( \frac{\xi - 2}{3} \right) \cos 2kT \left( \frac{\sin kT}{k} \right)^2 \right] - \frac{3}{4}\beta^2, \\
8\pi \rho &= \frac{1}{4M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ \frac{(M^2 - L^2)}{4} \cos^2 2kT \right] + \frac{3}{4}\beta^2.
\end{align*}
\] (32)

From Eq. (14) we have
\[
\frac{3}{2} \dot{\beta} + \beta \dot{\beta} + \frac{3}{2} \beta^2 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0,
\]
which gives either \( \beta = 0 \) or \( \dot{\beta} + \beta \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0 \).

Therefore
\[
\frac{\dot{\beta}}{\beta} = - \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right),
\] (34)

which reduces to
\[
\frac{\dot{\beta}}{\beta} = -2k \cot kT.
\] (35)

Integrating above Eq. (35), we obtain
\[
\beta = \frac{1}{\sin^2 kT}. \quad (36)
\]
For the specification of $\xi$, we assume that the fluid obeys an equation of state of the form

$$p = \gamma \rho,$$

(37)

where $\gamma(0 \leq \gamma \leq 1)$ is a constant. Thus, given $\xi(t)$ we can solve for the cosmological parameters. In most of the investigation involving bulk viscosity it is assumed to be a simple power function of the energy density [45]–[49]

$$\xi(t) = \xi_0 \rho^n,$$

(38)

where $\xi_0$ and $n$ are constants. For small density, $n$ may even be equal to unity as used in Murphy’s work [35] for simplicity. If $n = 1$, (38) may correspond to a radiative fluid [49]. Near the big bang, $0 \leq n \leq \frac{1}{2}$ is a more appropriate assumption [50] to obtain realistic models.

On using (38) in (32), we obtain

$$8\pi p = \frac{1}{4M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ M(M - L) \sin^2 kT \cos 2kT ight.$$

$$- \frac{(L + M)}{4} \cos 2kT \{ (L + M) \cos 2kT - 2M \}$$

$$+ 16\pi M^2 \left( \xi_0 \rho^n - \frac{2}{3} a \right) \cos 2kT \left( \frac{\sin kT}{k} \right)^2 \left\{ 3 - \frac{3}{4} \beta^2 \right\}.$$  

(39)

For simplicity and realistic models of physical importance, we consider the following two cases ($n = 0, 1$):

3.1.1 Model I: Solution for $n = 0$

When $n = 0$, (38) reduces to $\xi = \xi_0 = \text{constant}$. Using (33) and (37) in Eq. (39) leads to

$$64\pi M^2(1 + \gamma) \rho = \left( \frac{k}{\sin kT} \right)^4 \cos 2kT \left[ M^2 - L^2 + 2(L^2 + M^2) \sin^2 kT ight.$$  

$$+ 32\pi M^2 \left( \xi_0 - \frac{2}{3} a \right) \left( \frac{\sin kT}{k} \right)^2 \left\{ 3 - \frac{3}{4} \beta^2 \right\}.$$  

(40)

3.1.2 Model II: Solution for $n = 1$

When $n = 1$, (38) reduces to $\xi = \xi_0 \rho$. Using (33) and (37) in Eq. (39) leads to

$$16\pi \rho = \frac{k^4 \cos 2kT}{M^2[2(1 + \gamma) \sin^2 kT - k^2 \xi_0 \cos 2kT] \sin^2 kT}$$

$$\times \left[ \frac{M^2 - L^2}{2} + (L^2 + M^2) \sin^2 kT - \frac{8\pi a}{3} \cos 2kT \left( \frac{k}{\sin kT} \right)^2 \right].$$  

(41)
Some Geometric and Physical Aspects of the Models

With regard to the kinematic properties of the velocity vector $v^i$ in the metric model (31), a straight forward calculation leads to the following expressions for the scalar of the expansion $\theta$, shear scalar $\sigma^2$, deceleration parameter $q$, and proper volume $V^3$ of the fluid.

$$\theta = \frac{1}{2} \cos 2kT \left( \frac{k}{\sin kT} \right)^2,$$

(42)

$$\sigma^2 = \frac{1}{48} \left( 1 + \frac{3L^2}{M^2} \right) \cos^2 2kT \left( \frac{k}{\sin kT} \right)^4,$$

(43)

$$q = -\frac{\ddot{V}}{V} \frac{\dot{V}}{V^2} = \frac{1}{2} (3 \tan^2 kT - 1),$$

(44)

$$V^3 = \sqrt{-g} = \frac{\sin^2 kT}{k^2}$$

(45)

The expressions for $\sigma \theta$ is found to be

$$\frac{\sigma}{\theta} = \frac{1}{2 \sqrt{3}} \left( 1 + \frac{3L^2}{M^2} \right)^{\frac{1}{2}} = \text{constant}.$$  

(46)

The rotation $\omega$ is identically zero.

The non-vanishing components of conformal curvature tensor are obtained as

$$C_{12}^{12} = \frac{\cos 2kT}{48M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ M^2 - L^2 - 2L(3M - L) \sin^2 kT \right],$$

(47)

$$C_{13}^{13} = \frac{\cos 2kT}{48M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ M^2 - L^2 + 2L(3M + L) \sin^2 kT \right],$$

(48)

$$C_{14}^{14} = \frac{\cos 2kT}{24M^2} \left( \frac{k}{\sin kT} \right)^4 \left[ L^2 - M^2 - 2L^2 \sin^2 kT \right].$$

(49)

The rate of expansion $H_i$ (Hubble parameters) in the direction of X, Y, Z are given by

$$H_1 = 0,$$

(50)

$$H_2 = \frac{(M + L)}{4MT^2},$$

(51)

$$H_3 = \frac{(M - L)}{4MT^2}.$$  

(52)

Now since

$$\int_{t_0}^{t} \frac{dt}{V(t)} = k^\frac{3}{2} \int_{t_0}^{t} \sin^{-\frac{3}{2}kT}dt,$$

(53)

which is a convergent integral, so the particle horizon exists.

The models represent shearing, non-rotating and Petrov type I non-degenerate in general, in which the flow is geodetic. However, if $L = 0$ then space-time reduces to Petrov
type D. The model starts expanding at $T \geq 0$ but the initial expansion is slow. When $T$ is closer to $\pi/2k$, it has stiff rise in the expansion and then decreases. This shows the case of $T = 0$ or $T = \pi/k$. The large values of $\theta$ near $T = \pi/2k$ is reflection of trigonometric property. But expansion remains finite. As $T$ increases the proper volume also increases. It is observed from Eq. (44) that $q < 0$ when $\tan kT < \frac{1}{\sqrt{3}}$ which implies an accelerating model of the universe. Recent observations of type Ia supernovae [51, 52] reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range $-1 < q \leq 0$. It follows that our models of the universe are consistent with recent observations. From Eq. (46), it can be observed that shear is proportional to scalar of expansion $\theta$ in the models and the models do not approach isotropy.

Halford [10] has pointed out that the displacement field $\phi_i$ in Lyra’s geometry plays the role of cosmological constant $\Lambda$ in the normal general relativistic treatment. From Eq. (36), it is observed that the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford as well as with the recent observations [51, 52] leading to the conclusion that $\Lambda(t)$ is a decreasing function of $t$.

3.2 Case II: Let $\eta = b\theta$, where $b$ is an arbitrary constant

In this case Eq. (18) reduces to

$$\mu \ddot{\mu} + 16\pi b \mu^2 = 0, \quad (54)$$

which on integration leads to

$$\mu = \left[(1 + 16\pi b)(k_1 t + k_2)\right]^{1/(1+16\pi b)}, \quad (55)$$

where $k_1$ and $k_2$ are constants of integration.

Equation (16) reduces to

$$\mu \frac{d}{dt} \left(\frac{\dot{\nu}}{\nu}\right) = -(1 + 16\pi b) \frac{\ddot{\mu}}{\mu}, \quad (56)$$

which on integration leads to

$$\nu = k_3(k_1 t + k_2)^{k_4/(k_1(1+16\pi b))}, \quad (57)$$

where $k_3$ and $k_4$ are constants of integration.

From Eqs. (55) and (57), we obtain

$$B^2 = \mu \nu = k_3k_5(k_1 t + k_2)^{k_6(k_1+k_4)}, \quad (58)$$

$$C^2 = \frac{\mu}{\nu} = \frac{k_5}{k_3}(k_1 t + k_2)^{k_6(k_1-k_4)}, \quad (59)$$

where

$$k_5 = (1 + 16\pi b)^{1/(1+16\pi b)},$$
\[ k_6 = \frac{1}{k_1(1 + 16\pi b)}. \]

Hence the metric (1) reduces to

\[ ds^2 = -dt^2 + dx^2 + k_3k_5(k_1t + k_2)^{k_6(k_1+k_4)}dy^2 + \frac{k_5}{k_3}(k_1t + k_2)^{k_6(k_1-k_4)}dz^2. \]  

(60)

After suitable transformation of coordinates, the metric (60) takes the form

\[ ds^2 = -\frac{1}{k_1^2}dT^2 + dX^2 + T^{k_6(k_1+k_4)}dY^2 + T^{k_6(k_1-k_4)}dZ^2. \]  

(61)

The pressure and density for the model (61) are given by

\[ 8\pi p = -\frac{k_1^2k_6^2}{4T^2}(k_1 + k_4)(k_1k_6 + k_4k_6 - 2) + \frac{8\pi k_1^2k_6}{T} \times \left[ \frac{bk_1k_6}{3T}(k_1 - k_4) + \xi \right] - \frac{k_2}{2k_5^2T^{2k_1k_6}} - \frac{3}{4}\beta^2, \]  

(62)

\[ 8\pi \rho = \frac{k_1^2k_6^2(k_1^2 - k_4^2)}{4T^2} + \frac{3}{4}\beta^2. \]  

(63)

In this case using the values of \( B \) and \( C \) in Eq. (34), we obtain

\[ \frac{\dot{\beta}}{\beta} = -\frac{k_1k_6}{T}, \]  

(64)

which on integration gives

\[ \beta = \frac{1}{T^{k_1k_6}}. \]  

(65)

On using (38) in (62), we obtain

\[ 8\pi p = -\frac{k_1^2k_6^2}{4T^2}(k_1 + k_4)(k_1k_6 + k_4k_6 - 2) + \frac{8\pi k_1^2k_6}{T} \times \left[ \frac{bk_1k_6}{3T}(k_1 - k_4) + \xi_0\rho^n \right] - \frac{k_2}{2k_5^2T^{2k_1k_6}} - \frac{3}{4}\beta^2, \]  

(66)

3.2.1 Model I: Solution for \( n = 0 \)

When \( n = 0 \), (38) reduces to \( \xi = \xi_0 = \text{constant} \). Using (63) and (37) in Eq. (66) leads to

\[ 8\pi(1 + \gamma)\rho = \frac{k_1^2k_6}{2T^2}(k_1 + k_4)(1 - k_1k_6) + \frac{8\pi k_1^2k_6}{T} \times \left[ \frac{bk_1k_6}{3T}(k_1 - k_4) + \xi_0 \right] - \frac{k_2}{2k_5^2T^{2k_1k_6}}. \]  

(67)
3.2.2 Model II: Solution for $n = 1$

When $n = 1$, (38) reduces to $\xi = \xi_0 \rho$. Using (63) and (37) in Eq. (66) leads to

$$8\pi \left[ 1 + \gamma - \frac{k_1^2 k_6 \xi_0}{T} \right] \rho = \frac{k_2^2 k_6^2}{6T^2} \left[ 3(k_1 + k_4)(1 - k_4 k_6) + 16\pi b k_1(k_1 - 3k_4) \right] - \frac{k_2}{2k_3^2 T^{2k_1 k_6}}. \quad (68)$$

Some Geometric and Physical Aspects of the Models

The expressions for the scalar of the expansion $\theta$, shear scalar $\sigma^2$, deceleration parameter $q$, and proper volume $V^3$ for the model (61) are given by

$$\theta = \frac{k_1^2 k_6}{T}, \quad (69)$$

$$\sigma^2 = \frac{k_1^2 k_6^2 (k_1^2 + k_2^2)}{2T^2}. \quad (70)$$

The expressions for $\frac{\sigma}{\theta}$ and $\frac{\rho}{\theta^2}$ are found to be

$$\frac{\sigma}{\theta} = \frac{(k_1^2 + 3k_4^2)^{1/2}}{2\sqrt{3}k_1} = \text{constant}, \quad (71)$$

$$\frac{\rho}{\theta^2} = \frac{1}{32\pi k_1^2 k_6^2} \left[ (k_1^2 - k_2^2)k_1^2 k_6 + 3T^{2(1-k_1 k_6)} \right]. \quad (72)$$

$$q = \frac{(k_1 k_6 - 3)}{k_1 k_6}. \quad (73)$$

$$V^3 = T^{k_1 k_6}. \quad (74)$$

The rotation $\omega$ is identically zero.

The non-vanishing components of conformal curvature tensor are obtained as

$$C_{12}^{12} = \frac{k_1^2 k_6}{12T^2} [k_1 - 3k_4 + k_4 k_6 (3k_1 - k_4)], \quad (75)$$

$$C_{13}^{13} = \frac{k_1^2 k_6}{12T^2} [k_1 + 3k_4 - k_4 k_6 (3k_1 + k_4)], \quad (76)$$

$$C_{14}^{14} = \frac{k_1^2 k_6}{6T^2} (k_1^2 k_6 - k_1). \quad (77)$$

The models represent an expanding, shearing but non-rotating universe in general. The models explode with a big bang at $T = 0$ and the expansion in the models stops at $T = \infty$. When $k_1 = 0$ then $\theta = 0$, which implies that $\eta = 0$. Therefore, viscosity is due to expansion in the model. We take $k_1 \neq 0$. The space-time is Petrov type I non-degenerate. However, if $k_4 = 0$, the space-time reduces to Petrov type ID. For large values of $T$, the space-time is conformally flat. Since $\frac{\sigma}{\theta} = \text{constant}$, hence the models do
not approach isotropy at all times.

The rate of expansion $H_i$ (Hubble parameters) in the direction of $X$, $Y$, $Z$ are given by

$$H_1 = 0,$$

$$H_2 = \frac{k_1k_6(k_1 + k_4)}{2T},$$

$$H_3 = \frac{k_1k_6(k_1 - k_4)}{2T}.\tag{78}$$

As $T$ increases the proper volume also increases. It is observed from Eq. (73) that $q < 0$ when $k_1k_6 < 3$ which implies an accelerating model of the universe. Recent observations of type Ia supernovae [51, 52] reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range $-1 < q \leq 0$. It follows that our models of the universe are consistent with recent observations. From Eq. (71), it can be observed that shear is proportional to scalar of expansion $\theta$ in the models and the models do not approach isotropy.

From Eq. (65), it is observed that the displacement vector $\beta(t)$ is a decreasing function of time which is corroborated with Halford [10] as well as with the recent observations [51, 52] leading to the conclusion that $\Lambda(t)$ is a decreasing function of $t$.

4. **Solution in Absence of Shear Viscosity**

When $\eta \to 0$, then the metric (31) leads to

$$ds^2 = -16T^2dT^2 + dX^2 + T(2 + \frac{2L}{M})dY^2 + T(2 - \frac{2L}{M})dZ^2.\tag{81}$$

The pressure and the density for the model (81) are given by

$$8\pi p = \frac{1}{16M^2T^4}[64\pi M^2\xi T^2 + M^2 - L^2] - \frac{3}{4}\beta^2,\tag{82}$$

$$8\pi \rho = \frac{(M^2 - L^2)}{16M^2T^4} + \frac{3}{4}\beta^2.\tag{83}$$

In this case Eq. (34) reduces to

$$\frac{\dot{\beta}}{\beta} = -\frac{1}{2T},\tag{84}$$

which on integration gives

$$\beta = \frac{1}{T^2}.\tag{85}$$

4.1 Model I: Solution for $n = 0$

When $n = 0$, (38) reduces to $\xi = \xi_0 = \text{constant}$. Hence in this case Eq. (82), with the use of (37) and (83), leads to

$$8\pi(1 + \gamma)\rho = \frac{1}{8M^2T^4}[32\pi M^2\xi_0 T^2 + M^2 - L^2].\tag{86}$$
4.2 Model II: Solution for \( n = 1 \)

When \( n = 1 \), (38) reduces to \( \xi = \xi_0 \rho \). Hence in this case Eq. (82), with the use of (37) and (83), leads to

\[
[2(1 + \gamma)T^2 - \xi_0] \rho = \frac{M^2 - L^2}{32\pi M^2 T^2}.
\]

(87)

Some Geometric and Physical Aspects of the Models

The expressions for the scalar of the expansion \( \theta \), shear scalar \( \sigma^2 \), deceleration parameter \( q \), and proper volume \( V^3 \) for the model (81) are given

\[
\theta = \frac{1}{2T^2},
\]

(88)

\[
\sigma^2 = \frac{(M^2 + 3L^2)}{48M^2 T^4},
\]

(89)

\[
q = \frac{1}{2},
\]

(90)

\[
V^3 = T^2.
\]

(91)

The expressions for \( \frac{\sigma}{\theta} \) and \( \frac{\rho}{\theta^2} \) are found to be

\[
\frac{\sigma}{\theta} = \frac{1}{2\sqrt{3M}} (M^2 + 3L^2)^{\frac{1}{2}},
\]

(92)

\[
\frac{\rho}{\theta^2} = \frac{1}{32\pi M^2} [M^2 - L^2 + 12M^2 T^3].
\]

(93)

The rotation \( \omega \) is identically zero.

The non-vanishing components of conformal curvature tensor are obtained as

\[
C_{12}^{12} = C_{13}^{13} = \frac{(M^2 - L^2)}{48M^2 T^4},
\]

(94)

\[
C_{14}^{14} = \frac{(L^2 - M^2)}{24M^2 T^4}.
\]

(95)

The models represent an expanding, shearing but non-rotating universe in general. The models explode with a big bang at \( T = 0 \) and the expansion in the models stops at \( T = \infty \). Since \( \frac{\sigma}{\theta} = \text{constant} \), hence the models do not approach isotropy at all times. As \( T \) increases the proper volume also increases.

5. Discussion and Concluding Remarks

In this paper, we have presented a new class of exact solutions of Einstein’s field equations for Bianchi type-I space-time with bulk viscous fluid distribution within the framework of Lyra’s geometry for time dependent displacement field. Generally, the models represent shearing, non-rotating and Petrov type D universe in which the flow vector is geodetic.
In all these models, we observe that they do not approach isotropy for large values of time.

It is possible to discuss entropy in our universe. In thermodynamics the expression for entropy is given by

\[ TdS = d(\rho V^3) + \bar{p}(dV^3), \]  

(96)

where \( V^3 = BC \) is the proper volume in our case. To solve the entropy problem of the standard model, it is necessary to treat \( dS > 0 \) for at least a part of evolution of the universe. Hence Eq. (96) reduces to

\[ TdS = \dot{\rho} + (\rho + \bar{p}) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) > 0. \]  

(97)

The conservation equation \( T^i_{ij} = 0 \) for (1) leads to

\[ \dot{\rho} + (\rho + \bar{p}) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{3}{2} \beta^2 + \frac{3}{2} \beta^2 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0. \]  

(98)

Therefore, Eqs. (97) and (98) lead to

\[ \frac{3}{2} \dot{\beta} + \frac{3}{2} \beta^2 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) < 0. \]  

(99)

which gives to \( \beta < 0 \). Thus, the displacement vector \( \beta(t) \) affects entropy because for entropy \( dS > 0 \) leads to \( \beta(t) < 0 \).

It is observed that the displacement vector \( \beta(t) \) coincides with the nature of the cosmological constant \( \Lambda \) which has been supported by the work of several authors as discussed in the physical behaviour of the model in Sections 3 and 4. In recent time \( \Lambda \)-term has attracted theoreticians and observers for many a reason. The nontrivial role of the vacuum in the early universe generates a \( \Lambda \)-term that leads to inflationary phase. Observationally, this term provides an additional parameter to accommodate conflicting data on the values of the Hubble constant, the deceleration parameter, the density parameter and the age of the universe (for example, see Refs. [53] and [54]). In recent past there is an upsurge of interest in scalar fields in general relativity and alternative theories of gravitation in the context of inflationary cosmology [55, 56, 57]. Therefore the study of cosmological models in Lyra’s geometry may be relevant for inflationary models. There seems a good possibility of Lyra’s geometry to provide a theoretical foundation for relativistic gravitation, astrophysics and cosmology. However, the importance of Lyra’s geometry for astrophysical bodies is still an open question.

The effect of bulk viscosity is to produce a change in perfect fluid and hence exhibit essential influence on the character of the solution. The effect is clearly visible on the \( p \) effective (see details in previous sections). We also observe that Murphy’s conclusion [35]
about the absence of a big bang type singularity in the infinite past in models with bulk viscous fluid, in general, is not true. The results obtained by Myung and Cho [58] also show that, it is, in general, not valid, since for some cases big bang singularity occurs in finite past.

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Geometrical Behaviours of LRS Bianchi Type-I Cosmological Model

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Abstract: By using Einstein’s theory of general relativity some properties of spatially homogeneous locally rotationally symmetric (LRS) Bianchi type-I space-time are investigated in empty space. The concept of Riemannian curvature tensor, Ricci tensor and Ricci scalar has been used to discuss the geometrical behavior of the space-time. It is shown that, LRS Bianchi type-I has always flat geometry in empty space. Also we have shown that the vacuum model does not have singularity when time goes to zero.

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Keywords: Cosmological Models; LRS Bianchi Type-I models; Curvature Tensor; Ricci Tensor
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1. Introduction

The Bianchi cosmologies play an important role in theoretical cosmology and have been much studied since the 1960s. Bianchi cosmological models are spatially homogenous space-time. More precisely, they are manifolds of the form $M = I \times G$ where $I \subset \mathbb{R}$ is an interval and $G$ is a Lie group, endowed with a Lorentzian metric of the form $-dt^2 + g_t$ where $(g_t)_{t \in I}$ is a family of left-invariant Riemannian metrics on $G$.

The physical content of a space-time $M$ is encoded by a non-linear partial differential equation (PDE) on its Lorentz metric: The so called Einstein equation. For Bianchi

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cosmological models, this PDE (Einstein field equations) reduce to a set of second order ordinary differential equation on the family of metrics \((g_t)_{t \in I}\). However, this is not the case in LRS Bianchi type cosmological models where Einstein field equation leads to a set of non-linear differential equations. A spatially homogenous Bianchi model necessarily has a three-dimensional group, which acts simply transitively on space-like three-dimensional orbits. For simplification and description of the large scale behaviour of the actual universe, locally rotationally symmetric [henceforth referred as LRS]Bianchi type-I space-time have widely studied [1]-[6]. When the Bianchi type-I space-time expands equally in two spatial directions it is called locally rotationally symmetric. Here we confine ourselves to a LRS model of Bianchi type-I. This model is characterized by three metric functions \(R_1(t), R_2(t)\) and \(R_3(t)\) such that \(R_1 \neq R_2 = R_3\).

A study of geometrical aspects of Bertotti-Robinson like space-time [7] has been done by Radojevic [8] and Mohanty et al [9]. Mohanty et al have studied some geometrical aspects of Bianchi type-I in the framework of general relativity [9]. In this latter, to study the geometrical aspects of the LRS Bianchi type-I space-time, we have calculated Riemannian curvature tensor, Ricci tensor, Ricci scalar. The solution of the Einstein’s vacuum field equations also has been derived.

2. Einstein’s Vacuum Solution

The Einstein’s field equations (in gravitational units \(c = 1, G = 1\)) for the gravitational field (metric) of a matter-energy configuration described by the energy-momentum tensor \(T_{\mu\nu}\).

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}
\]  

(1)

taking the trace of Eq(1) and substituting the result into Eq(1) again one obtains

\[
R_{\mu\nu} = (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda)
\]  

(2)

In particular, for the vacuum, \(T_{\mu\nu} = 0\), the Einstein equations are simply

\[
R_{\mu\nu} = 0
\]  

(3)

in two and three dimensions, vanishing of the Ricci tensor implies the vanishing of the Riemann tensor. Thus in these cases, the space-times are necessarily flat away from where there is matter, i.e at points at which \(T_{\mu\nu} = 0\). Thus there are no true gravitational fields and no gravitational waves.

In four dimensions, however, the situation is completely different. The Ricci tensor has 10 independent components whereas the Riemann tensor has 20. Thus there are 10 components of the Riemann tensor which can curve the vacuum.
3. The Metric and Field Equations

We consider the LRS Bianchi type-I metric in the form

$$ds^2 = dt^2 - A^2 dx^2 - B^2 (dy^2 + dz^2)$$

(4)

where $A$ and $B$ are functions of $t$ only. Einstein's field equations (1) for line-element (4) leads to

$$R_{00} = -(\ddot{A}/A + 2\ddot{B}/B)$$

(5)

$$R_{11} = A(\ddot{A} + 2\dot{A}\dot{B}/B)$$

(6)

$$R_{22} = R_{33} = B(\ddot{B} + \dot{A}\dot{B}/A + \dot{B}^2/B)$$

(7)

where an overdot stands for the first and double overdot for the second derivative with respect to $t$. One can calculate the Ricci scalar; $R = g^\mu\nu R_{\mu\nu}$ as

$$R = -2(\ddot{A}/A + 2\ddot{B}/B + 2\dot{A}\dot{B}/AB + \dot{B}^2/B^2)$$

(8)

using the Einstein equation for vacuum dominate we get

$$\ddot{A} + 2\dot{A}\dot{B}/B = 0$$

(9)

$$\ddot{B} + \dot{A}\dot{B}/A + \dot{B}^2/B = 0$$

(10)

$$\dot{A}/A + 2\dot{B}/B = 0$$

(11)

solving eqs(9)-(11) one obtains

$$A = (3\alpha_At + 3\alpha_A)^{\frac{1}{3}}, B = (\frac{3}{2}\alpha_Bt + \frac{3}{2}\alpha_B)^{\frac{2}{3}}$$

(12)

by assuming $\alpha_{1A} = \frac{\alpha_{AB}}{2}, \alpha_{2A} = \frac{\alpha_{3B}}{2}$ and after a suitable coordinate transformations the vacuum model for the metric (4) reduces as

$$ds^2 = dT^2 - T^{2n_1} dx^2 - T^{2n_2}(dy^2 + dz^2)$$

(13)

we remark that it is follows from Eq(12) that $n_1$ and $n_2$ in Eq(13) can only have the value of $\frac{1}{3}$ and $\frac{2}{3}$ respectively. This may effect the claim made by Mohanty et al. that for Bianchi type-I, $n_1$, $n_2$ and $n_3$ could have any value under the conditions $\sum_{i=1}^{n} n_i = 1$ and $\sum_{i,j=1}^{3} n_in_j = 0$ if $i \neq j$. we see that this model has the same properties of Bianchi type-I cosmological model. i.e

1- the model is not asymptotically flat at infinite future as in the case of other symmetrical space times and

2- the model collapses at initial epoch.
4. Space of Constant Curvature

A Space is a space of constant curvature if the condition

\[ R_{\mu\nu\rho\sigma} = K(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \]  

(14)

holds. Here \( K \) shows the curvature of Riemannian manifold. Non vanishing components of Eq(14) for line-element(4) are

\[ R_{1010} = -KA^2, \quad R_{2020} = R_{3030} = -KB^2 \]  

(15)

\[ R_{1212} = R_{1313} = KA^2B^2, \quad R_{2323} = KB^4 \]  

(16)

In other hand one can finds the components of Riemannian curvature tensor via

\[ R_{\sigma\rho\mu\nu} = \frac{1}{2}(g_{\sigma\mu,\rho\nu} + g_{\rho\nu,\sigma\mu} - g_{\sigma\nu,\rho\mu} - g_{\rho\mu,\sigma\nu}) + \Gamma_{\lambda\sigma\mu}^{\lambda} \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\lambda\sigma\nu}^{\lambda} \Gamma_{\rho\mu}^{\lambda} \]  

(17)

For line-element (4) the above equation reduces as

\[ R_{0i00} = -\frac{1}{2}\frac{\partial^2 g_{ii}}{\partial x_0^2} + g_{ii}(\Gamma_{0i}^i)^2, \quad i, j = 1, 2, 3 \]  

(18)

and

\[ R_{ijij} = -g_{00}\Gamma_{ii}^0\Gamma_{jj}^0, \quad i, j = 1, 2, 3 \]  

(19)

So the non vanishing components of Eq(17) for line-element (4) can be calculated as

\[ R_{1010} = A\ddot{A}, \quad R_{2020} = R_{3030} = B\ddot{B} \]  

(20)

\[ R_{1212} = R_{1313} = -AB\ddot{A}B, \quad R_{2323} = -B^2\ddot{B}^2 \]  

(21)

Matching the components of Riemannian curvature tensor from Eqs(15)-(16) and Eqs(20)-(21)

\[ K = -\frac{1}{6}(2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}) \]  

(22)

comparing this result with the result which obtain in Eq(8) we can write

\[ K = -\frac{1}{6}R \]  

(23)

Although the above equation shows that in general, LRS Bianchi type-I is not a space of constant curvature, however, from Eqs(9)-(11) it is easy to show that the only value of curvature constant, \( K \) in empty space which can be obtain from Eq(23), is zero. This result is interesting because we see that the behavior of LRS Bianchi type-I in empty space-time is almost similar to the behavior of de Sitter model.
5. Symmetric Space

A space satisfying

\[ R_{\mu \nu \rho \sigma ; \lambda} = 0 \]  \hspace{1cm} (24)

is called a symmetric space. Where the semicolon shows the covariant differentiation. For non vanishing components of Riemannian curvature tensor of metric (4), Eq (24) reads to

\[ R_{1212;0} = \frac{\partial}{\partial x^0}(A\dot{A}\dot{B}) - 2(A\dot{A} + B\dot{B})A\dot{A}\dot{B} = 0 \]  \hspace{1cm} (25)

\[ R_{1010;0} = \frac{\partial}{\partial x^0}(A\ddot{A}) - 2\ddot{A}A = 0 \]  \hspace{1cm} (26)

\[ R_{2020;0} = \frac{\partial}{\partial x^0}(B\ddot{B}) - 2\ddot{B}B = 0 \]  \hspace{1cm} (27)

the solution of Eqs (25)-(27) gives

\[ A = e^t \]  \hspace{1cm} (28)

\[ B = e^t \]  \hspace{1cm} (29)

This special choice of metric components makes the line-element (4) to be a symmetric space. In view of Eqs (28)-(29) the metric (4) takes the form

\[ ds^2 = dt^2 - e^{2t}(dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (30)

After a proper choices of coordinates Eq (31) can be transformed to

\[ ds^2 = dT^2 - \exp[2T(dx^2 + dy^2 + dz^2)] \]  \hspace{1cm} (31)

It is interesting to note that the vacuum model (31) has no singularity at \( T = 0 \).

6. Conclusion

In this paper it is shown that the LRS Bianchi type-I space-time described by the line-element (4) is a space-time of constant curvature in empty space and hence it is an Einstein space. The importance of this result is that although the equality of metric components is the necessary condition for a metric to describe a flat space-time, we found an flat space-time with different time-dependence metric components \( A \neq B \). Also we have shown that with the special choose of metric components i.e. \( A = B = \exp(t) \), the metric (4) becomes symmetric and hence the vacuum model is free from singularity.

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Bianchi Type V Bulk Viscous Cosmological Models with Time Dependent $\Lambda$-Term

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Abstract: Spatially homogeneous and anisotropic Bianchi type V space-time with bulk viscous fluid source and time-dependent cosmological term are considered. Cosmological models have been obtained by assuming a variation law for the Hubble parameter which yields a constant value of deceleration parameter. Physical and kinematical parameters of the models are discussed. The models are found to be compatible with the results of cosmological observations.

Keywords: Cosmological models; Bianchi space-time; Hubble’s parameter; Constant deceleration parameter; Bulk viscosity; Variable $\Lambda$-term

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1. Introduction

Observations during the last few years provided increasingly strong evidence that the universe at present is expanding with acceleration [3, 33, 36, 37]. In Einstein’s theory of general relativity, to account for such an expansion, one needs to introduce some new energy density with a large negative pressure in the present universe, in addition to the usual relativistic or non-relativistic matter. This exotic matter causing cosmic acceleration is known as dark energy. The nature of dark energy is unknown and many radically different models related to this dark energy have been proposed [29, 38].

The simplest explanation of dark energy is provided by the cosmological constant $\Lambda$, but it needs to be severely fine-tuned due to the problem associated with its energy scale. The vacuum energy density observed today falls below the value of the vacuum energy density predicted by quantum field theory by many order of magnitude [54]. To explain the decay of the density, a number of dynamical models have been suggested

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in which cosmological term $\Lambda$ varies with cosmic time $t$. These models give rise to an effective cosmological term which as long as the universe expands, decays from a huge value at initial times to the small value observed at present. Cosmological models with different decay laws for the variation of cosmological term were investigated during last two decades [1, 7, 9, 30, 45, 46, 49, 51].

In the investigation of most of the cosmological models, the source of the gravitational field is assumed to be a perfect fluid. But these models do not explain satisfactorily the early stages of evolution. Viscosity may be important in cosmology for a number of reasons. Dissipative mechanisms responsible for smoothing out initial isotropies and the observed high entropy per baryon in the present state of the universe can be explained by involving some kind of dissipative mechanisms e.g. bulk viscosity [52, 53]. Dissipative effects including bulk viscosity are supposed to play a very important role in the early evolution of the universe. During the neutrino decoupling stage, apart from streaming neutrinos moving with fundamental velocity, there is a part behaving like a viscous fluid co-moving with matter. Decoupling of radiation and matter during the recombination era is also expected to give rise to viscous effects. Moreover, a combination of cosmic fluid with bulk dissipative pressure can generate accelerated expansion [28]. Influence of viscosity on the nature of the initial singularity and on the formation of galaxies have been investigated [16, 28]. It has been shown that the coincidence problem can be solved by taking viscous effects into account [12, 13]. Bulk viscosity leading to an accelerated phase of the universe today has been studied by Fabris et al. [17]. Santos et al. [39] have derived exact solution with bulk viscosity by considering the bulk viscous coefficient as power function of mass density. Johri and Sudarshan [21] have investigated the effect of bulk viscosity on the evolution of Friedmann models. Cosmological models with bulk viscosity have also been studied by Burd and Coley [8], Maartens [24], Pavon and Zimdahl [31], Pavon et al. [32].

In the construction of a cosmological model, assumption of homogeneity and isotropy of the universe are motivated by the cosmological principle and mathematical tractability of the resulting FRW models. However, the observed universe is obviously neither homogeneous nor isotropic. So these symmetries can only be approximate. There are theoretical arguments [11, 27] and recent experimental data regarding cosmic background radiation anisotropies which support the existence of an anisotropic phase that approaches an isotropic one [22]. These observations led us to consider more general anisotropic cosmologies, whilst retaining the assumption of (large scale) spatial homogeneity. Spatially homogeneous and anisotropic cosmological models which provide a richer structure, both geometrically and physically, than the FRW model play significant role in the description of early universe. Bianchi type V models being anisotropic generalization of open FRW models are interesting to study. These models are favoured by the available evidences for low density universes. Bianchi type V cosmological models have been investigated by Collins [15], Farnsworth [18], Maartens and Nel [25], Wainwright et al. [50]. Coley [14] has investigated Bianchi type V imperfect cosmological model. Bianchi type V bulk viscous cosmological models have also been studied by Bali and Singh [4], Pradhan and
Yadav [35], Singh and Chaubey [47].

In the present paper, we examine the possibility of the following three cases of phenomenological decay of \( \Lambda \) in the background of Bianchi type V space-time with bulk viscous fluid source:

**Case 1:** \( \Lambda \sim H^2 \)
**Case 2:** \( \Lambda \sim H \)
**Case 3:** \( \Lambda \sim \rho \).

Here \( H \) and \( \rho \) are, respectively, the Hubble parameter and matter energy density of the Bianchi type V space-time. The dynamical laws for decay of \( \Lambda \) have been widely studied by Arbab [1, 2], Carvalho et al. [9], Chen and Wu [10], Schutzhold [40, 41], Vishwakarma [48] to name only a few.

### 2. Metric and Field Equations

We consider the Bianchi type V space-time in orthogonal form represented by the line element

\[
ds^2 = -dt^2 + A^2(t)dx^2 + e^{2\alpha x} \left\{ B^2(t)dy^2 + C^2(t)dz^2 \right\}.
\]  

(1)

We assume the cosmic matter consisting of bulk viscous fluid given by the energy-momentum tensor

\[
T_{ij} = (\rho + \bar{p})v_iv_j + \bar{p}g_{ij},
\]

(2)

with

\[
\bar{p} = p - \zeta v_i^i,
\]

(3)

where \( \rho \) is the energy density of matter, \( p \) is the isotropic pressure, \( \zeta \) is the coefficient of bulk viscosity and \( v_i \), the four-velocity vector of the fluid satisfying \( v_i v^i = -1 \). The semicolon stands for covariant differentiation. On thermodynamical grounds bulk viscous coefficient \( \zeta \) is positive, assuring that the viscosity pushes the dissipative pressure \( \bar{p} \) towards negative values. But correction to the thermodynamical pressure \( p \) due to bulk viscous pressure is very small. Therefore, the dynamics of cosmic evolution does not change fundamentally by the inclusion of viscous term in the energy momentum tensor.

The Einstein’s field equations with time-varying cosmological term \( \Lambda (t) \) are given by

\[
R^i_j - \frac{1}{2} R g^i_j = -8\pi G T^i_j + \Lambda g^i_j.
\]

(4)

We use comoving system of reference so that \( v_i = -\delta_{i4} \). The field equations (4) for the Bianchi type V space-time lead to

\[
8\pi G \bar{p} - \Lambda = \frac{\alpha^2}{A^2} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} - \frac{\dot{B}\dot{C}}{BC},
\]

(5)

\[
8\pi G \bar{p} - \Lambda = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\dot{C}}{C} - \frac{\dot{A}\dot{C}}{AC},
\]

(6)

\[
8\pi G \bar{p} - \Lambda = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{A}\dot{B}}{AB},
\]

(7)
\[ 8\pi G \rho + \Lambda = \frac{3\alpha^2}{A^2} + \frac{\dot{A} B}{AB} + \frac{\dot{B} C}{BC} + \frac{\dot{A} C}{AC}, \]  
(8)

\[ 0 = \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}, \]  
(9)

where an overhead dot (\(\cdot\)) denotes ordinary differentiation with respect to cosmic time \(t\).

Covariant divergence of (4) gives

\[ \dot{\rho} + (\rho + \bar{p}) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{\dot{\Lambda}}{8\pi G} = 0. \]  
(10)

We observe that for a constant \(\Lambda\), equation (10) reduces to the equation of continuity. In view of energy conservation, equation (10) shows that a decaying vacuum term \(\Lambda\) transfers energy continuously to the matter component. The effective time-dependent cosmological term is regarded as second fluid component with energy density \(\rho_v = \frac{\Lambda(t)}{8\pi G}\), where \(\rho_v\) is the vacuum energy density. We assume that the non-vacuum component of matter obeys the equation of state

\[ p = \omega \rho, \quad \omega \in [0, 1]. \]  
(11)

To write metric functions explicitly, we introduce the average scale factor \(R\) of Bianchi type V space-time defined by \(R^3 = ABC\). From equations (5)–(7) and (9), we obtain

\[ \frac{\dot{A}}{A} = \frac{\dot{R}}{R}, \]  
(12)

\[ \frac{\dot{B}}{B} = \frac{\dot{R}}{R} - \frac{k_1}{R^3}, \]  
(13)

\[ \frac{\dot{C}}{C} = \frac{\dot{R}}{R} + \frac{k_1}{R^3}, \]  
(14)

where \(k_1\) is constant of integration. Equations (12)–(14), on integration yield

\[ A = m_1 R, \]  
(15)

\[ B = m_2 R \exp \left( -k_1 \int \frac{dt}{R^3} \right), \]  
(16)

\[ C = m_3 R \exp \left( k_1 \int \frac{dt}{R^3} \right), \]  
(17)

where \(m_1, m_2\) and \(m_3\) are constants of integration satisfying \(m_1 m_2 m_3 = 1\). Using suitable coordinate transformations, constants \(m_2\) and \(m_3\) can be absorbed. Therefore, \(m_2\) and \(m_3\) can be taken to be 1 implying \(m_1 = m_2 = m_3 = 1\).

We introduce the dynamical scalars such as volume expansion \(\theta\) and shear scalar \(\sigma\) as usual

\[ \theta = v^i_i, \quad \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}, \]  
(18)
where $\sigma_{ij}$ is the shear tensor defined by
\[
\sigma_{ij} = \frac{1}{2} \left( v_{i;\alpha} h_j^{\alpha} + v_{j;\alpha} h_i^{\alpha} \right) - \frac{1}{3} \theta h_{ij}.
\] (19)

Here $h_{ij}$ is the projection tensor given by
\[
h_{ij} = g_{ij} + v_i v_j.
\] (20)

For the Bianchi type V metric, the dynamical scalars have the form
\[
\theta = 3 \frac{\dot{R}}{R},
\] (21)
\[
\sigma = k_1 \frac{\dot{R}}{R^3}.
\] (22)

In analogy with FRW universe, we define generalized Hubble parameter $H$ and generalized deceleration parameter $q$ as
\[
H = \frac{\dot{R}}{R},
\] (23)
\[
q = - \frac{\ddot{R}}{R \dot{H}^2}.
\] (24)

We can write equations (5)–(8) and (10) in terms of $H$, $\sigma$ and $q$ as
\[
8\pi G \bar{p} - \Lambda = \frac{\alpha^2}{R^2} + H^2 (2q - 1) - \sigma^2,
\] (25)
\[
8\pi G \rho + \Lambda = - \frac{3\alpha^2}{R^2} + 3H^2 - \sigma^2,
\] (26)
\[
\dot{\rho} + 3(\rho + \bar{p}) H + \frac{\dot{\Lambda}}{8\pi G} = 0.
\] (27)

From equation (26), we get
\[
\frac{\sigma^2}{\theta^2} = \frac{1}{3} - \frac{8\pi G \rho}{\theta^2} - \frac{3\alpha^2}{R^2 \theta^2} - \frac{\Lambda}{\theta^2}.
\] (28)

Therefore $0 < \frac{\sigma^2}{\theta^2} < \frac{1}{3}$ and $0 < \frac{8\pi G \rho}{\theta^2} < \frac{1}{3}$ for $\Lambda \geq 0$. Thus a positive $\Lambda$ restricts the upper limit of anisotropy whereas a negative $\Lambda$ will increase the anisotropy. From the equations (25) and (26), we obtain
\[
\frac{d\theta}{dt} = -4\pi G (\rho + 3p) - 2\sigma^2 - \frac{\theta^2}{3} + 12\pi G \zeta \theta + \Lambda,
\] (29)

which is the Raychaudhuri equation for the given distribution. We observe that for $\Lambda \leq 0$ and $\zeta = 0$, the universe will always be in decelerating phase provided the strong energy condition [19] holds. In this case, we have
\[
\frac{d\theta}{dt} \leq -\frac{\theta^2}{3},
\] (30)
which integrates to give
\[
\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{t}{3},
\]
(31)
where \(\theta_0\) is the initial value of \(\theta\). If \(\theta_0 < 0\), \(\theta\) will diverge \((\theta \to -\infty)\) for \(t < \frac{3}{|\theta_0|}\). From equation (29), one also concludes that the presence of viscosity and a positive \(\Lambda\) will slow down the rate of decrease of volume expansion. Again from equation (22), we get
\[
\dot{\sigma} = -3\sigma H.
\]
(32)
Thus, the energy density associated with the anisotropy \(\sigma\) decays rapidly in an evolving universe and it becomes negligible for infinitely large values of \(R\). From equations (25) and (26), we obtain
\[
\frac{\dot{R}}{R} = -\frac{4}{3}\pi G(\rho + 3p) - \frac{2}{3}\sigma^2 + 4\pi G\zeta\theta + \frac{\Lambda}{3}.
\]
(33)
We observe that the positive cosmological term and bulk viscosity contribute positively in driving the acceleration of the universe. Also from equation (26), we get
\[
\frac{3\dot{R}^2}{R^2} = \frac{3\alpha^2}{R^2} + \sigma^2 + 8\pi G\rho + \Lambda.
\]
(34)
When \(\Lambda \geq 0\), each term on the right hand side of (34) is non-negative. Thus \(\dot{R}\) does not change sign and we get ever-expanding models. For \(\Lambda < 0\), however, we can get universes that expand and then recontract. From equation (10), we obtain
\[
R^{-3(\omega+1)} \frac{d}{dt} \left\{ \rho R^{3(\omega+1)} \right\} = 9\zeta H^2 - \frac{\dot{\Lambda}}{8\pi G}.
\]
(35)
Thus, decaying vacuum energy and viscosity of the fluid lead to matter creation.

Over the years, it has been difficult and fascinating problem for cosmologists to explain the expansion history of the universe. To describe the dynamics of the universe, Hubble parameter \(H\) and deceleration parameter \(q\) are important observational quantities. The present value \(H_0\) of Hubble parameter sets the present time scale of the expansion while \(q_0\), the present day value of deceleration parameter tells us that the expansion of the present universe is accelerating rather than going to decelerate as expected before the supernovae of Ia observations [3, 33, 36, 37]. From equations (12)–(14), we observe that scale factors are completely characterized by the Hubble parameter \(H\). Therefore, we assume a relation between Hubble parameter \(H\) and average scale factor \(R\) given by
\[
H = k R^{-m}
\]
(36)
where \(k > 0\) and \(m \geq 0\) are constants. Such a relation has already been discussed by Berman [5], Berman and Gomide [6] in case of FRW models that yields a constant value of deceleration parameter. Models with constant deceleration parameter have also been studied by a number of authors [20, 26, 34, 42–44] for FRW and Bianchi cosmology. For the relation (36), deceleration parameter \(q\) comes out to be constant i.e.
\[
q = m - 1.
\]
(37)
The equation (37) shows that the universe is decelerating for \( m > 1 \) and it represents an accelerating universe for \( m < 1 \). When \( m = 1 \), we obtain \( H = \frac{1}{T} \) and \( q = 0 \). Therefore galaxies move with constant speed and the model represents anisotropic Milne universe [23] for \( m = 1 \). For \( m = 0 \), we get \( H = k \) and \( q = -1 \). Thus Hubble parameter \( H \), being constant in time, equals to its present value \( H_0 \) and the model describes accelerated phase of the universe.

Equation (36) integrates to give

\[
R = (mkt + t_1)^\frac{1}{m} \quad \text{for} \quad m \neq 0
\]

and

\[
R = \exp\{k(t - t_0)\} \quad \text{for} \quad m = 0,
\]

where \( t_1 \) and \( t_0 \) are constants of integration.

Equation (38) along with (15)–(17) gives

\[
A = (mkt + t_1)^\frac{1}{m},
\]

\[
B = (mkt + t_1)^\frac{1}{m} \exp\left\{ -\frac{k_1(mkt + t_1)^{\frac{m-3}{m}}}{k(m-3)} \right\},
\]

\[
C = (mkt + t_1)^\frac{1}{m} \exp\left\{ \frac{k_1(mkt + t_1)^{\frac{m-3}{m}}}{k(m-3)} \right\}.
\]

For this solution, the metric (1) assumes the following form after suitable transformation of coordinates

\[
ds^2 = -dT^2 + (mkT)^\frac{2}{m}dX^2
\]
\[
+ \exp\left\{ 2\alpha X - \frac{2k_1(mkT)^{\frac{m-3}{m}}}{k(m-3)} \right\} dY^2
\]
\[
+ \exp\left\{ 2\alpha X + \frac{2k_1(mkT)^{\frac{m-3}{m}}}{k(m-3)} \right\} dZ^2.
\]

Equations (15)–(17) with the use of equation (39) yield

\[
A = \exp\{k(t - t_0)\},
\]

\[
B = \exp\left\{ k(t - t_0) + \frac{k_1}{3k}e^{-3k(t-t_0)} \right\},
\]

\[
C = \exp\left\{ k(t - t_0) - \frac{k_1}{3k}e^{-3k(t-t_0)} \right\}.
\]

The line-element (1) for this solution can be written as

\[
ds^2 = -dT^2 + e^{2kT}dX^2
\]
\[
+ \exp\left( 2kT + 2\alpha X + \frac{2k_1}{3k}e^{-3kT} \right) dY^2
\]
\[
+ \exp\left( 2kT + 2\alpha X - \frac{2k_1}{3k}e^{-3kT} \right) dZ^2.
\]
3. Discussion

We now discuss the models resulting from different dynamical laws for the decay of $\Lambda$.

3.1

For the model (43), average scale factor $R$ is given by

$$R = (mT)^{-\frac{1}{m}}.$$  

(48)

Volume expansion $\theta$, Hubble parameter $H$ and shear scalar $\sigma$ for this model are:

$$\theta = 3H = \frac{3}{mT},$$  

(49)

$$\sigma = k_1(mT)^{-\frac{3}{m}}.$$  

(50)

We observe that the model is not tenable for $m = 0$ and $m = 3$. For $m < 3$, $\frac{\sigma}{\theta} \to 0$ as $T \to \infty$. Therefore, the model approaches isotropy asymptotically.

3.1.1 Case 1:

We consider

$$\Lambda = 3\beta H^2,$$  

(51)

where $\beta$ is a constant. From equations (11), (25) and (26), we obtain

$$8\pi G \rho = \frac{3 - 3\beta}{m^2T^2} - \frac{3\alpha^2}{(mT)^{\frac{2}{m}}} - \frac{k_1^2}{(mT)^{\frac{2}{m}}},$$  

(52)

$$24\pi G \zeta = \frac{3(1 + \omega)(1 - \beta) - 2m}{mT} - \frac{(1 + 3\omega)m\alpha^2T}{(mT)^{\frac{2}{m}}} + \frac{(1 - \omega)k_1^2mT}{(mT)^{\frac{2}{m}}},$$  

(53)

$$\Lambda = \frac{3\beta}{m^2T^2}.$$  

(54)

We observe that the model has singularity at $T = 0$. It starts with a big bang from its singular state at $T = 0$ and continues to expand till $T = \infty$. At $T = 0$, $\rho$, $p$, $\Lambda$, $\zeta$ are all infinite and they become negligible for large values of $T$. Therefore, for large times, the model represents a non-rotating, shearing and expanding universe having big bang start and approaches isotropy asymptotically.

3.1.2 Case 2:

We assume

$$\Lambda = aH,$$  

(55)
where $a$ is a positive constant. For this choice, we obtain

\[
8\pi G\rho = \frac{3}{m^2T^2} - \frac{3\alpha^2}{(mkT)^{\frac{2}{n}}} - \frac{k_1^2}{(mkT)^{\frac{2}{n}}} - \frac{a}{mT},
\]

(56)

\[
24\pi G\zeta = \frac{3(1 + \omega) - 2m}{mT} - \frac{(1 + 3\omega)m\alpha^2T}{(mkT)^{\frac{2}{n}}} + \frac{(1 - \omega)mk_1^2T}{(mkT)^{\frac{2}{n}}} - (1 + \omega)a,
\]

(57)

\[
\Lambda = \frac{a}{mT}.
\]

(58)

This model also has singularity at $T = 0$. It evolves from its singular state at $T = 0$ with $\rho$, $p$, $\Lambda$, $\zeta$ all diverging and expansion in the model becomes zero for $T \to \infty$. We observe that the vacuum energy in this case decays slowly than the case 1.

### 3.1.3 Case 3:

We now consider

\[
\Lambda = 8\pi G\gamma\rho,
\]

(59)

where $\gamma$ is a constant. In this case, we obtain

\[
8\pi G(1 + \gamma)\rho = \frac{3}{m^2T^2} - \frac{3\alpha^2}{(mkT)^{\frac{2}{n}}} - \frac{k_1^2}{(mkT)^{\frac{2}{n}}},
\]

(60)

\[
24\pi G(1 + \gamma)\zeta = \frac{3(1 + \omega) - 2m(1 + \gamma)}{mT} - \frac{(1 + 3\omega - 2\gamma)m\alpha^2T}{(mkT)^{\frac{2}{n}}} + \frac{(1 - \omega + 2\gamma)mk_1^2T}{(mkT)^{\frac{2}{n}}},
\]

(61)

\[
\left(1 + \frac{1}{\gamma}\right)\Lambda = \frac{3}{m^2T^2} - \frac{3\alpha^2}{(mkT)^{\frac{2}{n}}} - \frac{k_1^2}{(mkT)^{\frac{2}{n}}}.
\]

(62)

This model also starts expanding with a big bang at $T = 0$ with $\rho$, $p$, $\Lambda$, $\zeta$ all infinite and expansion in the model ceases at $T = \infty$. The bulk viscosity coefficient, being infinitely large at the initial singularity decreases with time. Matter density $\rho$ and cosmological term $\Lambda$ also decrease in the course of expansion to become zero for large times.

### 3.2

For the model (47), average scale factor $R$, expansion scalar $\theta$, Hubble parameter $H$, shear scalar $\sigma$ and deceleration parameter $q$ are given by

\[
R = e^{kT},
\]

(63)
\[ \theta = 3H = 3k, \]  
\[ \sigma = k_1 e^{-3kT}, \]  
\[ q = -1. \]  

The energy density \( \rho \) and bulk viscosity \( \zeta \) have the expressions:

\[ 8\pi G\rho = 3k^2 - 3\alpha^2 e^{-2kT} - k_1^2 e^{-6kT} - \Lambda \]  
\[ 24\pi G k\zeta = 3(\omega + 1)k^2 - (3\omega + 1)\alpha^2 e^{-2kT} + (1 - \omega)k_1^2 e^{-6kT}(\omega + 1)\Lambda. \]

We observe that \( \Lambda \sim H^2 \) and \( \Lambda \sim H \), give \( \Lambda \) to be constant because Hubble parameter \( H \) is constant. Therefore, in these cases we obtain the model similar to the model (40) considered by Singh and Baghel [44].

For the case \( \Lambda = 8\pi G\gamma\rho \), from equations (67) and (68), we obtain

\[ 8\pi G (1 + \gamma)\rho = 3k^2 - 3\alpha^2 e^{-2kT} - k_1^2 e^{-6kT}, \]  
\[ 24\pi G (1 + \gamma)k\zeta = 3(1 + \omega)k^2 - (1 + 3\omega - 2\gamma)\alpha^2 e^{-2kT} + (1 - \omega + 2\gamma)k_1^2 e^{-6kT}, \]  
\[ \left(1 + \frac{1}{\gamma}\right) \Lambda = 3k^2 - 3\alpha^2 e^{-2kT} - k_1^2 e^{-6kT}. \]  

The model has no initial singularity. Expansion in the model starts at \( T = 0 \) with \( \rho \), \( \theta \), \( \sigma \), \( \Lambda \) and \( \zeta \) all finite. The expansion scalar \( \theta \) is constant throughout the expansion. Therefore the model represents uniform expansion. For large values of \( T \), matter density \( \rho \), bulk viscosity \( \zeta \) and cosmological term \( \Lambda \) remain non-zero constants and anisotropy \( \sigma/\theta \) becomes zero. Therefore the model approaches isotropy. For this model, deceleration parameter \( q = -1 \). Therefore, the model represents an accelerating universe. Thus, the model behaves in accordance with cosmological observations which indicate that the universe has entered a phase of accelerating expansion. This model represents a non-singular, shearing and accelerating universe which becomes isotropic for large times.

**Conclusion**

Anisotropic Bianchi type V cosmological models with bulk viscous fluid and time varying cosmological term are investigated by assuming a variation law for the Hubble’s parameter that yields a constant value of deceleration parameter. Universe models for \( m \neq 0 \) and \( m = 0 \) have been derived. Three different decay laws for the cosmological term have been discussed in the context of models obtained. We observe that for \( m \neq 0 \), the model starts with a big bang at \( T = 0 \) where cosmological parameters diverge. It becomes isotropic for large values of \( T \), provided \( m < 3 \). The cosmological term \( \Lambda \) being infinite at the initial singularity becomes negligible for large times.
When \( m = 0 \), we obtain a non-singular model representing accelerated phase of the universe. It evolves with finite values of kinematical parameters and expands uniformly. The model approaches isotropy for large values of \( T \). In this model, matter density \( \rho \), bulk viscosity \( \zeta \) and cosmological term \( \Lambda \) remain non-zero for \( T \to \infty \).

From equation (37), one concludes that for \( m > 1 \), the model represents a decelerating universe and for \( 0 \leq m < 1 \), it gives rise to an accelerating universe. When \( m = 1 \), we obtain \( H = \frac{1}{T} \) and \( q = 0 \) so that every galaxy moves with constant speed. Therefore, for \( m = 1 \), we recovers an anisotropic Milne model [23].

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**References**

Corrections to Massive Neutrino Masses, Caused by Vacuum Polarisation in Strong Coulomb Field of Daughter Nuclei in Weak Decays Of Heavy Ions

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Abstract: We calculate corrections to masses of massive neutrino mass–eigenstates, caused by vacuum polarization in the strong Coulomb fields of daughter heavy nuclei in the K–shell electron capture decays (EC) and positron (β+) decays of highly ionized heavy ions, investigated experimentally at GSI in Darmstadt. Some applications of the obtained results are discussed. © Electronic Journal of Theoretical Physics. All rights reserved.

Keywords: Neutrino Masses; Neutrino Interactions; Weak-interaction; Solar Neutrinos

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1. Introduction

Nowadays the existence of massive neutrinos, neutrino–flavour mixing and neutrino oscillations is well established experimentally and elaborated theoretically [1].

The K–shell electron capture (EC) decays and positron (β+) decays of the H–like heavy ions [2]–[5] is a nice tool for the investigation of the hyperfine structure of highly

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ionized heavy ions [6]–[8]. In these decays massive neutrinos in the final state appear in a coherent superposition of the electron neutrino $|\nu_e\rangle = \sum_j U_{ej}^* |\nu_j\rangle$, where $U_{ej}$ are elements of the unitary mixing matrix $U$ and $|\nu_j\rangle$ are neutrino mass–eigenstates with masses $m_j$ [1]. The reactions of the EC–decays and $\beta^+$–decays of the H–like heavy ions can be conventionally defined as follows

$$A X^{(Z-1)+} \rightarrow A Y^{(Z-1)+} + \nu_e,$$
$$A X^{(Z-1)+} \rightarrow A Y^{(Z-2)+} + e^+ + \nu_e,$$  

for example, $^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+} + \nu_e$ and $^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{57+} + e^+ + \nu_e$ and so on [2]–[5].

The description of the electron neutrino in terms of a coherent superposition of neutrino mass–eigenstates $|\nu_e\rangle = \sum_j U_{ej}^* |\nu_j\rangle$ implies that in the final state massive neutrinos move in the strong Coulomb field of the daughter ions $A Y^{(Z-1)+}$ and $A Y^{(Z-2)+}$. According to modern theory of weak interactions [1], neutrino mass–eigenstates can be virtually in the dissociated states $\nu_j \rightarrow \sum_\ell U_{j\ell} \ell^- W^+$, where $\ell^-$ is the electron $e^-$, negatively charged muon $\mu^-$ or $\tau^-$–lepton and $W^+$ is the $W$–boson of the Weinberg–Salam theory of electroweak interactions. The interaction of electrically charged particles with the Coulomb fields of the daughter ions $A Y^{(Z-1)+}$ and $A Y^{(Z-2)+}$ can be expressed in terms corrections to masses of neutrino mass–eigenstates. This effect is similar to vacuum polarization induced in a strong Coulomb field [9]–[11].

In this paper we calculate corrections to masses of massive neutrinos caused by strong Coulomb fields of highly charged daughter ions and discuss one of the possible applications of these results - the calculation of masses of neutrino mass–eigenstates, using the experimental data on the antineutrino oscillations by KamLAND [12] and experimental data on a time modulation of the EC–decay rates by GSI [3]–[5].

2. Corrections to Massive Neutrino Masses From Vacuum Polarisation in Strong Coulomb Field of Highly Charged Nuclei

A virtual dissociation of massive neutrinos into $\ell^- W^+$ pairs leads to self–energy corrections, which can be defined by Feynman diagrams shown in Fig. 1.

Since the $W$–boson is a very heavy particle and its mass is much greater than exchange energies, one can use the heavy–boson approximation replacing the propagator of the $W$–boson by the $\delta$–function and shrinking the line of a virtual $W$–boson into a point. Since we investigate a dissociation of massive neutrinos into $\ell^- W^+$ pairs in strong Coulomb fields of highly charged daughter ions, in a heavy–boson approximation the analysis of an influence of a strong Coulomb field on massive neutrinos reduces to a research of virtual leptons $\ell^-$, moving in a strong Coulomb field. This effect is similar to vacuum polarization caused by a strong attractive Coulomb field, leading to an induced electric charge [9, 10, 11]. In the case of massive neutrinos, vacuum polarization in strong Coulomb fields of highly charged heavy daughter ions, produced in weak decays of highly ionized heavy ions, leads to corrections to masses of massive neutrinos. Following [10, 11], we describe virtual
negatively charged leptons $\ell^-$ by exact Green functions in a strong attractive Coulomb field.

For the calculation of the diagrams in Fig. 1 we use weak leptonic interaction [1]

$$\mathcal{L}_W(x) = -\frac{G_F}{\sqrt{2}} \sum_j \sum_{\ell} U_{j\ell} U_{\ell j}^* \left[ \bar{\psi}_{\nu_j}(x) \gamma^\mu (1 - \gamma^5) \psi_{\ell}(x) \right] \left[ \bar{\psi}_\ell(x) \gamma_\mu (1 - \gamma^5) \psi_{\nu_j}(x) \right],$$

(2)

defined by the W–boson exchange, where $x = (t, \vec{r})$, $G_F$ is the Fermi constant, $\psi_{\nu_j}(x)$ and $\psi_{\ell}(x)$ are operators of the neutrino $\nu_j$ and lepton fields $\ell = e^-, \mu^-$ and $\tau^-$, respectively, and $U_{ij}$ are the elements of the unitary neutrino–flavour mixing matrix $U$ [1]. In our analysis neutrinos $\nu_j (j = 1, 2, 3)$ are Dirac particles with masses $m_j (j = 1, 2, 3)$, respectively [1].

The self–energy correction for the neutrino mass–eigenstate $\nu_j$ due to weak leptonic interaction Eq.(2) is defined by [13]

$$-\delta \Sigma_{\nu_j} = \lim_{T \to \infty} \frac{d^4 x}{T} \langle \nu_j(\vec{k}_j, \sigma_j) | \mathcal{L}_W(x) | \nu_j(\vec{k}_j, \sigma_j) \rangle. \tag{3}$$

Making Fierz transformation and calculating vacuum expectation values of field operators of negatively charged leptons [13] we get the following expression for the matrix element $\langle \nu_j(\vec{k}_j, \sigma_j) | \mathcal{L}_W(x) | \nu_j(\vec{k}_j, \sigma_j) \rangle$ of the $\nu_j \to \nu_j$ transition

$$\langle \nu_j(\vec{k}_j, \sigma_j) | \mathcal{L}_W(x) | \nu_j(\vec{k}_j, \sigma_j) \rangle =$$

$$= -\sqrt{2} G_F \sum_{\ell} U_{j\ell} \bar{\psi}_{\nu_j}(\vec{r}, 0) \psi_{\nu_j}(\vec{r}, 0) i \int \frac{dE}{2\pi} \text{tr} \{ \gamma^0 G_{\ell}(\vec{r}, \vec{r}; E) \}, \tag{4}$$

where $\psi_{\nu_j}(\vec{r}, 0)$ is a wave function of a massive neutrino and $G_{\ell}(\vec{r}, \vec{r}; E)$ is the energy–dependent Green function of a negatively charged lepton $\ell^-$ in a strong Coulomb field, produced by a positive electric charge $Ze$ of a daughter ion [10, 11].

Substituting the matrix element Eq.(4) into the r.h.s. of Eq.(3) we arrive at the following expression for the self–energy correction

$$\delta \Sigma_{\nu_j} = \int d^3 x \bar{\psi}_{\nu_j}(\vec{r}, 0) \delta m_j(r) \psi_{\nu_j}(\vec{r}, 0), \tag{5}$$

where $\delta m_j(r)$, the correction to mass of massive neutrino $\nu_j$ caused by vacuum polarization in a strong Coulomb field, is given by

$$\delta m_j(r) = \sum_{\ell} U_{j\ell} U^*_{\ell j} \mathcal{M}_{\ell}(r) \tag{6}$$
with $\mathcal{M}_\ell(r)$ defined by

$$
\mathcal{M}_\ell(r) = i \sqrt{2} G_F \frac{m_\ell}{\pi^2 r^2} \sum_{n=1}^{\infty} n \int_0^\infty dx \int_0^\infty dy \ e^{-2m_\ell r \sqrt{x^2+1} \coth y} \left\{ 2Z \alpha y \cos \left( \frac{2Z \alpha xy}{\sqrt{x^2+1}} \right) \right. \\
\times \tilde{I}_{2\nu} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) - \sin \left( \frac{2Z \alpha xy}{\sqrt{x^2+1}} \right) \left[ 2m_\ell r \tilde{I}_{2\nu+1} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) + \frac{2\nu x}{\sqrt{x^2+1}} \right] \\
\times \tilde{I}_{2\nu} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) \right\},
$$

Taking into account the results, obtained in [11], we get

$$
\mathcal{M}_\ell(r) = \sqrt{2} G_F \frac{m_\ell}{\pi^2 r^2} \sum_{n=1}^{\infty} n \int_0^\infty dx \int_0^\infty dy \ e^{-2m_\ell r \sqrt{x^2+1} \coth y} \left\{ 2Z \alpha y \cos \left( \frac{2Z \alpha xy}{\sqrt{x^2+1}} \right) \right. \\
\times \tilde{I}_{2\nu} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) - \sin \left( \frac{2Z \alpha xy}{\sqrt{x^2+1}} \right) \left[ 2m_\ell r \tilde{I}_{2\nu+1} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) + \frac{2\nu x}{\sqrt{x^2+1}} \right] \\
\times \tilde{I}_{2\nu} \left( \frac{2m_\ell r \sqrt{x^2+1}}{\sinh y} \right) \right\}, \tag{8}
$$

where $\nu = \sqrt{n^2 - (Z\alpha)^2}$, $\tilde{I}_{2\nu+1}(z) = I_{2\nu+1}(z) - I_{2n+1}(z)$ and $\tilde{I}_{2\nu}(z) = I_{2\nu}(z) - I_{2n}(z)$ and $I_{\nu}(z)$ are modified Bessel functions [14], Using the integral representations for the Bessel functions we determine $\tilde{I}_{2\nu}(z)$ as

$$
\tilde{I}_{2\nu}(z) = I_{2\nu+1}(z) - I_{2\nu}(z) = \\
= \frac{2}{\pi} \int_0^\pi e^{z \cos \theta} \sin \left( (n + \sqrt{n^2 - (Z\alpha)^2}) \theta \right) \sin \left( \frac{(Z\alpha)^2}{n + \sqrt{n^2 - (Z\alpha)^2}} \theta \right) d\theta \\
- \frac{(-1)^n}{\pi} \sin \left( \frac{(Z\alpha)^2}{n + \sqrt{n^2 - (Z\alpha)^2}} \right) \int_0^\infty e^{-z \cosh \xi - \xi \sqrt{n^2 - (Z\alpha)^2}} d\xi \sim O((Z\alpha)^2). \tag{9}
$$

Following [9, 11] we have renormalised the self–energy correction having subtracted the contribution of order of $O(Z\alpha)$. This is similar to renormalisation of a total induced vacuum–polarization “charge” in a strong attractive Coulomb field [9, 11]. This results in an induced vacuum polarisation leading to mass–corrections to massive neutrino masses of order of $O((Z\alpha)^3)$ [9, 11].

A total renormalised mass of neutrino mass–eigenstate $\nu_j$ is equal to $m_j(r) = m_j + \delta m_j(r)$, where $m_j$ is a renormalised proper mass of massive neutrino $\nu_j$ and $\delta m_j(r)$ is a neutrino mass–correction, induced by vacuum polarisation in a strong Coulomb field. It vanishes at $Z\alpha \to 0$ as $O((Z\alpha)^3)$. As a result a total energy of massive neutrino $\nu_j$, moving with a momentum $\vec{k}_j$ in a strong Coulomb field of highly charged daughter ions, produced in weak decays of highly ionized heavy ions, is $E_j(r) = \sqrt{\vec{k}_j^2 + m_j^2(r)}$.

### 3. Numerical Values of Mass–Corrections of Massive Neutrinos, Caused by Coulomb Interaction

Using the definition of matrix elements $U_{\ell j}$ of mixing matrix $U$ in terms of mixing angles [1] we obtain the following expressions for the corrections to massive neutrino masses

$$
\delta m_1(r) = \cos^2 \theta_{12} \mathcal{M}_{e-}(r) + \sin^2 \theta_{12} \cos^2 \theta_{23} \mathcal{M}_{\mu-}(r) + \sin^2 \theta_{23} \mathcal{M}_{\tau-}(r), \\
\delta m_2(r) = \sin^2 \theta_{12} \mathcal{M}_{e-}(r) + \cos^2 \theta_{12} \cos^2 \theta_{23} \mathcal{M}_{\mu-}(r) + \sin^2 \theta_{23} \mathcal{M}_{\tau-}(r), \tag{10}
$$
Fig. 2 The corrections to the neutrino masses, caused by a strong nuclear Coulomb field, where $\delta m_1(r)$ and $\delta m_2(r)$ are presented by the solid and dotted line, respectively.

where $\theta_{12}$ and $\theta_{23}$ are mixing angles [1]. The corrections to the neutrino masses Eq.(8) are defined for the mixing angle $\theta_{13} = 0$, which is very close to the experimental value, defined by upper limit $\sin^2\theta_{13} < 0.032$ [1], and usually used in theoretical analysis of neutrino oscillations [17]. For numerical estimates we set $\theta_{12} = 34^0$ and $\theta_{23} = 45^0$ [1] (see also [18]). The neutrino mass–corrections $\delta m_j(r)$, induced by strong Coulomb fields of highly charged daughter ions in the weak decays of the H–like heavy ions $^{140}\text{Pr}^{58+}$, are shown in Fig. 2.

In weak decays of heavy nuclei neutrinos are emitted from the surface of nuclei at nuclear radius $R$ [19, 15]. Hence, according to [19, 15], corrections to neutrino masses should be calculated at $r = R$. Setting $R = 1.1 \times A^{1/3}$ [20] (see also [7]) we obtain the corrections to neutrino masses $\delta m_1(R)$ and $\delta m_2(R)$ adduced in Table 1.

<table>
<thead>
<tr>
<th>$^{AX(Z-1)+}$</th>
<th>$\delta m_1(R) \times 10^4\text{ eV}/c^2$</th>
<th>$\delta m_2(R) \times 10^4\text{ eV}/c^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{122}\text{I}^{52+}$</td>
<td>$-12.42$</td>
<td>$-6.94$</td>
</tr>
<tr>
<td>$^{140}\text{Pr}^{58+}$</td>
<td>$-14.74$</td>
<td>$-8.22$</td>
</tr>
<tr>
<td>$^{142}\text{Pm}^{60+}$</td>
<td>$-16.20$</td>
<td>$-9.00$</td>
</tr>
</tbody>
</table>

Table 1 Numerical values for neutrino mass–corrections, caused by vacuum polarisation in the strong Coulomb field of daughter ions, calculated for $R = 1.1 \times A^{1/3}$ [20] (see also [7]).

The mass–corrections are calculated for massive neutrinos produced in the $EC$–decays
and $\beta^+$-decays of the H–like heavy ions

\begin{align}
^{122}\text{I}\,^{52+} & \rightarrow ^{122}\text{Te}\,^{52+} + \nu_e, \\
^{140}\text{Pr}\,^{58+} & \rightarrow ^{140}\text{Ce}\,^{58+} + \nu_e, \\
^{142}\text{Pm}\,^{60+} & \rightarrow ^{142}\text{Nd}\,^{60+} + \nu_e, \\
^{142}\text{Pm}\,^{60+} & \rightarrow ^{142}\text{Te}\,^{52+} + \nu_e,
\end{align}

measured currently at GSI. The accuracy of the corrections to neutrino masses, given in Table 1, is of about 4%. It is caused by the experimental uncertainties of the mixing angles $\theta_{12} = 33.9^{+2.4}_{-2.2}$ degrees and $\theta_{23} \leq 45$ degrees [1]. The obtained results can be used for the analysis of neutrino energy spectra in weak decays of highly ionised heavy ions.

4. Neutrino Masses From GSI and KamLAND Experiments

One of the interesting applications of the obtained corrections to massive neutrino masses, caused by vacuum polarisation in strong Coulomb fields of highly charged daughter ions, is a calculation of neutrino masses by using the experimental data on a difference of squared values of neutrino masses, measured by KamLAND [12], and the experimental data on a time modulation of the rates of the number of daughter ions, produced in the \( EC \)–decays of the H–like heavy ions \( ^{142}\text{Pm}\,^{60+}, \, ^{140}\text{Pr}\,^{58+} \) and \( ^{122}\text{I}\,^{52+} \), measured at GSI [3]–[5].

Using the reactor antineutrinos $\bar{\nu}_e$ the experimental group at KamLAND has measured the antineutrino energy spectrum accounting for the $\bar{\nu}_e \leftrightarrow \bar{\nu}_e$ oscillations defined by the probability

\begin{equation}
P_{\bar{\nu}_e \leftrightarrow \bar{\nu}_e}(E_{\bar{\nu}_e}) = 1 - \sin^2(2\theta_{12}) \sin^2 \left( \frac{\Delta m_{21}^2 L}{4E_{\bar{\nu}_e}} \right),
\end{equation}

where $\Delta m_{21}^2 = m_2^2 - m_1^2$ is determined by the proper neutrino masses, $L = 180$ km is a distance between a source and a detector of antineutrinos and $E_{\bar{\nu}_e}$ is an antineutrino energy. Antineutrinos have been detected by the reaction $\bar{\nu}_e + p \rightarrow n + e^+$, having a threshold at $E_{\bar{\nu}_e}^{(th)} = 1.8$ MeV. The experimental value of $\Delta m_{21}^2 = m_2^2 - m_1^2$, deduced from the KamLAND experiment data on the antineutrino spectrum, which we call below as $(\Delta m_{21}^2)_{KL}$, is equal to $(\Delta m_{21}^2)_{KL} = 7.59(21) \times 10^{-5}$ eV\(^2\) [12].

The experimental investigation of the K–shell electron capture (\( EC \)) decays of the H–like heavy ions \( ^{142}\text{Pm}\,^{60+}, \, ^{140}\text{Pr}\,^{58+}, \) and \( ^{122}\text{I}\,^{52+} \)

\begin{align}
^{142}\text{Pm}\,^{60+} & \rightarrow ^{142}\text{Nd}\,^{60+} + \nu_e, \\
^{140}\text{Pr}\,^{58+} & \rightarrow ^{140}\text{Ce}\,^{58+} + \nu_e, \\
^{122}\text{I}\,^{52+} & \rightarrow ^{122}\text{Te}\,^{52+} + \nu_e,
\end{align}

has been recently carried out in the Experimental Storage Ring (ESR) at GSI in Darmstadt [3]–[5]. The measurements of the rates $dN_{EC}^d(t)/dt$ of the number $N_d^{EC}$ of daughter ions \( ^{142}\text{Nd}\,^{60+}, \, ^{140}\text{Ce}\,^{58+} \) and \( ^{122}\text{Te}\,^{52+} \) showed time modulation of the exponential decay
with periods and amplitudes

\[ T_{EC} = \begin{cases} 
7.10(22)s & a_{EC} = 0.23(4) \\
7.06(8)s & 1^{42}\text{Pm}^{58+} \rightarrow 1^{142}\text{Nd}^{60+} + \nu_e \\
6.11(3)s & 1^{140}\text{Pr}^{58+} \rightarrow 1^{140}\text{Ce}^{58+} + \nu_e \\
6.11(3)s & 1^{122}\text{I}^{52+} \rightarrow 1^{122}\text{Te}^{52+} + \nu_e.
\end{cases} \tag{14} \]

Since the rates of the number of daughter ions are defined by

\[ \frac{dN^E_{EC}(t)}{dt} = \lambda_{EC}(t) N_m(t), \tag{15} \]

where \( \lambda_{EC}(t) \) is the \( EC \)-decay rate and \( N_m(t) \) is the number of mother ions, time modulation of \( dN^E_{EC}(t)/dt \) implies a periodic time–dependence of the \( EC \)-decay rate \( \lambda_{EC}(t) \) \cite{3}–\cite{5}, which can be represented by the form

\[ \lambda_{EC}(t) = \lambda_{EC}(1 + a_{EC} \cos(\omega_{EC}t + \phi_{EC})), \tag{16} \]

where \( a_{EC}, T_{EC} = 2\pi/\omega_{EC} \) and \( \phi_{EC} \) are an amplitude, a period and a phase of a time–dependent term \cite{3}–\cite{5}.

As has been proposed in \cite{21} (see also \cite{8}), such a periodic time–dependence of the \( EC \)-decay rates can be explained by the differences of squared masses of the neutrino mass–eigenstates. A period of a time modulation \( T_{EC} \) has been obtained as

\[ T_{EC} = \frac{4\pi\gamma M_m}{\Delta m^2_{21}}, \tag{17} \]

where \( M_m \) is a mass of a mother ion, \( \gamma = 1.43 \) is the Lorentz factor of H–like heavy ions moving in the ESR \cite{3}. Then, \( \Delta m^2_{21} = \tilde{m}_2^2 - \tilde{m}_1^2 \) is a difference of squared neutrino masses \( \tilde{m}_2 \) and \( \tilde{m}_1 \), which can, in principle, include mass–corrections induced by strong Coulomb fields of highly charged daughter ions, i.e. \( \tilde{m}_2 = m_2 + \delta m_2(R) \) and \( \tilde{m}_1 = m_1 + \delta m_1(R) \).

Using experimental values of mother ion masses and experimental values of periods of time modulation Eq.(14) one can calculate \( \Delta m^2_{21} \), which we call \( \Delta m^2_{21}\)\text{GSI}. They are

\[ \Delta m^2_{21}\text{GSI} = \begin{cases} 
2.20(7) \times 10^{-4} \text{eV}^2 & 1^{142}\text{Pm}^{60+} \rightarrow 1^{142}\text{Nd}^{60+} + \nu_e \\
2.18(3) \times 10^{-4} \text{eV}^2 & 1^{140}\text{Pr}^{58+} \rightarrow 1^{140}\text{Ce}^{58+} + \nu_e \\
2.19(1) \times 10^{-4} \text{eV}^2 & 1^{122}\text{I}^{52+} \rightarrow 1^{122}\text{Te}^{52+} + \nu_e,
\end{cases} \tag{18} \]

which can be approximated well by an averaged value \( \Delta m^2_{21}\)\text{GSI} \( = 2.19 \times 10^{-4} \text{eV} \). Since \( M_m \simeq 931.494 \text{ A} \), where \( A \) is a mass number of a mother nucleus, periods of time modulation of the \( EC \)–decay rates in Eq.(17) can be defined by \( T_{EC} = A/20 \text{s} \), where the coefficient \( 1/20 \) is calculated as \( \kappa = 4\pi\gamma M_m/A(\Delta m^2_{21})\text{GSI} \simeq 1/20 \). It is seen that the formula \( T_{EC} = A/20 \text{s} \) fits well the experimental values of periods of time modulation.

The observed discrepancy between \( \Delta m^2_{21}\)\text{KL} \( = 7.59(21) \times 10^{-5} \text{eV}^2 \) and \( \Delta m^2_{21}\)\text{GSI} \( = 2.19 \times 10^{-4} \text{eV} \) can be really reconciled by taking into account corrections to neutrino
masses, induced by vacuum polarisation in strong Coulomb fields of highly charged daughter ions.

Since in GSI experiments on the $EC$-decays $^A X^{(Z-1)+} \rightarrow ^A Y^{(Z-1)+} + \nu_e$ massive neutrinos move in strong Coulomb fields of daughter ions $^A Y^{(Z-1)+}$, their masses should be corrected by vacuum polarisation induced by strong Coulomb fields. This means that $(\Delta m_{21}^2)_{\text{GSI}}$ should be taken in the form $(\Delta m_{21}^2)_{\text{GSI}} = (m_2 + \delta m_2(R))^2 - (m_1 + \delta m_1(R))^2$ as we have assumed above. In turn, in the KamLAND experiment a difference $(\Delta m_{21}^2)_{\text{KL}}$ is defined by the proper neutrino masses, i.e. $(\Delta m_{21}^2)_{\text{KL}} = m_2^2 - m_1^2$.

Combining these results together we arrive at a system of algebraical equations for $m_1$ and $m_2$

$$
\begin{align*}
(m_2 + \delta m_2(R))^2 - (m_1 + \delta m_1(R))^2 &= (\Delta m_{21}^2)_{\text{GSI}}, \\
m_2^2 - m_1^2 &= (\Delta m_{21}^2)_{\text{KL}},
\end{align*}
$$

the solution of which for the experimental values of $(\Delta m_{21}^2)_{\text{GSI}}$ and $(\Delta m_{21}^2)_{\text{KL}}$ gives the following values of neutrino masses, averaged over GSI experimental data: $m_1 = 0.11429$ eV, $m_2 = 0.11463$ eV and $m_3 = 0.12476$ eV.

The mass $m_3$ one can calculate using the experimental value $\Delta m_{32}^2 = m_3^2 - m_2^2 = 2.40 \times 10^{-3}$ eV$^2$, deduced from experimental data on atmospheric neutrino oscillations [18] (see also [1]).

The neutrino masses $m_1 = 0.11429$ eV and $m_2 = 0.11463$ eV reproduce the experimental data by GSI with an accuracy of about 6%. This agrees well with the accuracy of the calculation of neutrino mass–corrections in Table 1, which is of about 4%.

The sum of the calculated neutrino masses $\sum_j m_j \simeq 0.35$ eV satisfies well the cosmological constraint $\sum_j m_j < 2$ eV [1].

**Conclusion**

We have shown that an interaction of virtually produced $\ell^-W^+$ pairs $\nu_j \rightarrow \sum_i U_{ij} \ell^-W^+$ of massive neutrinos $\nu_j$ with strong Coulomb fields of highly charged daughter ions, produced in weak decays of highly ionized heavy ions, can induce certain corrections to neutrino masses due to vacuum polarization. For the calculation of corrections to neutrino masses we have taken into account the contribution of the $W^-$–boson exchange only. The contribution of the $Z$–boson exchange is proportional to the constant $g_V = -0.040 \pm 0.015$ [1]. This means that the corrections to neutrino masses, caused by the $Z$–boson exchanges, are smaller compared with corrections, which can be caused by experimental uncertainties of the mixing angles $\theta_{12} = 33.9^{+2.4}_{-2.2}$ degrees and $\theta_{23} \leq 45$ degrees [1]. As a result a total energy of massive neutrino $\nu_j$, moving with a momentum $\vec{k}_j$ in a strong Coulomb field of highly charged daughter ions, produced in weak decays of highly ionized heavy ions, takes the form $E_j(r) = \sqrt{\vec{E}_j^2 + m_j^2(r)}$, where $m_j(r) = m_j + \delta m_j(r)$ with a renormalised proper neutrino mass $m_j$ and a correction to neutrino mass $\delta m_j(r)$, caused by vacuum polarization in a strong Coulomb field of daughter ions vanishing as $O((Z\alpha)^3)$.
for $Z \alpha \rightarrow 0$.

Using corrections to neutrino masses, produced by vacuum polarization in strong Coulomb fields, the experimental data by KamLAND on the antineutrino oscillations and the experimental data by GSI on the periods of time modulation of the K–shell electron capture decay rates and following the hypothesis, that such a time modulation is a consequence of the interference of massive neutrinos, we have calculated the masses of massive neutrino mass–eigenstates: $m_1 = 0.11429 \text{ eV}$, $m_2 = 0.11463 \text{ eV}$ and $m_3 = 0.12476 \text{ eV}$. For the calculation of $m_3 = 0.12476 \text{ eV}$ we have used the experimental value $\Delta m_{32}^2 = 2.40 \times 10^{-3} \text{ eV}^2$, deduced for experimental data on atmospheric neutrino oscillations [1, 18].

The obtained values of neutrino masses agree well with a cosmological constraint on the sum of neutrino masses $\sum_j m_j < 2 \text{ eV}$ [1] and reproduce the experimental data by GSI and KamLAND with an accuracy of about 6%.

The neutrino masses, which we have calculated above, taking into account corrections to neutrino masses, can be used for the analysis of weak decays of light ions and estimates of upper bounds on neutrinoless double–$\beta^-$ decay rates [1].

References


Neutrino Mass Differences and Nonunitarity of Neutrino Mixing Matrix from Interfering Recoil Ions

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Abstract: We show that the recent observation of the time modulation of two-body weak decays of heavy ions reveals the mass content of the electron neutrinos via interference patterns in the recoiling ion wave function. From the modulation period we derive the difference of the square masses $\Delta m^2 \approx 22.5 \times 10^{-5} \text{eV}^2$, which is about 2.8 times larger than that derived from a combined analysis of KamLAND and solar neutrino oscillation experiments. It is, however, compatible with a data regime to which the KamLAND analysis attributes a smaller probability. The experimental results displayed in Fig. 1 imply that the neutrino mixing matrix violates unitarity by about 10%.

Keywords: Neutrino Mass and Mixing; Interfering Recoil Ions; Weak-interaction
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1. Introduction

At the GSI in Darmstadt, the experimental storage ring ESR permits observing completely ionized heavy atoms $I$ or hydrogen-like heavy ions $I_H$ over a long time [1, 2] and thus to measure the time dependence of their weak two-body decays $I_H \rightarrow I + \nu_e$ or $I \rightarrow I_H + \bar{\nu}_e$. The first is the well-known electron-capture (EC) process. The virtue of such experiments is that the properties of the neutrino or antineutrino can

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be deduced from measurements of the time dependence of the transition observing only the initial and final ions. The special efficiency of these experiments becomes clear in the Dirac sea interpretation of the second process, where the initial ion simply absorbs a negative-energy antineutrino in the vacuum. Since the vacuum has all negative-energy states filled, the vacuum is a source of negative-energy neutrinos of maximally possible current density, i.e., the best possible neutrino source in the universe. This is why the ESR experiments yield information on neutrino properties with great precision even if the targets and exposure times are quite small, in particular much smaller than the $2.44 \times 10^{32}$ proton-yrs (2881 ton-yrs) in the famous KamLAND experiments [3], which are only sensitive to the much less abundant positive-energy neutrinos produced by nuclear reactors.

Apart from the neutrino mass difference, the experiment reveals also another important property of the presently popular neutrino mixing scheme: the matrix which expresses the neutrino flavor states into fixed-mass states must be nonunitary to explain the data. The measurement determines the degree of nonunitarity to be roughly 10%.

2. Two-Neutrino Mixing

To illustrate this we consider here at first only the two lightest neutrinos. According to Pontecorvo [4, 5], the Dirac fields of the physical electron and muon-neutrinos $\nu_f = (\nu_e, \nu_\mu)$, the so-called flavor fields, are superpositions of neutrino fields $\nu_i = (\nu_1, \nu_2)$ of masses $m_1$ and $m_2$:

$$\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \quad \nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta,$$

where $\theta$ is a mixing angle. This is, of course, the neutrino analog of the famous Cabibbo mixing of up and down quarks. The free Dirac action has the form

$$\mathcal{A} = \sum_f \int d^4x \bar{\nu}_f(x) (i\gamma^\mu \partial_\mu - \mathcal{M}) \nu_f(x),$$

where $\gamma^\mu$ are the Dirac matrices, and $\mathcal{M}$ is a mass matrix, whose diagonal and off-diagonal elements are $m_f = (m_e, m_\mu)$ and $m_{e\mu} = m_{\mu e}$, respectively. The eigenvalues $m_i = (m_1, m_2)$ are related to $m_f$ by [4, 5, 6, 7],

$$m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \quad m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta,$$

$$m_{e\mu} = m_{\mu e} = (m_2 - m_1) \sin \theta \cos \theta.$$

The weak transition between the electron $e$ and its neutrino $\nu_e$ is governed by the interaction

$$\mathcal{A}_{\text{int}} = g \int d^4x W^-_\mu(x) J^{+\mu}(x) + \text{h.c.} \equiv g \int d^4x W^-_\mu(x) \bar{\nu}(x) \gamma^\mu (1 - \gamma_5) \nu_e(x) + \text{h.c.},$$

where $\gamma_5$ is the product of Dirac matrices $i\gamma^0\gamma^1\gamma^2\gamma^3$. 
Since the interaction (3) involve only the flavor fields (1), the states of masses $m_i$ will always be produced as coherent superpositions. The weakness of the interaction will allow us to calculate the shape of the mixed wave packet from perturbation theory. Consider the decay $I \rightarrow I_H + \bar{\nu}_e$ which is a superposition of the states of masses $m_1$ and $m_2$. The formulas will be applicable for electron capture if we exchange $M_H$ by the mass $M$ of the bare ion and deal with outgoing neutrinos.

In the center-of-mass (CM) frame of the initial bare ion of mass $M$, the final $H$-like ion has the same momentum as the antineutrino $\bar{\nu}_i$ ($i = 1, 2$), whose energy is $\omega_i \equiv \omega_{k,i} = \sqrt{k_i^2 + m_i^2}$ determined by

$$M \equiv M_H + Q = \omega_i + \sqrt{M_H^2 + k_i^2} = \omega_i + \sqrt{M_H^2 + \omega_i^2 - m_i^2}, \quad i = 1, 2,$$

so that

$$\omega_i = [(2M_H + Q)Q + m_i^2]/2(M_H + Q). \quad (5)$$

Subtracting $\omega_2$ and $\omega_1$ from each other we find the energy difference

$$\Delta \omega \equiv \omega_2 - \omega_1 = \frac{m_2^2 - m_1^2}{2M} = \frac{\Delta m^2}{2M}. \quad (6)$$

The denominator $M$ is of the order of 100 GeV and much larger than $\Delta m^2$, so that $\Delta \omega$ is extremely small. It is the difference of the recoil energies transferred to the outcoming ion by the antineutrinos of masses $m_1$ and $m_2$. Without recoil, we would have found the four orders of magnitude larger energy difference at the same momentum $\Delta \omega_k = \omega_{k,2} - \omega_{k,1} = (\Delta m^2 + \omega_{k,1}^2)^{1/2} - \omega_{k,1} \approx \Delta m^2/2\omega_{k,1} \approx \Delta m^2/2Q$. This is the frequency with which the incoming negative-energy neutrino current of momentum $k$ oscillates in the vacuum. Note that although $\Delta \omega$ is small, the momentum difference $\Delta k \equiv k_2 - k_1$ associated with the energies $\omega_{1,2}$ is as large as $\Delta \omega_k$, but has the opposite sign.

3. Experiments

The best experimental results are available for the EC-processes reported in Ref. [1], where an electron is captured from the K-shell and converted into an electron-neutrino which runs off to infinity. On the average, the decay is exponential with a rate expected from a standard-model calculation. In addition, however, the decay rate shows modulations with a frequency $\Delta \omega$. The experimental results are [8]

$$^{140}_{58}\text{Pr}^{58+} \rightarrow ^{140}_{58}\text{Ce}^{58+} : \quad \Delta \omega \approx 0.890(11) \text{ sec}^{-1} \quad (Q = 3.386 \text{ keV}), \quad (7)$$

$$^{142}_{61}\text{Pm}^{60+} \rightarrow ^{142}_{60}\text{Nd}^{60+} : \quad \Delta \omega \approx 0.885(31) \text{ sec}^{-1} \quad (Q = 4.470 \text{ keV}). \quad (8)$$

In both cases, the period of modulation is roughly 7 sec, and scales with $M$ (see Fig. 1). The decay rate has the form $\lambda(t) = \lambda(0)(1 + a \cos(\Delta \omega t + \Delta \phi))$ with a modulation amplitude of $a = 0.18(3)$. 
Modulations of decay rate for the processes $^{140}\text{Pr}^{58+} \rightarrow ^{140}\text{Ce}^{58+}$ and $^{142}\text{Pm}^{60+} \rightarrow ^{142}\text{Nd}^{60+}$. The period in both cases roughly 7 sec. The inserts show the frequency analyses. Plots are from Ref. [1]. The decay rate is modulated by a factor $1 + a \cos(\Delta \omega t + \Delta \phi)$ with $a = 0.18(3)$.

Figure 2 The upper KamLAND regime of 2006 [10] is compatible with our result $\Delta m^2 \approx 22.5 \times 10^{-5} \text{eV}^2$.

We expect these modulations to be associated with the frequency $\Delta \omega$ of Eq. (6), and thus to give information on $\Delta m^2$. Inserting the experimental numbers for $\Delta \omega$ into Eq. (6) and taking into account that the particles in the storage ring run around with 71% of the light velocity with a Lorentz factor $\gamma \approx 1.43$, we find for both processes [9]

$$\Delta m^2 \approx 22.5 \times 10^{-5} \text{eV}^2. \quad (9)$$

This is by a factor $\approx 2.8$ larger than the result $\Delta m^2 \approx 7.58^{+0.3}_{-0.3} \times 10^{-5} \text{eV}^2$ favored by the KamLAND experiment [3, 11], but it lies close to their less favored result [10], which the authors excluded by $2.2\sigma$ in 2005, and now by $6\sigma$ [3] (see Fig. 2).

So far we do not yet understand the origin of this discrepancy. One explanation has been attempted in Ref. [12] where the authors investigate the influence of the strong Coulomb field around the ion upon the process.
4. Entangled Wavefunction

For a theoretical explanation of the modulations, we first simplify the situation and ignore all spins and the finite size of the ions. Then the decay of the initial ion $I$ into the ion $I_H$ plus an electron-antineutrino $\bar{\nu}_e$. can be described by an effective interaction for this process is

$$\mathcal{A}_{\text{int}} = g \int d^3x I_H^\dagger(x)\nu_e(x)I(x)$$

$$= g \int d^3x \left[ \cos \theta I_H^\dagger(x)\nu_1(x)I(x) + \sin \theta I_H^\dagger(x)\nu_2(x)I(x) \right] , \quad (10)$$

where $I(x)$, $\nu_e(x)$, and $I_H(x)$ are the field operators of the involved particles. In the CM frame, the initial ion is at rest, the final moves nonrelativistically. The outgoing wave is spherical. The role of the antineutrino creation operators in $\nu_1$ and $\nu_2$ is simply to create a coherent superposition of two such waves with the two different $k$- and $\omega$-values calculated above. The combined outgoing wave function will be

$$\int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \left[ \cos \theta e^{-iE_p t+i p x} |I_H(p)\rangle e^{-i\omega_1 k_1 t+i k x} |\bar{\nu}_1(k)\rangle + \sin \theta e^{-iE_p t+i p x} |I_H(p)\rangle e^{-i\omega_2 k_2 t+i k x} |\bar{\nu}_2(k)\rangle \right] . \quad (11)$$

The states $|\bar{\nu}_1(k)\rangle$, $|\bar{\nu}_2(k)\rangle$, in turn, can be reexpressed in terms of the electron- and muon-neutrino states as

$$|\bar{\nu}_1(k)\rangle = \cos \theta |\bar{\nu}_e(k)\rangle - \sin \theta |\bar{\nu}_\mu(k)\rangle, \quad |\bar{\nu}_2(k)\rangle = \sin \theta |\bar{\nu}_e(k)\rangle + \cos \theta |\bar{\nu}_\mu(k)\rangle . \quad (12)$$

Thus we find for the transition to an electron-neutrino of any momentum the effective action

$$\int \frac{d^3k}{(2\pi)^3} \langle \bar{\nu}_e(-k) | \mathcal{A}_{\text{int}} | 0 \rangle = g \int d^3x I_H(x)I(x)\nu_e(x) \quad (13)$$

with a spacetime-dependent potential

$$v_{\nu_e}(x) = \int \frac{d^3k}{(2\pi)^3} \left[ \cos^2 \theta e^{i\omega_1 k_1 t+i k x} + \sin^2 \theta e^{i\omega_2 k_2 t+i k x} \right] . \quad (14)$$

In Born approximation we find from this the scattering state of the recoiling ion $I_H$:

$$\langle x | \psi^{(+)}(t) \rangle^{\nu_e} = -\frac{g}{r} \left[ \cos^2 \theta e^{i(k_1 r - \omega_1 t)} + \sin^2 \theta e^{i(k_2 r - \omega_2 t)} \right] . \quad (15)$$

This wave carries a radial current density of ions $I_H$

$$j^{\nu_e}_r = \frac{g^2}{M_H r^2} \left[ \cos^4 \theta k_1 + \sin^4 \theta k_2 + \sin^2 \theta \cos^2 \theta (k_1 + k_2) \cos(\Delta k r - \Delta \omega t) \right] . \quad (16)$$

In order to find the decay rate we integrate this over a sphere of radius $R$ surrounding the initial ion, choosing for $R$ any size $\ll 1/\Delta k \approx 10^4$ m. For this surface we find the outgoing probability current density

$$\dot{P} = 4\pi g^2 \frac{k}{M} \left[ 1 - \frac{1}{2} \sin^2(2\theta) + \frac{1}{2} \sin^2(2\theta) \cos(\Delta \omega t) \right] , \quad (17)$$
where we have approximated \( k_1 \) and \( k_2 \) by their average \( \bar{k} \).

This \( \dot{P} \) can explain directly the observed modulations of the decay rate of the initial ions. The is only one problem: the amplitude of modulations are predicted to be 

\[
a = \frac{1}{2} \sin^2(2\theta) / [1 - \frac{1}{2} \sin^2(2\theta)] \equiv 0.72.
\]

Experimentally, however, \( a \) is much smaller. It has the value 0.18(3).

The discrepancy is explained by a missing contribution to the decay rate. So far we have only included the contribution of the effective action (10). There is, however, also a second effective action which is generated by the matrix elements

\[
\int \frac{d^3k}{(2\pi)^3} \langle \bar{\nu}_\mu(-k)|A_{\text{int}}|0 \rangle = g \int d^3x I_H(x) I(x) v_{\nu_\nu}(x) \tag{18}
\]

where

\[
v_{\nu_\nu}(x) = \int \frac{d^3k}{(2\pi)^3} \sin \theta \cos \theta \left[ -e^{i\omega_1 k_1 t + ikx} + e^{i\omega_2 k_2 t + ikx} \right]. \tag{19}
\]

This can be derived directly from Eqs. (12) and (13). Here the Born approximation yields the scattering state of the recoiling ion \( I_H \):

\[
\langle x|\psi^{(+)}; t \rangle \bar{\nu}_\mu \equiv \frac{g}{r} \sin \theta \cos \theta \left[ e^{i(k_1 r - \omega_1 t)} - e^{i(k_2 r - \omega_2 t)} \right]. \tag{20}
\]

Its radial current density is now

\[
j_r^{\nu_\nu} = \frac{g^2}{M_H r^2} \sin^2 \theta \cos^2 \theta [k_1 + k_2] [1 - \cos(\Delta k r - \Delta \omega t)]. \tag{21}
\]

Now we have another problem: the modulations of this current cancel the modulations of the current (17). We may suspect that this has to do with the fact that there are three neutrinos which we must take into consideration.

5. Three-Neutrino Mixing

Let us now include all three known neutrinos \( \nu_e, \nu_\mu, \nu_\tau \). Their fields are denoted by \( \nu_\sigma \) with \( \sigma = e, \mu, \tau \). These fields are combinations of three fields with definite mass

\[
\nu_\sigma = U_{\sigma i} \nu_i, \quad \sigma = (e, \mu, \tau), \tag{22}
\]

The mixing matrix \( U_{\sigma i} \) is called Maki-Nakagawa-Sakata matrix, or short MNS-matrix, the neutrino analog of the 3 × 3 Cabibbo-Kobayashi-Maskawa matrix for the mixing of the quarks \( d, s, b \). It is commonly assumed to be unitary, i.e., to satisfy the relation

\[
\sum_\sigma U_{\sigma j}^* U_{\sigma l} = \delta_{jl}. \tag{23}
\]

Its standard parametrization is the following product of four simple unitary matrices

\[
U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24}
\]
where \( s_{ij} \equiv \sin \theta_{ij}, \) \( c_{ij} \equiv \cos \theta_{ij} \). For quarks, the unitarity relation (23) is presently in the focus of experimental and theoretical studies in many research groups [14]. For leptons, the data have so far been insufficient to test it.

Generalizing (13), (14) and (18), (19), we have from each flavor \( \sigma \) an effective potential

\[
\int \frac{d^3k}{(2\pi)^3} \langle \bar{\nu}_\sigma(-\mathbf{k})|A_{\text{int}}|0 \rangle = g \int d^3x I_H(x)I(x)v_{\bar{\nu}_\sigma}(x)
\]

with a potential

\[
v_{\bar{\nu}_\sigma}(x) = \int \frac{d^3k}{(2\pi)^3} U_{e\sigma j}^* U_{\sigma l} e^{i\omega_{k,j} t + i\mathbf{k} \cdot \mathbf{x}}.
\]

This produces an outgoing wave function of the ion \( I_H \) in the center-of-mass frame due to the potential \( v_{\bar{\nu}_\sigma} \) is

\[
\langle \mathbf{x}|\psi^{(+)};t\rangle_{\bar{\nu}_\sigma} = -\frac{g}{r} \sum_{j=1}^{3} U_{e\sigma j}^* e^{i(k_j r - \omega_j t)},
\]

with an ion current density

\[
j_{r_{\bar{\nu}_\sigma}} = \frac{g^2}{M_H r^2} \sum_{j,l=1}^{3} \sum_{\sigma=1}^{3} U_{e\sigma j}^* U_{\sigma l}^* k_j e^{i[(k_j - k_l) r - (\omega_j - \omega_l) t]}.
\]

If we sum over all flavors of the antineutrino and use the unitarity relation (23), we obtain the total radial current density

\[
j_r = \sum_{\sigma} j_{r_{\bar{\nu}_\sigma}} = \frac{g^2}{M_H r^2} \sum_{j=1}^{3} U_{e}^j U_{e j}^* k_j.
\]

As previously for two flavors, the modulations in (28) disappear.

However, the GSI experiments did observe modulations with an amplitude \( a \approx 0.18(3) \). Thus we must conclude that the unitarity relation (23) must be violated. Since so far only the lowest possible modulation frequency \( \Delta \omega = \omega_2 - \omega_1 \) between the two lightest neutrinos has been measured, we may parametrize the right-hand side of the unitarity violation by

\[
\sum_{\sigma} U_{e\sigma j}^* U_{\sigma l} = u_0 \delta_{jl} + u_{21}(\delta_{j,2} \delta_{l,1} + \delta_{j,1} \delta_{l,2}) + \ldots,
\]

and find

\[
j_r = \frac{g^2}{M_H r^2} \left\{ u_0 S_0 \delta_{ij} + 2u_{21} S_{21} \cos[\Delta k r - \Delta \omega t + \Delta \phi] \right\},
\]

where

\[
S_0 \equiv \sum_{j=1}^{3} U_{e j}^* U_{e j}^* k_j, \quad S_{21} \equiv |U_{e 2} U_{e 1}^*| (k_2 + k_1)/2, \quad \Delta \phi \equiv \arg U_{e 2} U_{e 1}^*.
\]
Assuming that the violation of unitarity is small, the sums $S_0$ and $S_1$ are close to unity. Then we deduce from the experimental result $a \approx 0.18(3)$ that
\[
\frac{u_{21}}{u_0} \approx 10\%.
\] (33)

A possible origin of this unitarity violation could be that there are more than three families of leptons in nature and that universality of weak interaction is not valid for all of them. If the symmetry between quarks and leptons of the standard model persists to higher energies, we do not expect more than eight lepton families to exist—more than eight quark families would ruin asymptotic freedom and thus confinement. Thus there is room for more than the three quark and lepton families observed so far. Indeed, a fourth set of families is under intense discussion [15] in connection with the new accelerator LHC at CERN. So far, there are only weak bounds on their masses from different sources [3]:
\[
m_{\nu'} \geq 256 \text{ GeV}, \quad m_{\nu'} \geq 128 \text{ GeV}, \quad m_{\nu'} \geq 100.8 \text{ GeV}, \quad m_{\nu_{\tau'}} \geq 90.3 \text{ GeV}.
\] (34)

If any of the heavier leptons is coupled with a coupling constant that does not fit into the CKM scheme, unitarity will certainly be violated. More data will be needed to decide precisely how.

6. Comments

It is noteworthy that this analysis, in which we extract the properties of the unobserved antineutrino from the behavior of the ion, corresponds precisely to the usual entanglement analysis of decay processes such as $\pi^0 \rightarrow \gamma + \gamma$. There the measurement of the polarization of one photon tells us immediately the polarization properties of the other, unobserved photon.

A few comments are in place on several recent publications [16, 17, 18, 19, 20, 21] which deny the relation between neutrino oscillations and the nonexponential decay seen in the GSI experiment for various reasons. In Ref. [16], the basic argument is that the antineutrino oscillations set in after their emission, so that they cannot be observed in the GSI experiment. The present discussion shows that although the first part of this argument is true, the conclusion depends on the unitary assumption of the mixing matrix.

Finally we should point out that similar oscillation phenomena in the associate production of particles together with an oscillating partner have been proposed and controversially discussed before by many authors in the production of muons together with antineutrinos in the decay $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ [23], and in the production of $\Lambda$ hyperons together with neutral Kaons [24]. In the latter case the oscillation would come from a nonunitarity of the quark mixing matrix, which seems to be much smaller than that of the neutrino mixing matrix reported here.
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Appendix: Properties of Outgoing Wave

The interaction is time-dependent and we must adapt the scattering theory to this situation. Recall briefly the theory for a time-independent interaction, where the scattering amplitude is obtained from the standard limiting formula

\[ \langle p'|\hat{S}|p\rangle = \lim_{t \to \infty} \langle p'|\hat{U}_t(t,0)|p^{(+)}\rangle = \lim_{t \to \infty} \langle p'|e^{i\hat{H}_0 t}e^{-i(\hat{H}_0+\hat{V})t}|p^{(+)}\rangle = \lim_{t \to \infty} e^{i(E_{p'}-E_p)t} \langle p'|\hat{p}^{(+)}\rangle. \tag{35} \]

Here \(|p'|\) denotes an eigenstate of the free Hamiltonian \(\hat{H}_0\) with momentum \(p'\) and energy \(E_{p'} = p'^2/2M\), and \(|p^{(+)}\) is an eigenstate of the interacting Hamiltonian \(\hat{H}_0+\hat{V}\) with momentum \(p\) and energy \(E_p\). It solves the Lippmann-Schwinger equation:

\[ |p^{(+)}\rangle = |p\rangle + \frac{1}{E_p - \hat{H}_0 + i\eta} \hat{V} |p^{(+)}\rangle, \tag{36} \]

which is verified by multiplying both sides by \(E - \hat{H}_0\) from the left. Inserting this into (35) leads to

\[ \langle p'|\hat{S}|p\rangle = \langle p'|p\rangle + \lim_{t \to \infty} \frac{e^{i(E_{p'}-E_p)t}}{E_p - E_{p'} + i\eta} \langle p'|\hat{V}|p^{(+)}\rangle, \tag{37} \]

where \(\eta > 0\) is an infinitesimally number. The second term contains the \(T\)-matrix \(T_{p'p} \equiv \langle p'|\hat{V}|p^{(+)}\rangle\) which describes true scattering. In the absence of neutrino oscillations, \(|p\rangle\) is simply the initial ion at rest, and \(|p'|\) the state with the ion \(I_H\) with momentum \(p+k\) and the antineutrino \(\bar{\nu}_e\) with momentum \(-k\). The limit \(t \to \infty\) in the prefactor can simply be taken after rewriting it as \(-i \int_{-\infty}^{t} dt e^{(E_{p'}-E_p-in)t}\), which obviously tends to \(-2\pi i \delta(E_{p'} - E_p)\) in the limit \(t \to \infty\). The \(\delta\)-function ensures the conservation of energy in the process. This is, of course, the standard derivation of Fermi’s Golden Rule which we repeated here to clarify that it is applicable only to processes in which the final state is an eigenstate of the free-particle Hamiltonian operator \(\hat{H}_0\).

Another way of deriving this result is based on the spatial wave function associated with the state \(|p^{(+)}\rangle\). One multiplies Eq. (36) by the state \(\langle x|\) from the left and obtains the wave function

\[ \langle x|p^{(+)}\rangle = \langle x|p\rangle + \int d^3x'G(E_p; x, x')\hat{V}(x')\langle x'|p^{(+)}\rangle, \tag{38} \]
where
\[
G(E; \mathbf{x}, \mathbf{x}') \equiv \langle \mathbf{x}' | \frac{1}{E - \hat{H}_0 + i\eta} | \mathbf{x} \rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{e^{i\mathbf{p}'(\mathbf{x} - \mathbf{x}')}}{E - \mathbf{p}'^2/2M + i\eta} \approx -2M \frac{e^{ip'r}}{4\pi r} e^{-ip\hat{x}x}\]
with \( r \equiv |\mathbf{x} - \mathbf{x}'|, \ \hat{x} \equiv \mathbf{x}/r \) and \( p'_{E} = \sqrt{2ME} \).

In Born approximation, one inserts on the right-hand side of Eq. (38) a plane wave \( \langle \mathbf{x}'|\mathbf{p}'(+)\rangle \approx \langle \mathbf{x}'|\mathbf{p} \rangle = e^{i\mathbf{x}\cdot\mathbf{p}}, \) and Eq. (38) becomes
\[
\langle \mathbf{x}|\mathbf{p}'(+)\rangle = \langle \mathbf{x}|\mathbf{p} \rangle - 2M \frac{e^{ip'r}}{4\pi r} \int d^3x'e^{-i(\mathbf{p}'-\mathbf{p})x'}V(x'),
\]
where \( \mathbf{p}' \) is short for the momentum of the outgoing particle of energy \( E_{p} \) in the direction of \( \mathbf{x}: \mathbf{p}' \equiv p'_E \hat{x}. \) Thus, in Born approximation, the amplitude for the final particle to emerge with momentum \( \mathbf{p}' \) is proportional the Fourier transform of the potential at the momentum transfer \( \Delta \mathbf{p} \equiv \mathbf{p}' - \mathbf{p}. \) If the potential is a plane wave of momentum \( -\mathbf{k}, \) i.e., if
\[
V(x) = \frac{g}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}
\]
then the final state has the wave function
\[
\langle \mathbf{x}|\mathbf{p}'(+)\rangle = \langle \mathbf{x}|\mathbf{p} \rangle - 2Mg \frac{e^{ip'r}}{4\pi r} \delta^{(3)}(\mathbf{p}' - \mathbf{p} + \mathbf{k}).
\]

Let us adapt this formalism to the oscillating situation. According to Eqs. (13), (14), the emission of an antineutrino of mass \( m_1 \) and momentum \( -\mathbf{k} \) is described by the time-dependent interaction potential
\[
v_{\nu u_1}(x, t) = \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^3} e^{i\omega_{k,1}t}, \quad \omega_{k,1} = \sqrt{k^2 + m_1^2},
\]
where we have dropped the factor \( \cos^2 \theta \) accompanying the coupling \( g, \) for brevity. As before, the incoming ion \( I \) has the momentum \( \mathbf{p}. \) Its energy is \( \mathbf{p}^2/2M. \) The outgoing ion \( I_H \) has the momentum \( \mathbf{p}' \) and an energy \( E_{p'} = M_H - M + \mathbf{p}^2/2M_H. \)

Let us first adapt the Lippmann-Schwinger approach. We introduce the time-dependent interacting state \( |\mathbf{p}'(t)\rangle \), which is an eigenstate of the full Hamiltonian, and satisfies the time-dependent Schrödinger equation
\[
i\partial_t|\mathbf{p}'(t)\rangle = [\hat{H}_0 + \hat{v}(x, t)]|\mathbf{p}'(t)\rangle, \quad \hat{H}_0 = \hat{p}^2/2M.
\]
The formal solution of this is
\[
|\mathbf{p}'(t)\rangle \equiv \hat{U}(t)|\mathbf{p}\rangle, \quad \hat{U}(t) \equiv \hat{T} e^{-i \int_0^t dt' [\hat{H}_0 + \hat{V}(x, t)]},
\]
An implicit expression for this state can be written, by analogy with (36), as
\[
|\mathbf{p}'(t)\rangle = |\mathbf{p}\rangle e^{-iE_{p}'t} + \frac{1}{i\partial_t - \hat{H}_0 + i\eta} \hat{V}(x, t)|\mathbf{p}'(t)\rangle, \quad E_{p}' = \mathbf{p}^2/2M.
\]
This can again be verified by multiplication from the left with $i\partial_t - \hat{H}_0$. Multiplying (46) by $e^{iE_p^0 t}$, we obtain

$$e^{iE_p^0 t} |p^{(+)}(t)\rangle = |p\rangle + \frac{1}{i\partial_t + E_p^0 - \hat{H}_0 + i\eta} e^{iE_p^0 t} \hat{V}(x, t) e^{-iE_p^0 t} |p^{(+)}(0)\rangle,$$  
(47)

To lowest approximation, we replace $|p^{(+)}(0)\rangle$ by $|p\rangle$ and insert (43) to find

$$e^{iE_p^0 t} |p^{(+)}(t)\rangle = |p\rangle + g \int_{-\infty}^{\infty} dt' \hat{G}(t, t') e^{ik\hat{x}} e^{i\omega_{k,1}(t' - t_0)} |p\rangle e^{-i(E_p - E_p^0)t'},$$  
(48)

where $\hat{G}(t, t')$ is the Fourier representation of the operator $(i\partial_t + E_p^0 - \hat{H}_0 + i\eta)^{-1}$:

$$\hat{G}(t, t') \equiv \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t - t')}}{E + E_p^0 - \hat{H}_0 + i\eta}.$$  
(49)

Performing the integral over $t'$ in (48) yields

$$e^{iE_p^0 t} |p^{(+)}(t)\rangle = |p\rangle + g \frac{e^{-i(E_p - E_p^0 - \omega_{k,1}t)} E_p - \hat{H}_0 - \omega_{k,1} + i\eta}{e^{i\omega_{k,1}t_0}} |p\rangle e^{i\omega_{k,1}t_0}.$$  
(50)

We multiply this equation from the left and insert in front of the right-hand state $|p\rangle$ a completeness relation $\int d^3x|x'\rangle\langle x'| = 1$. Approximating the matrix elements $\langle x|(E_p - \hat{H}_0 + i\eta)^{-1} |x'\rangle$ as usual in the large-$x$ regime by

$$\langle x|\frac{1}{E_p - \hat{H}_0 - \omega_{k,1} + i\eta} |x'\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{e^{ip'(x - x')}}{E_p - p'^2/2M - \omega_{k,1} + i\eta} \approx -2M e^{ip'x} \frac{4\pi r}{\omega_{k,1}} e^{-ip'x}$$  
(51)

where $p_k'$ is the momentum of the ion $I_H$ which conserves the energy, i.e., $p_k'^2/2M = E_p - \omega_{k,1}$. With this we obtain

$$\langle x|e^{iE_p^0 t} |p^{(+)}(t)\rangle = \langle x|p\rangle - 2M \frac{g}{(2\pi)^3} e^{ip'x} \frac{4\pi r}{4\pi} e^{-ip'x} e^{i(p + k)x} e^{-i\omega_{k,1}t_0}.$$  
(52)

We now perform the integral over $x'$. For this we assume the initial state to have zero momentum, $p + k = 0$. The integral over $x'$ forces the momentum of the outcoming ion $I_H$ to be equal to $k$. The integral over $x'$ creates a $\delta$-function $(2\pi)^3\delta^{(3)}(p_k' + k)$, so that we obtain

$$\langle x|e^{iE_p^0 t} |p^{(+)}(t)\rangle = \langle x|p\rangle - 2M \frac{g}{4\pi r} e^{-ip_k'x} \delta^{(3)}(p' - p + k) e^{-i\omega_{k,1}t_0}.$$  
(53)

Note that since energy and momentum are balanced, then $\omega_{k,1} = \omega_1$ of Eq. (5).

Consider now the case of two oscillating mass states and let us study the temporal behavior of the emerging energy distribution. The experiment does not explore the limit of very large times but measures the $t$-dependence starting from small $t$ after the ion enters the storage ring. Instead of the limiting energy conservation $\delta$-function $-2\pi i\delta(E_p' - E_p)$ in (37), it observes an approximation to it valid for short times.
To find it we insert, instead of (43), the mixed potential (14) into Eq. (47), so that the time-dependent factor in the resulting equation of type (48) has the form

\[ -i \int_0^t dt \left[ \cos^2 \theta e^{i(E_{1,k'} - E_k)t} + \sin^2 \theta e^{i(E_{2,k'} - E_k)t} \right], \]  

(54)

where \( E_{1,k'} = \sqrt{k'^2 + M_H^2 + \omega_{k',i}} \) and \( E_k = M_H + Q \). Since \( m_i^2 \ll Q \ll M_H \), we can approximate \( E_{1,k'} - E_k \approx \omega' - \omega_i \) where \( \omega_i \approx Q + m_i^2/2M_H \). Let us write \( \omega_{1,2} = \bar{\omega} \mp \frac{1}{2} \Delta m^2/2M_H = \bar{\omega} \pm \frac{1}{2} \Delta \omega \). Then (54) becomes, with the abbreviations \( C \equiv \cos^2 \theta \) and \( S \equiv \sin^2 \theta \),

\[ \left( C e^{i(\omega' - \omega_i)t/2} - 1 \right) \left( S e^{i(\omega' - \omega_2)t/2} - 1 \right) \right] + S e^{i(\omega' - \omega_2)t/2} \left[ \sin(\omega' - \omega_2)t/2 \right]/(\omega' - \omega_2)/2 \right). \]  

(55)

The absolute square of (55) multiplied by some factor \( |T|^2 \) determines probability \( P(t) \) to find the initial ions in the ring at the time \( t \). The integral over the final momenta is dominated by the immediate neighborhood of the poles at \( \omega_{1,2} \) where \( |k| \equiv Q \). There we may ignore the \( k \)-dependence of \( |T|^2 \), approximating it by a constant, and obtain the probability for the ion \( I_H \) to emerge with an energy \( M_H + Q - \omega' \) in the center-of-mass frame

\[ P^{\omega'}(t) \approx \left\{ C^2 s_1^2(\omega') + S^2 s_2^2(\omega') + 2CS \cos(\Delta \omega t/2) s_1(\omega') s_2(\omega') \right\} |T|^2, \]

(56)

where \( s_i(\omega') \equiv \sin[(\omega' - \omega_i)t/2]/[(\omega' - \omega_i)t/2] \). For large \( t \), the limiting relation \( \sin^2 at/a^2 \to t \pi \delta(a) \) allows us approximate \( s_i^2(\omega') \approx 2\pi t \delta(\omega' - \omega_i) \), and thus the first two terms in (56) by

\[ P_{12}^{\omega'}(t) \approx 2\pi t [C^2 \delta(\omega' - \omega_1) + S^2 \delta(\omega' - \omega_2)]|T|^2. \]

(57)

If this is integrated over \( \int d^3k'/(2\pi)^3 \approx Q^2 \int d\omega'/2\pi^2 \), the probabilities of \( \nu_1^- \) and \( \nu_2^- \)-decays simply add, thereby yielding the ordinary \( \beta \)-decay rate \( I \to I_H + \nu_e \), without mixing. Consider now the third term in (56). Here the integral over all \( \omega' \) yields

\[ \int d\omega' P_3^{\omega'}(t) \approx 2CS \cos(\Delta \omega t/2) 2\pi \frac{\sin(\Delta \omega t/2)}{\Delta \omega/2} |T|^2. \]

(58)

Thus we obtain for the total decay rate as a function of time

\[ \dot{P}(t) = \int d\omega' \left[ P_{12}^{\omega'}(t) + P_3^{\omega'}(t) \right] \approx 2\pi \left[ 1 + 2CS \cos(\Delta \omega t) \right] |T|^2. \]

(59)

It would be interesting to observe experimentally the predicted distribution (56) of antineutrino energies \( \omega' \) by measuring the recoil momenta \( k \) of the final ions \( I_H \). The distribution consists of two peaks associate with the emission of the antineutrinos \( \bar{\nu}_1 \) and \( \bar{\nu}_2 \). Centered between them lies the oscillating distribution proportional to \( s_1(\omega') s_2(\omega') \) shown in Fig. 1.
Note that the usual Feynman diagrams in momentum space cannot be used to describe the observed oscillations as done in Ref. [17], since they imply taking the limit $t \to \infty$ in which the oscillations disappear. Only diagrams in spacetime involving a propagator matrix in $\nu_1, \nu_2$ field space with off-diagonal matrix elements are applicable, and these reproduce the above-calculated oscillations.

References


[8] The right-hand superscript indicates the ionization degree of the associated atom.

[9] The isotope masses of the final ions are $M \approx 130.319$ GeV and $M \approx 132.186$ GeV, for $^{140}$Ce and $^{142}$Nd, respectively. Also recall that a frequency $1/\text{sec}$ corresponds to $\approx 6.0 \times 10^{-16}$ eV.

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Bifurcations of Fractional-order Diffusionless Lorenz System

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\begin{abstract}
Using the predictor-corrector scheme, the fractional-order diffusionless Lorenz system is investigated numerically. The effective chaotic range of the fractional-order diffusionless system for variation of the single control parameter is determined. The route to chaos is by period-doubling bifurcation in this fractional-order system, and some typical bifurcations are observed, such as the flip bifurcation, the tangent bifurcation, an interior crisis bifurcation, and transient chaos. The results show that the fractional-order diffusionless Lorenz system has complex dynamics with interesting characteristics.
\end{abstract}

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\section{Introduction}

Fractional calculus has a 300-year-old history, as old as calculus itself, but its applications to physics and engineering have just begun [1]. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [2], dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves. Many scientists have studied the properties of these fractional-order systems.

More recently, there has been growing interest in investigating the chaotic behavior and dynamics of fractional-order dynamic systems [3-20]. It has been shown that several chaotic systems can remain chaotic when their models become fractional [8]. In [4], it was shown that the fractional-order Chua's circuit with order as low as 2.7 can produce

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a chaotic attractor. In [5], it was shown that nonautonomous Duffing systems with order less than 2 can still behave in a chaotic manner, and in [6], chaos in a modified Duffing system exists for total system orders are 1.8, 1.9, 2.0, and 2.1. In [7], the fractional-order Wien bridge oscillator was studied, where it was shown that a limit cycle can be generated for any fractional-order, with an appropriate value of the amplifier gain. In [8], chaotic behavior of the fractional-order Lorenz system was studied, but unfortunately, the results presented in that paper are not correct. In [9], the chaotic behavior of a fractional-order “jerk” model was investigated, in which a chaotic attractor was obtained with the system order as low as 2.1, and chaos control of that fractional-order chaotic system was reported in [10]. In [11], Chaos in the fractional-order Rössler hyperchaotic system was studied, in which chaos was found in the fractional-order system with an order as low as 2.4 and hyperchaos was found with an order as low as 3.8. In [12], hyperchaotic behavior of an integer-order nonlinear system with unstable oscillators is preserved when the order becomes fractional. In [13-15], the chaotic behavior and its control in the fractional-order Chen system are investigated. Many other fractional-order nonlinear systems are chaotic, such as the fractional-order Arneodo’s system [16], the fractional-order Chen-Lee system [17], the fractional-order modified van der Pol system [19], and a fractional-order rotational mechanical system with a centrifugal governor [20]. Despite these many examples, the bifurcations of fractional-order nonlinear systems have not yet been well studied.

In this paper, we report the bifurcations that occur in a particularly simple, one-parameter version of the Lorenz model, called the diffusionless Lorenz equations (DLE) described in [21] and further investigated in [22]. The Kaplan-Yorke dimension of the diffusionless Lorenz system was calculated and used as a measure its complexity in [23]. However, the fractional-order variant of this system has not been studied, and it is an ideal candidate for examining bifurcations since it has a single bifurcation parameter. The paper is organized as follows. In Section 2, the numerical algorithm for the fractional-order diffusionless Lorenz system is presented. In Section 3, the chaotic behavior and bifurcations of the system are studied. Finally, we summarize the results and indicate future directions.

2. Fractional-order Derivative and its Numerical Algorithm

There are several definitions of fractional derivatives [24]. One popular definition involves a time-domain computation in which non-homogenous initial conditions are permitted and those values are readily determined [25]. The Caputo derivative definition [26] is given by

\[
\frac{d^\alpha f(t)}{dt^\alpha} = J^{n-\alpha} \frac{d^n f(t)}{dt^n}, \quad \text{or} \quad \frac{d^\alpha f(t)}{dt^\alpha} = J^{[\alpha]-\alpha} \frac{d^{[\alpha]} f(t)}{dt^{[\alpha]}},
\]  

(1)
where $n$ is the first integer which is not less than $\alpha$ and $J^{\theta}$ is the $\theta$-order Riemann-Liouville integral operator given by
\begin{equation}
J^{\theta}\varphi(t) = \frac{1}{\Gamma(\theta)} \int_{0}^{t} (t - \tau)^{\theta-1} \varphi(\tau) d\tau,
\end{equation}
where $\Gamma(\theta)$ is the gamma function with $0 < \theta \leq 1$.

The Diffusionless Lorenz equations are given by
\begin{equation}
\begin{aligned}
\dot{x} &= -y - x, \\
\dot{y} &= -xz, \\
\dot{z} &= xy + R,
\end{aligned}
\end{equation}
where $R$ is a positive parameter. When $R \in (0, 5)$, chaotic solutions occur. Equation (3) has equilibrium points at $(x^*, y^*, z^*) = (\pm \sqrt{R}, \mp \sqrt{R}, 0)$ with eigenvalues that satisfy the characteristic equation $\lambda^3 + \lambda^2 + R\lambda + 2R = 0$. Figure 1(a) shows the chaotic attractor for the diffusionless Lorenz system for the value of $R = 3.4693$ at which the Kaplan-Yorke dimension has its maximum value of 2.23542 [23].

Now, consider the fractional-order diffusionless Lorenz system given by
\begin{equation}
\begin{aligned}
\frac{d^{\alpha}x}{dt^{\alpha}} &= -y - x, \\
\frac{d^{\beta}y}{dt^{\beta}} &= -xz, \\
\frac{d^{\gamma}z}{dt^{\gamma}} &= xy + R,
\end{aligned}
\end{equation}
where $\alpha, \beta, \gamma$ determine the fractional-order, $0 < \alpha, \beta, \gamma \leq 1$. Figure 1(b) shows the chaotic attractor for the fractional-order diffusionless Lorenz system with $\alpha = \beta = \gamma = 0.95$ and $R = 3.4693$.

There are two ways to study fractional-order systems. One is through linear approximations. By using frequency-domain techniques based on Bode diagrams, one can...
obtain a linear approximation for the fractional-order integrator, the order of which de-

dpends on the desired bandwidth and the discrepancy between the actual and approximate
Bode diagrams. The other is the Adams-Bashforth-Moulton predictor-corrector scheme
[27-29], which is a time-domain approach and thus is more effective. Here, we derive a

generalization of the Adams-Bashforth-Moulton scheme appropriate for Eq. (4).

The following differential equation
\[
\begin{cases}
\frac{dx}{dt^\alpha} = f(t, x), & 0 \leq t \leq T \\
x^k(0) = x_0^{(k)} & k = 0, 1, 2, \ldots, \lfloor \alpha \rfloor - 1
\end{cases}
\tag{5}
\]
is equivalent to the Volterra integral equation [27]
\[
x(t) = \sum_{k=0}^{n-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) d\tau.
\tag{6}
\]

Obviously, the sum on the right-hand side is completely determined by the initial values,
and hence is known. In a typical situation, one has 0 < \alpha < 1, and hence the Volterra
equation (5) is weakly singular. In [27-29], the predictor-corrector scheme for equation
(5) is derived, and this approach can be considered to be an analogue of the classical
one-step Adams-Moulton algorithm.

Set \( h = T/N \) and \( t_j = jh (j = 0, 1, 2, \ldots, N) \) with \( T \) being the upper bound of the
interval on which we are looking for the solution. Then the corrector formula for Eq. (6)
is given by
\[
x_h(t_{n+1}) = \sum_{k=0}^{n-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, x_h(t_j)),
\tag{7}
\]
where
\[
a_{j,n+1} = \begin{cases} 
    n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha, & j = 0 \\
    (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1} - 2(n - j + 1)^{\alpha+1} & 1 \leq j \leq n
\end{cases}
\tag{8}
\]

By using a one-step Adams-Bashforth rule instead of a one-step Adams-Moulton rule,
the predictor \( x_h^p(t_{n+1}) \) is given by
\[
x_h^p(t_{n+1}) = \sum_{k=0}^{n-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x_h(t_j)),
\tag{9}
\]
where
\[
b_{j,n+1} = \frac{h^\alpha}{\alpha} ((n - j + 1)^{\alpha} - (n - j)^{\alpha}), \quad 0 \leq j \leq n.
\tag{10}
\]

Now, the basic algorithm for the fractional Adams-Bashforth-Moulton method is com-
pletely described by Eqs. (7) and (9) with the weights \( a_{j,n+1} \) and \( b_{j,n+1} \) being defined
according to (8) and (10), respectively.
The error estimate of this method is

$$e = \max_{j=0,1,\ldots,N} |x(t_j) - x_k(t_j)| = O(h^p)$$

(11)

where $p = \min(2, 1 + \alpha)$. Using this method, the fractional-order diffusionless Lorenz equations (4) can be written as

$$\begin{align*}
x_{n+1} &= x_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} \left\{ \left[-y_{n+1}^p - x_{n+1}^p\right] + \sum_{j=0}^{n} a_{1,j,n+1}(-y_j - x_j) \right\} \\
y_{n+1} &= y_0 + \frac{h^\beta}{\Gamma(\beta + 2)} \left\{ (-x_{n+1}^p z_{n+1}^p) + \sum_{j=0}^{n} a_{2,j,n+1}(-x_j z_j) \right\} \\
z_{n+1} &= z_0 + \frac{h^\gamma}{\Gamma(\gamma + 2)} \left\{ [x_{n+1}^p y_{n+1}^p + R] + \sum_{j=0}^{n} a_{3,j,n+1}(x_j y_j + R) \right\}
\end{align*}$$

(12)

where

$$\begin{align*}
x_{n+1}^p &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{1,j,n+1}(-y_j - x_j) \\
y_{n+1}^p &= y_0 + \frac{1}{\Gamma(\beta)} \sum_{j=0}^{n} b_{2,j,n+1}(-x_j z_j) \\
z_{n+1}^p &= z_0 + \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{n} b_{3,j,n+1}(x_j y_j + R)
\end{align*}$$

$$b_{1,j,n+1} = \frac{h^\alpha}{\alpha} ((n - j + 1)^\alpha - (n - j)^\alpha), \quad 0 \leq j \leq n$$

$$b_{2,j,n+1} = \frac{h^\beta}{\beta} ((n - j + 1)^\beta - (n - j)^\beta), \quad 0 \leq j \leq n$$

$$b_{3,j,n+1} = \frac{h^\gamma}{\gamma} ((n - j + 1)^\gamma - (n - j)^\gamma), \quad 0 \leq j \leq n$$

$$\begin{align*}
a_{1,j,n+1} &= \begin{cases} n^\alpha - (n - \alpha)(n+1)^\alpha & j = 0 \\ (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1}, & 1 \leq j \leq n \end{cases} \\
a_{2,j,n+1} &= \begin{cases} n^\beta - (n - \beta)(n+1)^\beta & j = 0 \\ (n - j + 2)^{\beta+1} + (n - j)^{\beta+1} - 2(n - j + 1)^{\beta+1}, & 1 \leq j \leq n \end{cases} \\
a_{3,j,n+1} &= \begin{cases} n^\gamma - (n - \gamma)(n+1)^\gamma & j = 0 \\ (n - j + 2)^{\gamma+1} + (n - j)^{\gamma+1} - 2(n - j + 1)^{\gamma+1}, & 1 \leq j \leq n \end{cases}
\end{align*}$$

3. Bifurcations of the Fractional-order Diffusionless Lorenz System

3.1 Fractional-order Diffusionless Lorenz Chaotic System

In our simulations, we have visually inspected the bifurcation diagrams to identify chaos. We also have confirmed these by calculating the largest Lyapunov exponent in some cases
using the Wolf algorithm [30]. Here, we let $\alpha = \beta = \gamma = q$, and the effective chaotic range of the fractional-order diffusionless system with different control parameter is found as shown in Fig. 2. Clearly there exist three different states corresponding to limit cycles, chaos, and convergence. If the integer-order system is chaotic, the effective chaotic range of the fractional-order system decreases slightly as control parameter increases. If the integer order system is a limit cycle, the fractional-order system can produce chaos, but the effective chaotic range of the fractional-order system shrinks significantly as the control parameter increases. The largest Lyapunov exponent of the fractional-order system with $R=8$ is shown in Fig. 3. It is consistent with the chaotic range at this value.

Fig. 2 Chaotic range of the fractional-order DLE. Fig. 3 Largest LE of the fractional-order DLE with $R=8$.

### 3.2 Bifurcations with different control parameter $R$

Here, the fractional-orders $\alpha$, $\beta$, $\gamma$ are equal and fixed at 0.95 while the control parameter $R$ is varied from 0.5 to 11. The initial states of the fractional-order diffusionless Lorenz system are $x(0) = -0.0249$, $y(0) = -0.1563$, and $z(0) = 0.9441$. For a step size in $R$ is 0.01 and the running time is 140s, the bifurcation diagram in Fig. 4 was obtained. It shows that the fractional-order DLE is chaotic with one periodic widow when the total order is $\alpha + \beta + \gamma = 2.85$. When the control parameter $R$ is decreased from 11, the fractional-order system enters into chaos by a period-doubling bifurcation as shown in Fig. 5(a) (with steps of 0.005). To observe the dynamic behavior, the periodic window is expanded with steps of 0.005, as shown in Fig. 5(b). There exists an interior crisis when $R \approx 5.57$, a flip bifurcation when $R \approx 6.02$, and a tangent bifurcation when $R \approx 6.55$. In the interior crisis, the chaotic attractor collides with an unstable periodic orbit or limit cycle within its basin of attraction. When the collision occurs, the attractor suddenly expands in size but remains bounded. For the tangent bifurcation, a saddle point and a stable node coalesce and annihilate one another, producing an orbit that has periods of chaos interspersed with periods of regular oscillation [31]. This same behavior occurs in the periodic windows of the logistic map, including the miniature windows within the larger windows [32]. In this periodic window, we also observe the route to chaos by a
period-doubling bifurcation.

Fig. 4 Bifurcation diagram of the fractional-order diffusionless Lorenz system with $R$ for $q=0.95$.

Fig. 5 Bifurcation diagram of the fractional-order diffusionless Lorenz system with $R$ for $q=0.95$.

(a) $R \in [9, 11]$  (b) $R \in [5, 7]$

3.3 Bifurcations with Different Fractional-orders

Now let $\alpha=\beta=\gamma=q$, and change the fractional-order $q$ from 0.9 to 1, but fix the control parameter at $R=8$. The initial states of the fractional-order diffusionless Lorenz system and running time kept the same as above, but with a variational step of $R$ set to 0.0005 gives the bifurcation diagram shown in Fig. 6. This case shows the route to chaos for the fractional-order DLE as the fractional-order decreases. It is interesting to note that a chaotic transient is observed when $q$ is less than 0.912. The state space trajectory is shown in Fig. 7(a) for $q=0.911$, which suddenly switches to a pattern of oscillation that decays to the right equilibrium point. The time history of $z(t)$ in Fig. 7(b) also shows eventual convergence to the fixed point. On the average, chaotic behavior switches to damped behavior after about 70 oscillations. For larger $q < q_0 \approx 0.912$, chaotic behavior persists longer. Similar behavior has been reported for the standard Lorenz system [33]. The fractional-order system gives way to chaos by a period-doubling bifurcation as shown in Fig. 8(a) with steps in $R$ of 0.0002. It is chaotic in the range of 0.92 to 0.962. When the
fractional-order is less than 0.92, the fractional-order system converges to a fixed point. Expanding the periodic window shows the dynamic behavior better as in Fig. 8(b) with steps of 0.0001. There exist three kinds of bifurcation, i.e., a tangent bifurcation, a flip bifurcation, and an interior crisis. The fractional-order parameter can be taken as a bifurcation parameter, just like the control parameter.

Fig. 6 Bifurcation diagram of the fractional-order diffusionless Lorenz system with $q$ for $R=8$.

Fig. 7 Transient chaotic behavior in system (4) with $R=8$ and $q=0.911$. (a) 3D view on the $x-y-z$ space (b) The time histories of variable $z$

3.4 Bifurcations with Different Fractional-order for the Three Equations

(1) Fix $\beta=\gamma=1$, $R=8$, and let $\alpha$ vary. The system is calculated numerically for $\alpha \in [0.4, 1]$ with an increment of $\alpha$ equal to 0.002. The bifurcation diagram is shown in Fig. 9(a). It is found that when $0.43 \leq \alpha \leq 0.94$, the fractional-order system is chaotic with one periodic window at $c \in (0.484, 0.503)$ as shown in Fig. 9(b) with steps of 0.0002. When $\alpha$ increases from 0.4, or decreases from 1, a period-doubling route to chaos is observed.

(2) Fix $\alpha=\gamma=1$, $R=8$, and let $\beta$ vary. The system is calculated numerically for $\beta \in [0.75, 1]$ with an increment of $\beta$ equal to 0.001. The bifurcation diagram is shown in Fig. 10. When $\beta$ decreases from 1, a period-doubling route to chaos is observed, and it
converges to a fixed point when $\beta$ is less than 0.8. Thus the total smallest order of the fractional-order diffusionless Lorenz chaotic system is 2.8 in this case.

(3) Fix $\alpha=\beta=1$, $R=8$, and let $\gamma$ vary. The system is calculated numerically for $\gamma \in [0.8, 1]$ with an increment of $\gamma$ equal to 0.001. The bifurcation diagram is shown in Fig. 11(a). Similar phenomena are found, but the chaotic range of the fractional-order system is much smaller than that of the previous two cases, and it enters into chaos by a period-doubling route, and then it converges to a fixed point when $\gamma=0.816$. The dynamic behaviors in the periodic window are similar with the case of (2) as shown in Fig. 11(b) with steps of 0.0002.
Fig. 10 Bifurcation diagrams of the fractional-order diffusionless Lorenz system with $\beta$ for $\alpha=\gamma=1$.

Fig. 11 Bifurcation diagrams of the fractional-order diffusionless Lorenz system with $\gamma$ for $\alpha=\beta=1$. (a) $\gamma \in [0.8, 1]$ (b) $\gamma \in [0.81, 0.86]$

Conclusions

In this paper, we have numerically studied the bifurcations and dynamics of the fractional-order diffusionless Lorenz system by varying the system parameter and the system order. Typical bifurcations such as period-doubling bifurcations, flip bifurcations, tangent bifurcations, and interior crisis bifurcations were observed when both the control parameter and the fractional-order are changed. Complex dynamic behaviors such as fixed point, periodic motion, transient chaos, and chaos, occur in this fractional-order system. Future work on the topic should include a theoretical analysis of the dynamics of the fractional-order system, as well as in-depth studies of chaos control and synchronization for the system.

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References

Underdeterminacy and Redundance in Maxwell’s Equations. Origin of Gauge Freedom - Transversality of Free Electromagnetic Waves - Gaugefree Canonical Treatment without Constraints

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Abstract: Maxwell’s (1864) original equations are redundant in their description of charge conservation. In the nowadays used, ‘rationalized’ Maxwell equations, this redundancy is removed through omitting the continuity equation. Alternatively, one can Helmholtz decompose the original set and omit instead the longitudinal part of the flux law. This provides at once a natural description of the transversality of free electromagnetic waves and paves the way to eliminate the gauge freedom. Poynting’s inclusion of the longitudinal field components in his theorem represents an additional assumption to the Maxwell equations. Further, exploiting the concept of Newtonian and Laplacian vector fields, the role of the static longitudinal component of the vector potential being not determined by Maxwell’s equations, but important in quantum mechanics (Aharonov-Bohm effect) is elucidated. Finally, extending Messiah’s (1999) description of a gauge invariant canonical momentum, a manifest gauge invariant canonical formulation of Maxwell’s theory without imposing any contraints or auxiliary conditions will be proposed as input for Dirac’s (1949) approach to special-relativistic dynamics.

Keywords: Electromagnetic Waves; Maxwell Equations; Helmholtz Decomposition; Gauge Theory; Poynting’s Theorem; Aharonov-Bohm Effect

PACS (2008): 41.20.-q; 41.20.Jb; 03.50.De; 03.50.-z; 11.15.-q; 73.23.-b

1. Introduction

Traditionally, there are two main approaches to classical electromagnetism (CEM), viz, (1) the experimental one going from the phenomena to the rationalized Maxwell equations (eg, Maxwell 1873, Mie 1941, Jackson 1999, Feynman, Leighton & Sands 2001);
the deductive one deriving the phenomena from the rationalized Maxwell equations (eg, Hertz 1889, Lorentz 1909, Sommerfeld 2001).

"Rationalized Maxwell equations" (Poynting 1884, Heaviside 1892) means Gauss’ laws for the magnetic (1a) and dielectric fields (1c) as well as Faraday’s induction (1b) and Ampère-Maxwell flux laws (1d). In SI units,

\[ \nabla \cdot \vec{B}(\vec{r}, t) = 0 \quad (1a) \]
\[ \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) = -\nabla \times \vec{E}(\vec{r}, t) \quad (1b) \]
\[ \nabla \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \quad (1c) \]
\[ \frac{\partial}{\partial t} \vec{D}(\vec{r}, t) = \nabla \times \vec{H}(\vec{r}, t) - \vec{j}(\vec{r}, t) \quad (1d) \]

For moving charges in vacuo, they can be simplified via \( \vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t) \), \( \vec{B}(\vec{r}, t) = \mu_0 \vec{H}(\vec{r}, t) \) to the microscopic Maxwell equations (Lorentz 1892).

\[ \nabla \cdot \vec{B}(\vec{r}, t) = 0 \quad (2a) \]
\[ \frac{\partial}{\partial t} \vec{B}(\vec{r}, t) = -\nabla \times \vec{E}(\vec{r}, t) \quad (2b) \]
\[ \nabla \cdot \vec{E}(\vec{r}, t) = \frac{1}{\varepsilon_0} \rho(\vec{r}, t) \quad (2c) \]
\[ \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) = \frac{1}{\varepsilon_0 \mu_0} \nabla \times \vec{B}(\vec{r}, t) - \frac{1}{\varepsilon_0} \vec{j}(\vec{r}, t) \quad (2d) \]

For both sets, two fundamental problems have to be clarified, viz,

1. the origin of the gauge freedom in the potentials, and
2. the origin of the transversality of free (unbounded) electromagnetic waves.

Stipulated by special relativity, all field variables are usually treated on equal footing. There are quite different types of field variables, however. This comes into play, in particular, when the boundary contains electrodes with fixed potential values, or when the domain under consideration is multiply connected. And Gauss’ laws,

\[ \nabla \vec{E}(\vec{r}, t) = \frac{1}{\varepsilon_0} \rho(\vec{r}, t) \quad (3) \]
\[ \nabla \vec{B}(\vec{r}, t) = 0 \quad (4) \]

"are not, properly speaking, equations of motion, but rather constraints imposed on the fields \( \mathcal{E} [\vec{E}] \) and \( \mathcal{H} [\vec{B}] \). They fix the longitudinal parts of these fields... In order to define the dynamical state of the system, it is therefore sufficient to specify the charge distributions and currents – that is, the positions and the velocities of the particles – on the one hand, and the transverse fields \( \mathcal{H} [\vec{B}] \) and \( \mathcal{E}_\perp [\vec{E}_T] \) on the other.” (Messiah 1999, XXI.22). This suggests to discriminate the transverse and longitudinal components of the field vectors from the very beginning, though this Helmholtz decomposition is not Lorentz covariant. However, for being compliant with special relativity, it is sufficient that an equation is Lorentz invariant (Barut 1964).

For this, I will – following the recommendation by Boltzmann (2001) – return to Maxwell’s (1864) original set of equations. Using the Helmholtz (1858) decomposition of
3D vector fields into transverse and longitudinal components, I will show that this set is both underdetermined and redundant (but not inconsistent). Remarkably enough, both deficiencies are related to longitudinal vector components.

Thus, to provide the formal basis, Section 2 relates the Helmholtz decomposition to Newtonian, Laplacian and vector fields in multiply connected domains. This will facilitate the understanding of the particular standing of the static longitudinal component of the vector potential, $\vec{A}_L(\vec{r})$, which is not accounted for in any variant of Maxwell equations and which does not enter the Maxwell-Lorentz force.

In Section 3, Maxwell’s 1864 set of equations are rewritten in terms of the transverse and longitudinal components of all fields. This reveals immediately, that the subset which deals with charge conservation is redundant. In the rationalized Maxwell equations used nowadays (Poynting 1884, Heaviside 1892), this redundancy is eliminated through removing the continuity law: $\nabla \vec{j} + \dot{\rho} = 0$, from the basic set of equations. In contrast, I will propose to eliminate this redundancy through removing the longitudinal component of Ampère-Maxwell’s flux law. A revised set of independent variables being free of underdeterminacy and redundancy will be proposed.

Moreover, this decomposition will discover the fact, that the incorporation of the longitudinal component of the electrical field strength, $\vec{E}_L$, in Poynting’s (1884) theorem represents an additional assumption, which, in turn, obscures the transversality of freely propagating electromagnetic waves. Thus, Poynting’s theorem separates into two theorems: one for the propagating transverse and one for the non-propagating longitudinal field parts, see Section 6.

Section 4 considers the role of $\vec{A}_L(\vec{r}, t)$ for the gauge freedom both in electromagnetism and in Schrödinger wave mechanics, where the latter provides a short-cut to a gauge invariant Hamiltonian.

Section 5 treats Mie’s approach to the rationalized Maxwell equations in terms of the Helmholtz decomposition. The decomposed Maxwell equations will be used in Section 6 to split Poynting’s (1884) theorem into a transverse one for the propagating and a longitudinal one for the non-propagating field momenta, respectively.

There seems to be an astounding difference between the Lagrangian and the Hamiltonian treatments of CEM. In virtually all CEM textbooks, both the Lagrangian and the Hamiltonian are explored for, (i), the motion of charged bodies subject to external electromagnetic fields and, (ii), electromagnetic fields with external charges and currents. In contrast, I’m not aware of a CEM textbook discussing not only the Lagrangian, but also the Hamiltonian for, (iii), closed systems of charges and fields. The latter is needed for the state description, not only in quantum physics (cf Dirac 1949), but also in classical physics. Indeed, many textbooks on quantum theory do contain such a Hamiltonian, but usually in momentum space.

As a matter of fact, it is not sufficient simply to insert the canonical momenta in the well-known total energy (84). And in contrast to the Lagrangian, where $L_{\text{tot}} = L_{\text{chg}} + L_{\text{field}} + L_{\text{int}}$, the Hamiltonian is not additive: $H_{\text{tot}} \neq H_{\text{chg}} + H_{\text{field}} + H_{\text{int}}$. Moreover, the fact, that – by virtue of the absence of a time-derivative – Gauss’ laws represent
Bergmannian constraints of 1st kind (Dirac 2001, p.8) rather than dynamical equations, prevents a standard treatment of the canonical theory. In Section 7, these difficulties will be overcome for the microscopic theory in common spacetime without invoking additional constraints, using Milton & Schwinger’s (2006) Lagrangian representation of the microscopic theory and extending it to a Hamiltonian representation.

Section 8 exploits these results for developing a manifest gauge-invariant, ie, gaugefree Lagrangian and Hamiltonian. This includes gaugefree canonical momenta for bodies and fields (following and extending the treatment by Messiah 1999).

Both the microscopic Maxwell equations and the Lagrangian equations of motion are easily written down in terms of Minkowski 4-scalars/vectors/tensors. In contrast, Hamilton’s equations of motion distinguish the time coordinate, what prevents a straightforward Lorentz covariant reformulation. Johns (2005) has put space and time variables on equal footing through extending the set of independent variables by an auxiliary parameter. Alternatively, there are proposals for a canonical field momentum density tensor, here, 

$$\Pi^\nu_\mu = \frac{\delta L}{\delta (\partial A^\mu/\partial x^\nu)} \quad (5)$$

This remains to be explored. – Anyway, as mentioned above, special-relativistic invariance is not bound to Lorentz covariance (Barut 1964). This is demonstrated in Dirac’s (1949) analysis of the possible forms of special-relativistic Hamiltonian dynamics (for a short review of the historical development and recent results, see Stefanovich 2008). Here, moreover, the unity of kinematics and dynamics is guaranteed from the very beginning in that dynamical variables are generators of kinematical transformations; thus, one goal of this contribution consists in providing a fully interacting starting Hamiltonian for that approach.

The main results will be summarized and discussed in Section 9.

2. Helmholtz Decomposition of 3D Vector Fields

In order to apply Helmholtz’s decomposition theorem appropriately, one has carefully to discriminate between certain types of vector fields, viz, Newtonian, Laplacian and vector fields in multiply connected domains.

2.1 Newtonian Vector Fields

Newtonian vector fields are vector fields in unbounded domains with a given distribution of sources and vortices (Schwab 2002). The classical example is Newton’s force of gravity. They are the actual subject of

Helmholtz’s decomposition theorem: Any sufficiently well-behaving 3D vector field, \( \vec{f}(\vec{r}) \), can uniquely be decomposed into a transverse or solenoidal, \( \vec{f}_T(\vec{r}) \), a longitudinal
or irrotational, \( \vec{f}_L(\vec{r}) \), and a constant components (which I will omit in what follows).

\[
\vec{f}(\vec{r}) = \iiint_V \vec{f}(\vec{r}') \delta(\vec{r} - \vec{r}') dV'; \quad \vec{r} \in V \setminus \partial V \tag{6}
\]

\[
= -\frac{1}{4\pi} \iiint_V \vec{f}(\vec{r}') \Delta \frac{1}{|\vec{r} - \vec{r}'|} dV' \tag{7}
\]

\[
= \frac{1}{4\pi} \nabla \times \nabla \times \iiint_V \vec{f}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dV' - \frac{1}{4\pi} \nabla \nabla \cdot \iiint_V \vec{f}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dV' \tag{8}
\]

\[
= \vec{f}_T(\vec{r}) + \vec{f}_L(\vec{r}) \tag{9}
\]

(These notions of longitudinal and transverse fields should not be confused with the notions of longitudinal and transverse waves in waveguides!)

It is thus most useful to introduce scalar, \( \phi_f(\vec{r}) \), and vector potentials, \( \vec{a}_f(\vec{r}) \), as

\[
\vec{f}_T(\vec{r}) = \nabla \times \vec{a}_f(\vec{r}); \quad \vec{f}_L(\vec{r}) = -\nabla \phi_f(\vec{r}) \tag{10}
\]

The minus sign is chosen to follow the definitions of the mechanical potential energy and the scalar potential in the electric field strength. \( \vec{a}_f \) is \textit{sourceless}; otherwise, one would increases the number of independent field variables.

As a consequence, each such vector field is uniquely determined by its sources, \( \phi_f \), and \textit{sourceless} vertices, \( \vec{f}_f \).

\[
\nabla \times \vec{f}(\vec{r}) = \nabla \times \vec{f}_T(\vec{r}) = \nabla \times \nabla \times \vec{a}_f(\vec{r}) = -\Delta \vec{a}_f(\vec{r}) = \vec{j}_f(\vec{r}) \tag{11}
\]

\[
\nabla \cdot \vec{f}(\vec{r}) = \nabla \cdot \vec{f}_L(\vec{r}) = -\Delta \phi_f(\vec{r}) = \rho_f(\vec{r}) \tag{12}
\]

Including the surface terms (Oughstun 2006, Appendix A), the potentials follow as

\[
\phi_f(\vec{r}) = \frac{1}{4\pi} \nabla \cdot \iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' \tag{13}
\]

\[
= \frac{1}{4\pi} \iint_V \rho_f(\vec{r}') |\vec{r} - \vec{r}'| \cdot d\vec{\sigma}' - \frac{1}{4\pi} \oint_{\partial V} \vec{f}(\vec{r}') \cdot d\vec{\sigma}' \tag{14}
\]

\[
\vec{a}_f(\vec{r}) = \frac{1}{4\pi} \nabla \times \iiint_V \frac{\vec{f}(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' \tag{15}
\]

\[
= \frac{1}{4\pi} \iint_V \frac{\vec{j}_f(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' + \frac{1}{4\pi} \oint_{\partial V} \frac{\vec{f}(\vec{r}')}{|\vec{r} - \vec{r}'|} \times d\vec{\sigma}' \tag{16}
\]

For the balance equations I will also need the

**Orthogonality theorem:** Integrals over mixed scalar products vanishes,

\[
\iiint_V \vec{f}_T \cdot \vec{g}_L dV = -\iiint_V \nabla \times \vec{a}_f \cdot \nabla \phi g dV \tag{17}
\]

\[
= -\iint_V \nabla \left( \phi g \nabla \times \vec{a}_f \right) dV = -\iint_V \nabla \left( \vec{a}_f \times \nabla \phi g \right) dV \tag{18}
\]

\[
= -\oint_{\partial V} \phi g \nabla \times \vec{a}_f \cdot d\vec{s} = -\oint_{\partial V} \vec{a}_f \times \nabla \phi g \cdot d\vec{s} = 0 \tag{19}
\]
if the surface, $\partial V$, lies infinitely away from the sources of the fields (cf Stewart 2008), or if the fields obey appropriate periodic boundary conditions on $\partial V$ (Heitler 1954, I.6.3).

If the Orthogonality theorem holds true, the integrals over the scalar products of two vectors separates as

$$\iiint_V \vec{f}(\vec{r}) \cdot \vec{g}(\vec{r}) dV = \iiint_V \vec{f}_T(\vec{r}) \cdot \vec{g}_T(\vec{r}) dV + \iiint_V \vec{f}_L(\vec{r}) \cdot \vec{g}_L(\vec{r}) dV$$  \hspace{1cm} (20)

In particular, both the Joule power and the electric field energy decompose into the contributions of the transverse and longitudinal components of the (di)electric field vectors.

The validity of this theorem will be assumed throughout this series of papers.

Notice that the electromagnetic vector potential, $\vec{A}$, is a vector potential in the sense of Helmholtz’s theorem only w.r.t. the magnetic induction, $\vec{B}$, not, however, w.r.t. the electric field strength, $\vec{E}$. As a consequence, both its transverse and longitudinal components are physically significant. The clue is thus to Helmholtz-decompose $\vec{A}$, too.

It should also be noted that the longitudinal and transverse components of a localized vector field are spread over the whole volume of definition. For instance, for a point-like body of charge $q$ moving along the trajectory $\vec{r}(t)$,

$$\rho(\vec{r}; t) = q \delta(\vec{r} - \vec{r}(t)); \quad \vec{j}(\vec{r}; t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}(t))$$  \hspace{1cm} (21)

one has ($\vec{r}_t \equiv \vec{r}(t)$; here, $t$ is merely a parameter)

$$\rho_j = \nabla \cdot \vec{j} = q \vec{v} \cdot \nabla \delta(\vec{r} - \vec{r}(t)) = -\frac{\partial \rho}{\partial t}$$  \hspace{1cm} (22a)

$$\vec{a}_j = \nabla \times \vec{j} = q \nabla \delta(\vec{r} - \vec{r}_t) \times \vec{v}$$  \hspace{1cm} (22b)

$$\phi_j = \frac{q}{4\pi} \nabla \cdot \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = -\frac{q}{4\pi} \frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3}$$  \hspace{1cm} (23a)

$$\vec{a}_j = \frac{q}{4\pi} \nabla \times \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = -\frac{q}{4\pi} \frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3}$$  \hspace{1cm} (23b)

$$\vec{j}_L = \frac{q}{4\pi} \nabla \frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3}; \quad \vec{j}_T = -\frac{q}{4\pi} \nabla \times \frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3}$$  \hspace{1cm} (24)

2.2 Laplacean Vector Fields

Laplacean vector fields are vector fields outside any sources and vortices, they (or their potentials) satisfy the Laplace equation and are essentially determined by the (inhomogeneous) boundary conditions (Schwab 2002). A typical example is the electric field between electrodes.

Since both their divergence and curl vanishes identically, Helmholtz’s theorem is not really useful for them.
2.3 Vector Fields in Multiply Connected Domains

Vector fields in multiply connected domains assume an 'androgyne' position in that they (or their potentials) satisfy the Laplace equation in certain, bounded domains, but not globally. A well known example is the magnetic field strength, $\vec{H}$, of a constant current, $I$, through an infinite straight conductor in vacuo. The 'magnetic ring voltage', $\oint \vec{H} \cdot d\vec{s}$, vanishes identically, as long as the path of integration lies entirely outside the conductor, so that no current flows through the area bounded by it. But it equals

$$n \int_\sigma (\nabla \times \vec{H}) \cdot d\vec{\sigma} = n \int_\sigma \vec{j} \cdot d\vec{\sigma} = nI$$

if the path surrounds the conductor $n$ times ($n$ integer). That means, that inside the conductor, $\vec{H}$ is a vortex field: $\nabla \times \vec{H} = \vec{j} \neq \vec{0}$, while outside the conductor, $\vec{H}$ is a gradient field: $\nabla \times \vec{H} = \vec{0}$. Obviously, Helmholtz’s theorem is only conditionally applicable, since the integral rather than the differential form of Ampère’s flux law is appropriate.

An analogous example is the vector potential in the Aharonov-Bohm (1959) setup. A constant current through an ideal straight infinite coil in vacuo with no spacing between its windings creates a magnetic field strength and induction being constant inside and vanishing outside the coil. However, by virtue of its continuity, the vector potential does not vanish outside the coil, but represents a gradient field there. I will return to this issue in Section 4.

3. Maxwell’s (1864) Original Equations Revisited

"He [Maxwell] would not have been so often misunderstood, if one would have started the study not with the treatise, while the specific Maxwellian method occurs much more clearly in his earlier essays." (Boltzmann 2001; cf also Sommerfeld 2001, §1) For this, let us return to Maxwell’s (1864) original set of "20 equations for the 20 variables" ($F, G, H = A$, $(\alpha, \beta, \gamma) = \vec{H}$, $(P, Q, R) = \vec{E}$, $(p, q, r) = \vec{j}$, $(f, g, h) = \vec{D}$, $(p', q', r') = \vec{J}$, $e = \rho$, $\psi = \Phi$). I will rewrite them in modern notation (the r.h.s. of the foregoing relations), SI units and together with their Helmholtz decomposition. For easier reference, Maxwell’s equation numbering is applied. In place of his eqs. (D) for moving conductors his eqs. (35) for conductors at rest is used. The signs in his eqs. (F) and (G) are changed according to the nowadays use.

3.1 Helmholtz Decomposition

A) The total current density, $\vec{J}$, is the sum of electric (conduction, convection) current density, $\vec{j}$, and displacement ('total polarization') current density, $\partial \vec{D}/\partial t$.

$$\vec{J}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{D}(\vec{r}, t) \quad (A)$$
This is Maxwell’s famous and crucial step to generalize Ampère’s flux law to open circuits and to convective currents. The time derivative is a precondition to obtain wave equations for the field variables.

The Helmholtz decomposition of this equation is obvious.

\[
\vec{J}_{T,L}(\vec{r},t) = \vec{j}_{T,L}(\vec{r},t) + \frac{\partial}{\partial t} \vec{D}_{T,L}(\vec{r},t) \tag{A_{T,L}}
\]

**B)** The "magnetic force" (induction, flux density), \( \mu \vec{H} \), is the vortex of the vector potential, \( \vec{A} \): \( \mu \vec{H} = \nabla \times \vec{A} \). Hence, it has got no longitudinal component.

\[
(\mu \vec{H})_T(\vec{r},t) = \nabla \times \vec{A}_T(\vec{r},t) \tag{B_T}
\]

\[
(\mu \vec{H})_L(\vec{r},t) \equiv 0 \tag{B_L}
\]

**C)** The total current density, \( \vec{J} \), is the vortex of the magnetic field strength, \( \vec{H} \).

\[
\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) \tag{C}
\]

Hence, it has got no longitudinal component, too.

\[
\vec{J}_T(\vec{r},t) = \nabla \times \vec{H}_T(\vec{r},t) \tag{C_T}
\]

\[
\vec{J}_L(\vec{r},t) = \vec{0} \tag{C_L}
\]

**D)** The "electromotive force" (electric field strength) equals

\[
\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t} \vec{A}(\vec{r},t) - \nabla \Phi(\vec{r},t) \tag{M-35}
\]

Therefore,

\[
\vec{E}_T(\vec{r},t) = -\frac{\partial}{\partial t} \vec{A}_T(\vec{r},t) \tag{M-35_T}
\]

\[
\vec{E}_L(\vec{r},t) = -\frac{\partial}{\partial t} \vec{A}_L(\vec{r},t) - \nabla \Phi(\vec{r},t) \overset{\text{def}}{=} -\nabla \phi\varepsilon(\vec{r},t) \tag{M-35_L}
\]

The longitudinal component consists of two terms, for which, however, there is no other equation. This makes the whole set to be underdetermined and is the origin of the gauge freedom in the potentials \( \vec{A} \) and \( \Phi \). Due to the redundancy in some equations below, it is not inconsistent, however.

This underdeterminacy is overcome, if one can work solely with \( \phi\varepsilon(\vec{r},t) \), the 'total scalar potential of \( \vec{E}(\vec{r},t) \)', or if one finds an additional equation for \( \vec{A}_L \) and \( \Phi \), respectively. An example is the boundary conditions in the Aharonov-Bohm (1959) setup, which determine \( \vec{A}_L \) outside the coil.

**E)** Electric field strength and dielectric displacement are related through the “equation of electric elasticity”.

\[
\vec{E}(\vec{r},t) = \frac{1}{\varepsilon} \vec{D}(\vec{r},t) \tag{E}
\]
Thus,\[ E_{T,L}(\vec{r}, t) = \frac{1}{\varepsilon} \vec{D}_{T,L}(\vec{r}, t) \] (E_{T,L})
if \( \varepsilon \) is a scalar constant.

**F)** Electric field strength and electric current density are related through the "equation of electric resistance" (\( \sigma \) being the specific conductivity).

\[ E(\vec{r}, t) = \frac{1}{\sigma} \vec{j}(\vec{r}, t) \] (F)

Thus,

\[ E_{T,L}(\vec{r}, t) = \frac{1}{\sigma} \vec{j}_{T,L}(\vec{r}, t) \] (F_{T,L})

if \( \sigma \) is a scalar constant.

For \( N \) point-like charges \( \{ q_a \} \) in vacuo (\( \sigma = 0 \)), eq. (F) is to be replaced with

\[ \sum_{a=1}^{N} q_a \vec{v}_a(t) \delta(\vec{r} - \vec{r}_a(t)) = \vec{j}(\vec{r}, t) \] (26)

**G)** The "free" charge density is related to the dielectric displacement through the "equation of free electricity".

\[ \rho(\vec{r}, t) - \nabla \vec{D}(\vec{r}, t) = 0 \] (G)

Obviously, it concerns the longitudinal component of \( \vec{D} \) only.

\[ \rho(\vec{r}, t) - \nabla D_L(\vec{r}, t) = 0 \] (G_{L})

**H)** In a conductor, there is – in analogy to hydrodynamics – "another condition", the "equation of continuity".

\[ \frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla \vec{j}(\vec{r}, t) = 0 \] (H)

It concerns the longitudinal component of \( \vec{j} \) only.

\[ \frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla j_L(\vec{r}, t) = 0 \] (H_{L})

At once, by virtue of eq.(C_{L}), it is merely a consequence of eq.(G_{L}). Here is the redundancy mentioned above.

With

\[ \vec{j} = \vec{j}_T + \vec{j}_L = \nabla \times \vec{a}_j - \nabla \phi_j \] (27)

one obtains the continuity equation in the form

\[ -\Delta \phi_j(\vec{r}, t) + \frac{\partial}{\partial t} \rho(\vec{r}, t) = 0 \] (28)

It has the advantage of being a single equation relating two scalar quantities one to another rather than four, as in its usual form (H).
3.2 Elimination of Underdeterminancy and Redundance

The underdeterminacy and redundance in Maxwell’s original set can be eliminated through removing \((\mu\vec{H})_L, \Phi\) and \(\vec{A}_L\) from the set of field variables, but retaining \(\phi_E = -\partial\phi_E/\partial t + \Phi\). I also remove the total current in view of its merely historical relevance. Then, it remains 18 equations for the 18 variables \((\mu\vec{H})_{(T)} = \vec{B}, \vec{H}, \vec{A}_T, \vec{D}, \vec{E}, \phi_E, \vec{j}\) and \(\rho\).

B’) The magnetic induction (flux density), \(\mu\vec{H}\), is solenoidal, since it is the vortex of the transverse component of the vector potential.

\[
\mu\vec{H} = (\mu\vec{H})_T = \nabla \times \vec{A}_T \tag{B’}
\]

C’) The transverse components of the conduction/convection and displacement current densities build the vortex of the transverse component, \(\vec{H}_T\), of the magnetic field strength, \(\vec{H}\).

\[
\nabla \times \vec{H}_T = \vec{j}_T + \frac{\partial}{\partial t} \vec{D}_T \tag{C’}
\]

D’) The electric field strength equals (and Helmholtz decomposes as)

\[
\vec{E} = -\frac{\partial}{\partial t} \vec{A}_T - \nabla \phi_E \tag{M-35’}
\]

E) Electric field strength and dielectric displacement are related through the “equation of electric elasticity” (E).

F) Electric field strength and electric current density are related through the “equation of electric resistance” (F).

G’) The ”free” charge density is related to the longitudinal component of the dielectric displacement through the ”equation of free electricity”.

\[
\rho(\vec{r}, t) - \nabla \vec{D}_L(\vec{r}, t) = 0 \tag{G’}
\]

H’) The conservation of charge is expressed through the equation of continuity.

\[
\frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla \vec{j}_L(\vec{r}, t) = 0 \tag{H’}
\]

Therefore, the redundance is removed in the flux law rather than eliminating the continuity equation from the set of basic equations. The continuity equation is retained, because it is a direct consequence of the fact, that – within this approach – the charge of a point-like body is a given, invariant property of it (like its mass). This also allows for an immediate explanation of the transversality of free electromagnetic waves (see below).

4. Gauge Freedom and the Role of \(\vec{A}_L\)

4.1 Classical Gauge Freedom

As mentioned after eq.(M-35_L) above, there is only one equation for the two fields \(\partial \vec{A}_L/\partial t\) and \(\Phi\). Hence, any change of the scalar and vector potentials such, that the expression
\[-\partial \phi_A / \partial t + \Phi \] remains unchanged, is without any physical effect within Maxwell’s theory.
In fact, the Helmholtz components and potentials of vector potential, \( \vec{A} \), and electrical field strength, \( \vec{E} \),
\[
\vec{A} = \vec{A}_T + \vec{A}_L = \nabla \times \vec{a}_A - \nabla \phi_A \\
\vec{E} = \vec{E}_T + \vec{E}_L = \nabla \times \vec{a}_E - \nabla \phi_E
\]
are known to be interrelated as
\[
\vec{E}_T = -\frac{\partial}{\partial t} \vec{A}_T; \quad \vec{a}_E = -\frac{\partial}{\partial t} \vec{a}_A \\
\vec{E}_L = -\frac{\partial}{\partial t} \vec{A}_L - \nabla \Phi; \quad \phi_E = -\frac{\partial}{\partial t} \phi_A + \Phi
\]
Hence, the gauge transformation,
\[
\vec{A} = \vec{A}’ - \nabla \chi; \quad \Phi = \Phi’ + \frac{\partial \chi}{\partial t}
\]
actually concerns only the scalar potential, \( \phi_A \), of \( \vec{A} \) as
\[
\phi_A = \phi_A’ + \chi
\]
but not the vector potential, \( \vec{a}_A \), of \( \vec{A} \).
In the Lorenz (1867) gauge used in Lorentz covariant formulations of the theory, one has
\[
\nabla \vec{A} = -\Delta \phi_A = -\frac{\partial \Phi}{\partial t}
\]
while in the Coulomb (transverse, radiation) gauge being popular in quantum electrodynamics,
\[
\nabla \vec{A} = -\Delta \phi_A = 0
\]
This all suggests to avoid the gauge indeterminacy at all through working solely with \( \vec{A}_T \) and \( \phi_E \). If necessary, \( \vec{A}_L \) can be determined as boundary value problem.

4.2 Quantum Gauge Freedom (Schrödinger Theory)

Although this series of papers deals with classical electromagnetism, it is enlightening and pedagogically useful to sidestep for looking at gauge freedom within Schrödinger wave mechanics.

In order to be independent of the interpretation of the quantum mechanical formalism, let me proceed as follows (Enders 2006, 2008a,b).
\[ |\psi|^2 \] and \( \langle \psi | \hat{H} | \psi \rangle \) are ‘Newtonian state functions’ of a non-relativistic quantum system as they are time-independent in stationary states and as their time-dependence is governed by solely the time-dependent part of the Hamiltonian. This suggests to extend Helmholtz’s (1847, 1911) explorations about the relationships between forces and energies to the question, which ‘external influences’ leave \( |\psi|^2 \) and \( \langle \psi | \hat{H} | \psi \rangle \) unchanged?
Obviously, $|\psi|^2$ is unchanged, if an external influence, $w$, affects only the phase, $\varphi$, of $\psi$. (Dirac 1931 required the phase to be independent of the state.)

$$\psi_w = \psi_0 e^{i\varphi(w)}; \quad \varphi(0) = 0$$

(36)

Then, if $\psi_0(\vec{r}, t)$ obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_0 = \frac{\hat{p}^2}{2m} \psi_0 + V \psi_0$$

(37)

$\psi_w(\vec{r}, t)$ obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_w = \hat{H}_w \psi_w = \frac{1}{2m} \left( \hat{p} - \hbar \nabla \varphi \right)^2 \psi_w + \left( V - \hbar \frac{\partial \varphi}{\partial t} \right) \psi_0$$

(38)

Consequently, in stationary states, $<\psi_w | \hat{H}_w | \psi_w>$ is independent of $w$, because $i\hbar \frac{\partial}{\partial t} \psi_w = E \psi_w$, where -- by the very definition of $w$ -- $E$ is independent of $w$. This is essentially the gauge invariance of the Schrödinger (Pauli 1926) and Dirac equations (Fock 1929) (see also Weyl 1929, 1931).

For influences caused by external electromagnetic fields, this quite general arguing leads to the following important observation, which will be exploited below when formulating a gaugefree canonical theory.

The common quasi-classical Schrödinger equation for a point-like charge, $q$, in an electromagnetic field reads

$$i\hbar \frac{\partial}{\partial t} \psi_{\vec{A}, \Phi}(\vec{r}, t) = \left[ \frac{1}{2m} \left( \hat{p} - q \vec{A}(\vec{r}, t) \right)^2 + q \Phi(\vec{r}, t) \right] \psi_{\vec{A}, \Phi}(\vec{r}, t)$$

(39)

Thus, the wave function

$$\psi_{\vec{A}_T, \phi_E}(\vec{r}, t) = \psi_{\vec{A}, \Phi}(\vec{r}, t) e^{i\frac{q}{\hbar} \phi_{\vec{A}}(\vec{r}, t)}$$

(40)

obeys a Schrödinger equation with a manifest gauge invariant Hamiltonian.

$$i\hbar \frac{\partial}{\partial t} \psi_{\vec{A}_T, \phi_E}(\vec{r}, t) = \left[ \frac{1}{2m} \left( \hat{p} - q \vec{A}_{\vec{T}}(\vec{r}, t) \right)^2 + q \phi_E(\vec{r}, t) \right] \psi_{\vec{A}_T, \phi_E}(\vec{r}, t)$$

(41)

This suggests that manifest gauge invariant theories can be obtained through replacing $\vec{A}$ with $\vec{A}_T$ and $\Phi$ with $\phi_E$.

It’s noteworthy that in both Hamiltonians the canonical momentum operator: $\hat{p} = -i\hbar \nabla$, is the same, while the corresponding classical canonical momenta are different.

It is noteworthy, that in multiply connected domains, notably outside an infinite coil, where the $\vec{B}$-field vanishes, $\phi_{\vec{X}}$ is not globally integrable. The phase of the wave function can acquire physical significance, as in the Aharonov-Bohm (1959) effect. This underpins the physical significance of the Helmholtz decomposition of the field variables.

Thus, the longitudinal component of a static vector potential, $\vec{A}_L(\vec{r})$, is classically not observable, because it does not contribute to the Maxwell-Lorentz force (Maxwell 1864, Lorentz 1892),

$$q \vec{E} + q \vec{v} \times \vec{B} = q \left( -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi + \vec{v} \times \nabla \times \vec{A} \right)$$

(42)
This suggests to remove $\vec{A}_L(\vec{r})$ from the classical theory altogether and to consider it to be a ‘quantum potential’ being proportional to Planck’s quantum of action, $h$. On the other hand, if one requires – for good reasons – $\vec{A}(\vec{r})$ to be continuous, $\vec{A}_L(\vec{r})$ can be finite even in the classical (limit) case.

Eq.(40) suggests to incorporate other non-dynamical fields not entering the Hamiltonian and being determined by Laplacean boundary-value problems, by means of appropriate phase factors, too.

"As emphasized by Yang [1974] the vector potential is an over complete specification of the physics of a gauge theory but the gauge covariant field strength underspecifies the content of a gauge theory. The Bohm-Aharonov [1959] effect is the most striking example of this, wherein there exist physical effects on charged particles in a region where the field strength vanishes. The complete and minimal set of variables necessary to capture all the physics are the non-integrable phase factors.” (Gross 1992, II.4) Because there are no such phase factors within classical electromagnetism, their classical limit is rather unclear. The complete and minimal set of classical variables obtained below is only loosely related to those. It is thus hoped that the gauge-free representation presented below will narrow this gap between classical and quantum theory.

5. Helmholtz Decomposition of the "Rationalized" Maxwell’s Equations within Mie’s Approach

Newton (1999, Definitions) has assumed that the mass is a constant property of a given body. Likewise, it is meaningful to consider the electric charge to be such a property. As a consequence, one has the continuity equation,

$$\nabla \vec{j}(\vec{r},t) + \frac{\partial}{\partial t} \rho(\vec{r},t) = 0$$

(43)

as a precondition of the dynamics of the fields created by the charges. This is in contrast with those approaches which see the continuity as contained in or even as a consequence of the rationalized Maxwell equations, but it is compatible with Maxwell’s original equations (see above).

Given that, Mie (1941) argues as follows (after Hehl & Obukhov 2003; see also Bopp 1962, Enders 2008).

1) Mathematically, to each given charge distribution, $\rho(\vec{r},t)$, there is a vector field, $\vec{D}(\vec{r},t)$, such, that

$$\nabla \vec{D}(\vec{r},t) = \rho(\vec{r},t)$$

(44)

According to Maxwell (1864, 1873), $\vec{D}(\vec{r},t)$ has got a physical meaning, viz, as (di)”electric displacement”.

Actually, there are infinitely many such vector fields, because $\vec{D}_T$ is not specified. Moreover, this conclusion is not unique as one can associate to $\rho(\vec{r},t)$ also a scalar field, $\phi(\vec{r},t)$, such, that the latter obeys the Helmholtz equation (Enders 2009)

$$\nabla^2 \phi(\vec{r},t) + \kappa \phi(\vec{r},t) = \rho(\vec{r},t)$$

(45)
2) By virtue of charge conservation (43),
\[
\frac{\partial}{\partial t} \nabla \vec{D}(\vec{r}, t) = \frac{\partial}{\partial t} \rho(\vec{r}, t) = -\nabla \vec{j}(\vec{r}, t)
\]
(46)
\[
0 = \nabla \left( \frac{\partial}{\partial t} \vec{D}(\vec{r}, t) + \vec{j}(\vec{r}, t) \right)
\]
(47)

Hence, mathematically, there is a vector field, \( \vec{H}(\vec{r}, t) \), such, that
\[
\nabla \times \vec{H}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{D}(\vec{r}, t)
\]
(48)

According to Maxwell (1864, 1873), \( \vec{H}(\vec{r}, t) \) has got a physical meaning, viz, as the magnetic field strength.

Actually, there are infinitely many such vector fields, because \( \vec{H}_L \) is not specified.

Thus, in the spirit of Helmholtz’s (1858) decomposition theorem, this approach can be formulated more precisely as follows.

1') Mathematically, to each given charge distribution, \( \rho(\vec{r}, t) \), there is a vector field, \( \vec{D}(\vec{r}, t) = \vec{D}_T(\vec{r}, t) + \vec{D}_L(\vec{r}, t) \), such, that \( \rho(\vec{r}, t) \) is the source of its scalar component.
\[
\nabla \vec{D}_L(\vec{r}, t) = \rho(\vec{r}, t)
\]
(49)

According to Maxwell, \( \vec{D}_L \) is the longitudinal component of the (di)"electric displacement”, if \( \rho(\vec{r}, t) \) represents the "free" charges.

2') By virtue of charge conservation,
\[
\nabla \left( \frac{\partial}{\partial t} \vec{D}_L(\vec{r}, t) + \vec{j}_L(\vec{r}, t) \right) = \nabla \vec{J}_L(\vec{r}, t) = 0
\]
(50)

Hence, the longitudinal component, \( \vec{J}_L \), of the total current, \( \vec{J} = \frac{\partial}{\partial t} \vec{D} + \vec{j} \), vanishes identically.
\[
\vec{J}_L(\vec{r}, t) = \frac{\partial}{\partial t} \vec{D}_L(\vec{r}, t) + \frac{\partial}{\partial t} \vec{D}_T(\vec{r}, t) \equiv \vec{0}
\]
(51)

Its transverse component can – as every solenoidal field – be written as the vortex of a vector field.
\[
\vec{J}_T(\vec{r}, t) = \frac{\partial}{\partial t} \vec{D}_T(\vec{r}, t) = \nabla \times \vec{H}_T(\vec{r}, t)
\]
(52)

According to Maxwell, \( \vec{H}_T \) is the transverse component of the magnetic field strength.

In other words, the longitudinal part (51) of Ampère-Maxwell’s flux law (48) merely duplicates the conservation of charge, whence its transverse part (52) becomes the effective flux law.

The two homogeneous rationalized Maxwell equations emerge from his 1864 set through, (i), setting \( \mu \vec{H} = \vec{B} \), the magnetic flux density (induction; Maxwell 1873, §604), and, (ii), eliminating the potentials.
\[
\nabla \vec{B}(\vec{r}, t) = 0
\]
(53)
\[
\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t)
\]
(54)
Here, the latter equation, nowadays called Faraday’s induction law, assumes a primary axiomatic position, while it was a secondary, to be derived feature in the original 1864 set of equations.

Actually, by virtue of \( \nabla \times \vec{E} = \nabla \times \vec{E}_T \), it contains solely transverse field components.

\[
\vec{B}_L(\vec{r}, t) \equiv \vec{0}
\]  

\[
\nabla \times \vec{E}_T(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}_T(\vec{r}, t)
\]  

Thus, the Helmholtz-decomposed ”rationalized” Maxwell equations represent a set of 6 equations for the 10 independent components of \( \vec{D}_L, \vec{D}_T, \vec{B}_L, \vec{B}_T, \vec{E}_T \) and \( \vec{H}_T \), where \( \rho \) and \( \vec{j}_T \) are considered to be external sources, independent variables. As for the full set, it can be closed through material equations; here, \( \vec{D}_T = \varepsilon \vec{E}_T \) and \( \vec{B}_T = \mu \vec{H}_T \).

\( \vec{E}_L \) and \( \vec{H}_L \) are needed to write the ”rationalized” Maxwell equations in a manifest Lorentz invariant manner; \( \vec{E}_L \) is also needed in the Maxwell-Lorentz force.

6. Transverse and Longitudinal Poynting Theorems

In the common derivations of Poynting’s (1884) theorem, it is discarded that both the l.h.s. of the flux law (48) and Faraday’s induction law (54) contain solely transverse field components, see eqs. (56) and (56), respectively. This fact is accounted for in the Transverse Poynting theorem:

\[
\iiint_V \vec{E}_T \cdot \vec{j}_T \, dV = -\iiint_V \left( \vec{E}_T \cdot \frac{\partial \vec{B}_T}{\partial t} + \vec{H}_T \cdot \frac{\partial \vec{B}_T}{\partial t} \right) \, dV - \oint_{\partial V} \left( \vec{E}_T \times \vec{H}_T \right) \cdot d\vec{\sigma} \tag{57}
\]

Indeed, (i), by virtue of the orthogonality theorem (17) and \( \vec{B} \equiv \vec{B}_T \),

\[
\iiint \vec{H} \cdot \left( \nabla \times \vec{E} \right) \, dV = \iiint \vec{H}_T \cdot \left( \nabla \times \vec{E}_T \right) \, dV 
= -\iiint \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \, dV = -\iiint \vec{H}_T \frac{\partial \vec{B}_T}{\partial t} \, dV \tag{58}
\]

and, (ii), due to \( \vec{j}_L + \frac{\partial}{\partial t} \vec{D}_L = \vec{0} \),

\[
\iiint \vec{E} \cdot \left( \nabla \times \vec{H} \right) \, dV = \iiint \vec{E}_T \cdot \left( \nabla \times \vec{H}_T \right) \, dV 
= \iiint \vec{E} \cdot \vec{j} \, dV + \iiint \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \, dV = \iiint \vec{E}_T \cdot \vec{j}_T \, dV + \iiint \vec{E}_T \cdot \frac{\partial \vec{D}_T}{\partial t} \, dV \tag{60}
\]

Its interpretation is quite analogous to the standard theorem.

• \( \iiint \vec{E}_T \cdot \vec{j}_T \, dV = \iiint \vec{E}_T \cdot \vec{j} \, dV \) is the Joule power of \( \vec{E}_T \) transfered from the field to the charged bodies.
\[ \int \int \int (\vec{E}_T \cdot \frac{\partial}{\partial t} \vec{D}_T + \vec{H}_T \cdot \frac{\partial}{\partial t} \vec{B}) \, dV \]

is the power of the transverse fields; for \( \int \vec{E}_T \cdot d\vec{D} \) is the work density done by the transverse electric field to produce the transverse displacement (cf. the arguing for \( \int \vec{E} \cdot d\vec{D} \) in Maxwell 1864, §72); analogously, \( \int \vec{H} \cdot d\vec{B} \) is the work density done by the magnetic field to produce the (always transverse) magnetic induction (flux density); note, that, \( \frac{1}{2} \int \int \int \vec{H} \cdot \vec{B} \, dV = \frac{1}{2} \int \vec{J} \cdot \vec{A} \, dV \), the work done by the (always transverse) total current density to produce the vector potential (cf. Maxwell 1864, §§33, 71).

\[ \oint_{\partial V} (\vec{E}_T \times \vec{H}_T) \cdot d\vec{\sigma} \]

is the power propagating through the surface \( \partial V \) into the exterior of the volume, \( V \), under consideration, where \( \vec{E}_T \times \vec{H}_T = \vec{S}^{(T)} \) is the ‘propagating part’ (not the transverse component!) of the Poynting vector, \( \vec{S} = \vec{E} \times \vec{H} \).

The energy balance for the longitudinal components is separately guaranteed through the **Longitudinal Poynting theorem**:

\[ \int \int \int_{V} \vec{E}_L \cdot \vec{j}_L \, dV = - \int \int \int_{V} \vec{E}_L \cdot \frac{\partial}{\partial t} \vec{D}_L \, dV \quad (62) \]

The Joule power of \( \vec{E}_L \) is balanced by the field power of the longitudinal (di)electric field vectors.

The ‘non-propagating part’ of the Poynting vector: \( \vec{S}^{(L)} = \vec{E}_L \times \vec{H}_T \) (this is not its longitudinal component!), does not contribute to the power/energy balance, since

\[ \oint_{\partial V} \vec{S}^{(L)} \cdot d\vec{\sigma} = 0. \]

This splitting of Poynting’s theorem into its longitudinal and transverse parts supports the view that the Helmholtz decomposition helps to discover the physical content of Maxwell’s theory.

It’s noteworthy that, energetically, \( \vec{E} \) and \( \vec{H} \) are a pair in both being intensive, driving quantities, while \( \vec{D} \) and \( \vec{B} \) are a pair in both being extensive and driven quantities.

### 7. Standard Canonical Classical Electromagnetism

#### 7.1 Standard Lagrangian

Milton & Schwinger (2006, 1.2) have formulated the microscopic theory in a most elegant manner through accounting explicitly for the fact, that the free-field Lagrangian is expressed much more concisely in terms of the fields \( \vec{E} \) and \( \vec{B} \) than in terms of the potentials \( \Phi \) and \( \vec{A} \). In SI units and for one body of mass \( m \) and charge \( q \) moving with velocity \( v < \ll c \) along the trajectory \( \vec{r}_m(t) \), the total Lagrangian reads

\[ L(t) = \int \int \int \mathcal{L} (\vec{r}, t) \, dV \]

\[ = \int \int \int \mathcal{L}_{\text{field}} (\vec{r}, t) \, dV + q \vec{v} \cdot \vec{A}(\vec{r}_m(t), t) - q \Phi(\vec{r}_m(t), t) + \frac{m}{2} \vec{v} \cdot \vec{v} \quad (64) \]

with the field’s Lagrange density

\[ \mathcal{L}_{\text{field}} (\vec{r}, t) = \frac{\varepsilon_0}{2} \vec{E}(\vec{r}, t)^2 - \frac{1}{2\mu_0} \vec{B}(\vec{r}, t)^2 \quad (65) \]
Throughout this paper it will be assumed that, for a point-like body, the discrete and
the continuum representations can be freely interchanged, where
\[ \rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_m(t)); \quad \vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_m(t)); \]
\[ \rho_m(\vec{r}, t) = m \delta(\vec{r} - \vec{r}_m(t)) \] (66)

Thus, the variational derivatives are, (i),
\[ \frac{\delta L}{\delta (\partial \vec{A}/\partial t)} = \frac{\partial \mathcal{L}}{\partial (\partial \vec{A}/\partial t)} - \frac{\partial \mathcal{L}}{\partial \vec{E}} = -\varepsilon_0 \vec{E} = \vec{\Pi} \vec{A} \] (67)
\[ \text{ie, the canonical momentum density of the vector potential;} \]
(ii),
\[ \frac{\delta L}{\delta \vec{A}} = \frac{\partial \mathcal{L}}{\partial \vec{A}} + \nabla \times \frac{\partial \mathcal{L}}{\partial \vec{B}} = \vec{j} - \nabla \times \frac{1}{\mu_0} \vec{B} = -\varepsilon_0 \frac{\partial}{\partial t} \vec{E} \] (68)
by virtue of the flux law (being the Lagrangian equation of motion for \( \vec{A} \)); (iii),
\[ \frac{\delta L}{\delta \Phi} = \frac{\partial \mathcal{L}}{\partial \Phi} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial \vec{E}} = -\rho + \nabla \varepsilon_0 \vec{E} \equiv 0 \] (69)
by virtue of Gauss’ law. Together with \( (\dot{\Phi} \equiv \partial \Phi/\partial t) \)
\[ \frac{\delta L}{\delta \Phi} = \frac{\partial \mathcal{L}}{\partial \Phi} \equiv 0, \] (70)
this means, that the scalar potential does not exhibit an own dynamics.

In order to obtain a manifest Lorentz covariant formulation of the theory, such a
dynamics is usually created by means of the Lorenz (1867) gauge
\[ \nabla \vec{A} + \dot{\Phi} = 0, \] (71)
providing an equation for \( \dot{\Phi} \) and adding the ‘nutritious zero’ \((-1/2\mu_0)(\nabla \vec{A} + \dot{\Phi})^2\) to the
Lagrange density (Heisenberg & Pauli 1929; see also Fock & Podolsky 1932, Dirac, Fock
& Podolsky 1932). Within quantum electrodynamics, this leads to the not observable
longitudinal and scalar (time-like) photons which are lateron projected out, however.

In contrast, within the gaugefree formulation, the scalar potential, \( \phi_{\vec{A}} \), of the vector
potential is combined with the common scalar potential, \( \Phi \), to the scalar potential of the
electric field strength, \( \phi_{\vec{E}} = \Phi - \partial \phi_{\vec{A}}/\partial t \). Via the Poisson equation,
\[ \Delta \phi_{\vec{E}} = -\frac{\rho}{\varepsilon_0}, \] (72)
in the dynamics of \( \phi_{\vec{E}} \) is tied to that of the charge density. A self-standing dynamics is
exhibited by the propagating transverse field components only. Its canonical theory will
be formulated in the next section.

The body’s dynamics is dealt with conventionally. (iv),
\[ \frac{\partial L}{\partial \vec{v}} = \frac{\partial}{\partial \vec{v}} \left( q \vec{v} \cdot \vec{A} + \frac{m}{2} v^2 \right) = m \vec{v}(t) + q \vec{A}(\vec{r}_m, t) = p(\vec{r}_m, t), \] (73)
ie, the canonical momentum of the body; (v),

\[
\frac{\partial L}{\partial \dot{r}_m} = \frac{\partial}{\partial \dot{r}_m} \left( q \mathbf{v} \cdot \mathbf{A}(r_m, t) - q\Phi(r_m, t) \right)
= q (\mathbf{v} \cdot \nabla_m) \mathbf{A} + q \mathbf{v} \times \nabla_m \times \mathbf{A} - q \nabla_m \Phi
\]

(74)

\[
= \frac{d}{dt} \left( mv(t) + q \mathbf{A}(r_m(t), t) \right)
\]

(75)

\[
= m \frac{d\mathbf{v}}{dt} + q (\mathbf{v} \cdot \nabla_m) \mathbf{A} + q \frac{\partial \mathbf{A}}{\partial t}
\]

(76)

This is the Newtonian equation of motion with the Maxwell-Lorentz force (42).

\[
m \frac{d\mathbf{v}}{dt} = -q \frac{\partial \mathbf{A}}{\partial t} - q \nabla_m \Phi + q \mathbf{v} \times \nabla_m \times \mathbf{A}
\]

(77)

\[
= q \mathbf{E} + q \mathbf{v} \times \mathbf{B}
\]

(78)

7.2 Hamiltonian and Total Energy

By definition, 'external influences' act upon a system without relevant back-reaction. The best known example is, perhaps, forced oscillations. In contrast, without external influences, the change rates are determined solely by the system’s own properties, while the origin of the system’s time scale plays no role. This implies, that the Lagrangian does not explicitly depend on time. Its implicit time-dependence is visible from

\[
\frac{dL}{dt} = \int \int \int \left( \frac{\delta L}{\delta \mathbf{A}} \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{\delta L}{\delta (\partial \mathbf{A}/\partial t)} \cdot \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) dV
+ \frac{\partial L}{\partial \dot{r}_m} \cdot \frac{d\dot{r}_m}{dt} + \frac{\partial L}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt}
= \frac{d}{dt} \int \int \int \left( \frac{\delta L}{\delta (\partial \mathbf{A}/\partial t)} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) dV + \frac{d}{dt} \left( \frac{\partial L}{\partial \mathbf{v}} \cdot \mathbf{v} \right)
\]

(81)

(cf Milton & Schwinger 2006, 1.3). Hence, the energy function,

\[
h = \int \int \int \left( \frac{\delta L}{\delta (\partial \mathbf{A}/\partial t)} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) dV + \frac{\partial L}{\partial \mathbf{v}} \cdot \mathbf{v} - L
\]

(82)

\[
= \int \int \int \left[ \frac{1}{2} E^2 + \varepsilon_0 \mathbf{E} \cdot \nabla \Phi + \frac{1}{2\mu_0} \mathbf{B}^2 \right] dV + q\Phi + \frac{m}{2} \mathbf{v}^2
\]

(83)

of closed systems is time-independent. Actually, if surface terms make no contribution (as assumed throughout this paper, except for the Poynting vector below), it equals the total energy,

\[
E = \int \int \int \left( \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right) dV + \frac{m}{2} \mathbf{v}^2 = \text{const}
\]

(84)
The total energy is conserved, if the external spacetime is homogeneous in time. And, as it should be, it is gauge invariant.

Replacing in the energy function (82) \( \vec{E} \) with \( -\vec{\Pi}_A / \varepsilon_0 \) and \( \vec{v} \) with \( (\vec{p} - q\vec{A}) / m \), one obtains the Hamiltonian,

\[
H = \iint \left[ \frac{1}{2 \varepsilon_0} \vec{\Pi}_A^2 - \vec{\Pi}_A \cdot \nabla \Phi + \frac{1}{2 \mu_0} \vec{B}^2 \right] dV + q\Phi + \frac{1}{2m} (\vec{p} - q\vec{A})^2 \tag{85}
\]

with the Hamilton density

\[
\mathcal{H} = \frac{1}{2\varepsilon_0} \vec{\Pi}_A^2 + \frac{1}{2\mu_0} \vec{B}^2 - \vec{\Pi}_A \cdot \nabla \Phi + \rho \Phi + \frac{1}{2\rho_m} \left( \vec{\pi} - \rho \vec{A} \right)^2 ;
\]

\[
\vec{\pi} = \rho_m \vec{v} + \rho \vec{A} \tag{87}
\]

This Hamiltonian is numerically gauge invariant, because the two \( \Phi \)-dependent terms cancel each another (see the total energy, \( E \), above), and \( \vec{p} - q\vec{A} = \vec{v} \) is gauge-independent, too. Notice, that it is necessary to keep explicitly those two terms, \( q\Phi \) and \( -\iint \vec{\Pi}_A \cdot \nabla \Phi dV \), in order to obtain the correct Hamiltonian equations of motion.

For the fields, these are, (i),

\[
\frac{\partial \Phi}{\partial t} = \frac{\delta H}{\delta \Pi_\Phi} \equiv 0 \tag{88}
\]

This is a formal equation (as \( \Pi_\Phi \equiv 0 \)); it indicates, again, that \( \Phi \) does not exhibit a dynamics on its own.

(ii),

\[
\frac{\partial \vec{A}}{\partial t} = \frac{\delta H}{\delta \Pi_\vec{A}} = \frac{\partial \mathcal{H}}{\partial \Pi_\vec{A}} = \frac{1}{\varepsilon_0} \vec{\Pi}_A - \nabla \Phi \tag{89}
\]

As usual (Goldstein 1950), the equations for the potentials merely reproduce the definitions of the canonical momenta.

(iii),

\[
\frac{\partial \Pi_\Phi}{\partial t} = -\frac{\delta H}{\delta \Phi} = -\frac{\partial \mathcal{H}}{\partial \Phi} + \nabla \cdot \frac{\partial \mathcal{H}}{\partial (\nabla \Phi)} = -\nabla \Pi_\Phi - \rho \equiv 0 \tag{90}
\]

by virtue of Gauss’ law. This, again, is not a dynamical equation, but a Bergmannian ”primary constraint” (Dirac 2001, p.8). In order to avoid this constraint, Goenner (2004, 5.2.6) has proposed to restrict \( \Pi_\Phi \) to this class of values in a rather formal manner. Below, I will show, that such restrictions are not necessary, when working with the Helmholtz-decomposed vector fields.

(iv),

\[
\frac{\partial \Pi_\vec{A}}{\partial t} = -\frac{\delta H}{\delta \vec{A}} = -\frac{\partial \mathcal{H}}{\partial \vec{A}} - \nabla \times \frac{\partial \mathcal{H}}{\partial \vec{B}} = -\frac{1}{\mu_0} \nabla \times \vec{B} + \rho \vec{v} \tag{91}
\]

This is the microscopic flux law.
8. Gaugefree Canonical Theory

Recall that the Hamiltonian in eq.(41) is *manifest gauge invariant*. This suggests, that manifest gauge invariant entities are obtained from their standard expressions through replacing \vec{A} with \vec{A}_T and \Phi with \phi_E. We will see, however, that it is not that simple.

8.1 The Helmholtz Decomposed Microscopic Maxwell Equations

The Helmholtz decomposed microscopic Maxwell equations follow from the Helmholtz decomposed macroscopic Maxwell equations (see above) in the same manner as the not decomposed ones do.

**Helmholtz decomposed Gauss’ law for the electrical field:** Gauss’ law for the electric field reduces to a Poisson equation for the scalar potential, \phi_E, of the electric field.

\[
\nabla \vec{E} = \nabla \vec{E}_L = -\Delta \phi_E = \frac{1}{\varepsilon_0} \rho
\]

**Helmholtz decomposed Gauss’ law for the magnetic field:** Gauss’ law for the magnetic field states that the latter is purely transverse.

\[
\nabla \vec{B} = \nabla \vec{B}_L = 0
\]

\[
\vec{B}_L \equiv \vec{0}; \quad \vec{B} = \vec{B}_T
\]

**Helmholtz decomposed Faraday’s induction law:** The induction law effectively connects solely transverse field components.

\[
\nabla \times \vec{E} = \nabla \times \vec{E}_T = -\frac{\partial}{\partial t} \vec{B}_{(T)}
\]

**Helmholtz decomposed Ampère-Maxwell’s flux law:** The flux law separates into a transverse and a longitudinal parts.

\[
\frac{1}{\mu_0} \nabla \times \vec{B}_{(T)} = \left( \vec{j}_L + \vec{j}_T \right) + \varepsilon_0 \frac{\partial}{\partial t} \left( \vec{E}_L + \vec{E}_T \right)
\]

Together with Gauss’ law (92), the longitudinal part,

\[
\vec{0} = \vec{j}_L + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}_L
\]

is equivalent to the continuity equation and, thus, can be dispensed in favour of that (see above). Consequently, the transverse one,

\[
\frac{1}{\mu_0} \nabla \times \vec{B}_{(T)} = \vec{j}_T + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}_T
\]

represents the *effective* flux law.

In contrast to the scalar potential, \(\Phi(\vec{r}, t)\), and to the vector potential, \(\vec{A}(\vec{r}, t)\), the (total) scalar potential of the electric field strength,

\[
\phi_E(\vec{r}, t) = \Phi(\vec{r}, t) - \frac{\partial}{\partial t} \phi_{\vec{A}}
\]
and the transverse component, $\vec{A}_T(\vec{r}, t)$, of the vector potential are gauge invariant, if not to say gaugefree. By virtue of the effective flux law (98), the latter one obeys the wave equation

$$ -\frac{1}{\mu_0} \Delta \vec{A}_T = \vec{j}_T - \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}_T $$

(100)

Manifest gauge invariant Lagrangian and Hamiltonian equations of motion have to reproduce this wave equation and $\Delta \phi_E = -\rho/\varepsilon_0$.

### 8.2 Gaugefree Canonical Particle Momentum

Obviously,

$$ \vec{p}^{(L)}(t) = m\vec{v}(t) + q\vec{A}_T(\vec{r}_m(t), t) $$

(101)

is a gauge invariant canonical momentum (cf Messiah 1999, XXI.23, p.1025, fn.1). More accurately, it is gaugefree, since $\vec{A}_T$ is not affected by gauge. The body’s gaugefree canonical momentum density thus equals

$$ \vec{\pi}^{(L)} = \rho_m \vec{v} + \rho \vec{A}_T $$

(102)

At first glance, it seems that the field term, $q\vec{A}_T$, represents the influence of the transverse field. However, via partial integration and vanishing of all fields outside the volume of integration, one obtains (reversing the calculations in Messiah 1999, XXI.23)

$$ \int \int \int \rho \vec{A}_T dV = -\varepsilon_0 \int \int \int (\Delta \phi_E) \vec{A}_T dV = -\varepsilon_0 \int \int \int \phi_E \Delta \vec{A}_T dV $$

(103)

$$ \varepsilon_0 \int \int \int \vec{E}_L \times \vec{B} dV = \frac{1}{c^2} \int \int \int \vec{S}^{(L)} dV = \vec{p}^{(L)}_{\text{field}}(t) $$

(105)

Hence, $q\vec{A}_T$ actually contains the contribution of the longitudinal electric field component to the field momentum (cf above). This is another backing for the view, that, cum grano salis, the motion of the longitudinal field, $\vec{E}_L$, is tied to the motion of the charged bodies, while the self-standing motion of the field is realized by solely the transverse fields, $\vec{E}_T$ and $\vec{B}$.

This way, the total momentum of the system charge & fields gains the intuitive expression

$$ \vec{p}_{\text{tot}} = \vec{p}^{(L)} + \vec{p}_{\text{prop}} $$

(106)

where

$$ \vec{p}_{\text{prop}} = \varepsilon_0 \int \int \int \vec{E}_T \times \vec{B} dV $$

(107)

is the momentum of the propagating part of the field (see above). Although it is mathematically equivalent to the standard expression,

$$ \vec{p}_{\text{tot}} = \vec{p}_{\text{kin}} + \vec{p}_{\text{field}} = m\vec{v} + \varepsilon_0 \int \int \int \vec{E} \times \vec{B} dV $$

(108)
it is physically preferable, because, in the latter one, neither the standard canonical momentum, \( \vec{p} = m \vec{v} + q \vec{A} \), nor the gauge invariant canonical momentum, \( \vec{p}^{(L)} \), have got a self-standing place. The difference
\[
\vec{p}_{\text{tot}} - \vec{p}_{\text{can}} = \vec{p}_{\text{field}} - q \vec{A} = \varepsilon_0 \iiint \vec{E}_T \times \vec{B} dV - q \vec{A}_L
\] (109)
has not got an own physical meaning.

8.3 Gaugefree Lagrangian

In terms of the Helmholtz decomposed and gaugefree field variables, the Lagrangian reads
\[
L^{gf}(t) = \iiint \mathcal{L}^{gf}(\vec{E}_T, \vec{E}_L, \vec{B}, \phi_\vec{E}, \vec{A}_T, \vec{v}) dV
\] (110)
\[
\mathcal{L}^{gf} = \frac{\varepsilon_0}{2} \vec{E}_T^2 + \frac{\varepsilon_0}{2} \vec{E}_L^2 - \frac{1}{2 \mu_0} \vec{B}^2 + \vec{j} \cdot \vec{A}_T - \rho \phi_\vec{E} + \frac{1}{2} \rho \vec{v}^2
\] (111)

Because there is no \( \vec{v}_T \), it is necessary to keep \( \vec{j} \cdot \vec{A}_T \), although \( \iiint \vec{j} \cdot \vec{A}_T dV = \iiint \vec{j}_T \cdot \vec{A}_T dV \) (see above). The difference to the standard Lagrangian is a total time-derivative.
\[
L^{gf}(t) - L(t) = \frac{d}{dt} q \phi_\vec{A}(\vec{r}(t), t)
\] (112)

For the field, there is a canonical momentum density only for the transverse component of the vector potential, \( \text{viz} \),
\[
\vec{\Pi}_{\vec{A}_T} = \frac{\delta L^{gf}}{\delta (\partial \vec{A}_T / \partial t)} = \frac{\partial \mathcal{L}^{gf}}{\partial \vec{A}_T} - \frac{\partial \mathcal{L}^{gf}}{\partial \vec{E}_T} = -\varepsilon_0 \vec{E}_T
\] (113)

The corresponding Lagrangian equation of motion is the transverse, effective part of the flux law.
\[
\frac{\delta L^{gf}}{\delta \vec{A}_T} = \frac{\partial \mathcal{L}^{gf}}{\partial \vec{A}_T} + \nabla \times \frac{\partial \mathcal{L}^{gf}}{\partial \vec{B}} = \vec{j}_T - \nabla \times \frac{1}{\mu_0} \vec{B} = -\varepsilon_0 \frac{\partial}{\partial t} \vec{E}_T
\] (114)

\( \phi_\vec{E} \) does not exhibit an own dynamics, because
\[
\frac{\delta L^{gf}}{\delta \phi_\vec{E}} = \frac{\partial \mathcal{L}^{gf}}{\partial \phi_\vec{E}} + \nabla \cdot \frac{\partial \mathcal{L}^{gf}}{\partial \vec{E}_L} = -\rho + \nabla \varepsilon_0 \vec{E}_L \equiv 0
\] (115)

by virtue of Gauss’ law (\( \nabla \vec{E} = \nabla \vec{E}_L \)) and
\[
\frac{\delta L^{gf}}{\delta \phi_\vec{E}} = \frac{\partial \mathcal{L}^{gf}}{\partial \phi_\vec{E}} \equiv 0,
\] (116)
(cf Messiah 1999, XXI.22). Since \( \phi_\vec{E} \) is gauge invariant, this cannot be changed (in contrast to \( \Phi \)).
8.4 Manifest Gauge Invariant Hamiltonian

Accordingly, the manifest gauge invariant, or gauge free Hamiltonian becomes

\[
H_{gf} = \vec{p}(L) \cdot \vec{v} + \iint \int \vec{\Pi} \cdot \frac{\partial \vec{A}_T}{\partial t} dV - L_{gf} = \iint \int S_{gf} dV
\]  

(117)

\[
S_{gf} = \frac{1}{2 \rho_m} \left( \vec{\pi}(L) - \rho \vec{A}_T \right)^2 + \rho \phi_{\vec{E}} + \frac{1}{2 \varepsilon_0} \vec{\Pi}^2 - \frac{\varepsilon_0}{2} \vec{E}_L^2 + \frac{1}{2 \mu_0} \vec{B}^2
\]  

(118)

\(H_{gf}\) equals numerically the total energy (84).

\[
H_{gf} - E = \iint \int \left( \rho \phi_{\vec{E}} - \varepsilon_0 \vec{E}_L^2 \right) dV = \iint \int \phi_{\vec{E}} \left( \rho - \varepsilon_0 \nabla \vec{E}_L \right) dV = 0
\]  

(119)

Thus, on comes from the total energy to the Hamiltonian also through accounting for Gauss’ law as a constraint.

\[
S_{gf} = \mathcal{E}_{gf} + \Lambda \left( \nabla \vec{E}_L - \frac{\rho}{\varepsilon_0} \right)
\]  

(120)

where

\[
\mathcal{E}_{gf} = \frac{\varepsilon_0}{2} \vec{E}_T^2 + \frac{\varepsilon_0}{2} \vec{E}_L^2 + \frac{1}{2 \mu_0} \vec{B}^2 + \frac{1}{2 \rho_m} \vec{v}^2
\]  

(121)

is the Helmholtz-decomposed total energy density. The Lagrangian multiplier, \(\Lambda\), follows from the Hamiltonian equations of motion. The advantages of this approach consists in that one needs not (more or less) guess the (not unique) Lagrangian, but, in this case, one can even derive one (together with \(\Lambda\)), for \(\mathcal{L}_{gf}\) and \(S_{gf}\) are bilinear in the dynamical variables.

The Hamiltonian equations of motion for the field variables (cf Heisenberg & Pauli 1929, Fock & Podolski 1932, Dirac, Fock & Podolsky 1932, Goldstein, eq.(11-56)) reproduce the Helmholtz decomposed microscopic Maxwell equations.

(1)

\[
\frac{\partial}{\partial t} \phi_{\vec{E}} = \frac{\partial S_{gf}}{\partial \phi_{\vec{E}}} \equiv 0
\]  

(122)

The absence of \(\Pi_{\phi_{\vec{E}}}\) means, firstly, that the longitudinal electric field component, \(\vec{E}_L\), does not have got a dynamics on its own, so that it depends on time only via the positions of the charges creating it. Consequently, not only \(\Phi\), but also \(\vec{A}_L\) does not represent an independent dynamical variable.

(2)

\[
\frac{\partial}{\partial t} \Pi_{\phi_{\vec{E}}} = - \frac{\partial S_{gf}}{\partial \phi_{\vec{E}}} + \nabla \cdot \frac{\partial S_{gf}}{\partial \vec{E}_L} = - \rho - \Delta \phi_{\vec{E}} \equiv 0
\]  

(123)

(124)

The absence of \(\Pi_{\phi_{\vec{E}}}\) means, secondly and again, that Gauss’ law for the electric field (here in Poisson’s form) is a constraint rather than an equation of motion.
\[ \frac{\partial}{\partial t} \vec{A}_T = \frac{\partial S^{gf}}{\partial \Pi_T} = \frac{1}{\varepsilon_0} \Pi_T = -\vec{E}_T \quad (125) \]

As above, this equation merely reproduces the definition of the canonical momentum density (and thus may be considered to be an identity rather than an equation of motion).

\[ \frac{\partial}{\partial t} \Pi_j^T = -\frac{\partial H_{gf}}{\partial A_j^T} + \sum_{k=1}^{3} \frac{\partial}{\partial r_k} \frac{\partial S^{gf}}{\partial (A_j^T/\partial r_k)}; \quad j = x, y, z \quad (126) \]

\[ = -\frac{\partial S^{gf}}{\partial A_j^T} - \nabla \times \frac{\partial S^{gf}}{\partial \vec{B}} = \vec{j}_T - \frac{1}{\mu_0} \nabla \times \vec{B} \quad (127) \]

By virtue of \( \Pi_T = \varepsilon_0 \partial \vec{A}_T / \partial t \), this is equivalent to the wave equation for \( \vec{A}_T \) (since \( \nabla \vec{A}_T = 0 \), we have \( \nabla \times \nabla \times \vec{A}_T = -\Delta \vec{A}_T \)).

\[ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}_T = \vec{j}_T + \frac{1}{\mu_0} \Delta \vec{A}_T \quad (128) \]

And this is equivalent to the transverse, effective flux law.

### 8.5 Time-dependence of the Gaugefree Hamiltonian, \( H^{gf} \): Conservation of Total Energy

The conservation law for the total energy has been shown above to separate into one for the transverse and one for the longitudinal field components. For this, it is sufficient here to consider the time-dependence of the gaugefree Hamiltonian, \( H^{gf} \). Let me rewrite \( H^{gf} \) as

\[ H^{gf}(t) = H_{body}(t) + H_{non-prop}(t) + H_{prop}(t) \]

Because a magnetic field does not change the kinetic energy of a charged body, the field-independent part, \( H_{body}(t) \), is – as in \( H(t) \) – effectively built by the (kinetic) energy of the free body.

\[ H_{body}(t) = \frac{1}{2m} \left( \vec{v}_T(t) - q \vec{A}_T(\vec{r}(t), t) \right)^2 = \frac{m}{2} v^2(t) \quad (129) \]

By virtue of the Maxwell-Lorentz force, the rate of its change equals the Joule power, \( P_{Joule} \).

\[ \frac{d}{dt} H_{body} = m \vec{v} \cdot \frac{d\vec{v}}{dt} = q \vec{v} \cdot \vec{E} \equiv P_{Joule} \quad (130) \]

\( H_{prop} \) contains the propagating field energy, \( ie \), that of the transverse field components.

\[ H_{prop}(t) = \iiint \left( \frac{\varepsilon_0}{2} \vec{E}_T(\vec{r}, t)^2 + \frac{1}{2\mu_0} \vec{B}(\vec{r}, t)^2 \right) dV \quad (131) \]
By virtue of the effective flux law (98) and the effective induction law (95), it is diminished by the transverse part of the Joule power (130) and by radiation out of the volume under consideration.

\[
\begin{align*}
\frac{dH_{\text{prop}}}{dt} &= \iiint \left( \varepsilon_0 \vec{E}_T \cdot \frac{\partial}{\partial t} \vec{E}_T + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial}{\partial t} \vec{B} \right) dV \\
&= \iiint \left( \vec{E}_T \cdot \frac{1}{\mu_0} \nabla \times \vec{B} - \vec{E}_T \cdot \vec{j}_T - \frac{1}{\mu_0} \vec{B} \cdot \nabla \times \vec{E}_T \right) dV \\
&= -q\vec{v} \cdot \vec{E}_T - \frac{1}{\mu_0} \oint \vec{E}_T \times \vec{B} \cdot d\vec{S}
\end{align*}
\]

As explained above, the radiation term contains \(\vec{E}_T\) rather than \(\vec{E}\).

\(H_{\text{non-\text{prop}}} \) contains the field energy of the longitudinal electric field.

\[
H_{\text{non-\text{prop}}} (t) = q\phi_E (\vec{r}(t)) - \frac{\varepsilon_0}{2} \iiint [\nabla \phi_E]^2 dV = \frac{\varepsilon_0}{2} \iiint \vec{E}_L^2 dV
\]

By virtue of the longitudinal part of Ampère-Maxwell’s flux law: \(\vec{0} = \vec{j}_L + \varepsilon_0 \frac{\partial}{\partial t} \vec{E}_L\), it is diminished by the longitudinal part of the Joule power (130).

\[
\frac{dH_{\text{non-\text{prop}}}}{dt} = \varepsilon_0 \iiint \vec{E}_L \cdot \frac{\partial \vec{E}_L}{\partial t} dV = -q\vec{v} \cdot \vec{E}_L
\]

Altogether,

\[
\frac{dH_{\text{gf}}}{dt} = -\frac{1}{\mu_0} \oint \vec{E}_T \times \vec{B} \cdot d\vec{S}
\]

In closed systems, \(H = E = \text{const}\), the total energy (84) of the system body & field.

External fields acting on the charged body, are to be added to \(\vec{A}_T\) in \(H_{\text{body}}\) and to \(q\phi_E\) in \(H_{\text{non-\text{prop}}}\) in the usual manner.

### Summary and Discussion

The theoretical classical electromagnetism rests essentially on the so-called ‘rational(ized)’ (1) and microscopic Maxwell equations (2), respectively. Both sets, however, hide the origins of gauge invariance and transversality of free electromagnetic waves. For this I returned to Maxwell’s (1864) original set of equations (A)...(H). According to Boltzmann (2001), they express the essence of Maxwell’s theory, while those subsequent modifications have led to rather misunderstand it.

As a matter of fact, when deriving the microscopic Maxwell equations (2) from a Helmholtzian analysis of the relationships between forces and energies, a factorization of the forces into geometrical and body-dependent quantities à la Newton and Hertz’s
interaction principle (Enders 2004, 2006, 2008, 2009), one arrives at the set
\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi \quad (138)
\]
\[
\vec{B} = \nabla \times \vec{A} \quad (139)
\]
\[
\nabla \vec{E} = \frac{\rho}{\varepsilon_0} \quad (140)
\]
\[
\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (141)
\]
As it contains the potentials explicitly, it is close to Maxwell’s (1864) original set (A)...(H).

In view of the Helmholtz (1858) decomposition of 3D vector fields, this set seems to
be not well-defined. The representation (M-35) of the electric field strength in terms of
the potentials contains two contributions to its longitudinal component, one from the
vector potential, \(-\partial \vec{A}_L/\partial t\), and one from the scalar potential, \(-\nabla \Phi\). For these two,
no other equation is established. Consequently, this set is actually not ”20 equations
for 20 variables” (Maxwell 1864, §70), but only 19 equations for 20 variables. It is not
inconsistent, however, because the equations being related to charge conservation are
redundant.

Both deficiencies can be removed in two different ways. The standard way results in
the ’rational(ized)’ set (1), where the potentials and the continuity equation have been
eliminated. Historically, this has led even to a principal underestimation of the potentials
(see, for instance, Drude 1906 referring to Heaviside, Hertz and Cohn).

An advantage of this concentration on the field equations is their Lorentz covariance
(after re-introducing the potentials). A disadvantage consists in that the experimentally
observed transversality of free electromagnetic waves does not naturally emerges out of
the theory.

The alternative way proposed here considers the conservation of charge to be a fund-
damental property of given bodies and, consequently, primary w.r.t. the fields created by
such bodies. In other words, the continuity equation represents a primary, independent
equation against the field equations. At once, the transversality of freely propagating
electromagnetic waves is natural consequence of this approach.

Accordingly, both the original and the rationalized sets of Maxwell equations effec-
tively contain
- the fact that a charge density creates a longitudinal (di)electric field (Gauss’ law);
- the continuity equation expressing the conservation of charge (Gauss’ law together
with the longitudinal part of Ampère-Maxwell’s flux law);
- the propagation of transverse electromagnetic waves (Faraday’s induction law to-
gether with the transverse part of Ampère-Maxwell’s flux law).

Consequently, a complete set of independent dynamical field variables contains 4
field components. For instance, 2 dynamically independent components of \(\vec{B}\) and of \(\vec{E}_T\)
each represent such a set; another example is given through 2 dynamically independent
components of \(\vec{A}_T\) and the corresponding 2 ones of \(\vec{\Pi}_T\).

It is perhaps no accident that the history of the electromagnetic potentials is even
more curvilinear than that of the field strengths. Maxwell (1861, 1862, 1864) saw the vector potential to represent Faraday’s ”electrotonic state” and the electromagnetic field momentum, respectively. Later, the potentials were considered to be superfluous or merely mathematical tools for solving the rationalized Maxwell’s equations. This mistake lived for a surprisingly long time, in spite of their appearance in the principle of least action (Schwartzschild 1903), in the Hamiltonian (Pauli 1926, Fock 1929) and, last but not least, in the Aharonov-Bohm (1959) effect. The double role of the vector potential, $\vec{A}$, in the electric field strength, $\vec{E}$, where $\partial \vec{A}/\partial t$ contributes to both the transverse and the longitudinal components, has surely hindered the clarification.

The Helmholtz decomposition of the ‘rationalized’ Maxwell equations also facilitates to understand Poynting’s (1884) theorem and the transversality of freely propagating electromagnetic waves. In the common treatments, the propagating field is connected with the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, which, however, contains both, transverse and longitudinal field components.

As a matter of fact, Poynting’s (1884) theorem rests on Faraday’s induction law (54) and Ampère-Maxwell’s flux law (48). The first one contains solely transverse field components. The same holds true for Ampère-Maxwell’s flux law after extraction of those parts which are related to charge conservation rather than to the interaction of electric and magnetic fields, see eq. (52). Consequently, free propagating electromagnetic fields contain solely Helmholtz-transverse field components. (Notice that the notation for waveguides is slightly different from that). The longitudinal electric field components ($\vec{E}_L, \vec{D}_L$) obey a separate energy balance with the kinetic energy of the charged bodies (‘Longitudinal Poynting’s theorem’).

In other words, the common derivation of Poynting’s theorem contains the additional assumption that the longitudinal (di)electric field components enter the radiation field, too. The vector identity

$$\vec{E} \cdot \left( \nabla \times \vec{H} \right) - \vec{H} \cdot \left( \nabla \times \vec{E} \right) \equiv \nabla \left( \vec{E} \times \vec{H} \right)$$

serves to interpret the Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, as propagating energy flux density, ie, as if all field components, both the longitudinal and the transverse ones, would propagate in the same manner towards infinity. Though even here, if surface contributions are absent, one has

$$\iiint \nabla \left( \vec{E} \times \vec{H} \right) dV = \iiint \left[ \vec{E} \cdot \left( \nabla \times \vec{H} \right) - \vec{H} \cdot \left( \nabla \times \vec{E} \right) \right] dV$$

$$= \iiint \left[ \vec{E}_T \cdot \left( \nabla \times \vec{H}_T \right) - \vec{H}_T \cdot \left( \nabla \times \vec{E}_T \right) \right] dV = \iiint \nabla \left( \vec{E}_T \times \vec{H}_T \right) dV$$

That means, again, that, in homogeneous isotropic media, the rationalized Maxwell equations actually contain the propagation of transverse fields only.

Within quantum electrodynamics, this additional hypothesis leads to the appearance of 4 equal photon states, where, actually only the 2 transverse ones are observable, while
the longitudinal and the scalar (time-like) ones are not. This seems to speak against that hypothesis. Its only justification consists in that it is necessary for the manifest Lorentz covariant formulation in Minkowski space. However, compatibility with special relativity can also be reached without this formulation (Barut 1964), in particular, by means of Dirac’s (1949) approach to relativistic canonical mechanics.

In order to avoid the difficulties just mentioned, I propose to treat the longitudinal and transverse field components from the very beginning as being physically different. Such an approach enables one to get manifest gauge invariant Lagrangians and Hamiltonians.

By virtue of Gauss’ law, the time-dependence of the longitudinal component of the electric field strength, $\vec{E}_L(\vec{r}, t)$, follows rigidly that of the charge density, $\rho(\vec{r}, t)$; hence, $\vec{E}_L$ is not an independent dynamical variable. This fact is not changed by any gauge. Thus, if one introduces via gauge new dynamical variables, these are finally unphysical (cf Pauli 2000, p.72). For instance, the Lorenz gauge allows for a separate wave equation for $\Phi$. This suggest both $\Phi$ and $\dot{\Phi}$ to be independent variables – however, $\dot{\Phi}$ is not, because $\dot{\Phi} = -\nabla \vec{A}$.

Littlejohn (2008) has stressed correctly, that the gauge transformation changes only the longitudinal component of the vector potential. His conclusion, however, that this is the ”nonphysical” part, while the transverse component is the physical one (Sect. 34.8), overlooks its role in the Aharonov-Bohm effect. Such contradictions have been avoided in this paper through, (i), working with combinations of $\Phi$ and $\vec{A}$, in which those ”nonphysical parts”, if present, cancel each another and, (ii), treating this gauge invariant combination separately from the dynamics of the other field components.

This represents a consequent development of Messiah’s (1999) treatment of the radiation field, where, however, the longitudinal field is ”eliminated” (loc. cit., XXI.22). In this paper, the longitudinal field is treated on equal footing with, though partly separately from the transverse field. Due to this modification, the results presented here are not bound to the radiation gauge, $\nabla \vec{A} = 0$, used by Messiah, but hold true for any gauge.

The approach presented in this paper benefits from the methodological advantages of the treatments by Newton, Euler and Helmholtz, where the subject under investigation (here, moving charged bodies and the electromagnetic fields created by them and acting back onto them) is defined before the mathematical formalism is developed (cf Suisky & Enders 2001; Enders & Suisky 2004, 2005; Enders 2006, 2008, 2009). This keeps the latter physically clear.

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The Proton as A Kerr-Newman Black Hole

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Abstract: The general equation governing the mass, spin and angular momentum of a Kerr-Newman black hole applies equally well to a proton when the gravitational coupling constant predicted by a discrete fractal paradigm is used in the equation, along with the standard mass, spin and angular momentum of the proton.

Keywords: Kerr-Newman Black Hole; Discrete Fractal Paradigm; Cosmology

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1. Introduction

A previous paper [1] demonstrated that the masses and radii of subatomic particles such as the proton and the alpha particle can be retrodicted to a first approximation using the basic physics of black holes, if the gravitational interactions within the subatomic particles are governed by a coupling constant that differs from the standard Newtonian constant by a factor of approximately $10^{38}$. This somewhat radical revision of the fundamental scaling properties of gravitational interactions is derived from a discrete fractal paradigm for nature’s global properties, which has been named the Self-Similar Cosmological Paradigm [2] (SSCP), or Discrete Scale Relativity [3]. According to this new paradigm, discrete scale invariance is a fundamental and universal symmetry principle that has not been adequately recognized in the past. The discrete self-similarity of nature’s hierarchically organized systems is identified as the direct physical manifestation of this global discrete scale invariance. In the present paper a previously proposed discrete self-similarity between hadrons and Stellar Scale Kerr-Newman black holes [1] will be explored in somewhat more detail for the proton.

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2. Discrete Fractal Scaling for Gravitation

The SSCP argues that nature’s unbounded hierarchical organization is divided into discrete cosmological Scales, of which we can currently observe the Atomic, Stellar and Galactic Scales. The discrete fractal scaling of the SSCP leads to the following expression for the coupling constants that characterize the gravitational interactions on differing cosmological Scales, as was discussed in a recent publication [3] on Discrete Scale Relativity:

\[ G_\Psi = [\Lambda^{1-D}]^\Psi G, \]  

(1)

where \( G \) is the conventional Newtonian gravitational coupling constant, which is appropriate for Stellar Scale calculations and is equal to \( 6.67 \times 10^{-8} \) cm\(^3\)/g sec\(^2\). The terms \( \Lambda (\approx 5.2 \times 10^{17}) \) and \( D (\approx 3.174) \) are the dimensionless scaling constants that are fundamental to the SSCP. The term \( \Psi (\equiv \ldots -2, -1, 0, 1, 2 \ldots) \) is a quantized index that designates specific cosmological Scales, and \( \Psi = 0 \) is assigned to the Stellar Scale.

Using Eq. (1) we can determine that the gravitational coupling constants on neighboring cosmological Scales are related by the general equation:

\[ G_{\Psi-1} = \Lambda^{2.174}G_\Psi. \]  

(2)

In the specific case of Atomic Scale systems (\( \Psi = -1 \)),

\[ G_{-1} = \Lambda^{2.174}G_0 = 2.18 \times 10^{31} \text{cm}^3/\text{gsec}^2. \]  

(3)

Having determined the value of the SSCP’s prediction for the appropriate gravitational coupling constant that applies within Atomic Scale systems, we can now show in a little more detail than was given before [1] that the mass/spin/angular momentum relationship of a Kerr-Newman black hole is in good agreement with an empirical mass/spin/angular momentum relationship for the proton.


Solutions of the Einstein field equations of General Relativity for spinning and charged black holes were achieved by Kerr [4] and Newman [5] several decades ago. An important and well known relationship [6] that applies to Kerr-Newman black holes is:

\[ J = a_r[G_\Psi M^2/c]. \]  

(4)

The symbol \( J \) designates the angular momentum of the object, \( a_r \) is referred to as the dimensionless spin parameter, \( G_\Psi \) is the appropriate gravitational coupling constant, \( M \) is the mass of the object, and \( c \) is the velocity of light. This equation, which was derived primarily for Stellar Scale black holes, also applies to the proton when the appropriate Atomic Scale values for \( J, G_\Psi \) and \( M \) are inserted.
4. The Proton as A Gravitational Black Hole

Conventional physics [7] has determined that the angular momentum of the proton \( J_p \) is:

\[
J_p = [j(j + 1)]^{1/2}h,
\]

(5)

where \( j \) is the proton’s dimensionless spin parameter, which equals \( \frac{1}{2} \), and \( h \) is Planck’s constant divided by \( 2\pi \). Given the standard equation for the Planck mass and the basic scaling rules of Discrete Scale Relativity, the SSCP asserts [8] that

\[
h = G_{-1}M^2/c,
\]

(6)

where \( M \) is the revised Planck mass based on \( G_{-1} \), and is equal to \( 1.20 \times 10^{-24} \text{ g} \). Therefore,

\[
J_p = [1/2(1/2 + 1)]^{1/2}[G_{-1}M^2/c] = 0.866[G_{-1}M^2/c].
\]

(7)

If the proton is correctly modeled in terms of a Kerr-Newman black hole, then the following relationship should hold true in accordance with Eq. (4):

\[
0.866[G_{-1}M^2/c] = a_*[G_{-1}m^2/c],
\]

(8)

where \( m \) is the mass of the proton. Eq. (8) can be simplified since the \( G_{-1} \) and \( c \) terms cancel out. We can then insert values for \( M (= 1.20 \times 10^{-24} \text{ g}) \) and \( m (= 1.67 \times 10^{-24} \text{ g}) \) into the remaining equation and solve for \( a_* \).

\[
a_* = 0.866(M/m)^2 = 0.866(0.72)^2 = 0.45.
\]

(9)

The fact that \( a_* = 0.45 \approx \frac{1}{2} \) is encouraging since this agrees with the proton’s empirically and theoretically determined dimensionless spin parameter at the 90% level.

A more accurate test can be achieved by the following method. Given \( G_{-1} \) as the correct gravitational coupling factor for geometrizing mass, charge and specific angular momentum, we may apply the full Kerr-Newman solution of the Einstein-Maxwell equations to the proton. Details of the geometrized methodology employed in the present paper can be found in chapter 33 of *Gravitation* by Misner, Thorne and Wheeler [9]. Calculations based on this method yield the following values for the radius and mass of the proton.

\[
r = m + [m^2 - q^2 - a^2 cos^2(\theta)]^{1/2} = 8.13 \times 10^{-14} \text{ cm}
\]

(10)

\[
m = \left\{ m_{tr}^2 + q^2/4m_{tr} \right\}^{1/2} + J^2/4(m_{tr})^2 = 1.67 \times 10^{-24} \text{ g}
\]

(11)

This demonstrates that the full Kerr-Newman solution of the Einstein-Maxwell equations accurately models the basic \( r, m, q \) and \( J \) properties of the proton when \( G_{-1} \) is adopted as the correct gravitational coupling factor within Atomic Scale systems. Small
but currently unavoidable uncertainties involved in determining the fundamental self-
similar scaling constants $\Lambda$ and D of the SSCP prevent a more exact quantitative test at
present, but such a test is in principle possible in the near future.

Conclusion

The main implication of the above results is that the equation

$$J_p = a_\ast [G_{\Psi} m^2 / c] \quad (12)$$

models the proton’s mass/spin/angular momentum relationship correctly when the
appropriate value of $G_{\Psi}$ is used in the calculations, in close analogy to Eq. (4). Within
the context of the SSCP, the proton appears to be an Atomic Scale Kerr-Newman black
hole.

References


28(12), 1503-1532, 1989; also see http://www.amherst.edu/~rloldershaw which is
the most comprehensive resource for studying the SSCP.

(DOI:10.107/s10509-007-9557-x); also available at


Self-Interacting Scalar Field and Galactic Dark Halos

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Abstract: We construct dark halo models which are supported by a self-interacting scalar field. The possibility that the energy density of such a field which could produce dark matter and dark energy inside and outside of the galactic dark halos is explored.

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1. Introduction

According to Newton’s laws, the rotation curve of spiral galaxies should decrease as \( \frac{1}{\sqrt{|r|}} \) with increasing distance from center in regions where the mass density almost vanishes. But the galaxies rotation curves are quite flat up to large distances from the center. According to various arguments, the mass of the visually luminous matter is not sufficient to bind clusters gravitationally, and significant amounts of non-luminous dark matter should exist as well.

The existence of dark matter in different scales of the universe, ranging from galaxies to cluster of galaxies up to cosmological scales have been asserted by astronomers. In 1933 Zwicky was the first who measured the radial velocity of eight galaxies of the Coma cluster and he deduced the total mass of clusters should be more than the amount of visible mass [1]. In the 1970’s majority of astronomers were convinced that similar situation exists for the spherical halos around individual galaxies [2, 3, 4].

Measurements of the rotation curves of high red-shift galaxies [5] and of gravitational lensing of light by clusters [6] show that dark matter lies within and particularly around...
galaxies, and not spread between them.

To date, various possibilities have been presented about baryonic dark matter. MA-CHO’s (Massive Compact Halo Objects) have been detected but they do not have sufficient density to explain flat rotation curves and cluster’s velocities. Indeed, baryonic matter comprises 4% out of 26% of what is need to make matter in the universe [7]. We should also mention investigations which try to explain the flat rotation curves by resorting to non-Newtonian gravity [8].

In addition, recent observations of the redshift-luminosity relation observed in Type-Ia supernovae [9, 10, 11, 12, 13, 14, 15, 16] and the cosmic microwave background radiation (CMBR)[7] indicate that our universe is accelerating. The cosmic fluid is dominated by an unknown component with negative pressure which is called dark energy. The observations also suggest that the present universe is almost flat ($\Omega_0 = 1.03 \pm 0.05$) with the energy density split into two main contributions: $\Omega_{\text{matter}} \simeq 0.26$ (baryonic and dark), and $\Omega_{D,E} \simeq 0.74$ (dark energy). Although the physical nature of dark energy is still a challenging problem, the cosmological constant is a first sight candidate. The cosmological constant energy density could originate from two sources. The first contribution is due to bare cosmological constant ($\Lambda_0$), which appears in Einstein’s equations and the second one is due to the vacuum energy density of quantum fields [18, 19, 20].

According to some observations, the dark energy density may be dynamical (vary slowly with time) in order to drive an accelerating universe. One possibility along this lines is the quintessence model, which suggests a scalar field $\phi$ slowing rolling down a potential (a process like the inflationary model, in which a phase transition occurs due to a scalar field in the very early universe.) In this model, the scalar field $\phi$ evolves very gradually with time, while being constant throughout the space.[see e.g. Peebles and Ratra 1998, Ratra and Peebles 1998, Wetterich 1998 Brax and Martin 1999,2000]. Quintessence is an interesting model that has attracted a great deal of attention (see also Refs. [26, 27, 28]).

On the other hand, some investigations have been carried out by invoking to a scalar field as the source of dark matter in spiral galaxies. The idea behind these works is to explore whether a scalar field can fluctuate along the history of the universe and thus form concentrations of scalar field energy density. Schunck (1998) has shown that an oscillating, massless, complex scalar field is capable of performing such a role [29]. More complicated systems with self-interaction potential, may also be considered for this purpose (see e.g. [30, 31, 32]).

In this paper, we would like to draw attention to the possibility of a unified picture to account for the existence of both dark matter and dark energy by a scalar field in current epoch, namely, after formation of objects such as galaxies.

We want to present a model in which the behavior of the $\phi$-field and in consequence $V(\phi)$, provides the corresponding energy density which could produce dark matter and dark energy inside and outside the halo, respectively. Along this lines, we will first consider a massive complex scalar field $\Phi$ with a second-order potential of the form $V(\Phi) \propto \Phi \Phi^*$. For this scenario we consider a scalar field which is not only varying with
time (same as in quintessence model) but also is varying with position. Next, a real scalar field $\phi$ undergoing spontaneous symmetry breaking will be considered. For this case, we assume that the real scalar field $\phi$ is a function only of radial coordinate $r$.

2. Preliminaries

Einstein equations can be derived from the action

$$A = \frac{1}{2\kappa} \int (R + \mathcal{L}_{\text{matt}}(\Phi, \partial \Phi)) \sqrt{|g|} d^4x,$$

in which $\mathcal{L}_{\text{matt}}$ is the Lagrangian density of matter, $R$ is the curvature scalar, $\kappa = 8\pi G$, $g$ the determinant of the metric tensor $g_{\mu\nu}$, and $\Phi$ is a complex scalar field. The variation of the action with respect to $g_{\mu\nu}$ leads to Einstein’s equations

$$G_{\mu\nu}^\alpha = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu}^\alpha.\quad (2)$$

The energy-momentum tensor in terms of the scalar field will be introduced shortly. Variation with respect to $\Phi$, will lead to equation of motion for the scalar field in the background geometry

$$\Box \Phi = 2 \frac{\partial}{\partial \Phi^*} V(\Phi, \Phi^*),\quad (3)$$

where $\Box$ is the invariant d’Alembertian defined by

$$\Box = \frac{1}{\sqrt{|g|}} \partial_\mu [\sqrt{|g|} g^{\mu\nu} \partial_\nu].\quad (4)$$

We first set up the metric for gravitational field within a spherically symmetric static dark halo. It reads

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2,\quad (5)$$

where $A(r)$ and $B(r)$ are arbitrary functions of the coordinate.

The non-vanishing components of the Einstein tensor for the metric (5) read

$$G_{tt}^t = \frac{A'}{rA^2} + \frac{1}{r^2} - \frac{1}{r^2A},\quad (6)$$

$$G_{rr}^r = \frac{-B'}{rAB} + \frac{1}{r^2} - \frac{1}{r^2A},\quad (7)$$

$$G_{\theta\theta}^\theta = -\frac{2ABB' - 2A'B^2 + 2rABB'' - rAB'^2 - rA'B'B}{4rA^2B^2},\quad (8)$$

$$G_{\phi\phi}^\phi = G_{\theta\theta}^\theta = G_{\phi\phi}^\phi.\quad (9)$$

The energy-momentum tensor required to support such a spacetime is in the form

$$T_{\nu}^\mu = \text{diag}(-\rho, P_r, P_t, P_t),\quad (10)$$
where \( \rho \) is the energy density and \( P_r \) and \( P_t \) are the radial and transverse pressures, respectively.

Applying Einstein’s equation (2) to the metric (5) gives

\[
\rho(r, t) = \frac{1}{8\pi G} \left[ \frac{A'}{rA^2} + \frac{1}{r^2} - \frac{1}{r^2 A} \right], \tag{11}
\]

\[
P_r(r, t) = -\frac{1}{8\pi G} \left[ \frac{-B'}{rAB} + \frac{1}{r^2} - \frac{1}{r^2 A} \right], \tag{12}
\]

\[
P_t(r, t) = -\frac{1}{8\pi G} \left[ \frac{-2ABB' - 2A'B^2 + 2rABB'' - rABr^2 - rA'B'B}{4rA^2B^2} \right]. \tag{13}
\]

Note that, in these equations dots and primes denote differentiation with respect \( r \) and \( t \), respectively, and we have used the system of units in which \( c = 1 \).

From equations (11) and (12) we obtain

\[
\frac{A'}{A} + \frac{B'}{B} = \kappa (\rho + P_r)rA, \tag{14}
\]

and

\[
\frac{A' r}{A^2} - \frac{1}{2A^2} = \kappa \rho \phi^2 - 1. \tag{15}
\]

The source is described by a complex scalar field with the Lagrangian

\[
\mathcal{L} = -\frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi^* - V(\Phi, \Phi^*), \tag{16}
\]

which leads to the following energy-momentum tensor

\[
T_{\mu}^\nu = \partial_{\mu} \Phi \partial_{\nu} \Phi^* - \delta_{\mu}^\nu [\frac{1}{2} \partial_{\lambda} \Phi \partial^{\lambda} \Phi^* + V(\Phi, \Phi^*)]. \tag{17}
\]

For the scalar field which evolves in the spherically symmetric background, we use the ansatz

\[
\Phi(r, t) = \phi(r)e^{-i\omega t}, \tag{18}
\]

where \( \omega \) is the frequency of the scalar field.

By using (5) and then (17) we obtain

\[
\rho = -T_0^0 = \frac{1}{2B} \phi^2 \omega^2 + \frac{1}{2A} \phi^2 - V(\Phi, \Phi^*), \tag{19}
\]

\[
P_r = T_1^1 = \frac{1}{2B} \phi^2 \omega^2 + \frac{1}{2A} \phi^2 - V(\Phi, \Phi^*), \tag{20}
\]

and

\[
P_t = T_2^2 = T_3^3 = \frac{1}{2B} \phi^2 \omega^2 - \frac{1}{2A} \phi^2 - V(\Phi, \Phi^*). \tag{21}
\]

Using (3) and (5), the differential equation governing the scalar field is obtained to be

\[
\phi'' + \frac{1}{2} \left( \frac{B'}{B} - \frac{A'}{A} + \frac{4}{r} \right) \phi' + A(\omega^2) \phi(r) = 2Ae^{i\omega t} \frac{\partial}{\partial \Phi^*} V(\Phi, \Phi^*). \tag{22}
\]
3. Solutions for the $\Phi^2$-Potential

In this section, we perform some applications of the equations derived in the last section. The type of potential that is used to construct the model is a second-order polynomial in the complex scalar field $\Phi$

$$V(\Phi) = m^2 \Phi \Phi^* + V_0,$$

where $m^2 > 0$ and $V_0$ is a constant. We proceed with the Newtonian approach, since the gravitational field due to the dark and luminous matter is sufficiently weak [33].

By using the Minkowski metric in spherical coordinates, namely

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta.$$

The scalar field equation (22) can be rewritten as

$$\phi'' + \frac{2}{r} \phi' + \left(\omega^2 - 2m^2\right)\phi(r) = 0,$$

and equations (19)-(21) take the forms

$$\rho = -T^0_0 = \frac{1}{2} \phi^2 \omega^2 + \frac{1}{2} \phi'^2 + m^2 \phi^2 + V_0,$$

$$P_r = T^1_1 = \frac{1}{2} \phi^2 \omega^2 + \frac{1}{2} \phi'^2 - m^2 \phi^2 - V_0,$$

$$P_t = T^2_2 = T^3_3 = \frac{1}{2} \phi^2 \omega^2 - \frac{1}{2} \phi'^2 - m^2 \phi^2 - V_0.$$

One can notice that, although the scalar field (18) is time dependent, the corresponding values of the non-vanishing components of the energy-momentum tensor are functions of the radial coordinate $r$ only.

Let us define the following dimensionless quantities

$$\theta = \frac{\phi}{\phi_0}, \quad x = \frac{r}{r_0}.$$

Then we can rewrite (25) and (26)-(28) as

$$\frac{d^2 \theta}{dx^2} + \frac{2}{x} \frac{d \theta}{dx} + \alpha \theta = 0,$$

$$\rho = \gamma \left[ \beta \theta^2 + \left(\frac{d \theta}{dx}\right)^2 \right] + V_0,$$

$$P_r = \gamma \left[ \theta^2 + \left(\frac{d \theta}{dx}\right)^2 \right] - V_0,$$

$$P_t = \gamma \left[ \theta^2 - \left(\frac{d \theta}{dx}\right)^2 \right] - V_0,$$
Fig. 1 The behavior of the dimensionless scalar field $\theta = \frac{\phi}{\phi_0}$ as a function of the dimensionless coordinate $x = \frac{r}{r_0}$.

in which $\alpha = r_0^2(\omega^2 - 2m^2)$, $\beta = \frac{\omega^2+2m^2}{\omega^2-2m^2} \neq 1$, $\gamma = \frac{1}{2} \frac{\phi_0^2}{r_0^2}$ are dimensionless parameters. For $\alpha = \gamma = 1$, equation (30) has the solution

$$\theta(x) = \frac{1}{x} [C \sin x + D \cos x],$$

(34)

where $C$ and $D$ are constants. The solutions that we are interested in are required to be non-singular at the origin, so we proceed with $D = 0$. The function $\theta(x)$ is plotted in figure 1.

From the solution (34), equations (31)- (33) can be rewritten in the following forms

$$\rho = \frac{C^2}{x^2} \left[ \beta \sin^2 x + \cos^2 x - \frac{\sin 2x}{x} + \frac{\sin^2 x}{x^2} \right] + V_0,$$

(35)

$$P_r = \frac{C^2}{x^2} \left[ 1 - \frac{\sin 2x}{x} + \frac{\sin^2 x}{x^2} \right] - V_0,$$

(36)

$$P_t = \frac{C^2}{x^2} \left[ - \cos 2x + \frac{\sin 2x}{x} - \frac{\sin^2 x}{x^2} \right] - V_0.$$

(37)

The energy density of the $\phi$-field depends sensitively on the value of the dimensionless quantity $x$. Inside the halo, the contribution of the energy density due to the first four
terms -corresponding to dark matter -prevails. By increasing the distance from the center of the halo, these terms diminish and the $\phi$-field goes to the vacuum state of the potential and its energy density reduces to the constant value $V_0$. Accordingly, outside the halo, the components of the stress-energy tensor will have the following forms

$$\rho = V_0,$$  \hspace{1cm} (38)

$$P_r = P_t = -V_0,$$ \hspace{1cm} (39)

and

$$P_r = P_t = -\rho.$$ \hspace{1cm} (40)

This is the familiar form of the equation of state for the cosmological constant, which is grossly considered as the origin of the dark energy in the present universe.

This implies that

$$V_0 = \frac{\Lambda}{8\pi G}.$$ \hspace{1cm} (41)

By using the mass function $m(x) = \int_0^x \rho(\xi)\xi^2d\xi$, the Newtonian mass of the dark matter is given by

$$M(x) = M' \left[ \beta\left(\frac{x}{2} - \frac{1}{4}\sin 2x\right) + \left(\frac{x}{2} + \frac{1}{4}\sin 2x\right) + \frac{\cos 2x - 1}{2x} \right],$$ \hspace{1cm} (42)
where $M' = 4\pi r^3_0 C^2$.

For a weak gravitational field, we have the Newtonian formula

$$v^2 = G\frac{M(r)}{r}.$$  \hfill (43)

Substituting (42) into this equation, yields

$$v_{rot}^2 = v^2 \left[ \beta \left( \frac{1}{2} x - \frac{1}{4} \sin 2x \right) + \frac{1}{x} \left( \frac{1}{2} x + \frac{1}{4} \sin 2x \right) - \frac{\sin^2 x}{x^2} \right].$$  \hfill (44)

The corresponding plot (for the tentative value $\beta = 10$) is presented in figure 2. Note that the coefficient $v^2 = 4\pi Gr^3 C^2$, which appears in equation (44), determines the halo characteristic mass. Subsequently, depending on the value of this coefficient, we can fit appropriate rotation curves for different galaxies.

For $\beta = 10$, the typical behavior of $M(x)/M'$ is shown in figure 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The Newtonian mass function $M(x)/M'$, for $\beta = 10$.}
\end{figure}

3.1 GR Solutions

From viewpoint of astrophysics, gravitational fields are often very weak, but sometimes the correction effects of general relativity become important (e.g. the precession of Mercury). Let us consider the general-relativistic effects on rotation curves of galaxies.
The nearly constant rotational velocity of spiral galaxies shows that the amount of dark matter grows continuously with increasing the distance from the center of the halo. For this configuration, the values of $A(r)$ and $B(r)$ are [35]

\begin{equation}
A(r) = \left[ 1 - \frac{2GM(r)}{r} \right]^{-1}, \tag{45}
\end{equation}

\begin{equation}
B(r) = B(r_1) \exp \left\{ - \int_r^{r_1} \frac{2G}{r'^2} [M(r') + 4\pi r'^3 P] A(r') \, dr' \right\}. \tag{46}
\end{equation}

In order to obtain the approximate value of these functions, we use the values of mass and radial pressure from the Newtonian solutions of the previous section, namely, equations (36) and (42). By inserting these equations into (45) and (46), we obtain

\begin{equation}
A(x) \approx 1 - 2v'^2 \left[ \frac{\beta}{x} \left( \frac{x}{2} - \frac{1}{4} \sin 2x \right) + \frac{1}{x} \left( \frac{x}{2} + \frac{1}{4} \sin 2x \right) + \frac{\cos 2x - 1}{2x^2} \right]^{-1}, \tag{47}
\end{equation}

Fig. 4 The metric function $A(x)$.
Fig. 5 The metric function \( B(x) \).

and

\[
B(x) = B(x_1) \exp \int_{x_1}^{x} \frac{-2 \nu^2}{x^2} \left[ \beta \left( \frac{x}{2} - \frac{1}{4} \sin 2x \right) + \left( \frac{x}{2} + \frac{1}{4} \sin 2x + \frac{\cos 2x - 1}{2x} \right) \right] \exp \left[ -x - \sin 2x + \frac{\sin^2 x}{x} \right] \left[ 1 - 2 \nu^2 \left( \frac{x}{2} - \frac{1}{4} \sin 2x \right) + \frac{1}{2} \left( \frac{x}{2} + \frac{1}{4} \sin 2x \right) + \frac{\cos 2x - 1}{2x^2} \right]^{-1}.
\] (48)

note that we have used the dimensionless variable \( x = \frac{r}{r_0} \). Figure 4 shows the behavior of \( A(x) \) as function of \( x \). In order to determine \( B(x) \), a numerical calculation was carried out. The result is illustrated in figure 5.

From equation (22), we have

\[
\phi'' + \frac{1}{2} \left( \frac{B'}{B} - \frac{A'}{A} + \frac{4}{r} \right) \phi' + A \left( \frac{\omega^2}{B} - 2m^2 \right) \phi = 0.
\] (49)

Let us simplify this equation. By inserting (6), (7) and (11), (12) into Einstein equation (2), we have

\[
\frac{B'}{rAB} - \frac{1}{r^2} + \frac{1}{r^2 A} = \kappa \left[ \frac{1}{2} A \phi'^2 + \frac{1}{2} B \omega^2 \phi'^2 - m^2 \phi^2 \right],
\] (50)

\[
\frac{A'}{rA^2} + \frac{1}{r^2} - \frac{1}{r^2 A} = \kappa \left[ \frac{1}{2} A \phi'^2 + \frac{1}{2} B \omega^2 \phi'^2 + m^2 \phi^2 \right].
\] (51)

Subtracting 6 from (51), we obtain

\[
\frac{B'}{B} - \frac{A'}{A} = \frac{2A}{r} - \frac{2}{r} - 2\kappa r Am^2 \phi^2,
\] (52)

substituting this equation into (49), yields

\[
\frac{d^2 \phi}{dr^2} + \frac{1}{r} (A + 1) \frac{d \phi}{dr} - \kappa r Am^2 \phi^2 \phi' + A \left( \frac{\omega^2}{B} - 2m^2 \right) \phi = 0.
\] (53)
From the dimensionless quantities (29), the preceding equation can be written in the alternative form

\[ \frac{d^2 \theta}{dx^2} + (A + 1) \frac{1}{x} \frac{d \theta}{dx} - \left( \frac{\beta - 1}{2} \right) \left[ -\xi x A \theta \frac{d \theta}{dx} + \left( \frac{A}{B} - A \right) \theta \right] + \frac{A}{B} \theta = 0, \]  

(54)

in which \( \xi = \kappa r_0^2 \) is a dimensionless parameter. As boundary condition, we apply \( \theta = 0 \), \( \frac{d \theta}{dx} = 0 \), at large enough \( x \), the numerical calculation determines the values of \( \theta \) at each point. The result is illustrated in figure 6.

For the static, spherically symmetric metric (5) the tangential velocity of a test particle obeys

\[ v_{\text{rot}}^2 = \frac{1}{2} r B'. \]  

(55)

From the dimensionless quantities (29), we have

\[ v_{\text{rot}}^2 = \frac{1}{2} x \frac{dB}{dx}, \]  

(56)

we have calculated \( v_{\text{rot}}/v_0 \) numerically. The outcome can be seen in figure 7. Note that, in this case \( v_0 \) is the value of \( v_{\text{rot}} \) at large \( x \). As expected, figures 6 and 7 show that the deviation from the Newtonian solution is not considerable.

4. Solutions for the \( \phi^4 \) - Potential

The flat rotation curve results of the \( \Phi^2 \)-potential, seem to be promising, but it is instructive to use a similar approach for another celebrated potential in the field theory, known as the \( \phi^4 \)-potential:

\[ V(\phi) = \frac{1}{2} \lambda (\phi^2 - \phi_0^2)^2, \]  

(57)
where $\lambda > 0$. In this case, we assume that the real scalar field $\phi$ is a function only of $r$. As illustrated in figure 8, the potential has one local maximum (false vacuum) at $\phi = 0$ and two global minima (true vacuums) at $\pm \phi_0$.

As pointed out before, the Newtonian limit is adequate for our investigation. So, we proceed in this framework.

For the real massless scalar field $\phi(r)$, the source is described with Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi). \quad (58)$$

The Euler-Lagrange equation leads to the differential equation for the scalar field

$$\Box \phi = \frac{dV(\phi)}{d\phi}. \quad (59)$$

For $\phi$ a function of $r$ only, we have

$$\frac{d^2 \phi(r)}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = \frac{dV(\phi)}{dr}. \quad (60)$$

Inserting (57) into this equation, we get

$$\frac{d^2 \phi(r)}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - 2\lambda(\phi^2 - \phi_0^2)\phi = 0. \quad (61)$$

The stress tensor $T_\nu^\mu$ related to Lagrangian (58) is

$$T_\nu^\mu = \partial^\mu \phi \partial_\nu \phi - \delta_\nu^\mu \left[ \frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi + \frac{1}{2} \lambda(\phi^2 - \phi_0^2)^2 \right]. \quad (62)$$
The non-vanishing components of the energy momentum tensor are

\[ \rho = -T_0^0 = \frac{1}{2}\phi'^2 + \frac{1}{2}\lambda(\phi^2 - \phi_0^2)^2, \] (63)

\[ P_r = T_1^1 = \frac{1}{2}\phi'^2 - \frac{1}{2}\lambda(\phi^2 - \phi_0^2)^2, \] (64)

and

\[ P_t = T_2^2 = T_3^3 = -\frac{1}{2}\phi'^2 - \frac{1}{2}\lambda(\phi^2 - \phi_0^2)^2, \] (65)

in which we have used the metric (24) for flat space. By using the dimensionless quantities (29), equations (61) and (63) can be written as

\[ \frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} - \alpha'(\theta^2 - 1)\theta = 0, \] (66)

and

\[ \rho = \beta'(\frac{d\theta}{dx})^2 + \gamma'(\theta^2 - 1)^2, \] (67)

where \( \alpha' = 2\lambda r_0^2 \phi_0^2, \ \beta' = \frac{1}{2}\frac{\phi_0^2}{r_0^2} \) and \( \gamma' = \frac{1}{2}\lambda \phi_0^4 \) are dimensionless parameters.

In order to get regular solutions for the differential equation (66), we put the initial conditions \( \theta(0) = 1 \) and \( \frac{d\theta}{dx}(0) = 0 \). By assuming \( \alpha' = 0.001 \) the numerical calculation was carried out and the function of \( \theta(x) \) is illustrated in figure 9. As it is seen, the scalar

**Fig. 8** Spontaneous symmetry breaking potential for the scalar field (the so-called \( \phi^4 \)-potential)
field is nearly at the true vacuum of the potential at origin. However, for large values of \(x\), the \(\phi\)-field fluctuates around its false vacuum.

Let us investigate some properties of the corresponding energy density \(\rho\) which makes it a particular and interesting case, by taking into account equation (67) and the behavior of \(\theta(x)\) which is plotted in figure 9. When the scalar field is near to the true vacuum of the potential \((\theta = 1)\), by a suitable choice of \(\beta'\), the first term of equation (67) prevails the second one, and we interpret this region as inside the halo and the corresponding value of the energy density is regarded as the dark matter energy density. However, for large values of \(x\), the first term can be ignored and this corresponds to outside the halo, where the scalar field goes to zero and the energy density (67), approaches the constant value

\[
\rho \to V(0) = \frac{1}{2} \lambda \phi_0^4.
\] (68)

In such a case, equations (64) and (65) become

\[
P_r = P_t = -V(0),
\] (69)
and $\rho = -P$, which is the equation of state of the cosmological constant. So we can write

$$V(\phi = 0) = \frac{1}{2} \lambda \phi_0^4 = \rho_\Lambda,$$

(70)

in which $\rho_\Lambda \simeq 10^{-9}$ joule m$^{-3}$. This equation determines the value of $\lambda$:

$$\lambda = \frac{2\rho_\Lambda}{\phi_0^4}.$$  

(71)

From the preceding equation, together with the value of $\alpha' = 2\lambda \phi_0^2 \phi_0^2$, we can easily obtain

$$\beta' = \frac{2\rho_\Lambda}{\alpha'}.$$  

(72)

So, equation (67) can be rewritten as

$$\rho = \rho_\Lambda \left[ \frac{2}{\alpha'} \left( \frac{d\theta}{dx} \right)^2 + (\theta^2 - 1)^2 \right].$$

(73)

The first term of this equation (which corresponds to the dark matter energy density) enables us to calculate rotation curve from the Newtonian formula (43). It has been obtained using numerical calculations and the result is shown in figure 10. It should be emphasized that the dimensionless parameter $\alpha'$ which appears in (73) determines the halo characteristics mass and density near the center. In other words, by suitable choice of $\alpha'$ we can get favorable rotation curves for different galaxies.

From (73), it is clear that for inside halo, the first term (energy density of dark matter) should be much greater than the second term (energy density of dark energy). For this purpose, we need $\alpha' < 1$. 

**Fig. 10** Rotation curve of the $\phi^4$-model for $\alpha' = 0.001$, $\theta(0) = 1$ and $\frac{d\theta(0)}{dx} = 0$, in which $v_0^2 = 4\pi G\rho_\Lambda \phi_0^2$. 


Conclusion

From the rotation curves of spiral galaxies and velocities of galaxies in clusters, the existence of dark matter has long been confirmed by astronomers.

On the other hand, reliable observational evidence from WMAP and Type-Ia supernovae results, not only confirm the existence of dark matter but also indicate that the present universe is almost flat, and consists of 4% baryons, 22% dark matter and 74% dark energy.

We presented a model, in which the energy density of a spherically symmetric scalar field could produce both the dark matter and dark energy inside and outside of the halo, respectively.

We started from a general, spherically symmetric line element. The \( \Phi \)-field was described by

\[
L = \frac{1}{2} \partial^\mu \Phi \partial^\nu \Phi^* - V(\Phi, \Phi^*),
\]

so that the corresponding stress tensor \( T^\mu_\nu \) has the form

\[
T^\mu_\nu = \text{diag}(-\rho, P_r, P_t, P_t).
\]

As a special case, we considered a complex scalar field \( \Phi \) with a second-order potential of the form \( V(\Phi) \propto \Phi \Phi^* \), in which \( \Phi(r,t) = \phi(r)e^{-i\omega t} \). In the Newtonian framework, we found the exact solution for \( \phi(r) \) and, consequently the corresponding energy density. The corresponding energy density was found to be high at the center and reduce to the constant value \( V_0 \) at large radii. Accordingly, the scalar field fluctuation could serve as dark matter and dark energy inside and outside the halo, respectively. By using these solutions, rotational velocity was computed and the corresponding rotation curve was obtained. By suitable choices for \( v' \) (halo characteristic parameter), we can fit appropriate rotation curves for different galaxies. The approximate GR solutions were also obtained, although the deviation from the Newtonian results were found to be quite small.

Furthermore, we presented a model with a real scalar field \( \phi \) as a function only of coordinate \( r \). We employed the \( \phi^4 \)-potential which is known to cause spontaneous symmetry breaking. We demonstrated how the scalar field can, in principle, reproduce dark matter energy density for inside and dark energy for outside the halo. Inside the halo, the rotational velocity was computed numerically and, shown to exhibit an almost flat behavior.

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References


Path Integral Quantization of the Electromagnetic Field Coupled to A Spinor

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Abstract: The Hamilton-Jacobi approach is applied to the electromagnetic field coupled to a spinor. The integrability conditions are investigated and the path integral quantization is performed using the action given by Hamilton-Jacobi approach. © Electronic Journal of Theoretical Physics. All rights reserved.

Keywords: Quantization of the Electromagnetic Fields; Path Integral; Hamilton-Jacobi Approach

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1. Introduction

The most common method for investigating the Hamilton treatment of constrained systems was initiated by Dirac[1-4]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Hence, equations of motion are obtained in terms of arbitrary parameters.

The starting point of the Hamilton-Jacobi approach [5-10] is the variational principle. The Hamiltonian treatment of constrained systems leads us to total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system.

Path integral quantization based on Hamilton-Jacobi method is developed in references [11-15].

Our aim in this paper is to quantize a system of electromagnetic field coupled to a spinor. The paper is arrange as follows: In Sec.2 the Hamilton-Jacobi formulation is briefly described. In Sec.3 we present a system of the electromagnetic field coupled to a spinor, which is quantized using Hamilton-Jacobi formulation. Sec.4 is the conclusion.

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2. Hamilton-Jacobi Formulation

One starts from singular Lagrangian \( L \equiv L(q_i, \dot{q}_i, t), i = 1, 2, \ldots, n \), with the Hess matrix

\[
A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j},
\]

of rank \((n - r), r < n\). The generalized momenta \( p_i \) corresponding to the generalized coordinates \( q_i \) are defined as

\[
p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \ldots, n - r, \quad (2)
\]

\[
p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}, \quad \mu = n - r + 1, \ldots, n. \quad (3)
\]

where \( q_i \) are divided into two sets, \( q_a \) and \( x_\mu \). Since the rank of the Hessian matrix is \((n - r)\), one may solve Eq. (2) for \( \dot{q}_a \) as

\[
\dot{q}_a = \dot{q}_a(q_i, \dot{x}_\mu, p_a; t). \quad (4)
\]

Substituting Eq. (4), into Eq. (3), we get

\[
p_\mu = -H_\mu(q_i, \dot{x}_\mu, p_a; t). \quad (5)
\]

The canonical Hamiltonian \( H_0 \) reads as

\[
H_0 = -L(q_i, \dot{x}_\nu, q_\alpha; t) + p_a \dot{q}_a - \dot{x}_\mu H_\mu, \quad \nu = 1, 2, \ldots, r. \quad (6)
\]

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

\[
H'_\alpha(x_\beta, q_\alpha, \frac{\partial S}{\partial q_\alpha}, \frac{\partial S}{\partial x_\beta}) = 0, \quad \alpha, \beta = 0, 1, \ldots, r, \quad (7)
\]

where

\[
H'_0 = p_0 + H_0, \quad (8)
\]

\[
H'_\mu = p_\mu + H_\mu. \quad (9)
\]

We define \( p_\beta = \partial S[q_a; x_a]/\partial x_\beta \) and \( p_a = \partial S[q_a; x_a]/\partial q_a \) with \( x_0 = t \) and \( S \) being the action.

The equations of motion are written as total differential equations in the form

\[
\begin{align*}
dq_a &= \frac{\partial H'_a}{\partial p_a} \, dx_a, \\
dp_a &= \frac{\partial H'_a}{\partial q_a} \, dx_a, \\
dp_\beta &= \frac{\partial H'_\beta}{\partial t_\beta} \, dx_a, \\
dz &= \left(-H_\alpha + p_a \frac{\partial H'_a}{\partial p_a}\right) \, dx_a,
\end{align*}
\]

\[
\quad (10) \quad (11) \quad (12) \quad (13)
\]
where \( Z = S(x_\alpha, q_a) \). These equations are integrable if and only if [12]

\[
dH'_0 = 0, \quad (14)
\]

and

\[
dH'_\mu = 0, \quad \mu = 1, 2, \ldots, r. \quad (15)
\]

If conditions (14), and (15) are not satisfied identically, one considers them as a new constraints and a gain consider their variations. Thus, repeating this procedure, one may obtain a set of constraints such that all variations vanish. Simultaneous solutions of canonical equations with all these constraints provide the set of canonical phase space coordinates \((q_a, p_a)\) as functions of \( t_a \); the canonical action integral is obtained in terms of the canonical coordinates. \( H'_\alpha \) can be interpreted as the infinitesimal generator of canonical transformations given by parameters \( t_\alpha \). In this case the path integral representation can be written as [14-16].

\[
\langle \text{Out} \mid S \mid \text{In} \rangle = \int \prod_{a=1}^{n-p} dq_a^* dp_a \exp \left[ i \int_{t_a}^{t'_a} \left( -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha \right], \quad (16)
\]

\[
a = 1, \ldots, n - p, \quad \alpha = 0, n - p + 1, \ldots, n.
\]

In fact, this path integral is an integration over the canonical phase space coordinates \((q^a, p^a)\).

### 3. Quantization of Electromagnetic Field Coupled to A Spinor

The Lagrangian of the electromagnetic field coupled to a spinor is given by [3,4].

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m \bar{\psi} \psi, \quad (17)
\]

where \( A_\mu \) are even variables while \( \psi \) and \( \bar{\psi} \) are odd ones. The electromagnetic tensor is defined as \( F_{\mu\nu} = \partial_\mu A^\nu - \partial_\nu A^\mu \), and we are adopting the Minkowski metric \( \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1) \).

The Lagrangian function (17) is singular, since the rank of the Hess matrix (1) is three.

The momenta variables conjugated, respectively, to \( A_i, A_0, \psi \) and \( \bar{\psi} \), are obtained as

\[
\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \quad (18)
\]

\[
\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0 = -H_1, \quad (19)
\]

\[
p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = -H_\psi, \quad (20)
\]

\[
p_\bar{\psi} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0 = -H_\bar{\psi}. \quad (21)
\]
where we must call attention to the necessity of being careful with the spinor indices. Considering, as usual, $\psi$ as a column vector and $\overline{\psi}$ as a row vector implies that $p_{\psi}$ will be a row vector while $p_{\overline{\psi}}$ will be a column vector.

With the aid of relation (18), the Lagrangian may be written as

$$L = -\frac{1}{2} \pi_i \pi^i - \frac{1}{4} F^{ij} F_{ij} + i \overline{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi - m \overline{\psi} \psi,$$

then the canonical Hamiltonian takes the form

$$H_0 = \pi^i \dot{A}_i + \frac{1}{2} \pi_i \pi^i + \frac{1}{4} F_{ij} F^{ij} - i \overline{\psi} (\gamma^\mu i e A_\mu + \gamma^i \partial_i) \psi + m \overline{\psi} \psi. \quad (23)$$

The velocities $\dot{A}_i$ can be expressed in terms of the momenta $\pi_i$ as

$$\dot{A}_i = -\pi_i + \partial_i A_0. \quad (24)$$

Therefore, the Hamiltonian is

$$H_0 = \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \frac{1}{4} \partial_i A_0 \psi - \pi^i (\gamma^\mu i e A_\mu + \gamma^0 \partial_i \psi + m \psi). \quad (25)$$

The set of Hamilton-Jacobi Partial Differential Equation (7) reads as

$$H'_0 = p_0 + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} \pi_i \pi^i + \pi^i \partial_i A_0 + \overline{\psi} \gamma^\mu e A_\mu \psi - \overline{\psi} (i \gamma^i \partial_i - m) \psi, \quad (26)$$

and

$$H'_1 = \pi^0 + H_1 = 0, \quad (27)$$

$$H'_\psi = p_\psi + H_\psi = 0, \quad (28)$$

$$H'_\overline{\psi} = p_\overline{\psi} + H_\overline{\psi} = 0. \quad (29)$$

Therefore, the total differential equations for the characteristic (9), (10) and (11), are obtained as

$$dA^i = \frac{\partial H'_0}{\partial \pi_i} dt + \frac{\partial H'_1}{\partial \pi_i} dA^0 + \frac{\partial H'_\psi}{\partial \pi_i} d\psi + \frac{\partial H'_\overline{\psi}}{\partial \pi_i} d\overline{\psi},$$

$$= -(\pi^i + \partial_i A_0) dt, \quad (30)$$

$$dA^0 = \frac{\partial H'_0}{\partial \pi_0} dt + \frac{\partial H'_1}{\partial \pi_0} dA^0 + \frac{\partial H'_\psi}{\partial \pi_0} d\psi + \frac{\partial H'_\overline{\psi}}{\partial \pi_0} d\overline{\psi},$$

$$= dA^0, \quad (31)$$

$$d\pi^i = -\frac{\partial H'_0}{\partial A_i} dt - \frac{\partial H'_1}{\partial A_i} dA^0 - \frac{\partial H'_\psi}{\partial A_i} d\psi - \frac{\partial H'_\overline{\psi}}{\partial A_i} d\overline{\psi},$$

$$= (\partial_i F^{ij} - e \overline{\psi} \gamma^i \psi) dt, \quad (32)$$

$$d\pi^0 = -\frac{\partial H'_0}{\partial A_0} dt - \frac{\partial H'_1}{\partial A_0} dA^0 - \frac{\partial H'_\psi}{\partial A_0} d\psi - \frac{\partial H'_\overline{\psi}}{\partial A_0} d\overline{\psi},$$
\[ = (\partial_i \pi^i - e\bar{\psi}\gamma^0 \psi) \, dt, \quad (33) \]

\[
dp_\psi = -\frac{\partial H'_1}{\partial \psi} \, dt - \frac{\partial H'_1}{\partial \psi} \, dA^0 + \frac{\partial H'_\psi}{\partial \psi} \, d\psi + \frac{\partial H'_\psi}{\partial \psi} \, d\bar{\psi},
\]

\[ = -(i\gamma^i \partial_i + e\gamma^\mu A^\mu + m)\bar{\psi} \, dt, \quad (34) \]

and

\[
dp_\bar{\psi} = -\frac{\partial H'_1}{\partial \bar{\psi}} \, dt - \frac{\partial H'_1}{\partial \bar{\psi}} \, dA^0 + \frac{\partial H'_\bar{\psi}}{\partial \bar{\psi}} \, d\psi + \frac{\partial H'_\bar{\psi}}{\partial \bar{\psi}} \, d\psi,
\]

\[ = -(i\gamma^i \partial_i + e\gamma^\mu A^\mu + m)\psi \, dt - i\gamma^0 d\psi. \quad (35) \]

The integrability condition \((dH'_\alpha = 0)\) implies that the variation of the constraints \(H'_1, H'_\psi\) and \(H'_\bar{\psi}\) should be identically zero, that is

\[ dH'_1 = d\pi_0 = 0, \quad (36) \]

\[ dH'_\psi = dp_\psi - i\gamma^0 d\bar{\psi} = 0, \quad (37) \]

\[ dH'_\bar{\psi} = dp_{\bar{\psi}} = 0. \quad (38) \]

When we substituting from Eqs. (34) and (35) into Eqs.(37) and (38) respectively, the variation of

\[ dH'_\psi = 0, \quad (39) \]

and

\[ dH'_\bar{\psi} = 0, \quad (40) \]

are identically zero if the following relations are satisfied:

\[ i\bar{\psi}\gamma^\mu (\partial_\mu - ieA_\mu) + m\bar{\psi} = 0, \quad (41) \]

and

\[ i(\partial_\mu + ieA_\mu) \gamma^\mu \psi - m\psi = 0, \quad (42) \]

Then the set of equations \((30,32,33)\) is integrable and they are just ordinary differential equations which can be written in the forms

\[ \dot{A}^i = -\pi^i - \partial_i A^0, \quad (43) \]

\[ \dot{\pi}^i = \partial_i E^{ti} - e\bar{\psi}\gamma^i \psi, \quad (44) \]

\[ \dot{\pi}^0 = \partial_i \pi^i - e\bar{\psi}\gamma^0 \psi. \quad (45) \]

These are the equations of motion with full gauge freedom. It can be seen, from Eq. (31), that \(A^0\) is an arbitrary (gauge dependent) variable since its time derivative is arbitrary. Besides that, Eq. (43) shows the gauge dependence of \(A^i\) and it is clear that the curl of its vector form, leads to the well known Maxwell equations,

\[ \frac{\partial \vec{A}}{\partial t} = -\vec{E} - \vec{\nabla}(A_0 - \alpha) \Rightarrow \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}. \quad (46) \]
Writing $j^\mu = e\overline{\psi}\gamma^\mu \psi$ we get, from Eq. (44), the inhomogeneous Maxwell equation
\[ \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} - \vec{j}, \]
while the other inhomogeneous equation
\[ \vec{\nabla} \cdot \vec{E} = j^0, \] (48)
follows from Eq. (45). Expressions (41) and (42) are the known equations for the spinor $\psi$ and $\overline{\psi}$.

Eqs. (12) and (26-29) lead us to the canonical action integral
\[ Z = \int d^4x \left( -\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\pi_i\pi^i + \pi^i\dot{A}_i + \pi_i\partial_i A_0 + i\overline{\psi}\gamma^\mu (\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \right). \] (49)

Making use of equations (16) and (49), we obtain the path integral as
\[ \langle \text{out} | S | \text{In} \rangle = \int \prod_i dA_i d\pi^i d\psi d\overline{\psi} \exp \left[ i \left\{ \int d^4x \left( -\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\pi_i\pi^i ight. ight. ight. \\
\left. \left. \left. + \pi^i\dot{A}_i + \pi_i\partial_i A_0 + i\overline{\psi}\gamma^\mu (\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \right) \right\} \right]. \] (50)

Integration over $\pi_i$ gives
\[ \langle \text{out} | S | \text{In} \rangle = N \int \prod_i dA_i d\psi d\overline{\psi} \exp \left[ i \left\{ \int d^4x \left( \frac{1}{2}(\dot{A}^i + \partial_i A_0)^2 ight. \right. \right. \\
\left. \left. \left. - \frac{1}{4}F_{ij}F_{ij} + i\overline{\psi}\gamma^\mu (\partial_\mu + ieA_\mu)\psi - m\overline{\psi}\psi \right) \right\} \right]. \] (51)

which is an integration over the canonical phase space.

**Conclusion**

Path integral quantization of the electromagnetic field coupled to a spinor is obtained by the canonical path integral formalism based on Hamilton-Jacobi method [12-15]. The integrability conditions $dH'_1$, $dH'_\psi$ and $dH''_\psi$ are identically satisfied, and the system is integrable. Hence, the canonical phase space coordinates $(A^i, \pi^i), (\psi, p_\psi)$ and $(\overline{\psi}, p_{\overline{\psi}})$ are obtained in terms of the parameter $\tau$. If the system is integrable, then one can construct the reduced canonical phase-space. The path integral is obtained as an integration over the canonical phase-space coordinates $(A^i, \pi^i)$ and $(\psi, \overline{\psi})$ without using any gauge fixing condition. From the equations of motion of the system of electromagnetic field coupled to a spinor, we obtained the inhomogeneous Maxwell equations.

The advantage of this path integral formalism is that we have no need to enlarge the initial phase-space by introducing unphysical auxiliary field, no need to introduce Lagrange multipliers, no need to use delta functions in the measure as well as to use gauge fixing conditions; all that needed is the set of Hamilton-Jacobi partial differential equations and the equations of motion.
References


Neutrino Mixing and Cosmological Constant above GUT Scale

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Abstract: Neutrino mixing lead to a non zero contribution to the cosmological constant. We consider non renormalization $1/M_x$ interaction term as a perturbation of the neutrino mass matrix. We find that for the degenerate neutrino mass spectrum. We assume that the neutrino masses and mixing arise through physics at a scale intermediate between Planck Scale and the electroweak scale. We also assume, above the electroweak breaking scale, neutrino masses are nearly degenerate and their mixing is bimaximal. Quantum gravitational (Planck scale )effects lead to an effective $SU(2)_L \times U(10$ invariant dimension-5 Lagrangian involving neutrino and Higgs fields, which gives rise to additional terms in neutrino mass matrix. There additional term can be considered to be perturbation of the GUT scale bi-maximal neutrino mass matrix. We assume that the gravitational interaction is flavour blind and we study the neutrino mixing and cosmological constant due to physics above the GUT scale.

Keywords: Neutrino Mixing; Cosmological constant; GUT scale

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1. Introduction

The problem of cosmological constant is currently one of the most challenging open issue in theoretical physics and cosmology. The main difficulty comes from the mismatch between theoretical and accepted number. Cosmology constant may arise from neutrino mixing [1]. In this case of neutrinos, cosmological density related to the mixing and mass difference among the different generations. Phenomenological consequences of non-trivial condensate structure of the flavour vacuum have been studied for neutrino oscillations and Beta decay [2,3]. The nature of the cosmology constant $\Lambda$ is one of the most interesting issues in modern theoretical physics and cosmology. Experimental data

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coming from observation indicates that not only Λ is different from zero, Λ also dominates the universe dynamics driving an accelerated expansion [4,5]. In this paper, we study the neutrino mixing due to Planck scale and contribution to cosmological constant. In Section 2, we summarize the neutrino mixing due to Planck scale effects. In Section 3, we discuss the neutrino mixing and cosmological constant due to Planck scale effects. Section 4 is devoted to the conclusions.

2. Neutrino Oscillation Parameter due to Planck Scale Effects

The neutrino mass matrix is assumed to be generated by the see saw mechanism [6,7,8]. We assume that the dominant part of neutrino mass matrix arise due to GUT scale operators and the lead to bi-maximal mixing. The effective gravitational interaction of neutrino with Higgs field can be expressed as $SU(2)_L \times U(1)$ invariant dimension-5 operator [8],

$$L_{grav} = \frac{\lambda_{\alpha\beta}}{M_{pl}} (\psi_{A\alpha} \epsilon \psi_C) C^{-1} (\psi_{B\beta} \epsilon \psi_D) + h.c. \tag{1}$$

Here and every where we use Greek indices $\alpha, \beta$ for the flavour states and Latin indices $i,j,k$ for the mass states. In the above equation $\psi_{\alpha} = (\nu_{\alpha}, l_{\alpha})$ is the lepton doublet, $\phi = (\phi^+, \phi^0)$ is the Higgs doublet and $M_{pl} = 1.2 \times 10^{19} GeV$ is the Planck mass $\lambda$ is a $3 \times 3$ matrix in a flavour space with each elements $O(1)$. The Lorentz indices $a,b = 1,2,3,4$ are contracted with the charge conjugation matrix $C$ and the $SU(2)_L$ isospin indices $A,B,C,D = 1,2$ are contracted with $\epsilon = i\sigma_2, \sigma_m (m = 1,2,3)$ are the Pauli matrices. After spontaneous electroweak symmetry breaking the lagrangian in eq(1) generated additional term of neutrino mass matrix

$$L_{mass} = \frac{v^2}{M_{pl}} \lambda_{\alpha\beta} \nu_{\alpha} C^{-1} \nu_{\beta}, \tag{2}$$

where $v = 174 GeV$ is the VEV of electroweak symmetric breaking. We assume that the gravitational interaction is “flavour blind” that is $\lambda_{\alpha\beta}$ is independent of $\alpha, \beta$ indices. Thus the Planck scale contribution to the neutrino mass matrix is

$$\mu \lambda = \mu \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{3}$$

where the scale $\mu$ is

$$\mu = \frac{v^2}{M_{pl}} = 2.5 \times 10^{-6} eV. \tag{4}$$

We take eq(3) as perturbation to the main part of the neutrino mass matrix, that is generated by GUT dynamics. To calculate the effects of perturbation on neutrino
observable. The calculation developed in an earlier paper [8]. A natural assumption is that unperturbed (0th order mass matrix) $M$ is given by

$$M = U^* \text{diag}(M_i) U^\dagger,$$  \hspace{1cm} (5)

where, $U_{ei}$ is the usual mixing matrix and $M_i$ , the neutrino masses is generated by Grand unified theory. Most of the parameter related to neutrino oscillation are known, the major expectation is given by the mixing elements $U_{e3}$. We adopt the usual parametrization.

$$\begin{align*}
|U_{e2}| &= \tan\theta_{12}, \\
|U_{\mu3}| &= \tan\theta_{23}, \\
|U_{e3}| &= \sin\theta_{13}.
\end{align*}$$  \hspace{1cm} (6, 7, 8)

In term of the above mixing angles, the mixing matrix is

$$U = \text{diag}(e^{i\delta_f^1}, e^{i\delta_f^2}, e^{i\delta_f^3})R(\theta_{23})\Delta R(\theta_{13})\Delta^* R(\theta_{12}) \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1).$$  \hspace{1cm} (9)

The matrix $\Delta = \text{diag}(e^{i\theta_1^1}, e^{i\theta_2^1})$ contains the Dirac phase. This leads to CP violation in neutrino oscillation $a_1$ and $a_2$ are the so called Majoring phase, which effects the neutrino less double beta decay. $f_1$, $f_2$ and $f_3$ are usually absorbed as a part of the definition of the charge lepton field. Planck scale effects will add other contribution to the mass matrix that gives the new mixing matrix can be written as [8]

$$U' = U(1 + i\delta\theta),$$

$$\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
U_{e2}\delta\theta_{12}^* + U_{e3}\delta\theta_{23}^* , U_{e1}\delta\theta_{12} + U_{e3}\delta\theta_{23}^* , U_{e1}\delta\theta_{13} + U_{e3}\delta\theta_{23}^* \\
U_{\mu2}\delta\theta_{12}^* + U_{\mu3}\delta\theta_{23}^* , U_{\mu1}\delta\theta_{12} + U_{\mu3}\delta\theta_{23}^* , U_{\mu1}\delta\theta_{13} + U_{\mu3}\delta\theta_{23}^* \\
U_{\tau2}\delta\theta_{12}^* + U_{\tau3}\delta\theta_{23}^* , U_{\tau1}\delta\theta_{12} + U_{\tau3}\delta\theta_{23}^* , U_{\tau1}\delta\theta_{13} + U_{\tau3}\delta\theta_{23}^*
\end{pmatrix} + i \begin{pmatrix}
U_{e2}\delta\theta_{12} + U_{e3}\delta\theta_{23}^*, U_{e1}\delta\theta_{12} + U_{e3}\delta\theta_{23}^*, U_{e1}\delta\theta_{13} + U_{e3}\delta\theta_{23}^*
\end{pmatrix}.$$  \hspace{1cm} (10)

Where $\delta\theta$ is a hermitian matrix that is first order in $\mu[8,9]$. The first order mass square difference $\Delta M^2_{ij} = M^2_i - M^2_j$ get modified [8,9] as

$$\Delta M^2_{ij} = \Delta M^2_{ij} + 2(M_i Re(m_{ii}) - M_j Re(m_{jj})),$$  \hspace{1cm} (11)

where
\[ m = \mu U^\dagger \lambda U, \]

\[ \mu = \frac{v^2}{M_{pl}} = 2.5 \times 10^{-6} \text{eV}. \]

The change in the elements of the mixing matrix, which we parameterized by \( \delta \theta \)\[8,9\], is given by

\[
\delta \theta_{ij} = \frac{i \text{Re}(m_{jj})(M_i + M_j) - i \text{Im}(m_{jj})(M_i - M_j)}{\Delta M_{ij}^2}.
\]

The above equation determines only the off diagonal elements of matrix \( \delta \theta_{ij} \). The diagonal element of \( \delta \theta_{ij} \) can be set to zero by phase invariance. Using Eq(10), we can calculate neutrino mixing angle due to Planck scale effects,

\[
\frac{|U'_{e1}|}{|U'_{e2}|} = \tan \theta'_{12},
\]

\[
\frac{|U'_{\mu 3}|}{|U'_{\tau 3}|} = \tan \theta'_{23},
\]

\[
|U'_{e3}| = \sin \theta'_{13}.
\]

For degenerate neutrinos, \( M_3 - M_1 \approx M_3 - M_2 \gg M_2 - M_1 \), because \( \Delta_{31} \approx \Delta_{32} \gg \Delta_{21} \). Thus, from the above set of equations, we see that \( U'_{e1} \) and \( U'_{e2} \) are much larger than \( U'_{e3}, U'_{\mu 3} \) and \( U'_{\tau 3} \). Hence we can expect much larger change in \( \theta_{12} \) compared to \( \theta_{13} \) and \( \theta_{23} \)\[10\]. As one can see from the above expression of mixing angle due to Planck scale effects, depends on new contribution of mixing \( U' = U(1 + i \delta \theta) \).

3. Neutrino Mixing and Cosmological Constant Due to Planck Scale Effects

The connection between the vacuum energy density \( \langle \rho_{\text{vac}} \rangle \) and the cosmology constant \( \Lambda \) is provided by the well known relation

\[
\langle \rho_{\text{vac}} \rangle = \frac{\Lambda}{4\pi G},
\]

where \( G \) is the gravitational constant.

The expression of vacuum energy density \( \langle \rho_{\text{vac}}^{\text{mix}} \rangle \) due to neutrino mixing is given by \[11,12,13\]

\[
\langle \rho_{\text{vac}}^{\text{mix}} \rangle = 32\pi^2 \sin^2 \theta_{12} \int dk K^2 (\omega_{k,1} + \omega_{k,2})|V_k|^2,
\]

If we chose \( K \gg \sqrt{m_1 m_2} \), we obtain
\[ < \rho_{\text{mix}}^{\text{vac}} > \propto \sin^2 \theta_{12} (m_2 - m_1)^2 K^2 \int_0^k dk k^2 (\omega_{k,1} + \omega_{k,2}) |V_K|^2, \]  

(18)

For hierarchical neutrino model, for which \( m_2 > m_1 \), we have in this case \( K \gg \sqrt{m_1 m_2} \) and take into account the asymptotic properties of \( V_k \)

\[ |V_k|^2 \simeq \frac{(m_2 - m_1)^2}{4K^2}, \quad K \gg \sqrt{m_1 m_2} \]

We get

\[ < \rho_{\text{mix}}^{\text{vac}} > \propto \sin^2 \theta_{12} (m_2 - m_1)^2 \propto \frac{\Lambda}{4\pi G}, \]  

(19)

The new cosmological constant \( \Lambda \) due to Planck scale effects is given by

\[ \Lambda' = \propto \sin^2 \theta'_{12} (m_2' - m_1') \]  

(20)

where \( \theta'_{12} \) is given by eq(13)

We consider the Planck scale effects on neutrino mixing and we get the given range of mixing parameter of MNS matrix

\[ U' = R(\theta_{23} + \epsilon_3)U_{\text{phase}}(\delta)R(\theta_{13} + \epsilon_2)R(\theta_{12} + \epsilon_1). \]  

(21)

In Planck scale, only \( \theta_{12}(\epsilon_1 = \pm 3^o) \) have resonable deviation and \( \theta_{23}, \theta_{13} \) deviation is very small less than \( 0.3^o \)[10]. In the new mixing at Planck scale we get the cosmological density

\[ \Lambda' = \propto \sin^2 (\theta_{12} \pm \epsilon_1)(m_2'^2 - m_1'^2), \]  

(22)

The presence of a cosmological constant fluid has to be compatible with the structure formation, allow to set the upper bound \( \Lambda < 10^{-56} cm^{-2} \)[14]. Due to Planck scale effects mixing angle \( \theta_{12} \) deviated the cosmological constant \( \Lambda \).

Conclusions

We assume that the main part of neutrino masses and mixing from GUT scale operator. We considered these to be 0th order quanties. We further assume that GUT scale symmetry constrain the neutrino mixing angles to be bimaximal. The gravitational interaction of lepton field with S.M Higgs field give rise to a \( SU(2)_L \times U(1) \) invariant dimension-5 effective Lagrangian give originally by Weinberg [15]. On electroweak symmetry breaking this operators leads to additional mass terms. We considered these to be perturbation of GUT scale mass terms. We compute the first order correction to neutrino mass eigen value and mixing angles. In [10], it was shown that the change in \( \theta_{13}, \theta_{23} \) is very small (less then \( 0.3^o \))but the change in \( \theta_{12} \) can be substantial about \( \pm 3^o \). The change in all
the mixing angle are proportional to the neutrino mass eigenvalues. To maximizer the change, we assumed degenerate neutrino mass $2.0eV$. For degenerate neutrino masses, the change in $\theta_{13}, \theta_{23}$ are inversely proportional to $\Delta_{21}$. Since $\Delta_{31} \approx \Delta_{32} \gg \Delta_{21}$, the change in $\theta_{12}$ is much larger than the change in other mixing angle. In this paper, we write the cosmological constant above GUT scale in term of mixing angle for Majorana neutrinos, these expression in eq(x) for vacuum mixing. For Majorana neutrino, the expression is $\Lambda = \sin^2 \theta'_{12} (m_2^2 - m_1^2)$. In this paper, finally we wish make a important comment. Due to Planck scale effects mixing angle $\theta_{12}$ deviated the cosmological constant $\Lambda$.

References

The Restricted Three Body Problem with Quadratic Drag

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Abstract: When an asteroid, space-craft or another small object in the solar system is in the vicinity of a planet it is subjected to the gravitational forces of the Sun, the planet, the drag forces due to the solar wind and (possibly) the planet upper atmosphere. To determine the object trajectory we consider this problem within the context of the restricted three body problem in three dimensions with quadratic drag. In this setting we linearize the equations of motion of the object and cast them in a coordinate system with respect to the secondary (planet) which is assumed to be in a general Keplerian orbit around the primary (Sun). We then reduce them, to a simple system of three second order linear differential equations. These equations can be considered to be a generalization of Hill’s equations to general Keplerian orbits (of the secondary) with the addition of quadratic drag force acting on the third object in the system. We derive also ”approximate conservation laws” in three dimensions which represent a generalization of Jacobi’s integral in two dimensions and consider some special cases.

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1. Introduction

The present paper is being motivated by the emerging realization that many ”small” objects in the solar system whose size vary from few millimeters to few kilometers have orbits around the Sun which criss-cross the orbit of the various planets in the system. Although the smaller objects in this group might not pose any danger it is important to compute accurately their trajectories and the prospects of their capture when they are close enough to a planet. We note that it is not feasible (nor important from this point of view) to calculate these trajectories when they are far away from the planet as they

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are subjected then to many "small" perturbations.

From another point of view, the trajectory of a spacecraft sent from Earth to one of the planets should be the subject of similar considerations as it nears its destination.

It is obvious that this problem should be treated within the context of the restricted three-body problem (since the object mass is small). However in addition to the gravitational forces of the primary and secondary the object will be subjected to drag forces. The impact of these drag forces on the trajectory of these objects becomes more important as their size decreases. In the literature, drag forces on such objects have been modeled using various functional forms. Murray [11] used a simple drag force proportional to the particle velocity. On the other hand, Burns and Jackson et al [1, 8] derived and used a more general expression for this force which included radiation pressure and Poynting-Robertson drag. Elipe [2] considered this problem with "generalized" expressions for the drag forces. In this paper, we model this force as one which is proportional to the square of the velocity and in its direction. The choice of this expression is motivated by the fact we want to consider the trajectories of these particles for short duration of time near the secondary. Under this assumption, the prominent drag is provided by the solar wind and (possibly) the upper part of the planet atmosphere. Both of these contributions are modeled usually by a quadratic drag force. However, our treatment actually applies with minor modifications whenever the size of the drag force can be expressed as a function of the particle position and velocity and its direction is along the particle velocity. We observe that with the inclusion of a drag force, the system is energy dissipative. As a result, this problem cannot be treated by a Hamiltonian formalism.

It should be noted however that if Earth is the secondary, additional forces due to the \( J_2 \) effect and the moon have to be taken into consideration. Our treatment applies in the generic cases where these forces are weaker or absent.

To put this problem within a broader context, we note that for over a century the three-body problem has been one of the outstanding problems in Celestial mechanics with continuous contributions from numerous authors. (We mention here only a few seminal references [9, 12, 13, 14, 15]. For an extensive list of contributions see [9, 14].) Recently, however, there has been additional interest in this difficult problem and some new results were derived for some very special cases [10, 3, 5]. Similarly, the motion of satellites in the Earth atmosphere where they are subjected to the gravitational force of the Earth and a drag force had received wide attentions in the current literature [4, 7]. With this motivational background, it seems justifiable to treat the three-body problem with drag from a general mathematical and physical points of view.

It is our objective in this paper to consider the equations of the restricted three-body problem with the addition of quadratic drag force. In this setting, we derive reduced (approximate) equations for the trajectory of the object in a coordinates system centered at the secondary (planet) which is assumed to be in a general Keplerian orbit around the primary. The use of these coordinates (rather than barycentric coordinates) is due to their practical aspects as they obviate the need to "translate" the trajectory to a coordinate system attached to an observer on the planet. Thus these coordinates provide a natural framework to address the problem at hand. We shall show also that they lead to some major simplifications to the equations of motion and a new approximate expression for the Jacobi integral in three dimensions.
The plan of the paper is as follows: In Sec 2 we present the basic equations for the restricted three body problem with a quadratic drag term. The resulting equations of motion are linearized then under proper assumptions and recast in a coordinate system which is centered in the secondary. In Sec 3 these equations are simplified and reduced further. In Sec 4 we discuss adiabatic conservation laws and their relation to the Jacobi invariant [15]. Sec 5 considers some special cases and we derive simplified equations of motions and solutions under these assumptions. We also discuss the results of several simulations of the reduced equations and the impact of drag on the adiabatic invariants of motion. We end up in Sec 6 with summary and conclusions.

2. Basic Equations

In an inertial coordinate system whose origin is at the center of the central body (the primary) \( S \) we let \( R, \rho \) denote the positions of the secondary and the third object respectively. (We assume that, approximately, the center of mass of the primary coincides with the center of mass of the three bodies. Actually, this assumption is not necessary but we make it to simplify the presentation). Furthermore let \( r \) denote the relative position of the satellite with respect to the secondary. We then have (see Fig 1) \( \rho = R + r \) and the equation of motion of the third body is given by

\[
m_s \ddot{\rho} = - \frac{GM_S m_s}{\rho^3} \rho - \frac{GM_E m_s}{r^3} r - m_s g(\alpha, r)(\dot{r} \cdot \dot{r})^{1/2} \dot{r}.
\]  

(1)

In this equation \( \rho = |\rho|, \ r = |r|, \) and \( M_S, M_E, m_s \) denote respectively the masses of the primary (Sun), the secondary and the third body. \( G \) is the constant of gravity and dots denote differentiation with respect to time. The last term in this equation represents the drag which is being modeled, as a quadratic function of the satellite velocity with respect to secondary and \( \alpha \) is the drag coefficient (the factor \( m_s \) in front of \( g \) was added for convenience).

Throughout the paper we assume that \( r \ll R \) i.e the third body is in the vicinity of the secondary. Obviously this assumption restricts the validity of our results. However as noted in the introduction when this assumption is invalid there are too many other perturbations that affect the trajectory of the third object.

Assuming that \( r \ll R \) and using:

\[
\rho = (\rho \cdot \rho)^{1/2} = (R + r, R + r)^{1/2} = R \left[ 1 + \frac{2R \cdot r}{R^2} + \frac{r^2}{R^2} \right]^{1/2}
\]  

(2)

we can (by Taylor expansion) make the following approximation to eq. (1) [19]

\[
\dot{R} + \dot{r} = -\frac{GM_S}{R^3} \left[ R + r - \frac{3R \cdot r}{R^2} R \right] - \frac{GM_E}{r^3} r - g(\alpha, r)(\dot{r} \cdot \dot{r})^{1/2} \dot{r}.
\]  

(3)

Using the fact that at any point

\[
\dot{R} = -\frac{GM_S}{R^3} R
\]  

(4)

eq (3) reduces to

\[
\dot{r} = -\frac{GM_S}{R^3} \left[ r - \frac{3R \cdot r}{R^2} R \right] - \frac{GM_E}{r^3} r - g(\alpha, r)(\dot{r} \cdot \dot{r})^{1/2} \dot{r}.
\]  

(5)
Since the gravitational field of the third object is negligible it follows that the motion of the secondary is governed only by the gravitational field of $S$. As a result the motion of the secondary is in a fixed plane which we take as the $x - y$ plane. (Hence in polar coordinates the position of the secondary is given by $(R, \theta)$). From the law of angular momentum conservation we have

$$M_E R^2 \omega = L = \text{constant.} \quad (6)$$

where $\Omega$ is the angular velocity vector and $\Omega = \frac{d\theta}{dt} \mathbf{e}_z = (0, 0, \omega)$. Using the relation (6) to eliminate $R$ from eq. (5) we obtain

$$\ddot{\mathbf{r}} = -\frac{GM_S M_E^{3/2}}{L^{3/2}} \omega^{3/2} \left[ \mathbf{r} - \frac{3 \mathbf{R} \cdot \mathbf{r}}{R^2} \mathbf{R} \right] - \frac{GM_E}{r^3} \mathbf{r} - g(\alpha, \mathbf{r})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{1/2} \dot{\mathbf{r}}. \quad (7)$$

In a coordinate system rotating with the secondary (and fixed at its center) eq. (7) becomes:

$$\ddot{\mathbf{r}} + 2 \mathbf{\Omega} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \frac{d\mathbf{\Omega}}{dt} \times \mathbf{r} = k \Omega^2/2 \left[ \mathbf{r} - \frac{3 \mathbf{R} \cdot \mathbf{r}}{R^2} \mathbf{R} \right] - \frac{GM_E}{r^3} \mathbf{r} \quad (8)$$

where $k = \frac{GM_S M_E^{3/2}}{L^{3/2}}$. Here one should distinguish between $\mathbf{r}$ in eq. (7) which is a geocentric position vector with respect to an inertial frame while $\mathbf{r}$ in eq. (8) is the position vector in a rotating frame.

In this rotating system we now let the $x$-axis to be tangential but opposed to the motion of the secondary, the $y$-axis in the direction of $\mathbf{R}$ and the $z$-axis completes a right-handed system. (See Fig 2). In this frame $\mathbf{r} = (x, y, z)$, $\mathbf{R} = (0, 1, 0)R$ and $\mathbf{\Omega} = (0, 0, \dot{\theta}) = (0, 0, \omega)$. In component form eq. (8) then becomes;

$$\ddot{x} - 2\omega y - \omega^2 x - \omega y = -k \omega^{3/2} x - \frac{GM_E x}{r^3} - g(\alpha, \mathbf{r})(\dot{x} \cdot \dot{\mathbf{r}})^{1/2} (\dot{x} - \omega y) \quad (9)$$

$$\ddot{y} + 2\omega \dot{x} - \omega^2 y + \omega x = -k \omega^{3/2} y + 3k \omega^{3/2} y - \frac{GM_E y}{r^3} - g(\alpha, \mathbf{r})(\dot{y} \cdot \dot{\mathbf{r}})^{1/2} (\dot{y} + \omega x) \quad (10)$$

$$\ddot{z} = -k \omega^{3/2} z - \frac{GM_E z}{r^3} - g(\alpha, \mathbf{r})(\dot{z} \cdot \dot{\mathbf{r}})^{1/2} z \quad (11)$$

We now perform a change of variables from $t$ to $\theta(t)$ in these equations and divide the resulting equations by $\omega$. This leads to

$$\omega x'' + \omega' x' = [\omega - k \omega^{1/2}] x + \omega' y + 2\omega y' - \frac{\beta x}{r^3} - g(\alpha, \mathbf{r})\omega(\mathbf{r}' \cdot \mathbf{r}')^{1/2} [\mathbf{x}' - \mathbf{y}] \quad (12)$$

$$\omega y'' + \omega' y' = -\omega' x + [\omega + 2k \omega^{1/2}] y - 2\omega x' - \frac{\beta y}{r^3} - g(\alpha, \mathbf{r})\omega(\mathbf{r}' \cdot \mathbf{r}')^{1/2} (\mathbf{y}' + \mathbf{x}) \quad (13)$$

$$\omega z'' + \omega' z' = -k \omega^{1/2} z - \frac{\beta z}{r^3} - g(\alpha, \mathbf{r})\omega(\mathbf{r}' \cdot \mathbf{r}')^{1/2} z' \quad (14)$$

where $\beta = \frac{GM_E}{\omega}$ and primes denote derivatives with respect to $\theta$.

We now show that these equations can be simplified further by proper change of variables.
3. Derivation of the Reduced Equations

Since the secondary is in a Keplerian orbit around \( S \) we have [5]

\[
\frac{1}{R} = C(1 + e \cos \theta), \quad \omega = \frac{L C^2}{M E} (1 + e \cos \theta)^2
\]  

(15)

where \( C = \frac{G M_S M_E}{L^2} \) and \( e \) is the eccentricity of the orbit. (We place no restrictions on the value of \( e \). However in our context the secondary is in elliptic orbit around the primary.)

Substituting these expressions in eqs. (12)-(14) and dividing by \( \frac{L C^2}{M E} (1 + e \cos \theta) \) yields after some algebra:

\[
\frac{1}{(1 + e \cos \theta)^2} \left[ (1 + e \cos \theta)^2 \left( \frac{d^2}{dr^2} \right) \right] = \left( 1 + e \cos \theta - \frac{k M_E^{1/2}}{C L^{1/2}} \right) x
\]  

(16)

\[
+ 2[ (1 + e \cos \theta) y] - \frac{G M_E^3 x}{L^2 C^4 (1 + e \cos \theta)^3 r^3} - g_1(\alpha, r, \theta)(r' \cdot r')^{1/2} (x' - y)
\]

\[
\frac{1}{(1 + e \cos \theta)^2} \left[ (1 + e \cos \theta) \frac{d^2}{dy} \right] = \left( 1 + e \cos \theta + \frac{2 k M_E^{1/2}}{L^{1/2} C} \right) y
\]  

(17)

\[
- 2[ (1 + e \cos \theta) x'] - \frac{G M_E^3 y}{L^2 C^4 (1 + e \cos \theta)^3 r^3} - g_1(\alpha, r, \theta)(r' \cdot r')^{1/2} (y' + x)
\]

\[
\frac{1}{(1 + e \cos \theta)^2} \left[ (1 + e \cos \theta)^2 \frac{d^2}{dz} \right] = \frac{k M_E^{1/2}}{L^{1/2} C} z - \frac{G M_E^3 z}{L^2 C^4 (1 + e \cos \theta)^3 r^3} - g_1(\alpha, r, \theta)(r' \cdot r')^{1/2} z
\]  

(18)

Where \( g_1(\alpha, r, \theta) = \frac{g(\alpha, r) \omega M_E}{L C^2 (1 + e \cos \theta)} \).

We now introduce the variables

\[
u = (1 + e \cos \theta) x, \quad v = (1 + e \cos \theta) y, \quad w = (1 + e \cos \theta) z
\]  

(19)

Substituting these variables in eqs. (16)-(18) leads to;

\[
u'' = \frac{u}{1 + e \cos \theta} \left[ 1 - \frac{k M_E^{1/2}}{C L^{1/2}} \right] + 2v' - \frac{Av}{(1 + e \cos \theta) \sigma^3} - g_2(\alpha, r, \theta)(r' \cdot r')^{1/2} \left[ u' - v + \frac{ue \sin \theta}{(1 + e \cos \theta)} \right]
\]  

(20)

\[
v'' = \frac{v}{1 + e \cos \theta} \left[ 1 + 2 \frac{k M_E^{1/2}}{C L^{1/2}} \right] - 2u' - \frac{Av}{(1 + e \cos \theta) \sigma^3} - g_2(\alpha, r, \theta)(r' \cdot r')^{1/2} \left[ v' + u + \frac{ve \sin \theta}{(1 + e \cos \theta)} \right]
\]  

(21)

\[
w'' = -w - \frac{Aw}{(1 + e \cos \theta) \sigma^3} - g_2(\alpha, r, \theta)(r' \cdot r')^{1/2} \left[ w' + \frac{we \sin \theta}{(1 + e \cos \theta)} \right]
\]  

(22)
where \( \sigma = (u,v,w) \), \( A = \frac{GM^3}{L^2C^4} \) and \( g_2(\alpha,r,\theta) = \frac{g(\alpha,r)\omega M_E}{L C^2 (1 + e \cos \theta)^2} \). However it is now easy to see from the definitions of \( k, L, C \) and \( \omega \) that

\[
\frac{kM^4}{CL^{1/2}} = 1, \quad g_2(\alpha,r,\theta) = g(\alpha,r)
\]  

and hence the equations of motion reduce to;

\[
u'' = 2v' - \frac{Au}{(1 + e \cos \theta)^{3/2}} - g(\alpha,r)(r' \cdot r')^{1/2} \left[ u' - v + \frac{ue \sin \theta}{(1 + e \cos \theta)} \right]
\]  

\[
v'' = \frac{3v}{1 + e \cos \theta} - 2u' - \frac{Av}{(1 + e \cos \theta)^{3/2}} - g(\alpha,r)(r' \cdot r')^{1/2} \left[ v' + u + \frac{ve \sin \theta}{(1 + e \cos \theta)} \right]
\]  

\[
w'' = -w - \frac{Aw}{(1 + e \cos \theta)^{3/2}} - g(\alpha,r)(r' \cdot r')^{1/2} \left[ w' + \frac{we \sin \theta}{(1 + e \cos \theta)} \right]
\]  

Eqs. (24)-(26) represent a generalization of Hill’s equations, to include eccentricity in the orbit of the secondary and quadratic drag force acting on the third object in the system subject to the approximation made in eq. (2). In the past the three body problem has been treated extensively in the literature under various approximations (See e.g. [3, 6]). However we believe that our derivation of these equations in rendezvous coordinates (and arbitrary eccentricity) near the secondary with drag did not appear in the literature so far.

When \( g(\alpha,r) = 0 \) these equations resemble closely those that were derived for the rendezvous of a spacecraft and a satellite in [3, 5]. Furthermore in spite of the fact that these equations are nonlinear they are easily amenable to numerical computations. We observe also that if \( A << 1 \) and \( g(\alpha,r) << 1 \) then it is straightforward to obtain approximate solutions to these equations by first order perturbation expansion. (Since the solution of these equations with \( A = 0 \) and \( g(\alpha,r) = 0 \) is well known [7]).

4. Conservation Law

As we noted in the Introduction the addition of the drag term to the restricted three body problem renders this system to be dissipative. This precludes any exact conservation laws for the energy and angular momentum of the system. In spite of this fact it is useful (e.g as a check on the validity of numerical integration schemes) to derive equations for the evolution of these or related quantities under these circumstances. These equations are important for the treatment of weak drag forces and solutions of systems in three dimensions. (See next section).

To derive these "approximate conservation laws" for the angular momentum \( J \) and energy \( E \) that govern the motion of the third body we take the vector and scalar product of eq. (1) with \( \rho \) and \( \dot{\rho} \). After some simple algebraic manipulations we obtain

\[
\frac{dJ}{dt} = \frac{d}{dt}(\rho \times \dot{\rho}) = -\frac{GM_E}{r^3} R \times r - g(\alpha,r)(\dot{r} \cdot r)^{1/2}(R + r) \times \dot{r}
\]

\[
\frac{dE}{dt} = \frac{d}{dt}\left( \frac{\rho^2}{2} - \frac{GM_S}{\rho} - \frac{GM_E}{r} \right) = -\frac{GM_E}{r^3} R \cdot r - g(\alpha,r)(\dot{r} \cdot r)^{1/2}(\dot{\rho} \cdot \dot{r})
\]
These two equations show clearly that in general neither the angular momentum nor the energy of the third body are constant.

To derive the corresponding equation for the rate of change of the energy (with respect to $\theta$) from the reduced equations we multiply eqs. (24)-(26) by $u', v', w'$ and sum. The result can be expressed in the form;

$$\left\{ (\sigma')^2 - \frac{3v^2 + 2A}{1 + e \cos \theta} + w^2 \right\}' = - \left[ 3v^2 + \frac{2A}{\sigma} \right] \frac{e \sin \theta}{(1 + e \cos \theta)^2} - 2g(\alpha, r)(r' \cdot r')^{1/2} \left[ (\sigma')^2 + (uv' - vu') + \frac{e \sin \theta}{1 + e \cos \theta} \left( \frac{\sigma^2}{2} \right)' \right]$$

(29)

We see from this representation that if $e = 0$ then the rate of change (with respect to $\theta$) in the quantity on the left hand side of eq. (29) will be due solely to the dissipative effects of the drag. Furthermore if in addition $g(\alpha, \rho) = 0$ then eq. (29) represents an exact conservation law. However as this conservation law was derived from the approximate equations of motion of the third body (i.e. eq. (5)) it is also an approximate invariant of the exact equations of motion. Eq. (29) presents therefore the modifications to the Jacobi integral when one considers the linearized restricted three body problem in three dimensions with quadratic drag near the secondary. When $e$ and $g$ are small the left hand side of eq. (29) can be considered as an "adiabatic invariant" of the motion.

We can derive an expression for the rate of change of a related quantity if we multiply eqs. (24),(25) by $v$ and $u$ respectively and subtract. This leads to;

$$\left[ (uv' - vu') + (u^2 + v^2) \right]' = \frac{3uv}{(1 + e \cos \theta)} - g(\alpha, r)(r' \cdot r')^{1/2} [ (uv' - vu') + (v^2 + u^2) ]$$

(30)

This equation will be important when one considers the restricted three body system in two dimension.

5. Some Special Cases

In this section we consider some special cases where the equations of motion of the third body can be reduced further. The presentation is in order of difficulty. From the simplest to the most general case.

5.1 Two Dimensional Case with $e = 0$ and $g(\alpha, r) = 0$

To begin with we consider the two dimensional case where all the three bodies are in $x - y$ plane, the secondary is in a circular orbit around the primary and there is no drag.

Under these conditions eq.(29) is an exact invariant and eqs.(29)- (30) reduce to a system of two differential equations for $u, v$.

$$(u')^2 + (v')^2 - 3v^2 - \frac{2A}{\sigma} = \text{constant.}$$

(31)

$$\left[ (uv' - vu') + (u^2 + v^2) \right]' = 3uv$$

(32)
To simplify these equations we introduce
\[ u = \sigma \cos \phi, \quad v = \sigma \sin \phi \] (33)

Eqs. (31),(32) then become
\[ (\sigma')^2 + \sigma^2(\phi')^2 - \frac{2A}{\sigma} = 3\sigma^2 \sin^2 \phi + c_1 \] (34)
\[ [\sigma^2(\phi' + 1)]' = \frac{3}{2}\sigma^2 \sin(2\phi) \] (35)

where \( c_1 \) is a constant. Many solutions of Hill's equations under these restrictions appeared in the literature [25,27]. We note that eqs. (34),(35) admit a consistent solution with
\[ \sigma = \text{constant}, \quad (\phi')^2 = 3\sin^2 \phi + c \]

where \( c \) is a constant. However this is not a valid solution of eqs. (24),(25) since under these assumptions eqs.(31),(32) are not independent. In fact it is well known that Hill's problem does not admit "circular" \( \sigma = \text{const.} \) solution. However we can interpret this solution as one corresponding to the case where \( \sigma \) is almost constant viz \( |\sigma'| \ll 1 \).

We can reduce eqs. (34),(35) to a first order system by changing the independent variable from \( \theta \) to \( \phi \) and introduce \( p = \frac{d\phi}{d\theta} \) as a new dependent variable. After some algebra we obtain:
\[ p^2 \left[ \left( \frac{d\sigma}{d\phi} \right)^2 + \sigma^2 \right] - \frac{2A}{\sigma} = 3\sigma^2 \sin^2 \phi + c_1 \] (36)
\[ p \frac{d}{d\phi} [\sigma^2(p + 1)] = \frac{3}{2}\sigma^2 \sin(2\phi) \] (37)

where \( c_1 \) is a constant. These equations provide a formulation for the orbit of the satellite in terms of its local coordinates with respect to the secondary only.

5.2 Two Dimensional Case with \( e = 0 \)

Note that under these assumptions \( \sigma = r \) and eqs. (29)-(30) reduce to
\[ \left[ (u')^2 + (v')^2 - 3v^2 - \frac{2A}{\sigma} \right]' = -2g(\alpha, \sigma)(\sigma' \cdot \sigma')^{1/2} \left[ (uv' - vu') + (u^2 + v^2) \right] \] (38)
\[ \left[ (uv' - vu') + (u^2 + v^2) \right]' = 3uv - g(\alpha, \sigma)(\sigma' \cdot \sigma')^{1/2} \left[ (uv' - vu') + (u^2 + v^2) \right] \] (39)

Using eq. (33) we obtain
\[ \left[ (\sigma')^2 + \sigma^2(\phi')^2 - 3\sigma^2 \sin^2(\phi) - \frac{2A}{\sigma} \right]' = -2g(\alpha, \sigma)(\sigma' \cdot \sigma')^{1/2} \sigma^2(\phi' + 1) \] (40)
\[ [\sigma^2(\phi' + 1)]' = \frac{3}{2}\sigma^2 \sin(2\phi) - g(\alpha, \sigma)(\sigma' \cdot \sigma')^{1/2} [\sigma^2(\phi' + 1)] \] (41)

We can rewrite eq.(41) in the form
\[ \frac{\sigma'}{\sigma^2} = \frac{\frac{3}{2}\sin(2\phi) - \phi''}{\phi' + 1} - g(\alpha, \sigma)(\sigma' \cdot \sigma')^{1/2} \] (42)
This equation demonstrates the role that the drag term can play in many cases, especially when the first term on the right hand side of this equation is small. Under these circumstances the drag term can determine whether $\sigma^2$ (that is the distance between the secondary and the third object in the system) will increase or decrease and as a result change the long term nature of the third object trajectory.

Eliminating the drag forces from (41) and (40) we obtain

$$\left( (\sigma')^2 + \sigma^2(\phi')^2 - \frac{2A}{\sigma} - 2\sigma^2(\phi' + 1) \right) = 6\sigma\sigma' \sin^2 \phi$$

This implies that if $|\sigma'| \ll 1$ (that is the orbit of the third body is almost circular) then the left hand side represents an adiabatic invariant of the system.

5.3 Three Dimensional Case with $e \neq 0$

The relevant equations in this case are (24)-(26). Due to their complexity we can solve these (coupled) equations analytically only through the use of first order perturbation expansion. To this end we observe that from a practical point of view both $A$ and $g$ are small. Therefore we write

$$u = u_0 + \epsilon u_1, \quad v = v_0 + \epsilon v_1, \quad w = w_0 + \epsilon w_1.$$  (44)

Substituting these in (24)-(26) and neglecting terms containing $A\epsilon$ and $g(\alpha, r)\epsilon$ as being of second order we obtain to the zeroth order in $\epsilon$

$$u_0'' = 2v_0', \quad v_0'' = \frac{3v_0}{1 + \epsilon \cos \theta} - 2u_0', \quad w_0'' = -w_0.$$  (45)

A solution of the (resulting) equation for $v$ is

$$v(\theta) = C \sin \theta (1 + \epsilon \cos \theta)$$  (46)

Using this solution it is possible to write down explicitly the general solution to (45) (see Ref 16 for details). The first order equations in $\epsilon$ are the same as (24)-(26) except that in the terms containing $A$ and $g$, the variables $u, v, w$ should be replaced by $u_0, v_0, w_0$ while in the other terms these variables should be replaced by $u_1, v_1, w_1$. The general solution of this system of equations can be done then using standard methods for the solution of inhomogeneous system of linear equations and will not be presented here for brevity.

5.4 Model Validation

The equations of motion for the third body that were derived in the previous sections are highly nonlinear and therefore we have to resort to numerical methods to gauge their validity. (Actually it is well known that the three body problem, even without drag, has only few exact analytical solutions in very special cases [5,6]).

To evaluate the effect of the drag term on the trajectory of the third object in the system we simulated (for comparison purposes, using MATLAB) eqs. (24), (25) in two dimensions with $e = 0$ and $A = 1$. The initial values used were $u(0) = 15, v(0) =$
0, \( u'(0) = 0 \) and \( v'(0) = 0 \). For this setting the values of \( g \) were varied from \( g = 0 \) to \( g = 10^{-4} \) in steps of \( 2.5 \times 10^5 \) (the relative error in each integration step was set to \( 10^{-8} \)). Fig 3. presents the difference in the values of \( \sigma \) for the four orbits with drag from its value for the orbit without drag.

For each of these orbits we computed also the left hand side of eq.(38) along the orbit viz.

\[
J = (\sigma')^2 - 3v^2 - \frac{2A}{\sigma}
\]  

(47)

Fig 4. shows the difference between the values of \( J \) along the orbits with drag from its value for the orbit without drag. (For the orbit without drag \( J \) is a conserved quantity and its value in this case is \( -0.1333 \)). We note the similarity between Figs 3, 4 although they have different scales.

We compared also the (long term) trajectories predicted by our approximate equations eqs (24)-(26) with those that are obtained by direct numerical simulation of the exact equations of motion (1) (with \( g = 0 \)). To this end we assumed that the secondary is in circular orbit around the primary with \( R = 1 \text{AU} \). The initial conditions on the third object correspond to those that will let this object stay in circular orbit around the secondary if the primary was absent with \( r = 0.002 \text{AU} \). In these computations we set the step error tolerance to \( 10^{-12} \) for a total of \( 10^5 \) time steps. The difference in the values of \( r \) along the trajectories that follows from these two sets of equations for 100 days is presented in Fig 5. As expected this difference grows with time but the error throughout this time period remains bounded by \( 3 \times 10^{-7} \text{AU} \). This confirms the accuracy of eqs. (24)-(26).

Summary and Conclusions

In this paper we discussed the three body problem under two approximations. The first approximation results from neglecting the influence of the third body on the trajectory of the secondary. The second is due to the linearization of the expression for the gravitational force under the assumption \( r \ll R \). This approximation becomes asymptotically exact as \( r/R \to 0 \) i.e. we are assuming that the third object is close to the secondary. This is exactly the physical configuration of the system we wish to consider. However these equations and approximations will obviously become invalid when these assumptions are violated.

The results of this paper show that due to an "accidental" coincidence among the various constants that govern the restricted three body system (in three dimensions) the approximate equations of motion can be simplified analytically. As we pointed out in the last section this leads to further simplifications in some limiting cases. An adiabatic conservation law was derived for these equations which can be considered as a natural extension of the (exact) Jacobi integral for this problem in two dimensions. We demonstrated also that in some cases the trajectory of the third body can be computed by solving a reduced system of two nonlinear first order differential equations.

From another practical point of view the existence of many small objects in the solar system is still unknown. To gauge their potential for a possible collision (or fly by) with a planet it is important to include all known sources that might impact their trajectory
(as they near the planet). In this paper we derived proper equations for the inclusion of drag forces on the trajectory of these objects. We demonstrated also (analytically and by simulation) that these effects are important for the computation of the correct trajectory for these objects.

Finally we note in passing that a similar treatment can be made when the drag forces are linear in the third object velocity which might be important in some celestial contexts. A complete treatment of this case is available from the author.

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References


Fig. 1 A diagrammatic presentation of the three body problem discussed in this paper. We assume that $m_n \ll M_E \ll M_S$ and $r \ll R$.

Fig. 2 The frame attached to $M_E$ which is rotating anti-clockwise around $M_S$. 
Fig. 3 The differences $\sigma(t)-\sigma_0(t)$ for the trajectories with drag from $\sigma_0(t)$ which is the trajectory without drag with $\sigma_0(0) = 15$. All trajectories have the same initial condition. The values of $g$ used were $2.5 \times 10^{-5}$, $5 \times 10^{-5}$, $7.5 \times 10^{-5}$ and $10^{-4}$. These are represented respectively, by the solid, dashes, dashed-dot and dotted lines.

Fig. 4 The difference in the values of $J$ for trajectories with drag from the one without drag. The constant value of $J$ along the trajectory without drag is represented by the zero line. (The actual value of $J$ in this case is $-0.1333$). Same values for $g$ as in Fig 3.
Fig. 5 The difference in the values of $r(t)$ which are obtained from the numerical integration of eq. (1) and eqs. (24)-(26).