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Relativistic Effects on Quantum Bell States of Massive Spin $\frac{1}{2}$ Particles

J. P. Singh *

Department of Management Studies, Indian Institute of Technology Roorkee, Roorkee 247667, India

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Abstract: We examine the behaviour of the maximally entangled Bell state of two spin $\frac{1}{2}$ massive particles under relativistic transformations. On the basis of explicit calculations of the Wigner rotation and the use of transformation properties of Dirac spinors, we establish that the constituent particles of the Bell state undergo momentum dependent rotation of the spin orientations characterized by the Wigner angle $\phi_W = \tan^{-1} \frac{\sinh \varpi \sinh \tau}{\cosh \varpi + \cosh \tau}$. However, since local unitarity is retained in the process, the corresponding entanglement fidelity is not lost.

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1. Introduction

Potential applications of information encoded into the states of quantum systems are finding their way not only into revolutionizing computation but, more importantly, in achieving communication tasks with unprecedented efficiency e.g. quantum teleportation, entanglement enhanced communication, quantum clock synchronization and reference frame alignment, quantum enhanced global positioning etc. These applications warrant long distance signal transmission through quantum mechanical states. As such, in these functional areas of quantum information processing, relativistic effects could significantly contribute to the dynamics.

Even for quantum computing operations like quantum cryptography, error detection etc., it would be desirable to study relativistic implications since they may result in optimal processes and/or superior algorithms.

At a fundamental level, we also need to reconcile quantum information theory with

* jpsiitr@gmail.com, jatinfdm@iitr.ernet.in, jatinder_pal2000@yahoo.com

general relativity and other contemporary quantum theories like the quantum field theory, loop quantum gravity and string theory to examine where it fits in our overall quest for unification. Some of the literature that report work done in the direction of relativistic quantum information is listed in the references [1-40].

2. Prerequisites [41-43]

2.1 Unitary Representations of the Poincare Group

We start by briefly elucidating the salient features of the theory of unitary representations of the Poincare group as propounded by Wigner in his seminal article [41]. It is well known that the unitary representations of the Lorentz group $\left\{\hat{L}\left(\Lambda\right)\right\}$ and the four dimensional translational group $\left\{\hat{T}\left(a\right)\right\}$ (that together constitute the Poincare group) satisfy

$$\hat{T}\left(a\right)\hat{T}\left(b\right) = \hat{T}\left(a+b\right) \tag{1}$$

$$\hat{L}\left(\Lambda\right)\hat{L}\left(\Lambda'\right) = \hat{L}\left(\Lambda\Lambda'\right) \tag{2}$$

$$\hat{L}\left(\Lambda\right)\hat{T}\left(a\right) = \hat{T}\left(\Lambda a\right)\hat{L}\left(\Lambda\right) \tag{3}$$

The above equations constitute the defining equations for obtaining the unitary representations of the Poincare group. Some properties of the representation theory of the Poincare group are summarized below for use in the sequel. Proofs are accessible in any text on the subject e.g.[42,43].

(a) The parameter space of the Poincare group is spanned by a ten parameter basis, four of which relate to the four dimensional translation group $T_4 \equiv \left\{\hat{T}\left(a\right)\right\}$ (representing translations in the four directions of relativistic spacetime) and the remaining six relate to the Lorentz group $L \equiv \left\{\hat{L}\left(\Lambda\right)\right\}$ (three of which represent spatial rotations and the other three symbolize "boosts" i.e. spatiotemporal rotations along the three spatial axes).

(b) The parameter space corresponding to T_4 is the four dimensional Euclidean space that is simply connected so that irreducible representations (IRRs) of T_4 are single valued.

(c) Any element of $L \equiv \left\{\hat{L}\left(\Lambda\right)\right\}$ can be written as $L\left(\Lambda\right)^{\mu}_{\nu} = R^{\mu}_{\rho}B^{\rho}_{\sigma}\left(\tau\right)R'^{\sigma}_{\nu}$ where $\hat{R}$& $\hat{R}'$ are spatial rotations and $\hat{B}\left(\tau\right)$ is a Lorentz boost in the direction of the first spatial axis. Now, the parameter space of $\hat{B}\left(\tau\right)$ is a straight line. It is unidimensional and simply connected so that the IRRs are single valued. On the other hand, the rotation group admits both single valued and double valued IRRs. It follows that the unitary representations of the Poincare group could be either single valued or, at most, double valued depending on the character of the representation of the constituent rotation group.

(d) Infinitesimal Lorentz transformations can be written in terms of the generators as $\hat{L}\left(\varepsilon,\delta\right) = \hat{I} + i\varepsilon\hat{J} - i\delta\hat{K}$ where $\hat{J}$ & $\hat{K}$ are operators (generators of spatial rotations and boosts respectively) that satisfy the relations

$$\left[\hat{J}_i, \hat{J}_j\right] = i\sum_{k=1}^{3} \varepsilon_{ijk}\hat{J}_k \quad (4)$$

$$\left[\hat{J}_i, \hat{K}_j\right] = i\sum_{k=1}^{3} \varepsilon_{ijk}\hat{K}_k \quad (5)$$

$$\left[\hat{K}_i, \hat{K}_j\right] = -i\sum_{k=1}^{3} \varepsilon_{ijk}\hat{J}_k \quad (6)$$

For unitary representations, the operators $\hat{J}$ & $\hat{K}$ must necessarily be hermitian.

(e) Infinitesimal translations can be written in terms of the generators as $\hat{T}(a) = \hat{I} + ia^{\mu}\hat{P}_{\mu}$,$\mu = 0, 1, 2, 3$ where $\hat{P}_{\mu}$ are operators (generators of translations) that satisfy the relations

$$\left[\hat{P}^0, \hat{J}_n\right] = 0 \quad (7)$$

$$\left[\hat{P}_m, \hat{J}_n\right] = i\varepsilon^{mnl}\hat{P}_l \quad (8)$$

$$\left[\hat{P}_m, \hat{K}_n\right] = i\delta_{mn}\hat{P}^0 \quad (9)$$

$$\left[\hat{P}^0, \hat{K}_n\right] = i\hat{P}_n \quad (10)$$

Finite translations are, as usual, expressed in terms of the generators by exponentiation as

$$\hat{T}(a) = e^{i\hat{P}_{\mu}a^{\mu}} \quad (11)$$

2.2 Casimir Operators and Classification of IRRs

The operator $\hat{C}_1 = \hat{P}^{\mu}\hat{P}_{\mu} = \hat{P}_0^2 - \hat{P}^2$ commutes with all the generators of the Lie algebra of the Poincare group and hence constitutes a Casimir operator of the group. Hence, $\hat{C}_1 = \hat{P}^{\mu}\hat{P}_{\mu} = \hat{P}_0^2 - \hat{P}^2$ is invariant for an IRR. IRRs of the Poincare group can, therefore, be labeled by the eigenvalues of $\hat{C}_1 = \hat{P}^{\mu}\hat{P}_{\mu} = \hat{P}_0^2 - \hat{P}^2$. Noting that the generators of spatial translations are realized as the "momentum" operators and the generator of time translation as the "energy operator", we have, in terms of the respective eigenvalues, $c_1 = p^{\mu}p_{\mu} = p_0^2 - p^2 = M^2$ where M is the particle mass. Since the eigenvalues c_1 are not positive definite in the case of the Poincare group, we classify the IRRs as follows:-

(1) $[L]$ Null vector case corresponding to $c_1 = 0$, $p^0 = p = 0$
(2) $[M_{\pm}]$ Time like case corresponding $c_1 > 0$
(3) $[0_{\pm}]$ Light like case corresponding to $c_1 = 0$, $p \neq 0$
(4) $[T]$ Space like case corresponding to $c_1 < 0$.

We shall restrict ourselves to the time like case corresponding to particles with finite mass since that is relevant to the context. However, the treatment for the other cases follows on similar lines.

In establishing the Casimir nature of $\hat{C}_1$, we have simply used eqs. (7-10). However, eqs. (4-6) and hence, eq. (2) has not been considered. It follows that c_1 alone is not sufficient to completely identify the IRR of the Poincare group. Eq. (2) is the defining property of the homogeneous Lorentz group that is known to possess a Casimir operator$\hat{J}^2 - \hat{K}^2$. Therefore, involvement of eq. (2) in the analysis should lead to a second Casimir operator which we symbolize (for the time being) by $\hat{C}_2$ with the eigenvalue c_2. We can completely identify an IRR by these two eigenvalues c_1,c_2 and the energy sign $\varepsilon(p_0) = \frac{p_0}{|p_0|}$ since the energy sign also commutes with all the generators of the Poincare group corresponding to the classes of representations $[M_\pm]$ and $[0_\pm]$.

The second Casimir operator of the Poincare group is known as the Pauli-Lubanski vector $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ where $\hat{W} \equiv \left(\hat{W}^0, \hat{W}\right) = \left(\hat{P}.\hat{J}, \hat{P}^0\hat{J} - \hat{P} \times \hat{K}\right)$ that has the following properties:-

$$\hat{W}^\lambda \hat{P}_\lambda = 0 \tag{12}$$

$$\left[\hat{W}^\lambda, \hat{P}^\mu\right] = 0 \tag{13}$$

$$\left[\hat{W}^\lambda, \hat{J}^{\mu\nu}\right] = i\left(\hat{W}^\mu g^{\lambda\nu} - \hat{W}^\nu g^{\mu\lambda}\right) \tag{14}$$

$$\left[\hat{W}^\lambda, \hat{W}^\sigma\right] = i\varepsilon^{\lambda\sigma\mu\nu}\hat{W}_\mu \hat{P}_\nu \tag{15}$$

The fact that $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is a Casimir operator and commutes with all the generators of the group follows from

(i) Each component of $\hat{W}$is translationally invariant for $\left[\hat{W}^\lambda, \hat{P}^\mu\right] = 0$. Hence $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is also translationally invariant;

(ii) Further $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is the scalar product of a $4-$vector with itself. Hence, it is the square of the length of a 4vector which makes it invariant under homogeneous Lorentz transformations.

Thus, the massive class of representations of the Poincare group has two Casimir operators that are left invariant by group operations plus the energy sign. Hence, we label out the IRRs of the Poincare group by two indices corresponding to the eigenvalues of the two Casimir operators viz. $c_1 \equiv M^2$ and $\frac{1}{p^2}c_2 \equiv \frac{1}{p^2}W^2$ that is identified with the spin s in the case of massive particles $(M > 0)$. For a given IRR labeled by the eigenvalues of the two Casimir operators viz. M^2 and $\frac{1}{p^2}W^2$, the sign of the energy $\varepsilon(p_0)$ commutes with all the infinitesimal generators of the Poincare group. There are, thus, two IRRs for each combination of values of M^2 and $\frac{1}{p^2}W^2$, one for each sign of $\varepsilon(p_0)$.

2.3 Basis Vectors in an IRR Space

We, now, address the issue of the labeling of basis vectors within a given IRR. For the purpose, we note that the four dimensional translation group T_4 constitutes an invariant subgroup of the Poincare group. We also know that the basis vectors in a representation of the translation group can be labeled by the eigenvalues of the generators of the translation group. However, the specification of a basis vector using a single index corresponding to the generators of translations is incomplete. For a complete specification of the basis vectors, we need a second index that relates to the generators of "rotations" and is usually taken to be the eigenvalue of $\hat{J}_3$. A complete specification of a basis vector in the representation space $(M, s, \varepsilon(p_0))$ would, thus, consist of four indices, together with the energy sign $\varepsilon(p_0)$, two of them related to the eigenvalues of Casimir operators $\hat{C}_1$ & $\hat{C}_2$of the Poincare group and the energy sign $\varepsilon(p_0)$ for identifying the representation itself and the other two (p, m) that relate the eigenvalues of $\hat{P}_\mu$& $\hat{J}_3$ for specifying a basis vector within a representation. Now, all the basis vectors in a given representation space identified by $(M, s, \varepsilon(p_0))$ correspond to the same eigenvalue $c_1 \equiv M^2$. We also have the relativistic mass-energy relation $p_0 = \pm\sqrt{p^2 + M^2}$. It follows that we can as well use the eigenvalues p of 3-momentum in lieu of the eigenvalues $p \equiv (p_0, p)$ of 4-momentum for specification of the basis vectors.

2.4 Little Group Decomposition of the Poincare Group

It can be shown that the independent components of $\hat{W}$ are proportional to the generators of the group $SO(3)$ for, given a basis vector $|p, m\rangle$ in the $(M, s, \varepsilon(p_0))$representation of the Poincare group, we have (since these basis vectors are eigenstates of generators of translations $\hat{P}^\mu$), $\hat{P}^\mu|p, m\rangle = p^\mu|p, m\rangle$ where p^μ is the eigenvalue of $\hat{P}^\mu$ so that $\hat{W}^\lambda|p, m\rangle = \varepsilon^{\lambda\mu\nu\sigma}\hat{J}_{\mu\nu}\hat{P}_\sigma|p, m\rangle = \varepsilon^{\lambda\mu\nu\sigma}\hat{J}_{\mu\nu}p_\sigma|p, m\rangle$. For the standard vector, $p^0 = M$ and $p = 0$ whence $\hat{W}^0|0, m\rangle = \varepsilon^{0\mu\nu\sigma}\hat{J}_{\mu\nu}\hat{P}_\sigma|0, m\rangle = 0$ and $\hat{W}^i|0, m\rangle = \varepsilon^{i\mu\nu\sigma}\hat{J}_{\mu\nu}\hat{P}_\sigma|0, m\rangle = \frac{1}{2}\varepsilon^{ijk}\hat{J}_{jk}p^0|0, m\rangle = \frac{1}{2}\varepsilon^{ijk}\hat{J}_{jk}M|0, m\rangle$ which establishes our result.

Using eqs. (3-11), we obtain

$$e^{i\left[\hat{L}(\Lambda)\hat{P}_\mu\hat{L}(\Lambda)^{-1}\right]a^\mu} = \hat{L}(\Lambda)\, e^{i\hat{P}_\mu a^\mu}\hat{L}(\Lambda)^{-1} = \hat{L}(\Lambda)\,\hat{T}(a)\,\hat{L}(\Lambda)^{-1} = \hat{T}(\Lambda a) = e^{i\hat{P}_\nu \Lambda^\nu{}_\mu a^\mu}$$

whence we get the transformation rules for the covariant generators of translations as

$$\hat{L}(\Lambda)\,\hat{P}_\mu\hat{L}(\Lambda)^{-1} = \hat{P}_\nu\Lambda^\nu_\mu \quad \text{or equivalently} \quad \hat{L}(\Lambda)^{-1}\,\hat{P}_\mu\hat{L}(\Lambda) = \hat{P}_\nu\Lambda^\nu_\mu \tag{16}$$

Given a basis vector $|p, m\rangle$ in a representation $(M, s, \varepsilon(p_0))$, we have, on using eq.(16)

$\hat{P}_\mu\hat{L}(\Lambda)|p, m\rangle = \hat{L}(\Lambda)\,\hat{P}_\nu\Lambda^\nu{}_\mu|p, m\rangle = p_\nu\Lambda^\nu{}_\mu\hat{L}(\Lambda)|p, m\rangle$ thereby showing that $\hat{L}(\Lambda)|p, m\rangle$ is also an eigenvector of $\hat{P}_\mu$ with the eigenvalue $p_\nu\Lambda^\nu{}_\mu$. Given two basis vectors, in a representation space $|p, m\rangle$,$|p', m'\rangle$of the same energy sign, $\varepsilon(p_0)$, we can define an inner product by

$$\langle p, m \mid p', m' \rangle = \delta_{mm'} \omega_p \delta\left(p - p'\right) \tag{17}$$

where $\omega_p = \sqrt{p^2 + M^2}$. The Lorentz invariance of the integration element $\omega_k \delta\left(p - p'\right)$ is well known. The completeness of the space implies

$$\sum_m \int \frac{d\mathbf{p}}{\omega_p} |\mathbf{p}, m\rangle \langle \mathbf{p}, m| = 1 \tag{18}$$

Let us define the operator

$$\hat{S}\left(\Lambda\right) \quad \text{by} \quad \hat{S}\left(\Lambda\right) |p, m\rangle = |\Lambda p, m\rangle \tag{19}$$

$\hat{S}\left(\Lambda\right)$ is unitary because it leaves the inner product (17) invariant for

$$\langle \Lambda p', m' \mid \Lambda p, m \rangle = \langle p', m' | \hat{S}^{\dagger} \hat{S} | p, m \rangle = \delta_{mm'} \omega_{\Lambda p} \delta\left(\Lambda p - \Lambda p'\right)$$

$$= \delta_{mm'} \omega_p \delta\left(p - p'\right) = \langle p', m' \mid p, m \rangle \tag{20}$$

where the penultimate step follows from the Lorentz invariance of $\omega_k \delta\left(p - p'\right)$. Further

$$\hat{S}\left(\Lambda\right) \hat{S}\left(\Lambda'\right) = \hat{S}\left(\Lambda\Lambda'\right) \tag{21}$$

implying that $\hat{S}\left(\Lambda\right)$ is a unitary representation of the Lorentz group. Also being independent of m, it is diagonal with respect to the corresponding generator e.g. $\hat{J}_3$.

We also have$\hat{S}\left(\Lambda\right) \hat{T}\left(\Lambda^{-1} a\right) |p, m\rangle = \hat{S}\left(\Lambda\right) e^{i\hat{P}_\mu \left(\Lambda^{-1} a\right)^\mu} |p, m\rangle = \hat{S}\left(\Lambda\right) e^{ip_\mu \left(\Lambda^{-1} a\right)^\mu} |p, m\rangle$ $= e^{ip_\mu \left(\Lambda^{-1} a\right)^\mu} |\Lambda p, m\rangle = e^{i\left(\Lambda^{-1}\hat{P}\right)_\mu \left(\Lambda^{-1} a\right)^\mu} |\Lambda p, m\rangle = \hat{T}\left(a\right) |\Lambda p, m\rangle = \hat{T}\left(a\right) \hat{S}\left(\Lambda\right) |p, m\rangle$ whence, because of the completeness of the set of basis vectors, we can infer that

$$\hat{T}\left(\Lambda^{-1} a\right) \hat{S}\left(\Lambda\right)^{-1} = \hat{S}\left(\Lambda\right)^{-1} \hat{T}\left(a\right) \tag{22}$$

From eq. (3), we also infer that $\hat{L}\left(\Lambda\right) \hat{T}\left(a\right) = \hat{T}\left(\Lambda a\right) \hat{L}\left(\Lambda\right)$ so that $\hat{T}\left(\Lambda^{-1} a\right) \hat{L}\left(\Lambda\right)^{-1} = \hat{L}\left(\Lambda\right)^{-1} \hat{T}\left(a\right)$ which, together with eq. (22) gives $\left[\hat{L}\left(\Lambda\right) \hat{S}\left(\Lambda\right)^{-1}, \hat{T}\left(a\right)\right] = 0$. Since this equation holds for arbitrary a, it follows that $\hat{L}\left(\Lambda\right) \hat{S}\left(\Lambda\right)^{-1}$ commutes with the translation operators. This implies that it is diagonal with respect to the generators of translation $\hat{P}_\mu$. Writing it as $\hat{Q}\left(\Lambda, \hat{P}\right)$, we obtain

$$\hat{L}\left(\Lambda\right) = \hat{Q}\left(\Lambda, \hat{P}\right) \hat{S}\left(\Lambda\right) \tag{23}$$

Further, since both $\hat{L}\left(\Lambda\right)$, $\hat{S}\left(\Lambda\right)$ are unitary, it follows that $\hat{Q}\left(\Lambda, \hat{P}\right)$ is also unitary.

Now $\hat{Q}\left(\Lambda\Lambda', \hat{P}\right) \hat{S}\left(\Lambda\Lambda'\right) |p, m\rangle = \hat{L}\left(\Lambda\Lambda'\right) |p, m\rangle = \hat{L}\left(\Lambda\right) \hat{L}\left(\Lambda'\right) |p, m\rangle$

$$= \hat{Q}\left(\Lambda, \hat{P}\right) \hat{S}\left(\Lambda\right) \hat{Q}\left(\Lambda', \hat{P}\right) \hat{S}\left(\Lambda'\right) |p, m\rangle = \hat{Q}\left(\Lambda, \hat{P}\right) \hat{S}\left(\Lambda\right) \hat{Q}\left(\Lambda', \hat{P}\right) |\Lambda' p, m\rangle$$

$$= \hat{Q}\left(\Lambda, \hat{P}\right) \hat{S}\left(\Lambda\right) \hat{Q}\left(\Lambda', \Lambda' p\right) |\Lambda' p, m\rangle = \hat{Q}\left(\Lambda, \hat{P}\right) \hat{Q}\left(\Lambda', \Lambda' p\right) \hat{S}\left(\Lambda\right) |\Lambda' p, m\rangle$$

$$= \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda', \Lambda' p\right)|\Lambda\Lambda' p, m\rangle = \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda', \Lambda^{-1}\hat{P}\right)|\Lambda\Lambda' p, m\rangle$$

$$= \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda', \Lambda^{-1}\hat{P}\right)\hat{S}\left(\Lambda\Lambda'\right)|p, m\rangle$$

whence

$$\hat{Q}\left(\Lambda\Lambda', \hat{P}\right) = \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda', \Lambda^{-1}\hat{P}\right) \tag{24}$$

Using eq. (24), we obtain

$$|p, m\rangle = \hat{L}\left(I\right)|p, m\rangle = \hat{L}\left(\Lambda\Lambda^{-1}\right)|p, m\rangle = \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda^{-1}, \Lambda^{-1}\hat{P}\right)\hat{S}\left(\Lambda\Lambda^{-1}\right)|p, m\rangle$$

$$= \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda^{-1}, \Lambda^{-1}\hat{P}\right)|p, m\rangle = \hat{Q}\left(\Lambda, \hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)^{-1}|p, m\rangle$$

so that

$$\hat{Q}\left(\Lambda, \hat{P}\right)^{-1} = \hat{Q}\left(\Lambda^{-1}, \Lambda^{-1}\hat{P}\right) \tag{25}$$

Let us assume that there exists an operator $\hat{U}\left(\hat{P}\right)$ in the irreducible representation space that satisfies (i) unitarity, (ii) is diagonal with respect to $\hat{P}_\mu$, (iii) is single valued in $\hat{P}_\mu$ (so that it preserves the single valuedness of the state operator $|p, m\rangle$with respect to $\hat{P}_\mu$). Under the action of $\hat{U}\left(\hat{P}\right)$ (i) the translation operator $\hat{T}\left(a\right)$is not affected since $\left[\hat{T}\left(a\right), \hat{U}\left(\hat{P}\right)\right] = 0$, (ii) the Lorentz transformation $\hat{L}\left(\Lambda\right)$is transformed as $\hat{U}\left(\hat{P}\right)\hat{L}\left(\Lambda\right)\hat{U}\left(\hat{P}\right)^{-1}$. We have

$$\begin{aligned} &\hat{U}\left(\hat{P}\right)\hat{L}\left(\Lambda\right)\hat{U}\left(\hat{P}\right)^{-1}|p, m\rangle = \hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)\hat{S}\left(\Lambda\right)\hat{U}\left(\hat{P}\right)^{-1}|p, m\rangle \\ &= \hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)U\left(p\right)^{-1}\hat{S}\left(\Lambda\right)|p, m\rangle \\ &= \hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)U\left(p\right)^{-1}|\Lambda p, m\rangle \\ &= \hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)\hat{U}\left(\Lambda^{-1}\hat{P}\right)^{-1}|\Lambda p, m\rangle \\ &= \hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)\hat{U}\left(\Lambda^{-1}\hat{P}\right)^{-1}\hat{S}\left(\Lambda\right)|p, m\rangle \end{aligned} \tag{26}$$

It follows from eq. (26) that $\hat{Q}\left(\Lambda, \hat{P}\right)$ & $\hat{U}\left(\hat{P}\right)\hat{Q}\left(\Lambda, \hat{P}\right)\hat{U}\left(\Lambda^{-1}\hat{P}\right)^{-1}$ provide equivalent representations for given representations of $\hat{T}\left(a\right)$ & $\hat{S}\left(\Lambda\right)$.

To proceed further with the representation theory of the Poincare group, it is necessary at this point to introduce the concept of "little group". As mentioned above, the IRRs of the Poincare group are classified into timelike, null, lightlike and spacelike on the basis of the nature of the eigenvalues of the Casimir operator $\hat{P}^2$ and then by the sign of the energy eigenvalue. Thus, given an IRR, the corresponding identifying eigenvalue of $\hat{P}^2$ would be classified into one of the classes on the basis of being timelike, null, lightlike or spacelike with a sub classification within this on the basis of the energy sign. In other words, for a given IRR, the operator $\hat{P}^2$ and its eigenvalue $c_1 \equiv M^2$ would fall into one of $[M]$, $[0]$, $[L]$ or $[T]$ whereas the operator $\hat{P}_\mu$ and its associated eigenvalue p_μ would be classified into one of $[M_+]$,$[M_-]$, $[0_+]$, $[0_-]$, $[L]$ or $[T]$. In view of $\hat{P}_\mu \hat{L}(\Lambda)|p,m\rangle = p_\nu \Lambda^\nu{}_\mu \hat{L}(\Lambda)|p,m\rangle$, the eigenvalues p_μ within a class are connected with each other by a suitable Lorentz transformation. It is to be noted that in the case of class $[T]$ of representations we cannot subclassify the representations on the basis of the energy sign because the energy eigenvalue p_0 can change sign within the class itself by an appropriate Lorentz transformation. Physically, the class$[T]$ corresponds to imaginary mass $p^2 \equiv M^2 < 0$ which implies that the energy $p_0^2 = p^2 + M^2$ corresponding to such a representation could be made arbitrarily large negatively by an appropriate Lorentz transformation. Additionally, this representation also does not have a well defined and reasonable nonrelativistic limit. The class $[L]$ corresponds to the case of zero momentum and energy.

Let us denote by $\{p\}$, the set of eigenvalues p_μ of a particular class. Let us identify an element q_μ arbitrarily from the set $\{p\}$. As mentioned above, the various eigenvalues in $\{p\}$ are related to each other through suitable Lorentz transformations. Let $\{\eta\}$ be the set of Lorentz transformations that leave q_μ invariant i.e.

$$\eta^\mu_\nu q^\nu = q^\mu \quad \text{or} \quad \eta q = q \tag{27}$$

$\{\eta\}$ is a subset of the Lorentz group and is called a "little group" of the Lorentz group. Now, since the elements of $\{p\}$ are connected inter se by Lorentz transformation, given a $p_\mu \in \{p\}$, there would exist a Lorentz transformation κ_p such that

$$p = \kappa_p q \tag{28}$$

Corresponding to an arbitrary element $q_\mu \in \{p\}$, the Lorentz transformation $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p}$ is an element of the little group $\{\eta\}$ for, we have

$$\eta_p q = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p} q = \kappa_p^{-1}\Lambda\Lambda^{-1}p = \kappa_p^{-1}p = q \tag{29}$$

where we have used eq. (28). Using eqs. (24) & (29), we obtain

$$\begin{aligned}\hat{Q}\left(\Lambda,\hat{P}\right) &= \hat{Q}\left(\kappa_p\eta_p\kappa^{-1}_{\Lambda^{-1}p},\hat{P}\right) = \hat{Q}\left(\kappa_p,\hat{P}\right)\hat{Q}\left(\eta_p\kappa^{-1}_{\Lambda^{-1}p},\kappa_p^{-1}\hat{P}\right)\\ &= \hat{Q}\left(\kappa_p,\hat{P}\right)\hat{Q}\left(\eta_p,\kappa_p^{-1}\hat{P}\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p},\eta_p^{-1}\kappa_p^{-1}\hat{P}\right)\end{aligned} \tag{30}$$

Now, operating the right hand side of eq. (29) on the state $|p,m\rangle$, we obtain

$$\hat{Q}\left(\kappa_p,\hat{P}\right)\hat{Q}\left(\eta_p,\kappa_p^{-1}\hat{P}\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p},\eta_p^{-1}\kappa_p^{-1}\hat{P}\right)|p,m\rangle$$
$$= \hat{Q}\left(\kappa_p,\hat{P}\right)\hat{Q}\left(\eta_p,\kappa_p^{-1}\hat{P}\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p},\eta_p^{-1}\kappa_p^{-1}p\right)|p,m\rangle$$
$$= \hat{Q}\left(\kappa_p,\hat{P}\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p},\eta_p^{-1}\kappa_p^{-1}p\right)\hat{Q}\left(\eta_p,\kappa_p^{-1}\hat{P}\right)|p,m\rangle s$$

$= \hat{Q}\left(\kappa_p, \hat{P}\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p}, \eta_p^{-1}\kappa_p^{-1}p\right)\hat{Q}\left(\eta_p, \kappa_p^{-1}p\right)|p, m\rangle$
$= \hat{Q}\left(\kappa_p, \hat{P}\right)\hat{Q}\left(\eta_p, q\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p}, \eta_p^{-1}q\right)|p, m\rangle = \hat{Q}\left(\kappa_p, \hat{P}\right)\hat{Q}\left(\eta_p, q\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p}, q\right)|p, m\rangle$

whence

$$\hat{Q}\left(\Lambda, \hat{P}\right) = \hat{Q}\left(\kappa_p, \hat{P}\right)\hat{Q}\left(\eta_p, q\right)\hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p}, q\right) \tag{31}$$

Eq. (31) is identical in form to the expression $\hat{U}\left(\hat{P}\right)^{-1}\hat{Q}\left(\eta_p, q\right)\hat{U}\left(\Lambda^{-1}\hat{P}\right)$ if we identify $\hat{U}\left(\hat{P}\right)^{-1} = \hat{Q}\left(\kappa_p, \hat{P}\right)$ or $\hat{U}\left(\hat{P}\right) = \hat{Q}\left(\kappa_p, \hat{P}\right)^{-1} = \hat{Q}\left(\kappa_p^{-1}, \kappa_p^{-1}\hat{P}\right) = \hat{Q}\left(\kappa_p^{-1}, q\right)$ and
$\hat{U}\left(\Lambda^{-1}\hat{P}\right) = \hat{Q}\left(\kappa^{-1}_{\Lambda^{-1}p}, q\right)$. Since $\hat{U}\left(\hat{P}\right)^{-1}$ is unitary, it follows that $\hat{U}\left(\hat{P}\right)$ is unitary as well. It, therefore, follows from eq. (31) that $\hat{Q}\left(\eta_p, q\right)$ generates a representation of the Poincare group that is unitarily equivalent to $\hat{Q}\left(\Lambda, \hat{P}\right)$. Hence, in lieu of eq. (23), we can write the equivalent expression $\hat{L}\left(\Lambda\right) = \hat{Q}\left(\eta_p, q\right)\hat{S}\left(\Lambda\right)$ without any loss of generality. Further, using eq. (25), we get

$$\hat{Q}\left(\kappa_p, \hat{P}\right)^{-1} = \hat{Q}\left(\kappa_p^{-1}, \kappa_p^{-1}\hat{P}\right) = \hat{Q}\left(\kappa_p^{-1}, q\right) \tag{32}$$

Using eq. (24), we can also write $\hat{Q}\left(\eta_p\eta'_p, \hat{P}\right) = \hat{Q}\left(\eta_p, \hat{P}\right)\hat{Q}\left(\eta'_p, \eta_p^{-1}\hat{P}\right)$ and $\hat{Q}\left(\eta_p\eta'_p, q\right) = \hat{Q}\left(\eta_p, q\right)\hat{Q}\left(\eta'_p, \eta_p^{-1}q\right) = \hat{Q}\left(\eta_p, q\right)\hat{Q}\left(\eta'_p, q\right)$ where $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p}$ and $\eta'_p = \kappa_p^{-1}\Lambda'\kappa_{\Lambda'^{-1}p}$. This shows that $\hat{Q}\left(\eta_p, q\right)$ is a unitary representation of the relevant little group $\hat{Q}\left(\Lambda, \hat{P}\right)$. The operator $\hat{Q}\left(\eta_p, q\right)$is called the Wigner rotation.

2.5 Little Group and Wigner Rotation Corresponding to Massive Particles

Now, for the study of the representations corresponding to the massive $M_\pm$ and the identification of the corresponding "little group" let us define the "standard vector" as $q = \left(\varepsilon\left(p_0\right)M, 0, 0, 0\right)$, We have identified the standard vector in the "rest frame" so that all the spatial components of momentum are zero. The factor group of the Poincare group with respect to the subgroup of translations in four dimensions, T_4, (which is abelian) is the homogeneous Lorentz group. The maximal subgroup of the homogeneous Lorentz group that leaves $q = \left(\varepsilon\left(p_0\right)M, 0, 0, 0\right)$ invariant is the group of three dimensional rotations, identifiable with $SO\left(3\right)$ which, thus, constitutes the "little group" for this class of representations.

This may also be seen by invoking the fact that the generators of the little group of the standard vector e.g. $q = \left(\varepsilon\left(p_0\right)M, 0, 0, 0\right)$ are the independent components of the corresponding Pauli Lubanski vector $\hat{W} \equiv \left(\hat{W}^0, \hat{W}\right) = \frac{1}{2}\varepsilon^{\lambda\mu\nu\sigma}\hat{J}_{\mu\nu}\hat{P}_\sigma = \left(\hat{P}.\hat{J}, \hat{P}^0\hat{J} - \hat{P}\times\hat{K}\right)$ corresponding to the standard vector $q = \left(\varepsilon\left(p_0\right)M, 0, 0, 0\right)$. The second Casimir operator for massive particles is, therefore, $\hat{W}^{\mu}\hat{W}_{\mu} = W.W = \frac{1}{M^2}\hat{J}^2$. The corresponding Lie algebra takes the form $\left[\hat{W}^i, \hat{W}^j\right] = i\varepsilon^{ijk0}\hat{W}_k\hat{P}_0$ or equivalently $\left[\hat{J}^i, \hat{J}^j\right] = i\varepsilon^{ijk}\hat{J}_k$ which is the Lie algebra of the group of rotations in three dimensional space thereby confirming that the little group corresponding to massive representations is $SO\,(3)$.

As is common in the literature on relativity, we can segregate Lorentz transformation in two types with distinctly identifiable features viz. (i) the θ transformations that constitute spatial rotations about the three spatial axes respectively and do not involve any mixing of spatial and temporal coordinates and (ii) the τ transformations that involve Lorentz boosts along the three spatial axes that are in the nature of spatiotemporal rotations. The coordinate transformation equations under these transformations can be summarized as

$$\left.\begin{array}{c} \mathbf{x} \rightarrow \mathbf{x} + \mathbf{x} \times \theta \\ x^0 \rightarrow x^0 \end{array}\right\} \theta \text{ tranformation} \tag{33}$$

$$\left.\begin{array}{c} \mathbf{x} \rightarrow \mathbf{x} - \tau x^0 \\ x^0 \rightarrow x^0 - \tau\mathbf{x} \end{array}\right\} \tau \text{ tranformation} \tag{34}$$

We shall, now, obtain explicit expressions for the Wigner rotation for each of these transformations. Corresponding to $q = (\varepsilon\,(p_0)\,M, 0, 0, 0)$, the Lorentz transformation that satisfies the equation $p = \kappa_p q$ takes the explicit form

$$\kappa_p = \begin{pmatrix} \omega_p/M & \pm p_1/M & \pm p_2/M & \pm p_3/M \\ \pm p_1/M & 1 + \varpi_p p_1^2 & \varpi_p p_1 p_2 & \varpi_p p_1 p_3 \\ \pm p_2/M & \varpi_p p_2 p_1 & 1 + \varpi_p p_2^2 & \varpi_p p_2 p_3 \\ \pm p_3/M & \varpi_p p_3 p_1 & \varpi_p p_3 p_2 & 1 + \varpi_p p_3^2 \end{pmatrix} \tag{35}$$

where $\varpi_p = \frac{1}{p^2}\left(\frac{\omega_p}{M} - 1\right)$. In terms of the rapidity,$\varsigma = \cosh^{-1}\frac{\omega_p}{M} = \sinh^{-1}\frac{|p|}{M}$, eq. (35) may be written as $(\kappa_p)^i_j = \delta^i_j + \frac{p_i p_j}{p^2}(\cosh\varsigma - 1)$, $(\kappa_p)^i_0 = (\kappa_p)^0_i = \frac{p_i}{|p|}\sinh\varsigma$ and $(\kappa_p)^0_0 = \cosh\varsigma$.

Using this expression we can calculate the Wigner rotation $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p}$. Now, it is obvious from eq. (29) that $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p}$ is a member of the "little group" which in this case $(M_\pm)$ constitutes the group of three dimensional rotations. The action of $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p}$ on a four vector will, therefore, leave the time component invariant and we can write for $x \equiv (x^0, x)$, $\eta_p x = \eta_p\,(x^0, x) = (x^0, x + x \times \zeta)$

where $\zeta$is the rotation induced by the element $\eta_p = \kappa_p^{-1}\Lambda\kappa_{\Lambda^{-1}p} \in SO\,(3)$. For a θtransformation $\zeta = \theta$ and for a τtransformation $\zeta = \pm\frac{p\times\tau}{M+\omega_p}$ if we consider only first order terms in the angles.

2.6 Transformation rules for the Momentum - Spin Eigenstate Basis

The Hilbert space H of a massive particle can be represented as a direct sum of the eigenspaces H_p of the momentum operator i.e. $H = \oplus_{p \in \mathbb{R}^3} H_p$. The subspace that relates to the rest frame of the massive particle has the following properties:-

(i) it is invariant with respect to rotations for $\hat{P}\left(\hat{I} - i\hat{J}\theta\right)|0\rangle = \hat{P}|0\rangle - i\hat{P}\hat{J}\theta|0\rangle = 0$ showing that the rotated vector has also zero momentum and hence belongs to H_0. In arriving at this result, we have used the commutation relation $\left[\hat{P}_m, \hat{J}_n\right] = i\varepsilon^{mnl}\hat{P}_l$.

(ii) If we define the relativistic spin operator for positive energy massive particles as

$$\hat{S} = \frac{1}{M}\hat{P}^0\hat{J} - \frac{1}{M}\hat{P} \times \hat{K} - \frac{1}{\left(\hat{P}_0 + M\right)M}\hat{P}\left(\hat{P}.\hat{J}\right) \tag{36}$$

then for H_0, $\hat{S} \equiv \hat{J}$ for $\hat{S}_3|0\rangle = \left[\frac{1}{M}\hat{P}^0\hat{J}_3 - \frac{1}{M}\left(\hat{P}_1\hat{K}_2 - \hat{P}_2\hat{K}_1\right) - \frac{1}{\left(\hat{P}_0+M\right)M}\hat{P}_3\left(\hat{P}.\hat{J}\right)\right]|0\rangle$

$$= \left[\frac{1}{M}\hat{J}_3\hat{P}^0 - \frac{1}{M}\left(\hat{K}_2\hat{P}_1 - \hat{K}_1\hat{P}_2\right) - \frac{1}{\left(\hat{P}_0 + M\right)M}\left(\hat{P}.\hat{J}\right)\hat{P}_3\right]|0\rangle = \frac{1}{M}\hat{J}_3\hat{P}^0|0\rangle = \hat{J}_3|0\rangle \tag{37}$$

where we have used the commutators $\left[\hat{P}_0, \hat{J}_n\right] = 0$, $\left[\hat{P}_m, \hat{K}_n\right] = i\delta_{mn}\hat{P}^0$ and the fact that for a zero momentum positive energy particle $p_0 = M$.

(iii) A basis can be constructed in H_0 consisting of the eigenvectors of $\hat{S}_3$ and labeled by the eigenvalue of $\hat{S}_3$ e.g. $|0,\sigma\rangle$ where $\hat{S}_3|0,\sigma\rangle = \sigma|0,\sigma\rangle$. The action of the rotation operator on these basis vectors is

$$e^{-i\hat{J}\phi}|0,\sigma\rangle = e^{-i\hat{S}\phi}|0,\sigma\rangle = \sum_{\sigma'=-s}^{s} D^s_{\sigma'\sigma}(\phi)|0,\sigma'\rangle \tag{38}$$

where the D^s are $(2s+1) \times (2s+1)$ rotation matrices.

Our next step is to construct the basis for the remaining subspaces H_p with $p \neq 0$. The basis vectors $|0,\sigma\rangle$ of H_0 can be transformed by pure boost transformations to generate basis vectors $|p,\sigma\rangle$ of H_p. The appropriate boost transformation that does the trick is $\lambda_p \equiv e^{-i\hat{K}\theta_p} \approx \left(\hat{I} - i\hat{K}\theta_p\right)$ where $\theta_p = \frac{p}{|p|}\sinh^{-1}\frac{|p|}{M}$ so that, as a first order approximation $\theta_p \approx \frac{p}{M}$. Hence, $|p,\sigma\rangle = N(p)\lambda_p|0,\sigma\rangle = N(p)e^{-i\hat{K}\theta_p}|0,\sigma\rangle \approx N(p)\left(\hat{I} - i\hat{K}\theta_p\right)|0,\sigma\rangle$ with $N(p)$ being a normalization factor. We then, have,

$$\hat{P}|p,\sigma\rangle = N(p)\hat{P}e^{-i\hat{K}\theta_p}|0,\sigma\rangle = N(p)e^{-i\hat{K}\theta_p}e^{+i\hat{K}\theta_p}\hat{P}e^{-i\hat{K}\theta_p}|0,\sigma\rangle$$
$$\approx N(p)\left(\hat{I} - i\hat{K}\theta_p\right)\left[\left(\hat{I} + i\hat{K}\theta_p\right)\hat{P}\left(\hat{I} - i\hat{K}\theta_p\right)\right]|0,\sigma\rangle$$
$$\approx N(p)\left(\hat{I} - i\hat{K}\theta_p\right)\left[\hat{P} + i\left[\hat{K}\theta_p, \hat{P}\right]\right]|0,\sigma\rangle = N(p)\left(\hat{I} - i\hat{K}\theta_p\right)\left[\hat{P} + \hat{P}^0\theta_p\right]|0,\sigma\rangle$$
$$= N(p)\left(\hat{I} - i\hat{K}\theta_p\right)M\theta_p|0,\sigma\rangle \approx N(p)\left(\hat{I} - i\hat{K}\theta_p\right)p|0,\sigma\rangle = pN(p)\left(\hat{I} - i\hat{K}\theta_p\right)|0,\sigma\rangle = p|p,\sigma\rangle \tag{39}$$

confirming that $|p,\sigma\rangle$ is an eigenstate of the momentum operator with momentum p.

Now, consider

$\hat{S}_3|p,\sigma\rangle = N(p)\,\hat{S}_3 e^{-i\hat{K}\theta_p}|0,\sigma\rangle = N(p)\,e^{-i\hat{K}\theta_p}e^{+i\hat{K}\theta_p}\hat{S}_3 e^{-i\hat{K}\theta_p}|0,\sigma\rangle$

$\approx N(p)\left(\hat{I}-i\hat{K}\theta_p\right)\left[\left(\hat{I}+i\hat{K}\theta_p\right)\hat{S}_3\left(\hat{I}-i\hat{K}\theta_p\right)\right]|0,\sigma\rangle$

$\approx N(p)\left(\hat{I}-i\hat{K}\theta_p\right)\left(\hat{S}_3+i\left[\hat{K}\theta_p,\hat{S}_3\right]\right)|0,\sigma\rangle$

We have $\hat{S}_3|0,\sigma\rangle = \sigma|0,\sigma\rangle$ by definition of the basis vector, since it is labeled by the eigenvalue of $\hat{S}_3$. To determine $\left[\hat{K}\theta_p,\hat{S}_3\right]|0,\sigma\rangle$ we make use of the explicit representation of the spin operator as

$\hat{S}_3 = \frac{1}{M}\hat{P}^0\hat{J}_3 - \frac{1}{M}\left(\hat{P}_1\hat{K}_2-\hat{P}_2\hat{K}_1\right) - \frac{1}{\left(\hat{P}_0+M\right)M}\hat{P}_3\left(\hat{P}.\hat{J}\right).$

We have,

$\left[\hat{K}_1\theta_p^1,\hat{P}_0\hat{J}_3\right]|0,\sigma\rangle = \left\{\hat{P}_0\left[\hat{K}_1\theta_p^1,\hat{J}_3\right]+\left[\hat{K}_1\theta_p^1,\hat{P}_0\right]\hat{J}_3\right\}|0,\sigma\rangle = \left(-i\hat{P}_0\hat{K}_2\theta_p^1 - i\hat{P}_1\theta_p^1\hat{J}_3\right)|0,\sigma\rangle$

$= \left(-i\hat{P}_0\hat{K}_2\theta_p^1\right)|0,\sigma\rangle = -i\left(i\hat{P}_2+\hat{K}_2\hat{P}_0\right)\theta_p^1|0,\sigma\rangle = -iM\hat{K}_2\theta_p^1|0,\sigma\rangle$

$\left[\hat{K}_2\theta_p^2,\hat{P}_0\hat{J}_3\right]|0,\sigma\rangle = \left\{\hat{P}_0\left[\hat{K}_2\theta_p^2,\hat{J}_3\right]+\left[\hat{K}_2\theta_p^2,\hat{P}_0\right]\hat{J}_3\right\}|0,\sigma\rangle = \left(i\hat{P}_0\hat{K}_1\theta_p^2 - i\hat{P}_2\theta_p^2\hat{J}_3\right)|0,\sigma\rangle$

$= \left(i\hat{P}_0\hat{K}_1\theta_p^2\right)|0,\sigma\rangle = i\left(i\hat{P}_1+\hat{K}_1\hat{P}_0\right)\theta_p^2|0,\sigma\rangle = iM\hat{K}_1\theta_p^2|0,\sigma\rangle$

$\left[\hat{K}_3\theta_p^3,\hat{P}_0\hat{J}_3\right]|0,\sigma\rangle = \left\{\hat{P}_0\left[\hat{K}_3\theta_p^3,\hat{J}_3\right]+\left[\hat{K}_3\theta_p^3,\hat{P}_0\right]\hat{J}_3\right\}|0,\sigma\rangle = \left(-i\hat{P}_3\theta_p^3\hat{J}_3\right)|0,\sigma\rangle = 0$

so that $\left[\hat{K}\theta_p,\hat{P}_0\hat{J}_3\right]|0,\sigma\rangle = iM\left(\hat{K}_1\theta_p^2-\hat{K}_2\theta_p^1\right)|0,\sigma\rangle$

$\left[\hat{K}_1\theta_p^1,\hat{P}_1\hat{K}_2\right]|0,\sigma\rangle = \left\{\hat{P}_1\left[\hat{K}_1\theta_p^1,\hat{K}_2\right]+\left[\hat{K}_1\theta_p^1,\hat{P}_1\right]\hat{K}_2\right\}|0,\sigma\rangle = \left(-i\hat{P}_1\hat{J}_3\theta_p^1 - i\hat{P}_0\theta_p^1\hat{K}_2\right)|0,\sigma\rangle$

$= \left(-i\hat{P}_0\hat{K}_2\theta_p^1\right)|0,\sigma\rangle = -i\left(i\hat{P}_2+\hat{K}_2\hat{P}_0\right)\theta_p^1|0,\sigma\rangle = -iM\hat{K}_2\theta_p^1|0,\sigma\rangle$

$\left[\hat{K}_2\theta_p^2,\hat{P}_1\hat{K}_2\right]|0,\sigma\rangle = \left\{\hat{P}_1\left[\hat{K}_2\theta_p^2,\hat{K}_2\right]+\left[\hat{K}_2\theta_p^2,\hat{P}_1\right]\hat{K}_2\right\}|0,\sigma\rangle = 0$

$\left[\hat{K}_3\theta_p^3,\hat{P}_1\hat{K}_2\right]|0,\sigma\rangle = \left\{\hat{P}_1\left[\hat{K}_3\theta_p^3,\hat{K}_2\right]+\left[\hat{K}_3\theta_p^3,\hat{P}_1\right]\hat{K}_2\right\}|0,\sigma\rangle = \left(i\hat{P}_1\hat{J}_1\theta_p^3\right)|0,\sigma\rangle = 0$

so that $\left[\hat{K}\theta_p,\hat{P}_1\hat{K}_2\right]|0,\sigma\rangle = -iM\hat{K}_2\theta_p^1|0,\sigma\rangle$. Similarly, $\left[\hat{K}\theta_p,\hat{P}_2\hat{K}_1\right]|0,\sigma\rangle = -iM\hat{K}_1\theta_p^2|0,\sigma\rangle$ and $\left[\hat{K}\theta_p,\hat{P}_3\left(\hat{P}.\hat{J}\right)\right]|0,\sigma\rangle = \left\{\hat{P}_3\left[\hat{K}\theta_p,\left(\hat{P}.\hat{J}\right)\right]+\left[\hat{K}\theta_p,\hat{P}_3\right]\left(\hat{P}.\hat{J}\right)\right\}|0,\sigma\rangle = 0.$

Putting all these pieces together, we get

$$\hat{S}_3|p,\sigma\rangle = \sigma N(p)\left(\hat{I}-i\hat{K}\theta_p\right)|0,\sigma\rangle = \sigma|p,\sigma\rangle \tag{40}$$

showing that $|p,\sigma\rangle$ is an eigenstate of $\hat{S}_3$ with eigenvalue σ.

The effect of various transformations constituting the Poincare group on the basis vectors is summarized thus:-

(a) Translations - We have

$$e^{-iPa}|p,\sigma\rangle = e^{-ipa}|p,\sigma\rangle,\ e^{i\hat{H}x^0}|p,\sigma\rangle = e^{i\omega_p x^0}|p,\sigma\rangle \tag{41}$$

with $\omega_p^2 = p^2+M^2$.

(b) Spatial Rotations – We have $e^{-i\hat{J}\phi}|p,\sigma\rangle = N(p)\,e^{-i\hat{J}\phi}e^{-i\hat{K}\theta_p}|0,\sigma\rangle$

$$= N(p)\,e^{-i\hat{J}\phi}e^{-i\hat{K}\theta_p}e^{i\hat{J}\phi}e^{-i\hat{J}\phi}|0,\sigma\rangle = N(p)\,e^{-i\left(R(\phi)^{-1}\hat{K}\right)\theta_p}\sum_{\sigma'=-s}^{s}D_{\sigma'\sigma}(\phi)\,|0,\sigma'\rangle$$

$$= N\left(p\right) e^{-i\hat{K}R(\phi)\theta_p} \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}\left(\phi\right)|0,\sigma'\rangle = \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}\left(\phi\right)|R\left(\phi\right)p,\sigma'\rangle \tag{42}$$

where we have used $\hat{U}\left(R\right)\hat{P}_i|p\rangle = \hat{U}\left(R\right)\hat{P}_i\hat{U}\left(R\right)^{-1}\hat{U}\left(R\right)|p\rangle = \hat{U}\left(R\right)\hat{P}_i\hat{U}\left(R\right)^{-1}|p'\rangle$ $= \hat{U}\left(R\right)p_i|p\rangle = p_i|p'\rangle = \sum_j \left(R^{-1}\right)_i^j p'_j|p'\rangle = \sum_j \left(R^{-1}\right)_i^j \hat{P}_j|p'\rangle$ whence

$$\hat{U}\left(R\right)\hat{P}_i\hat{U}\left(R\right)^{-1} = \sum_j \left(R^{-1}\right)_i^j \hat{P}_j = \sum_j \left(R^T\right)_i^j \hat{P}_j.$$

(c) Lorentz Boosts - Let us apply a Lorentz boost Λ to a basis vector $|p,\sigma\rangle$ to obtain $\Lambda|p,\sigma\rangle = \Lambda N\left(p\right)\lambda_p|0,\sigma\rangle$. Now, the right hand side, being the product of two boosts, is also a Lorentz transformation and hence, can be represented by the product of a spatial rotation followed by a boost i.e. $\Lambda|p,\sigma\rangle = N\left(p\right)\Lambda\lambda_p|0,\sigma\rangle = N\left(p\right)\lambda_{p'}R\left(\phi_W\left(p,\Lambda\right)\right)|0,\sigma\rangle$ or equivalently $\lambda_{p'}^{-1}\Lambda\lambda_p|0,\sigma\rangle = R\left(\phi_W\left(p,\Lambda\right)\right)|0,\sigma\rangle$. Now, we have shown above that a rotation keeps invariant the subspace of zero momentum. It follows that the sequence of boosts on the left hand side must return each vector of zero momentum to the subspace of zero momentum. Now, $\lambda_p$transforms a vector of zero momentum to a vector of momentum p. Subsequent application of the boost Λwould transform this vector's momentum to Λp. It follows that the boost $\lambda_{p'}$ will transform the vector with momentum Λp back to zero i.e. $\lambda_{p'} = \lambda_{\Lambda p}$.Therefore, $e^{-i\hat{K}\theta}|p,\sigma\rangle = N\left(p\right)\Lambda\lambda_p|0,\sigma\rangle = N\left(p\right)\lambda_{\Lambda p}R\left(\phi_W\left(p.\Lambda\right)\right)|0,\sigma\rangle$

$$= N\left(p\right)\lambda_{\Lambda p} \sum_{\sigma'=-s}^{s} D^s_{\sigma'\sigma}\left(\phi_W\left(p,\Lambda\right)\right)|0,\sigma'\rangle = \frac{N\left(p\right)}{N\left(\Lambda p\right)} \sum_{\sigma'=-s}^{s} D^s_{\sigma'\sigma}\left(\phi_W\left(p,\Lambda\right)\right)|\Lambda p,\sigma'\rangle \tag{43}$$

where $D^s_{\sigma'\sigma}\left(\phi_W\left(p,\Lambda\right)\right)$ are the $(2s+1)\times(2s+1)$unitary rotation matrix representations of the Wigner rotations.

To determine the normalization factor $N\left(p\right)$, we invoke the requirement of the invariance of identity under Lorentz boosts i.e. $I = e^{-i\hat{K}\theta}Ie^{i\hat{K}\theta}$. Introducing the spectral resolution of the identity in the momentum representation $I = \int dp\psi|p\rangle\langle p|$ and omitting the spin indices for brevity, we have

$$I = e^{-i\hat{K}\theta}\left(\int dp|p\rangle\langle p|\right)e^{i\hat{K}\theta} = \int dp|\frac{N(p)}{N(\Lambda p)}|^2|\Lambda p\rangle\langle\Lambda p| = \int (\Lambda p)\, det|\frac{d(p)}{d(\Lambda p)}||\frac{N(p)}{N(\Lambda p)}|^2|\Lambda p\rangle\langle\Lambda p|$$

Since the Jacobian of the transformation $p \to \Lambda p$ viz. det $\left|\frac{dp}{d(\Lambda p)}\right|$ should not depend on the direction of the boost, we choose a boost along the direction of x^3 axis so that $p_1 = \left(\Lambda p\right)_1$, $p_2 = \left(\Lambda p\right)_2$, $p_3 = \left(\Lambda p\right)_3\cosh\theta - \omega_{\Lambda p}\sinh\theta$,
$\omega_p = \sqrt{M^2 + \left(\Lambda p\right)_1^2 + \left(\Lambda p\right)_2\left[\left(\Lambda p\right)_3\cosh\theta - \omega_{\Lambda p}\sinh\theta\right]^2}$
$= \omega_{\Lambda p}\cosh\theta - \left(\Lambda p\right)_3\sinh\theta$.
This gives

$$\det\left|\frac{dp}{d(\Lambda p)}\right| = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{(\Lambda p)_1\sinh\theta}{\omega_{(\Lambda p)}} & \frac{(\Lambda p)_2\sinh\theta}{\omega_{(\Lambda p)}} & \cosh\theta - \frac{(\Lambda p)_3\sinh\theta}{\omega_{(\Lambda p)}} \end{pmatrix}$$

$= \cosh\theta - \frac{(\Lambda p)_3\sinh\theta}{\omega_{\Lambda p}} = \frac{\omega_p}{\omega_{\Lambda p}}$ whence we have $N(p) = \omega_p^{-1/2}$.

2.7 Transformation rules for the Wavefunctions in the Momentum Representation

Two different eigenstates of the momentum operator, being different eigenstates of a hermitian operator must necessarily be orthogonal. i.e. $\langle p \mid p'\rangle = 0$ if $p \neq p'$. Had the spectrum of eigenvalues $Spec\{p\}$ being discrete, we would have the simple normalization $\langle p \mid p'\rangle = \delta_{pp'}$. However, such a normalization is obviously not possible for continuous spectra. To obviate this problem and obtain normalizable vectors, we introduce momentum space wavefunctions $\psi(p)$and write an arbitrary state vector as $|\Psi\rangle = \int dp\psi(p)|p\rangle$ with the normalization $\langle p' \mid p\rangle = \delta(p-p')$ so that

$\langle p \mid \Psi\rangle = \langle p|\int dp'\psi(p')|p'\rangle = \int dp'\psi(p')\langle p \mid p'\rangle = \int dp'\psi(p')\delta(p-p') = \psi(p)$ where we have, for the time being omitted the spin index and identified the eigenstates only by the momentum eigenvalue.

The transformation law for the momentum space wavefunction $\psi^m(p) = \langle p, m \mid \Psi\rangle$ under a Lorentz transformation $\hat{L}(\Lambda) = \hat{Q}(\eta_p, q)\hat{S}(\Lambda)$ can be written as

$$\psi^{m\prime}(p) = \langle p, m \mid \Psi\rangle' = \sum_{m'}\int\frac{dp'}{\omega_{p'}}\langle p, m|\hat{Q}(\eta_p, q)|p', m'\rangle\langle p, m|\hat{S}(\Lambda)|\Psi\rangle \tag{44}$$

where we have used the completeness property of the state vectors and where the bra and ket vectors have the same energy sign.

Using eq. (20) and the fact that $\hat{Q}(\eta_p, q)$ is diagonal with respect to the operator $\hat{P}_\mu$, we obtain

$$\langle p, m|\hat{Q}(\eta_p, q)|p', m'\rangle = \omega_p D^s\left[\hat{Q}(\eta_p, q)\right]^m_{m'}\delta(p-p') \tag{45}$$

Furthermore, since $\hat{S}(\Lambda)$ is unitary, we have $\hat{S}(\Lambda)^\dagger = \hat{S}(\Lambda)^{-1} = \hat{S}(\Lambda^{-1})$ whence

$$\langle p, m|\hat{S}(\Lambda)|\Psi\rangle = \left\langle\Lambda^{-1}p, m \mid \Psi\right\rangle = \psi^m\left(\Lambda^{-1}p\right) \tag{46}$$

Eq. (44) then becomes

$$\psi^{m'}(p) = \sum_{m'}\int\frac{dp'}{\omega_{p'}}\left\{\omega_p D^s\left[\hat{Q}(\eta_p, q)\right]^m_{m'}\delta(p-p')\right\}\left\langle\Lambda^{-1}p', m'|\Psi\right\rangle$$

$$= \sum_{m'}\omega_p D^s\left[\hat{Q}(\eta_p, q)\right]^m_{m'}\left\langle\Lambda^{-1}p, m' \mid \Psi\right\rangle = \sum_{m'}D^s\left[\hat{Q}(\eta_p, q)\right]^m_{m'}\psi^{m'}\left(\Lambda^{-1}p\right) \tag{47}$$

The inner product of two state vectors with same energy signs takes the form

$$\langle p_1, m_1 \mid p_2, m_2 \rangle = \sum_m \int \frac{dp}{\omega_p} \psi^{m^*}(p_1) \psi^m(p_2) \tag{48}$$

For the case of infinitesimal rotations induced by $\hat{Q}(\eta_p, q)$ (which is a unitary representation of the three dimensional rotation operator $\eta_p = \kappa_p^{-1} \Lambda \kappa_{\Lambda^{-1}p} \in SO(3)$ inducing a rotation by an angle ζ), we have

$$\psi'^m(p) = \sum_{m'} \left(I + i\hat{S}\zeta\right)_{m'}^m \psi^m(p - p \times \zeta) = \sum_{m'} \left[I + i\zeta\left(\frac{1}{i} p \times \nabla_p + \hat{S}\right)\right]_{m'}^m \psi^m(p) \tag{49}$$

where we have made used of the differential representation of the translation operator. Defining the generators of spatial rotations, $\hat{J}$, and Lorentz boosts, $\hat{K}$ by

$$\hat{J} = -i(p \times \nabla_p) + \hat{S} \tag{50}$$

and

$$\hat{K} = p_0 \left(i\nabla_p + \frac{p \times \hat{S}}{\omega_p (M + \omega_p)} \right) \tag{51}$$

we can write the transformation laws for the θ transformation and τ transformation as

$$\psi'^m(p) = \begin{cases} \sum_{m'} \left(I + i\hat{J}\theta\right)_{m'}^m \psi^{m'}(p) \ (\theta \ \text{transformation}) \\ \sum_{m'} \left(I - i\hat{K}\tau\right)_{m'}^m \psi^{m'}(p) \ (\tau \ \text{transformation}) \end{cases} \tag{52}$$

In deriving the above expressions, we have used e.g.

$$\hat{K}_1 \psi(p) = iLim_{\theta \to 0} \frac{d}{d\theta} \left[e^{-i\hat{K}_1\theta} \psi(p, \sigma) \right] == iLim_{\theta \to 0} \frac{d}{d\theta} \left[\langle p, \sigma | e^{-i\hat{K}_1\theta} | \Psi \rangle \right]$$

$$= iLim_{\theta \to 0} \frac{d}{d\theta} \left[\langle p, \sigma | e^{\left(i\hat{K}_1\theta\right)^\dagger} | \Psi \rangle \right] = iLim_{\theta \to 0} \frac{d}{d\theta} \left[\langle p, \sigma | \left(e^{i\hat{K}_1\theta}\right)^\dagger | \Psi \rangle \right]$$

$$= iLim_{\theta \to 0} \frac{d}{d\theta} \left[\sqrt{\frac{\omega_{\Lambda^{-1}p}}{\omega_p}} \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}(-\phi_W(p, -\Lambda)) \left\langle \Lambda^{-1}p, \sigma \mid \Psi \right\rangle \right]$$

$$= iLim_{\theta \to 0} \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}(-\phi_W(p, -\Lambda)) \frac{d}{d\theta} \left[\sqrt{\frac{\omega_{\Lambda^{-1}p}}{\omega_p}} \psi\left(\Lambda^{-1}p, \sigma\right) \right]$$

$$= iLim_{\theta \to 0} \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}(-\phi_W(p, -\Lambda)) \frac{d}{d\theta} \left[\begin{array}{c} \sqrt{\frac{\sqrt{p^2+M^2}\cosh\theta - p_1 \sinh\theta \omega_{\Lambda^{-1}p}}{\sqrt{p^2+M^2}}} \\ \psi\left(p_1 \cosh\theta - \sqrt{p^2+M^2}\sinh\theta, p_2, p_3, \sigma\right) \end{array} \right]$$

$$= -i \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma}\left(-\phi_W\left(p,-\Lambda\right)\right)\left[\omega_p \frac{d}{dp_1} + \frac{p_1}{2\omega_p}\right]\psi\left(p,\sigma\right) \tag{53}$$

3. Lorentz Transformation of Bell States

We write the state vector corresponding to two massive spin $\frac{1}{2}$ particles in the momentum representation as

$$|\Psi_{AB}\rangle = \sum_{\sigma_1\sigma_2}\int\int \frac{dp_1 dp_2}{\left(2\omega_{p_1}\right)\left(2\omega_{p_2}\right)}\psi_{\sigma_1,\sigma_2}\left(p_1,p_2\right)|p_1,\sigma_1\rangle_A |p_2,\sigma_2\rangle_B \tag{54}$$

where the two particle wavefunctions satisfy the condition

$$\sum_{\sigma_1\sigma_2}\int\int \frac{dp_1 dp_2}{\left(2\omega_{p_1}\right)\left(2\omega_{p_2}\right)}|\psi_{\sigma_1,\sigma_2}(p_1,p_2)|^2 = 1 \tag{55}$$

Since, a multiparticle state transforms as the direct product of single particle states, the relativistic spin entangled Bell state $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$ can be expressed in terms of Dirac spinors as:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(u_A\left(p,\frac{1}{2}\right)\otimes u_B\left(-p,\frac{1}{2}\right) + u_A\left(p,-\frac{1}{2}\right)\otimes u_B\left(-p,-\frac{1}{2}\right)\right) \tag{56}$$

To study the effect of relativistic transformations on entangled states, we take the Bell state as a prototype. Further, since quantum multiparticle states transform as direct products of single particle states, it is sufficient for us to study the behaviour of the Dirac spinors $u\left(p,s\right)$ under such transformations.

The Dirac spinors $u_{A(B)}\left(\pm p,\pm\frac{1}{2}\right)$ representing particles in motion with momenta $\pm p$can be obtained from the corresponding rest frame spinors $u_{A(B)}\left(0,\frac{1}{2}\right) = \begin{pmatrix} 1 & 0 & 1 & 0\end{pmatrix}^T$ and $u_{A(B)}\left(0,-\frac{1}{2}\right) = \begin{pmatrix} 0 & 1 & 0 & 1\end{pmatrix}^T$ by imparting a suitable boost transformation as explained below.

Consider the motion of a wave particle traveling with a velocity $v \equiv \begin{pmatrix} v_1 & v_2 & v_3\end{pmatrix}$in an arbitrary direction represented by the unit vector $n \equiv \begin{pmatrix} n_1 & n_2 & n_3\end{pmatrix}$ i.e. i.e.$v = |v|\, n = |v|\begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma\end{pmatrix}$. The generators of unit Lorentz transformations along the direction of the unit vector $n \equiv \begin{pmatrix} n_1 & n_2 & n_3\end{pmatrix}$ are given by:-

$$(I_n)^{\nu}_{\ \mu} = \begin{pmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix} \tag{57}$$

and the infinitesimal Lorentz transformations along $n \equiv \begin{pmatrix} n_1 \ n_2 \ n_3 \end{pmatrix}$ are

$$x'^{\nu} = a^{\nu}_{\ \mu} x^{\mu} = x^{\nu} + \Delta\vartheta^{\nu}_{\ \mu} x^{\mu} = x^{\nu} + \Delta\vartheta \, (I_n)^{\nu}_{\ \mu} \, x^{\mu} \tag{58}$$

The spinor transformation operator $\hat{S}$, corresponding to the above Lorentz transformation is:-

$$\hat{S} = \exp\left[-\frac{i}{4}\vartheta\hat{\sigma}_{\mu\nu}\,(I_n)^{\mu\nu}\right] = \exp\left\{-\frac{i\vartheta}{4}\left[\left(\hat{\sigma}_{01}I_n^{01} + \hat{\sigma}_{02}I_n^{02} + \hat{\sigma}_{03}I_n^{03}\right) + \left(\hat{\sigma}_{10}I_n^{10} + \hat{\sigma}_{20}I_n^{20} + \hat{\sigma}_{30}I_n^{30}\right)\right]\right\}$$

because all the other elements of I_n vanish. Now, since $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ and $I_n^{10} = g^{00}\,(I_n)^1_{\ 0} = (I_n)^1_{\ 0} = (I_n)^0_{\ 1} = -g^{11}\,(I_n)^0_{\ 1} = -I_n^{01}$ etc., we have

$$\hat{S} = \exp\left[-\frac{i\vartheta}{2}\left(\hat{\sigma}_{01}I_n^{01} + \hat{\sigma}_{02}I_n^{02} + \hat{\sigma}_{03}I_n^{03}\right)\right] \tag{59}$$

$$= \exp\left[-\frac{i\vartheta}{2}\left(\hat{\sigma}_{01}\cos\alpha + \hat{\sigma}_{02}\cos\beta + \hat{\sigma}_{03}\cos\gamma\right)\right] \tag{60}$$

where we have used $I_n^{01} = g^{11}\,(I_n)^0_{\ 1} = -\,(I_n)^0_{\ 1} = \cos\alpha$etc. This gives

$$\hat{S} = \exp\left(-\frac{\vartheta}{2}\alpha.n\right) = \exp\left(-\frac{\vartheta}{2}\frac{\alpha.v}{|v|}\right) \tag{61}$$

since $\sigma_{0i} = \frac{i}{2}\left[\gamma_0, \gamma_i\right] = i\gamma_0\gamma_i = i\gamma^0\gamma^i g_{ii} = -i\gamma^0\gamma^i = -i\alpha_i$.

Now,

$$(\alpha.v)^2 = \left(\alpha^i v_i\right)\left(\alpha^j v_j\right) = \alpha^i\alpha^j v_i v_j = \frac{1}{2}\left(\alpha^i\alpha^j v_i v_j + \alpha^j\alpha^i v_j v_i\right) = \frac{1}{2}\left(\alpha^i\alpha^j + \alpha^j\alpha^i\right)v_i v_j = \delta^{ij}v_i v_j = v^2 \tag{62}$$

Therefore, $\frac{(\alpha.v)^2}{|v|^2} = 1$ and on expanding the exponential of eq. (61) as a power series in $\left(-\frac{\vartheta}{2}\frac{\alpha.v}{|v|}\right)$ and collecting even and odd powers in $\frac{\vartheta}{2}$ we obtain

$$\hat{S} = \exp\left(-\frac{\vartheta}{2}\frac{\alpha.v}{|v|}\right) = I\left[1 + \left(-\frac{1}{2}\vartheta\right)\left(\frac{\alpha.v}{|v|}\right) + \frac{1}{2!}\left(-\frac{1}{2}\vartheta\right)^2\left(\frac{\alpha.v}{|v|}\right)^2 + ...\right]$$

$$= I\left[1 + \frac{1}{2!}\left(\frac{1}{2}\vartheta\right)^2 + ...\right] - \left(\frac{\alpha.v}{|v|}\right)\left[\left(\frac{1}{2}\vartheta\right) + \frac{1}{3!}\left(\frac{1}{2}\vartheta\right)^3 + ...\right] = I\cosh\left(\frac{\vartheta}{2}\right) - \left(\frac{\alpha.v}{|v|}\right)\sinh\left(\frac{\vartheta}{2}\right)$$

$$= I\cosh\left(\frac{\vartheta}{2}\right) - \left(\frac{\alpha.p}{|p|}\right)\sinh\left(\frac{\vartheta}{2}\right) \tag{63}$$

Using

$$\alpha.v = \sum_{k=1}^{3}\alpha^k v_k = \sum_{k=1}^{3}\begin{pmatrix} 0 & \sigma^k v_k \\ \sigma^k v_k & 0 \end{pmatrix}, \frac{v_k}{|v|} = \frac{p_k}{|p|}, \cosh\left(\frac{\vartheta}{2}\right) = \sqrt{\frac{p_0+M}{2M}}, \sinh\left(\frac{\vartheta}{2}\right) = -\sqrt{\frac{p_0-M}{2M}}$$

we get

$$\hat{S} = \frac{1}{\sqrt{2M}}\begin{pmatrix} (p_0+M)^{1/2} & 0 & \frac{p_3(p_0-M)^{1/2}}{|p|} & \frac{(p_1-ip_2)(p_0-M)^{1/2}}{|p|} \\ 0 & (p_0+M)^{1/2} & \frac{(p_1+ip_2)(p_0-M)^{1/2}}{|p|} & \frac{-p_3(p_0-M)^{1/2}}{|p|} \\ \frac{p_3(p_0-M)^{1/2}}{|p|} & \frac{(p_1-ip_2)(p_0-M)^{1/2}}{|p|} & (p_0+M)^{1/2} & 0 \\ \frac{(p_1+ip_2)(p_0-M)^{1/2}}{|p|} & \frac{-p_3(p_0-M)^{1/2}}{|p|} & 0 & (p_0+M)^{1/2} \end{pmatrix}$$

so that $u\left(p,\frac{1}{2}\right) = \frac{1}{\sqrt{2M}}\begin{pmatrix} (p_0+M)^{1/2} + \frac{p_3(p_0-M)^{1/2}}{|p|} \\ \frac{(p_1+ip_2)(p_0-M)^{1/2}}{|p|} \\ \frac{p_3(p_0-M)^{1/2}}{|p|} + (p_0+M)^{1/2} \\ \frac{(p_1+ip_2)(p_0-M)^{1/2}}{|p|} \end{pmatrix} = \begin{pmatrix} \cosh\frac{\vartheta}{2} + \frac{p_3}{|p|}\sinh\frac{\vartheta}{2} \\ \frac{(p_1+ip_2)}{|p|}\sinh\frac{\vartheta}{2} \\ \cosh\frac{\vartheta}{2} + \frac{p_3}{|p|}\sinh\frac{\vartheta}{2} \\ \frac{(p_1+ip_2)}{|p|}\sinh\frac{\vartheta}{2} \end{pmatrix}$ and

$$u\left(p,-\frac{1}{2}\right) = \frac{1}{\sqrt{2M}}\begin{pmatrix} \frac{(p_1-ip_2)(p_0-M)^{1/2}}{|p|} \\ (p_0+M)^{1/2} - \frac{p_3(p_0-M)^{1/2}}{|p|} \\ \frac{(p_1-ip_2)(p_0-M)^{1/2}}{|p|} \\ -\frac{p_3(p_0-M)^{1/2}}{|p|} + (p_0+M)^{1/2} \end{pmatrix} = \begin{pmatrix} \frac{(p_1-ip_2)}{|p|}\sinh\frac{\vartheta}{2} \\ \cosh\frac{\vartheta}{2} - \frac{p_3}{|p|}\sinh\frac{\vartheta}{2} \\ \frac{(p_1-ip_2)}{|p|}\sinh\frac{\vartheta}{2} \\ \cosh\frac{\vartheta}{2} - \frac{p_3}{|p|}\sinh\frac{\vartheta}{2} \end{pmatrix} \tag{64}$$

Our objective here is to examine the impact of a Lorentz transformation on the Bell state given by eq. (56).

Let us, now, impart a boost transformation,

$$\Lambda = \begin{pmatrix} \cosh\tau & \sinh\tau & 0 & 0 \\ \sinh\tau & \cosh\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{65}$$

directed along the x^1 axis and identified by the rapidity $\tau = -\tanh v_1$ on the spinor $u\,(p,s)$. Under this transformation, the components of the momentum vector transform as $p'^{\mu} = (\Lambda p)^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$ so that

$$p' \equiv (p_0 \cosh\tau + p_1 \sinh\tau, p_0 \sinh\tau + p_1 \cosh\tau, p_2, p_3) \tag{66}$$

and

$$p" = \Lambda^{-1} p = (p_0 \cosh\tau - p_1 \sinh\tau, -p_0 \sinh\tau + p_1 \cosh\tau, p_2, p_3) \tag{67}$$

Under this transformation, the spinors $u\,(p,\sigma)$ will transform as

$$\begin{pmatrix} u'\left(p, \frac{1}{2}\right) \\ u'\left(p, -\frac{1}{2}\right) \end{pmatrix} = D\left(\phi_W\,(p,\Lambda)\right) \begin{pmatrix} u\left(\Lambda^{-1} p, \frac{1}{2}\right) \\ u'\left(\Lambda^{-1} p, -\frac{1}{2}\right) \end{pmatrix} \tag{68}$$

where Λp is the spatial component of Λp, $D\left(\phi_W\,(\pm p,\Lambda)\right)$ are the unitary representations of the three dimensional Wigner rotation corresponding to the Lorentz transformation Λ. $u\,(\pm\Lambda^{-1}p,\sigma)$ can be calculated by using eqs. (64) & (67). Our task is, therefore, now confined to obtaining the Wigner rotation corresponding to Λ.

Hence, our next step is to construct the Wigner rotation and identify the Wigner angle corresponding to the Lorentz boost given by eq. (65). To keep the calculations as simple as possible so as not to obscure the physical content, we consider the standard vector as $l \equiv (M,0,0,0)$ and subject it to a Lorentz boost κ_k directed along the x^3 axis and identified by the rapidity parameter ϖ. Its matrix representation is

$$\kappa_k = \begin{pmatrix} \cosh\varpi & 0 & 0 & \sinh\varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\varpi & 0 & 0 & \cosh\varpi \end{pmatrix} \tag{69}$$

Under the effect of this transformation, the standard vector transforms as $k \equiv \kappa_k l = (M\cosh\varpi, 0, 0, M\sinh\varpi)$. We, now, have

$$\kappa_k^{-1} = \begin{pmatrix} \cosh\varpi & 0 & 0 & -\sinh\varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\varpi & 0 & 0 & \cosh\varpi \end{pmatrix} \tag{70}$$

$$\kappa_k^{-1}\Lambda = \begin{pmatrix} \cosh\varpi\cosh\tau & \cosh\varpi\sinh\tau & 0 & -\sinh\varpi \\ \sinh\tau & \cosh\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\varpi\cosh\tau & -\sinh\varpi\sinh\tau & 0 & \cosh\varpi \end{pmatrix} \tag{71}$$

and

$$\Lambda^{-1}k = (M\cosh\varpi\cosh\tau, -M\cosh\varpi\sinh\tau, 0, M\sinh\varpi)$$

so that

$$\kappa_{\Lambda^{-1}k} = \begin{pmatrix} \cosh\varpi\cosh\tau & -\cosh\varpi\sinh\tau & 0 & \sinh\varpi \\ -\cosh\varpi\sinh\tau & 1+\alpha\left(M\cosh\varpi\sinh\tau\right)^2 & 0 & -\alpha M^2\cosh\varpi\sinh\varpi\sinh\tau \\ 0 & 0 & 1 & 0 \\ \sinh\varpi & -\alpha M^2\cosh\varpi\sinh\varpi\sinh\tau & 0 & 1+\alpha M^2\sinh^2\varpi \end{pmatrix} \tag{72}$$

where

$$\alpha = \frac{(\cosh\varpi\cosh\tau - 1)}{M^2\left(\cosh^2\varpi\cosh^2\tau - 1\right)} = \frac{1}{M^2\left(\cosh\varpi\cosh\tau + 1\right)} \tag{73}$$

Instead of making explicit calculations of the Wigner rotation $\eta_k = \kappa_k^{-1}\Lambda\kappa_{\Lambda^{-1}k}$ which are quite cumbersome, we examine the impact of η_k on the spatial basis vectors $(i, j, k) \equiv \left((0,1,0,0)^T, (0,0,1,0)^T, (0,0,0,1)^T\right)$. The results are

$$\eta_k i = \eta_k(0,1,0,0) = \kappa_k^{-1}\Lambda\left(-\cosh\varpi\sinh\tau, 1+\alpha M^2\cosh^2\varpi\sinh^2\tau, 0, -\alpha M^2\cosh\varpi\sinh\varpi\sinh\tau\right)$$

$$= \left(0, \frac{\cosh\varpi + \cosh\tau}{\cosh\varpi\cosh\tau + 1}, 0, -\frac{\sinh\varpi\sinh\tau}{\cosh\varpi\cosh\tau + 1}\right) \tag{74}$$

$$\eta_k j = \eta_k(0,0,1,0) = (0,0,1,0) \tag{75}$$

$$\eta_k k = \eta_k(0,0,0,1) = \kappa_k^{-1}\Lambda\left(\sinh\varpi, -\alpha M^2\cosh\varpi\sinh\varpi\sinh\tau, 0, 1+\alpha M^2\sinh^2\varpi\right)$$

$$= \left(0, \frac{\sinh\varpi\sinh\tau}{\cosh\varpi\cosh\tau + 1}, 0, \frac{\cosh\varpi + \cosh\tau}{\cosh\varpi\cosh\tau + 1}\right) \tag{76}$$

The above expressions for the three transformed spatial basis vectors vindicate that the Wigner rotation is a spatial rotation that takes the form of a rotation about the x^2 axis and further, identify the Wigner angle as

$$\phi_W = \tan^{-1} \frac{\sinh \varpi \sinh \tau}{\cosh \varpi + \cosh \tau} \tag{77}$$

Using eq. (77), we can write the unitary representation of the three dimensional Wigner rotation as

$$D^{1/2} \left(\phi_W \left(p, \Lambda \right) \right) = \begin{pmatrix} \cos \phi_W & -\sin \phi_W \\ \sin \phi_W & \cos \phi_W \end{pmatrix} \tag{78}$$

From eqs. (68) and (78), we arrive at expression for the transformation of the Dirac spinors and hence, of the Bell state as

$$u' \left(p, \frac{1}{2} \right) = \cos \phi_W u \left(\Lambda^{-1} p, \frac{1}{2} \right) - \sin \phi_W u \left(\Lambda^{-1} p, -\frac{1}{2} \right) \tag{79}$$

$$u' \left(p, -\frac{1}{2} \right) = \sin \phi_W u \left(\Lambda^{-1} p, \frac{1}{2} \right) + \cos \phi_W u \left(\Lambda^{-1} p, -\frac{1}{2} \right) \tag{80}$$

Conclusions

We have, thus, shown in this paper that the constituent particles of the maximally entangled Bell states, when subject to Lorentz boosts undergo a momentum dependent Wigner rotation (which is a rotation in spatial coordinates and hence, is unitary). The entanglement fidelity is, therefore, preserved under such transformations Explicit calculation for the Wigner rotation corresponding to a boost along the x^1 axis is performed and the it is shown to be a rotation about the x^2 axis through an angle that is momentum dependent and is given by eq. (77). As is shown, the rotation matrix, being unitary, restores unitarity in the transformation. It is also shown that the spins in the Bell state undergo a reorientation in the direction of the boost when observed from a frame moving with a constant velocity with respect to the rest frame. The point to be noted here is that since a multi particle state transforms as the direct product of single particle states, the transformed multiparticle state consists of a linear combination of spin states with the transformed momenta. It follows that if we obtain the density matrix of the "reduced state" by tracing over one of the states, we will still get a maximally entangled (mixed) density matrix implying that the entanglement fidelity is not disturbed under such Lorentz transformations as are considered here.

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