

Nonholonomic Ricci Flows and Parametric Deformations of the Solitonic pp–Waves and Schwarzschild Solutions

Sergiu I. Vacaru*

*Faculty of Mathematics, University "Al. I. Cuza" Iași 700506, Iași, Romania
The Fields Institute for Research in Mathematical Science, 222 College Street, 2d Floor,
Toronto M5T 3J1, Canada*

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Abstract: We study Ricci flows of some classes of physically valuable solutions in Einstein and string gravity. The anholonomic frame method is applied for generic off–diagonal metric ansatz when the field/ evolution equations are transformed into exactly integrable systems of partial differential equations. The integral varieties of such solutions, in four and five dimensional gravity, depend on arbitrary generation and integration functions of one, two and/ or three variables. Certain classes of nonholonomic frame constraints allow us to select vacuum and/or Einstein metrics, to generalize such solutions for nontrivial string (for instance, with antisymmetric torsion fields) and matter field sources. A very important property of this approach (originating from Finsler and Lagrange geometry but re–defined for semi–Riemannian spaces) is that new classes of exact solutions can be generated by nonholonomic deformations depending on parameters associated to some generalized Geroch transforms and Ricci flow evolution. In this paper, we apply the method to construct in explicit form some classes of exact solutions for multi–parameter Einstein spaces and their nonholonomic Ricci flows describing evolutions/interactions of solitonic pp–waves and deformations of the Schwarzschild metric. We explore possible physical consequences and speculate on their importance in modern gravity.

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* the affiliation for Fields Institute is for a former visiting position;

Sergiu.Vacaru@gmail.com; <http://www.scribd.com/people/view/1455460-sergiu>

1. Introduction

This is the fifth paper in a series of works on nonholonomic Ricci flows of metrics and geometric objects subjected to nonintegrable (nonholonomic) constraints [1, 2, 3, 4]. It is devoted to explicit applications of new geometric methods in constructing exact solutions in gravity and Ricci flow theory. Specifically, we shall consider a set of particular solutions with solitonic pp-waves and anholonomic deformations of the Schwarzschild metric, defined by generic off-diagonal metrics¹, describing nonlinear gravitational interactions and gravitational models with effective cosmological constants, for instance, induced by string corrections [5, 6] or effective approximations for matter fields. We shall analyze how such gravitational configurations (some of them generated as exact solutions by geometric methods in Section 4 of Ref. [7]) may evolve under Ricci flows; on previous results and the so-called anholonomic frame method of constructing exact solutions, see works [8, 9, 10, 11, 12, 13] and references therein.

Some of the most interesting directions in modern mathematics are related to the Ricci flow theory [14, 15], see reviews [16, 17, 18]. There were elaborated a set of applications with such geometric flows, following low dimensional or approximative methods to construct solutions of evolution equations, in modern gravity and mathematical physics, for instance, for low dimensional systems and gravity [19, 20, 21, 22] and black holes and cosmology [23, 24]. One of the most important tasks in such directions is to formulate certain general methods of constructing exact solutions for gravitational systems under Ricci flow evolution of fundamental geometric objects.

In Refs. [25, 26, 27], considering the Ricci flow evolution parameter as a time like, or extra dimension, coordinate, we provided the first examples when physically valuable Ricci flow solutions were constructed following the anholonomic frame method. A quite different scheme was considered in Ref. [3] with detailed proofs that the information on general metrics and connection in Riemann–Cartan geometry, and various generalizations to nonholonomic Lagrange–Finsler spaces, can be encoded into bi-Hamilton structure and related solitonic hierarchies. There were formulated certain conditions when nonholonomic solitonic equations can be constrained to extract exact solutions for the Einstein equations and Ricci flow evolution equations.

The previous partner paper [4] was devoted to the geometry of parametrized nonholonomic frame transforms as superpositions of the Geroch transforms (generating exact vacuum gravitational solutions with Killing symmetries, see Refs. [28, 29]) and the anholonomic frame deformations and oriented to carry out a program of generating off-diagonal exact solutions in gravity [7] and Ricci flow theories. The goal of this work is to show how such new classes of parametric nonholonomic solutions, formally constructed for the Einstein and string gravity [7], can be generalized to satisfy certain geometric evolution equations and define Ricci flows of physically valuable metrics and connections.

The structure of the paper is the following: In section 2, we outline the necessary formulas for nonholonomic Einstein spaces and Ricci flows. There are introduced the general

¹ which can not diagonalized by coordinate transforms

ansatz for generic off–diagonal metrics (for which, we shall construct evolution/ field exact solutions) and the primary metrics used for parametric nonholonomic deformations to new classes of solutions.

In section 3, we construct Ricci flow solutions of solitonic pp–waves in vacuum Einstein and string gravity.

Section 4 is devoted to a study of parametric nonholonomic transforms (defined as superpositions of the parametric transforms and nonholonomic frame deformations) in order to generate (multi-) parametric solitonic pp–waves for Ricci flows and in Einstein spaces.

Section 5 generalizes to Ricci flow configurations the exact solutions generated by parametric nonholonomic frame transforms of the Schwarzschild metric. There are analyzed deformations and flows of stationary backgrounds, considered anisotropic polarizations on extra dimension coordinate (possibly induced by Ricci flows and extra dimension interactions) and examined five dimensional solutions with running of parameters on nonholonomic time coordinate and flow parameter.

The paper concludes with a discussion of results in section 6.

The Appendix contains some necessary formulas on effective cosmological constants and nonholonomic configurations induced from string gravity.

2. Preliminaries

We work on five and/or four dimensional, (5D and/ or 4D), nonholonomic Riemannian manifolds \mathbf{V} of necessary smooth class and conventional splitting of dimensions $\dim \mathbf{V} = n + m$ for $n = 3$, or $n = 2$ and $m = 2$, defined by a nonlinear connection (N–connection) structure $\mathbf{N} = \{N_i^a\}$, such manifolds are also called N–anholonomic [8]. The local coordinates are labelled in the form $u^\alpha = (x^i, y^a) = (x^1, \hat{x}^i, y^4 = v, y^5)$, for $i = 1, 2, 3$ and $\hat{i} = 2, 3$ and $a, b, \dots = 4, 5$. Any coordinates from a set u^α can be for a three dimensional (3D) space, time, or extra dimension (5th coordinate). Ricci flows of geometric objects will be parametrized by a real $\chi \in [0, \chi_0]$. Four dimensional (4D) spaces, when the local coordinates are labelled in the form $u^{\hat{\alpha}} = (\hat{x}^i, y^a)$, i. e. without coordinate x^1 , are defined as a trivial embedding into 5D ones. In general, we shall follow the conventions and methods stated in Refs. [4, 7] (the reader is recommended to consult those works on main definitions, denotations and geometric constructions).

2.1 Ansatz for the Einstein and Ricci flow equations

A nonholonomic manifold \mathbf{V} , provided with a N–connection (equivalently, with a locally fibred) structure and a related preferred system of reference, can be described in equivalent forms by two different linear connections, the Levi Civita ∇ and the canonical distinguished connection, d–connection $\widehat{\mathbf{D}}$, both completely defined by the same metric

structure

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) \mathbf{e}^a \otimes \mathbf{e}^b, \\ e^i &= dx^i, \quad \mathbf{e}^a = dy^a + N_i^a(u) dx^i, \end{aligned} \quad (1)$$

in metric compatible forms, $\nabla \mathbf{g} = 0$ and $\widehat{\mathbf{D}} \mathbf{g} = 0$. It should be noted that, in general, $\widehat{\mathbf{D}}$ contains some nontrivial torsion coefficients induced by N_i^a , see details in Refs. [1, 2, 3, 4, 8, 9, 10, 11, 12, 13]. For simplicity, we shall omit "hats" and indices, or other labels, and write, for instance, $u = (x, y)$, $\partial_i = \partial/\partial x^i$ and $\partial_a = \partial/\partial y^a, \dots$ if such simplifications will not result in ambiguities.

In order to consider Ricci flows of geometric objects, we shall work with families of ansatz ${}^x \mathbf{g} = \mathbf{g}(\chi)$, of type (1), parametrized by χ ,

$$\begin{aligned} {}^x \mathbf{g} &= g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3, \chi) dx^2 \otimes dx^2 + g_3(x^2, x^3, \chi) dx^3 \otimes dx^3 \\ &\quad + h_4(x^k, v, \chi) {}^x \delta v \otimes {}^x \delta v + h_5(x^k, v, \chi) {}^x \delta y \otimes {}^x \delta y, \\ {}^x \delta v &= dv + w_i(x^k, v, \chi) dx^i, \quad {}^x \delta y = dy + n_i(x^k, v, \chi) dx^i, \end{aligned} \quad (2)$$

for $g_1 = \pm 1$, with corresponding flows for N-adapted bases,

$${}^x \mathbf{e}_\alpha = \mathbf{e}_\alpha(\chi) = ({}^x \mathbf{e}_i = \mathbf{e}_i(\chi) = \partial_i - N_i^a(u, \chi) \partial_a, e_a), \quad (3)$$

$${}^x \mathbf{e}^\alpha = \mathbf{e}^\alpha(\chi) = (e^i, {}^x \mathbf{e}^a = \mathbf{e}^a(\chi) = dy^a + N_i^a(u, \chi) dx^i) \quad (4)$$

defined by $N_i^4(u, \chi) = w_i(x^k, v, \chi)$ and $N_i^5(u, \chi) = n_i(x^k, v, \chi)$. For any fixed value of χ , we may omit the Ricci flow parametric dependence.

The frames (4) satisfy certain nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad (5)$$

with anholonomy coefficients

$$W_{ia}^b = \partial_a N_i^b \quad \text{and} \quad W_{ji}^a = \Omega_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a). \quad (6)$$

A local basis is holonomic (for instance, the local coordinate basis) if $W_{\alpha\beta}^\gamma = 0$ and integrable, i.e. it defines a fibred structure, if the curvature of N-connection $\Omega_{ij}^a = 0$.

We can elaborate on a N-anholonomic manifold \mathbf{V} (i.e. on a manifold provided with N-connection structure) a N-adapted tensor and differential calculus if we decompose the geometric objects and basic equations with respect to N-adapted bases (4) and (3) and using the canonical d-connection $\widehat{\mathbf{D}} = \nabla + \mathbf{Z}$, see, for instance, the formulas (A.17) in Ref. [4] for the components of distortion tensor \mathbf{Z} , which contains certain nontrivial torsion coefficients induced by "off-diagonal" N_i^a . Two different linear connections, ∇ and $\widehat{\mathbf{D}}$, define respectively two different Ricci tensors, ${}_1 R_{\alpha\beta}$ and $\widehat{\mathbf{R}}_{\alpha\beta} = [\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{bj}, \widehat{S}_{ab}]$ (see (A.12) and related formulas in [1]). Even, in general, $\widehat{\mathbf{D}} \neq \nabla$, for certain classes of ansatz and N-adapted frames, we can obtain, for some nontrivial coefficients, relations of type $\widehat{\mathbf{R}}_{\alpha\beta} = {}_1 R_{\alpha\beta}$. This allows us to constrain some classes of solutions of the Einstein

and/ or Ricci flow equations constructed for a more general linear connection $\widehat{\mathbf{D}}$ to define also solutions for the Levi Civita connection ∇ .²

For a 5D space initially provided with a diagonal ansatz (1), when $g_{ij} = \text{diag}[\pm 1, g_2, g_3]$ and $g_{ab} = \text{diag}[g_4, g_5]$, we considered [25, 4] the nonholonomic (normalized) evolution equations (parametrized by an ansatz (2))

$$\frac{\partial}{\partial \chi} g_{ii} = -2 \left[\widehat{R}_{ii} - \lambda g_{ii} \right] - h_{cc} \frac{\partial}{\partial \chi} (N_i^c)^2, \quad (7)$$

$$\frac{\partial}{\partial \chi} h_{aa} = -2 \left(\widehat{R}_{aa} - \lambda h_{aa} \right), \quad (8)$$

$$\widehat{R}_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta, \quad (9)$$

with the coefficients defined with respect to N–adapted frames (3) and (4). This system of constrained (nonholonomic) evolution equations in a particular case is related to families of metrics ${}^\lambda \mathbf{g} = {}^\lambda g_{\alpha\beta} \mathbf{e}^\alpha \otimes \mathbf{e}^\beta$ for nonholonomic Einstein spaces, considered as solutions of

$$\widehat{\mathbf{R}}_{\alpha\beta} = \lambda \mathbf{g}_{\alpha\beta}, \quad (10)$$

with effective cosmological constant λ (in more general cases, we can consider effective, locally anisotropically polarized cosmological constants with dependencies on coordinates and χ : such solutions were constructed in Refs. [10, 8]). For any solution of (10) with nontrivial

$$\mathbf{g}_{\alpha\beta} = [\pm 1, g_{2,3}(x^2, x^3), h_{4,5}(x^2, x^3, v)] \text{ and } w_{2,3}(x^2, x^3, v), n_{2,3}(x^2, x^3, v),$$

we can consider nonholonomic Ricci flows of the horizontal metric components, $g_{2,3}(x^2, x^3) \rightarrow g_{2,3}(x^2, x^3, \chi)$, and of certain N–connection coefficients $n_{2,3}(x^2, x^3, v) \rightarrow n_{2,3}(x^2, x^3, v, \chi)$, constrained to satisfy the equation (7), i.e.

$$\frac{\partial}{\partial \chi} [g_{2,3}(x^2, x^3, \chi) + h_5(x^2, x^3, v) (n_{2,3}(x^2, x^3, v, \chi))^2] = 0. \quad (11)$$

Having constrained an integral variety of (10) in order to have $\widehat{\mathbf{R}}_{\alpha\beta} = {}_\perp R_{\alpha\beta}$ for certain subclasses of solutions³, the equations (11) define evolutions of the geometric objects just for a family of Levi Civita connections ${}^\lambda \nabla$.

Computing the components of the Ricci and Einstein tensors for the metric (2) (see details on tensors components' calculus in Refs. [10, 8]), one proves that the equations

² we note that such equalities are obtained by deformation of the nonholonomic structure on a manifold which change the transformation laws of tensors and different linear connections

³ this imposes certain additional restrictions on $n_{2,3}$ and $g_{2,3}$, see discussions related to formulas (A.16)–(A.20) and explicit examples for Ricci flows stated by constraints (49), or (74) in Ref. [4]; in the next sections, we shall consider explicit examples

(10) transform into a parametric on χ system of partial differential equations:

$$\widehat{R}_2^2 = \widehat{R}_3^3(\chi) = \frac{1}{2g_2(\chi)g_3(\chi)} \left[\frac{g_2^\bullet(\chi)g_3^\bullet(\chi)}{2g_2(\chi)} + \frac{(g_3^\bullet(\chi))^2}{2g_3(\chi)} - g_3^{\bullet\bullet}(\chi) + \frac{g_2'(\chi)g_3'(\chi)}{2g_3(\chi)} + \frac{(g_2'(\chi))^2}{2g_2(\chi)} - g_2''(\chi) \right] = -\lambda, \quad (12)$$

$$\widehat{S}_4^4 = \widehat{S}_5^5(\chi) = \frac{1}{2h_4(\chi)h_5(\chi)} \times \left[h_5^*(\chi) \left(\ln \sqrt{|h_4(\chi)h_5(\chi)|} \right)^* - h_5^{**}(\chi) \right] = -\lambda, \quad (13)$$

$$\widehat{R}_{4i}(\chi) = -w_i(\chi) \frac{\beta(\chi)}{2h_5(\chi)} - \frac{\alpha_i(\chi)}{2h_5(\chi)} = 0, \quad (14)$$

$$\widehat{R}_{5i}(\chi) = -\frac{h_5(\chi)}{2h_4(\chi)} [n_i^{**}(\chi) + \gamma(\chi)n_i^*(\chi)] = 0, \quad (15)$$

where, for $h_{4,5}^* \neq 0$,

$$\alpha_i = h_5^* \partial_i \phi, \quad \beta = h_5^* \phi^*, \quad (16)$$

$$\gamma = \frac{3h_5^*}{2h_5} - \frac{h_4^*}{h_4}, \quad \phi = \ln \left| \frac{h_5^*}{\sqrt{|h_4 h_5|}} \right|, \quad (17)$$

when the necessary partial derivatives are written in the form $a^\bullet = \partial a / \partial x^2$, $a' = \partial a / \partial x^3$, $a^* = \partial a / \partial v$. In the vacuum case, we shall put $\lambda = 0$. Here we note that the dependence on χ can be considered both for classes of functions and integrations constants and functions defining some exact solutions of the Einstein equations or even for any general metrics on a Riemann-Cartan manifold (provided with any compatible metric and linear connection structures).

For an ansatz (2) with $g_2(x^2, x^3, \chi) = \epsilon_2 e^{\psi(x^2, x^3, \chi)}$ and $g_3(x^2, x^3, \chi) = \epsilon_3 e^{\psi(x^2, x^3, \chi)}$, we can restrict the solutions of the system (12)–(15) to define Ricci flows solutions with the Levi Civita connection if the coefficients satisfy the conditions

$$\begin{aligned} \epsilon_2 \psi^{\bullet\bullet}(\chi) + \epsilon_3 \psi''(\chi) &= \lambda \\ h_5^* \phi / h_4 h_5 &= \lambda, \\ w_2' - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n_2'(\chi) - n_3^\bullet(\chi) &= 0, \end{aligned} \quad (18)$$

for

$$w_{\widehat{i}} = \partial_{\widehat{i}} \phi / \phi^*, \quad \text{where } \varphi = -\ln \left| \frac{\sqrt{|h_4 h_5|}}{|h_5^*|} \right|, \quad (19)$$

for $\widehat{i} = 2, 3$, see formulas (49) in Ref. [4].

2.2 Five classes of primary metrics

We introduce a list of 5D quadratic elements, defined by certain primary metrics, which will be subjected to parametrized nonholonomic transforms in order to generate new

classes of exact solutions of the Einstein and Ricci flow equations, i.e. of the system (12)–(15) and (10) with possible additional constraints in order to get geometric evolutions in terms of the Levi Civita connection ∇ .

The first type quadratic element is taken

$$\delta s_{[1]}^2 = \epsilon_1 d\boldsymbol{\varkappa}^2 - d\xi^2 - r^2(\xi) d\vartheta^2 - r^2(\xi) \sin^2 \vartheta d\varphi^2 + \varpi^2(\xi) dt^2, \quad (20)$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$\begin{aligned} x^1 &= \boldsymbol{\varkappa}, x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t, \\ \check{g}_1 &= \epsilon_1 = \pm 1, \check{g}_2 = -1, \check{g}_3 = -r^2(\xi), \check{h}_4 = -r^2(\xi) \sin^2 \vartheta, \check{h}_5 = \varpi^2(\xi), \end{aligned} \quad (21)$$

for

$$\xi = \int dr \left| 1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2} \right|^{1/2} \quad \text{and} \quad \varpi^2(r) = 1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}.$$

For the constants $\varepsilon \rightarrow 0$ and μ being a point mass, the element (20) defines just a trivial embedding into 5D (with extra dimension coordinate $\boldsymbol{\varkappa}$) of the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$.⁴

The second quadratic element is

$$\delta s_{[2]}^2 = -r_g^2 d\varphi^2 - r_g^2 d\check{\vartheta}^2 + \check{g}_3(\check{\vartheta}) d\check{\xi}^2 + \epsilon_1 d\chi^2 + \check{h}_5(\xi, \check{\vartheta}) dt^2, \quad (22)$$

where the local coordinates are

$$x^1 = \varphi, x^2 = \check{\vartheta}, x^3 = \check{\xi}, y^4 = \chi, y^5 = t,$$

for

$$d\check{\vartheta} = d\vartheta / \sin \vartheta, \quad d\check{\xi} = dr / r \sqrt{|1 - 2\mu/r + \varepsilon/r^2|},$$

and the Schwarzschild radius of a point mass μ is defined $r_g = 2G_{[4]}\mu/c^2$, where $G_{[4]}$ is the 4D Newton constant and c is the light velocity. The nontrivial metric coefficients in (22) are parametrized

$$\begin{aligned} \check{g}_1 &= -r_g^2, \check{g}_2 = -r_g^2, \check{g}_3 = -1/\sin^2 \vartheta, \\ \check{h}_4 &= \epsilon_1, \check{h}_5 = [1 - 2\mu/r + \varepsilon/r^2] / r^2 \sin^2 \vartheta. \end{aligned} \quad (23)$$

The quadratic element defined by (22) and (23) is a trivial embedding into 5D of the Schwarzschild quadratic element multiplied to the conformal factor $(r \sin \vartheta / r_g)^2$. We emphasize that this metric is not a solution of the Einstein or Ricci flow equations but it will be used in order to construct parametrized nonholonomic deformations to such solutions.

⁴ For simplicity, we consider only the case of vacuum solutions, not analyzing a more general possibility when $\varepsilon = e^2$ is related to the electric charge for the Reissner–Nordström metric (see, for example, [30]). In our further considerations, we shall treat ε as a small parameter, for instance, defining a small deformation of a circle into an ellipse (eccentricity).

We shall use a quadratic element when the time coordinate is considered to be "anisotropic",

$$\delta s_{[3]}^2 = -r_g^2 d\varphi^2 - r_g^2 d\vartheta^2 + \check{g}_3(\vartheta) d\xi^2 + \check{h}_4(\xi, \vartheta) dt^2 + \epsilon_1 d\kappa^2 \quad (24)$$

where the local coordinates are

$$x^1 = \varphi, \quad x^2 = \vartheta, \quad x^3 = \xi, \quad y^4 = t, \quad y^5 = \kappa,$$

and the nontrivial metric coefficients are parametrized

$$\begin{aligned} \check{g}_1 &= -r_g^2, \quad \check{g}_2 = -r_g^2, \quad \check{g}_3 = -1/\sin^2 \vartheta, \\ \check{h}_4 &= [1 - 2\mu/r + \varepsilon/r^2] / r^2 \sin^2 \vartheta, \quad \check{h}_5 = \epsilon_1. \end{aligned} \quad (25)$$

The formulas (24) and (25) are respective reparametrizations of (22) and (23) when the 4th and 5th coordinates are inverted. Such metrics will be used for constructing new classes of exact solutions in 5D with explicit dependence on time like coordinate.

The fourth quadratic element is introduced by inverting the 4th and 5th coordinates in (20)

$$\delta s_{[4]}^2 = \epsilon_1 d\kappa^2 - d\xi^2 - r^2(\xi) d\vartheta^2 + \varpi^2(\xi) dt^2 - r^2(\xi) \sin^2 \vartheta d\varphi^2 \quad (26)$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$\begin{aligned} x^1 &= \kappa, \quad x^2 = \xi, \quad x^3 = \vartheta, \quad y^4 = t, \quad y^5 = \varphi, \\ \check{g}_1 &= \epsilon_1 = \pm 1, \quad \check{g}_2 = -1, \quad \check{g}_3 = -r^2(\xi), \quad \check{h}_4 = \varpi^2(\xi), \quad \check{h}_5 = -r^2(\xi) \sin^2 \vartheta. \end{aligned} \quad (27)$$

Such metrics can be used for constructing exact solutions in 4D gravity and Ricci flows with anisotropic dependence on time coordinate.

Finally, we consider

$$\delta s_{[5]}^2 = \epsilon_1 d\kappa^2 - dx^2 - dy^2 - 2\kappa(x, y, p) dp^2 + dv^2 / 8\kappa(x, y, p), \quad (28)$$

where the local coordinates are

$$x^1 = \kappa, \quad x^2 = x, \quad x^3 = y, \quad y^4 = p, \quad y^5 = v,$$

and the nontrivial metric coefficients are parametrized

$$\begin{aligned} \check{g}_1 &= \epsilon_1 = \pm 1, \quad \check{g}_2 = -1, \quad \check{g}_3 = -1, \\ \check{h}_4 &= -2\kappa(x, y, p), \quad \check{h}_5 = 1/8 \kappa(x, y, p). \end{aligned} \quad (29)$$

The metric (28) is a trivial embedding into 5D of the vacuum solution of the Einstein equation defining pp-waves [38] for any $\kappa(x, y, p)$ solving

$$\kappa_{xx} + \kappa_{yy} = 0,$$

with $p = z + t$ and $v = z - t$, where (x, y, z) are usual Cartesian coordinates and t is the time like coordinates. The simplest explicit examples of such solutions are

$$\kappa = (x^2 - y^2) \sin p,$$

defining a plane monochromatic wave, or

$$\begin{aligned} \kappa &= \frac{xy}{(x^2 + y^2)^2 \exp[p_0^2 - p^2]}, \text{ for } |p| < p_0; \\ &= 0, \text{ for } |p| \geq p_0, \end{aligned}$$

defining a wave packet travelling with unit velocity in the negative z direction.

3. Solitonic pp–Waves and String Torsion

Pp–wave solutions are intensively exploited for elaborating string models with nontrivial backgrounds [31, 32, 33]. A special interest for pp–waves in general relativity is related to the fact that any solution in this theory can be approximated by a pp–wave in vicinity of horizons. Such solutions can be generalized by introducing nonlinear interactions with solitonic waves [12, 34, 35, 36, 37] and nonzero sources with nonhomogeneous cosmological constant induced by an ansatz for the antisymmetric tensor fields of third rank, see Appendix. A very important property of such nonlinear wave solutions is that they possess nontrivial limits defining new classes of generic off–diagonal vacuum Einstein spacetimes and can be generalized for Ricci flows induced by evolutions of N–connections.

In this section, we use an ansatz of type (2),

$$\begin{aligned} \delta s_{[5]}^2 &= \epsilon_1 d\boldsymbol{\varkappa}^2 - e^{\psi(x,y,\chi)} (dx^2 + dy^2) \\ &\quad - 2\kappa(x, y, p) \eta_4(x, y, p) \delta p^2 + \frac{\eta_5(x, y, p)}{8\kappa(x, y, p)} \delta v^2 \\ \delta p &= dp + w_2(x, y, p) dx + w_3(x, y, p) dy, \\ \delta v &= dv + n_2(x, y, p, \chi) dx + n_3(x, y, p, \chi) dy \end{aligned} \tag{30}$$

where the local coordinates are

$$x^1 = \boldsymbol{\varkappa}, \quad x^2 = x, \quad x^3 = y, \quad y^4 = p, \quad y^5 = v,$$

and the nontrivial metric coefficients and polarizations are parametrized

$$\begin{aligned} \check{g}_1 &= \epsilon_1 = \pm 1, \quad \check{g}_2 = -1, \quad \check{g}_3 = -1, \\ \check{h}_4 &= -2\kappa(x, y, p), \quad \check{h}_5 = 1/8\kappa(x, y, p), \\ \eta_1 &= 1, \quad g_\alpha = \eta_\alpha \check{g}_\alpha. \end{aligned}$$

For trivial polarizations $\eta_\alpha = 1$ and $w_{2,3} = 0, n_{2,3} = 0$, the metric (30) is just the pp–wave solution (28).

3.1 Ricci flows of solitonic pp–wave solutions in string gravity

Our aim is to define such nontrivial values of polarization functions when $\eta_5(x, y, p)$ is defined by a 3D soliton $\eta(x, y, p)$, for instance, as a solution of solitonic equation

$$\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta \eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1, \quad (31)$$

and $\eta_2 = \eta_3 = e^{\psi(x, y, \chi)}$ is a family solutions of (12) transformed into

$$\psi^{\bullet\bullet}(\chi) + \psi''(\chi) = \frac{\lambda_H^2}{2}. \quad (32)$$

The solitonic deformations of the pp–wave metric will define exact solutions in string gravity with H –fields, see in Appendix the equations (A.3) and (A.4) for the string torsion ansatz (A.5), when with $\lambda = \lambda_H$.⁵

Introducing the above stated data for the ansatz (30) into the equation (13),⁶ we get two equations relating $h_4 = \eta_4 \check{h}_4$ and $h_5 = \eta_5 \check{h}_5$,

$$\eta_5 = 8 \kappa(x, y, p) \left[h_{5[0]}(x, y) + \frac{1}{2\lambda_H^2} e^{2\eta(x, y, p)} \right] \quad (33)$$

and

$$|\eta_4| = \frac{e^{-2\phi(x, y, p)}}{2\kappa^2(x, y, p)} \left[\left(\sqrt{|\eta_5|} \right)^* \right]^2, \quad (34)$$

where $h_{5[0]}(x, y)$ is an integration function. Having defined the coefficients h_a , we can solve the equations (14) and (15) expressing the coefficients (16) and (17) through η_4 and η_5 defined by pp– and solitonic waves as in (34) and (33). The corresponding solutions are

$$w_1 = 0, w_2 = (\phi^*)^{-1} \partial_x \phi, w_3 = (\phi^*)^{-1} \partial_x \phi, \quad (35)$$

for $\phi^* = \partial\phi/\partial p$, see formulas (19) and

$$n_1 = 0, n_{2,3} = n_{2,3}^{[0]}(x, y, \chi) + n_{2,3}^{[1]}(x, y, \chi) \int \left| \eta_4 \eta_5^{-3/2} \right| dp, \quad (36)$$

where $n_{2,3}^{[0]}(x, y, \chi)$ and $n_{2,3}^{[1]}(x, y, \chi)$ are integration functions, restricted to satisfy the conditions (11),

$$\begin{aligned} & \frac{\partial}{\partial \chi} \left[-e^{\psi(x, y, \chi)} + \eta_5(x, y, p) \check{h}_5(x, y, p) (n_{2,3}^{[0]}(x, y, \chi) \right. \\ & \left. + n_{2,3}^{[1]}(x, y, \chi) \int \left| \eta_4(x, y, p) \eta_5^{-3/2}(x, y, p) \right| dp \right]^2 = 0. \end{aligned} \quad (37)$$

We note that the ansatz (30), without dependence on χ and with the coefficients computed following the equations and formulas (32), (34), (33), (35) and (36), defines a class

⁵ as a matter of principle we can consider that ϕ is a solution of any 3D solitonic, or other, nonlinear wave equation.

⁶ such solutions can be constructed in general form (see, in details, the formulas (26)–(28) in Ref. [4], for corresponding reparametrizations)

of exact solutions (depending on integration functions) of gravitational field equations in string gravity with H -field. For corresponding families of coefficients evolving on χ and constrained to satisfy the conditions (37) we get solutions of nonholonomic Ricci flow equations (7)–(9) normalized by the effective constant λ_H induced from string gravity.

Putting the above stated functions ψ, k, ϕ and η_5 and respective integration functions into the corresponding ansatz, we define a class of evolution and/or gravity field solutions,

$$\begin{aligned} \delta s_{[sol2]}^2 &= \epsilon_1 d\mathcal{X}^2 - e^{\psi(\chi)} (dx^2 + dy^2) \\ &\quad + \frac{\eta_5}{8\kappa} \delta p^2 - \kappa^{-1} e^{-2\phi} \left[\left(\sqrt{|\eta_5|} \right)^* \right]^2 \delta \nu^2(\chi), \\ \delta p &= dp + (\phi^*)^{-1} \partial_x \phi dx + (\phi^*)^{-1} \partial_y \phi dy, \\ \delta v(\chi) &= dv + \left\{ n_2^{[0]}(\chi) + \widehat{n}_2^{[1]}(\chi) \int k^{-1} e^{2\phi} \left[\left(|\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} dx \\ &\quad + \left\{ n_3^{[0]}(\chi) + \widehat{n}_3^{[1]}(\chi) \int k^{-1} e^{2\phi} \left[\left(|\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} dy, \end{aligned} \quad (38)$$

where some constants and multiples depending on x and y are included into $\widehat{n}_{2,3}^{[1]}(x, y, \chi)$ and we emphasize the dependence of coefficients on Ricci flow parameter χ . Such families of generic off-diagonal metrics possess induced both nonholonomically and from string gravity torsion coefficients for the canonical d-connection (we omit explicit formulas for the nontrivial components which can be computed by introducing the coefficients of our ansatz into (A.2)). This class of solutions describes nonlinear interactions of pp-waves and 3D solutions in string gravity in Ricci flow theory.

The term $\epsilon_1 d\mathcal{X}^2$ can be eliminated in order to describe only 4D configurations. Nevertheless, in this case, there is not a smooth limit of such 4D solutions for $\lambda_H^2 \rightarrow 0$ to those in general relativity, see the second singular term in (33), proportional to $1/\lambda_H^2$.

Finally, note that explicit values for the integration functions and constants can be defined (for a fixed system of reference and coordinates) from certain initial value and boundary conditions. In this work, we shall analyze the properties of the derived classes of solutions and their multi-parametric transforms and geometric flows working with general forms of generation and integration functions.

3.2 Solitonic pp-waves in vacuum Einstein gravity and Ricci flows

In this section, we show how the anholonomic frame method can be used for constructing 4D metrics induced by nonlinear pp-waves and solitonic interactions for vanishing sources and the Levi Civita connection. For an ansatz of type (30), we write

$$\eta_5 = 5\kappa b^2 \text{ and } \eta_4 = h_0^2(b^*)^2/2\kappa.$$

A 3D solitonic solution can be generated if b is subjected to the condition to solve a solitonic equation, for instance, of type (31), or other nonlinear wave configuration. We chose a parametrization when

$$b(x, y, p) = \check{b}(x, y)q(p)k(p),$$

for any $\check{b}(x, y)$ and any pp-wave $\kappa(x, y, p) = \check{\kappa}(x, y)k(p)$ (we can take $\check{b} = \check{\kappa}$), where $q(p) = 4 \tan^{-1}(e^{\pm p})$ is the solution of "one dimensional" solitonic equation

$$q^{**} = \sin q. \quad (39)$$

In this case,

$$w_2 = [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}| \text{ and } w_3 = [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}|. \quad (40)$$

The final step in constructing such vacuum Einstein solutions is to chose any two functions $n_{2,3}(x, y)$ satisfying the conditions $n_2^* = n_3^* = 0$ and $n_2' - n_3^\bullet = 0$ which are necessary for Riemann foliated structures with the Levi Civita connection, see discussion of formulas (42) and (43) in Ref. [4] and conditions (18). This mean that in the integrals of type (36) we shall fix the integration functions $n_{2,3}^{[1]} = 0$ but take such $n_{2,3}^{[0]}(x, y)$ satisfying $(n_2^{[0]})' - (n_3^{[0]})^\bullet = 0$.

We can consider a trivial solution of (12), i.e. of (32) with $\lambda = \lambda_H = 0$.

Summarizing the results, we obtain the 4D vacuum metric

$$\begin{aligned} \delta s_{[sol2a]}^2 &= - (dx^2 + dy^2) - h_0^2 \check{b}^2 [(qk)^*]^2 \delta p^2 + \check{b}^2 (qk)^2 \delta v^2, \\ \delta p &= dp + [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}| dx + [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}| dy, \\ \delta v &= dv + n_2^{[0]} dx + n_3^{[0]} dy, \end{aligned} \quad (41)$$

defining nonlinear gravitational interactions of a pp-wave $\kappa = \check{\kappa}k$ and a soliton q , depending on certain type of integration functions and constants stated above. Such vacuum Einstein metrics can be generated in a similar form for 3D or 2D solitons but the constructions will be more cumbersome and for non-explicit functions, see a number of similar solutions in Refs. [12, 8].

Now, we generalize the ansatz (41) in a form describing normalized Ricci flows of the mentioned type vacuum solutions extended for a prescribed constant λ necessary for normalization. We chose

$$\begin{aligned} \delta s_{[sol2a]}^2 &= - (dx^2 + dy^2) - h_0^2 \check{b}^2(\chi) [(qk)^*]^2 \delta p^2 + \check{b}^2(\chi) (qk)^2 \delta v^2, \\ \delta p &= dp + [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}| dx + [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}| dy, \\ \delta v &= dv + n_2^{[0]}(\chi) dx + n_3^{[0]}(\chi) dy, \end{aligned} \quad (42)$$

where we introduced the parametric dependence on χ ,

$$b(x, y, p, \chi) = \check{b}(x, y, \chi) q(p) k(p)$$

which allows us to use the same formulas (40) for $w_{3,4}$ not depending on χ . The values $\check{b}^2(\chi)$ and $n_{2,3}^{[0]}(\chi)$ are constrained to be solutions of

$$\frac{\partial}{\partial \chi} [\check{b}^2 (n_{2,3}^{[0]})^2] = -2\lambda \text{ and } \frac{\partial}{\partial \chi} \check{b}^2 = 2\lambda \check{b}^2 \quad (43)$$

in order to solve, respectively, the equations (7) and (8). As a matter of principle, we can consider a flow dependence as a factor $\psi(\lambda)$ before $(dx^2 + dy^2)$, i.e. flows of the

h–components of metrics which will generalize the ansatz (42) and constraints (43). For simplicity, we have chosen a minimal extension of vacuum Einstein solutions in order to describe nonholonomic flows of the v–components of metrics adapted to the flows of N–connection coefficients $n_{2,3}^{[0]}(\chi)$. Such nonholonomic constraints on metric coefficients define Ricci flows of families of vacuum Einstein solutions defined by nonlinear interactions of a 3D soliton and a pp–wave.

4. Parametric Transforms and Flows and Solitonic pp–Wave Solutions

There are different possibilities to apply parametric and frame transforms and define Ricci flows and nonholonomic deformations of geometric objects. The first one is to perform a parametric transform of a vacuum solution and then to deform it nonholonomically in order to generate pp–wave solitonic interactions. In the second case, we can subject an already nonholonomically generated solution of type (41) to a one parameter transforms. Finally, in the third case, we can derive two parameter families of nonholonomic soliton pp–wave interactions. For simplicity, Ricci flows will be considered after certain classes of exact solutions of field equations will have been constructed.

4.1 Flows of solitonic pp–waves generated by parametric transforms

Let us consider the metric

$$\delta s_{[5a]}^2 = -dx^2 - dy^2 - 2\check{\kappa}(x, y) dp^2 + dv^2/8\check{\kappa}(x, y) \quad (44)$$

which is a particular 4D case of (28) when $\kappa(x, y, p) \rightarrow \check{\kappa}(x, y)$. It is easy to show that the nontrivial Ricci components $R_{\alpha\beta}$ for the Levi Civita connection are proportional to $\check{\kappa}^{\bullet\bullet} + \check{\kappa}''$ and the non–vanishing components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ are of type $R_{a1b1} \simeq R_{a2b2} \simeq \sqrt{(\check{\kappa}^{\bullet\bullet})^2 + (\check{\kappa}^{\bullet'})^2}$. So, any function $\check{\kappa}$ solving the equation $\check{\kappa}^{\bullet\bullet} + \check{\kappa}'' = 0$ but with $(\check{\kappa}^{\bullet\bullet})^2 + (\check{\kappa}^{\bullet'})^2 \neq 0$ defines a vacuum solution of the Einstein equations. In the simplest case, we can take $\check{\kappa} = x^2 - y^2$ or $\check{\kappa} = xy/\sqrt{x^2 + y^2}$ like it was suggested in the original work [38], but for the metric (44) we do not consider any multiple $q(p)$ depending on p .

Subjecting the metric (44) to a parametric transform, we get an off–diagonal metric of type

$$\begin{aligned} \delta s_{[2p]}^2 &= -\eta_2(x, y, \theta)dx^2 + \eta_3(x, y, \theta)dy^2 \\ &\quad -2\check{\kappa}(x, y) \eta_4(x, y, \theta, \chi)\delta p^2 + \frac{\eta_5(x, y, \theta, \chi)}{8\check{\kappa}(x, y)}\delta v^2, \\ \delta p &= dp + w_2(x, y, \theta)dx + w_3(x, y, \theta)dy, \\ \delta v &= dv + n_2(x, y, \theta, \chi)dx + n_3(x, y, \theta, \chi)dy \end{aligned} \quad (45)$$

which may define Ricci flows, or vacuum solutions of the Einstein equations, if the coefficients are restricted to satisfy the necessary conditions. Such parametric transforms consist a particular case of frame transforms when the coefficients $g_{\underline{\alpha}\underline{\beta}}$ are defined by the coefficients of (44) and ${}^{\circ}\tilde{g}_{\alpha\beta}$ are given by the coefficients (45). The polarizations $\eta_{\hat{a}}(x, y, \theta, \chi)$ and N-connection coefficients $w_{\hat{i}}(x, y, \theta)$ and $n_{\hat{i}}(x, y, \theta, \chi)$ determine the coefficients of the matrix of parametric, or Geroch, transforms (for details on nonholonomic generalizations and Geroch equations, see section 4.2 and Appendix B in Ref. [4] and sections 2.2 and 3 in Ref. [7]; we note that in this work we have an additional to θ Ricci flow parameter χ).

Considering that $\eta_2 \neq 0$,⁷ we multiply (45) on conformal factor $(\eta_2)^{-1}$ and redefining the coefficients as $\check{\eta}_3 = \eta_3/\eta_2$, $\check{\eta}_a = \eta_a/\eta_2$, $\check{w}_a = w_a$ and $\check{n}_a = n_a$, for $\hat{i} = 2, 3$ and $a = 4, 5$, we obtain

$$\begin{aligned} \delta s_{[2a]}^2 &= -dx^2 + \check{\eta}_3(x, y, \theta) dy^2 \\ &\quad - 2\check{\kappa}(x, y) \check{\eta}_4(x, y, \theta, \chi) \delta p^2 + \frac{\check{\eta}_5(x, y, \theta, \chi)}{8\check{\kappa}(x, y)} \delta v^2 \\ \delta p &= dp + \check{w}_2(x, y, \theta) dx + \check{w}_3(x, y, \theta) dy, \\ \delta v &= dv + \check{n}_2(x, y, \theta, \chi) dx + \check{n}_3(x, y, \theta, \chi) dy \end{aligned} \quad (46)$$

which is not an exact solution but can be nonholonomically deformed into exact vacuum solutions by multiplying on additional polarization parameters. Firstly, we first introduce the polarization $\eta_2 = \exp \psi(x, y, \theta, \chi)$ when $\eta_3 = \check{\eta}_3 = -\exp \psi(x, y, \theta, \chi)$ are defined as families of solutions of $\psi^{\bullet\bullet}(\chi) + \psi''(\chi) = \lambda$. Then, secondly, we redefine $\check{\eta}_a \rightarrow \eta_a(x, y, p, \chi)$ (for instance, multiplying on additional multiples) by introducing additional dependencies on "anisotropic" coordinate p such a way when the ansatz (46) transform into

$$\begin{aligned} \delta s_{[2a]}^2 &= -e^{\psi(x, y, \theta, \chi)} (dx^2 + dy^2) \\ &\quad - 2\check{\kappa}(x, y) k(p) \eta_4(x, y, p, \theta) \delta p^2 + \frac{\eta_5(x, y, p, \theta)}{8\check{\kappa}(x, y) k(p)} \delta v^2 \\ \delta p &= dp + w_2(x, y, p, \theta) dx + w_3(x, y, p, \theta) dy, \\ \delta v &= dv + n_2(x, y, \theta, \chi) dx + n_3(x, y, \theta, \chi) dy. \end{aligned} \quad (47)$$

The "simplest" Ricci flow solutions induced by flows of the h-metric and N-connection coefficients are

$$w_1 = 0, w_2(\theta) = (\phi^*)^{-1} \partial_x \phi, w_3(\theta) = (\phi^*)^{-1} \partial_x \phi, \quad (48)$$

for $\phi^* = \partial\phi/\partial p$, see formulas (19), and

$$n_{2,3} = n_{2,3}^{[0]}(x, y, \theta, \chi) + n_{2,3}^{[1]}(x, y, \theta, \chi) \int \left| \eta_4(\theta) \eta_5^{-3/2}(\theta) \right| dp, \quad (49)$$

where $n_{2,3}^{[0]}(x, y, \theta, \chi)$ and $n_{2,3}^{[1]}(x, y, \theta, \chi)$ are constrained as (11),

$$\begin{aligned} \frac{\partial}{\partial \chi} [-e^{\psi(x, y, \theta, \chi)} + \eta_5(x, y, p, \theta) \check{h}_5(x, y, p) (n_{2,3}^{[0]}(x, y, \theta, \chi) \\ + n_{2,3}^{[1]}(x, y, \theta, \chi) \int \left| \eta_4(x, y, p, \theta) \eta_5^{-3/2}(x, y, p, \theta) \right| dp)^2] = 0. \end{aligned} \quad (50)$$

⁷ $\eta_2 \rightarrow 1$ and $\eta_3 \rightarrow 1$ for infinitesimal parametric transforms

The difference of formulas (48), (49) and (50) and respective formulas (35), (36) and (37) is that the set of coefficients defining the nonholonomic Ricci flow of metrics (47) depend on a free parameter θ associated to some 'primary' Killing symmetries like it was considered by Geroch [28]. The analogy with Geroch's (parametric) transforms is more complete if we do not consider dependencies on χ and take the limit $\lambda \rightarrow 0$ which generates families, on θ , of vacuum Einstein solutions, see formula (105) in Ref. [7].

In order to define Ricci flows for the Levi Civita connection, with $g_4 = -2\check{\kappa}k\eta_4$ and $g_5 = \eta_5/8\check{\kappa}k$ from (47), the coefficients of this metric must solve the conditions (18), when the coordinates are parametrized $x^2 = x, x^3 = y, y^4 = p$ and $y^5 = v$. This describes both parametric nonholonomic transform and Ricci flows of a metric (44) to a family of evolution/ field exact solutions depending on parameter θ and defining nonlinear superpositions of pp-waves $\kappa = \check{\kappa}(x, y)k(p)$.

It is possible to introduce solitonic waves into the metric (47). For instance, we can take $\eta_5(x, y, p, \theta) \sim q(p)$, where $q(p)$ is a solution of solitonic equation (39). We obtain nonholonomic Ricci flows of a family of Einstein metrics labelled by parameter θ and defining nonlinear interactions of pp-waves and one-dimensional solitons. Such solutions with prescribed $\psi = 0$ can be parametrized in a form very similar to the ansatz (41).

4.2 Parametrized transforms and flows of nonholonomic solitonic pp-waves

We begin with the ansatz (41) defining a vacuum off-diagonal solution. That metric does not depend on variable v and possess a Killing vector $\partial/\partial v$. It is possible to apply the parametric transform writing the new family of metrics in terms of polarization functions,

$$\begin{aligned} \delta s_{[sol2\theta]}^2 &= -\eta_2(\theta') dx^2 + \eta_3(\theta') dy^2 - \eta_4(\theta') h_0^2 \check{b}^2 [(qk)^*]^2 \delta p^2 \\ &\quad + \eta_5(\theta') \check{b}^2 (qk)^2 \delta v^2, \\ \delta p &= dp + \eta_2^4(\theta') [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}| dx \\ &\quad + \eta_3^4(\theta') [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}| dy, \\ \delta v &= dv + \eta_2^5(\theta') n_2^{[0]} dx + \eta_3^5(\theta') n_3^{[0]} dy, \end{aligned} \quad (51)$$

where all polarization functions $\eta_{\hat{\alpha}}(x, y, p, \theta')$ and $\eta_{\hat{\alpha}}^g(x, y, p, \theta')$ depend on anisotropic coordinate p , labelled by a parameter θ' . The new class of solutions contains the multiples $q(p)$ and $k(p)$ defined respectively by solitonic and pp-waves and depends on certain integration functions like $n_i^{[0]}(x, y)$ and integration constant h_0^2 . Such values can be defined exactly by stating an explicit coordinate system and for certain boundary and initial conditions.

It should be noted that the metric (51) can not be represented in a form typical for nonholonomic frame vacuum ansatz for the Levi Civita connection (i.e. it can not be represented, for instance, in the form (78) with the coefficients satisfying the conditions (79) in Ref. [7]). This is obvious because in our case η_2 and η_3 may depend on anisotropic coordinates p , i.e. our ansatz is not similar to (2), which is necessary for the anholonomic

frame method. Nevertheless, such classes of metrics define exact vacuum solutions as a consequence of the Geroch method (for nonholonomic manifolds, we call it also the method of parametric transforms). This is the priority to consider together both methods: we can parametrize different type of transforms by polarization functions in a unified form and in different cases such polarizations will be subjected to corresponding type of constraints, generating anholonomic deformations or parametric transforms.

Nevertheless, we can generate nonholonomic Ricci flows solutions from the very beginning, considering such flows, at the first step of transforms, for the metric (41), prescribing a constant λ necessary for normalization, which result in (42) and at the second step to apply the parametric transforms. After first step, we get an ansatz

$$\begin{aligned} \delta s_{[sol2a,\chi]}^2 &= - [\eta_2(\chi)dx^2 + \eta_3(\chi)dy^2] \\ &\quad - h_0^2 \check{b}^2(\chi) [(qk)^*]^2 \delta p^2 + \check{b}^2(\chi) (qk)^2 \delta v^2, \\ \delta p &= dp + [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}(\chi)| dx \\ &\quad + [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}(\chi)| dy, \\ \delta v &= dv + n_2^{[0]}(\chi) dx + n_3^{[0]}(\chi) dy, \end{aligned} \quad (52)$$

where we introduced the parametric dependence on χ ,

$$b(x, y, p, \chi) = \check{b}(x, y, \chi) q(p) k(p)$$

which allows us to use the same formulas (40) for $w_{3,4}$ not depending on χ . The values $\check{b}^2(\chi)$ and $n_2^{[0]}(\chi)$ are constrained to be solutions of

$$\frac{\partial}{\partial \chi} \left[-\eta_{2,3}(x, y, \chi) + \check{b}^2(x, y, \chi) (q(p)k(p))^2 (n_{2,3}^{[0]}(x, y, \chi))^2 \right] = 0,$$

obtained by introducing (52) into (7). The second step is to introduce polarizations functions $\eta_{\hat{\alpha}}(x, y, p, \theta')$ and $\eta_i^a(x, y, p, \theta')$ for a parametric transform, which is possible because we have a Killing symmetry on $\partial/\partial v$ and (52) is an Einstein metric (we have to suppose that such parametric transforms can be defined as solutions of the Geroch equations [7, 28] for the Einstein spaces, not only for vacuum metrics, at least for small prescribed cosmological constants). Finally, we get a two parameter metric, on θ' and χ ,

$$\begin{aligned} \delta s_{[sol2a,\chi]}^2 &= - [\eta_2(\theta')dx^2 + \eta_3(\theta')dy^2] - \eta_4(\theta') h_0^2 \check{b}^2(\chi) [(qk)^*]^2 \delta p^2 \\ &\quad + \eta_5(\theta') \check{b}^2(\chi) (qk)^2 \delta v^2, \\ \delta p &= dp + \eta_2^4(\theta') [(\ln |qk|)^*]^{-1} \partial_x \ln |\check{b}| dx \\ &\quad + \eta_3^4(\theta') [(\ln |qk|)^*]^{-1} \partial_y \ln |\check{b}| dy, \\ \delta v &= dv + \eta_2^5(\theta') n_2^{[0]}(\chi) dx + \eta_3^5(\theta') n_3^{[0]}(\chi) dy, \end{aligned} \quad (53)$$

with the nonholonomic Ricci flow evolution equations

$$\begin{aligned} &\frac{\partial}{\partial \chi} \{ -\eta_{2,3}(x, y, p, \chi, \theta') + \\ &\eta_5(x, y, p, \theta') \check{b}^2(x, y, \chi) (q(p)k(p))^2 \left[\eta_3^5(x, y, p, \theta') n_{2,3}^{[0]}(x, y, \chi) \right]^2 \} = 0, \end{aligned}$$

to which one reduces the equation (7) by introducing (52). Such parametric nonholonomic Ricci flows can be constrained for the Levi Civita connection if we consider coefficients satisfying certain conditions equivalent to (18) and (19), imposed for the coefficients of auxiliary metric (52), when

$$\begin{aligned} \eta_2(x, y, \chi) &= g_3(x, y, \chi) = -e^{\psi(x, y, \chi)}, \\ h_4 &= -h_0^2 \check{b}^2(\chi) [(qk)^*]^2, h_5 = \check{b}^2(\chi) (qk)^2, \\ w_{\check{i}} &= \partial_{\check{i}} \phi / \phi^*, \text{ where } \varphi = -\ln \left| \sqrt{|h_4 h_5|} / |h_5^*| \right|, \\ n_{2,3}(\chi) &= n_{2,3}^{[0]}(x, y, \chi) \end{aligned}$$

are constrained to

$$\begin{aligned} \psi^{\bullet\bullet}(\chi) + \psi''(\chi) &= -\lambda \\ h_5^* \phi / h_4 h_5 &= \lambda, \\ w'_2 - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n'_2(\chi) - n_3^\bullet(\chi) &= 0. \end{aligned}$$

In a more general case, we can model Killing—Ricci flows for the canonical d–connections.

4.3 Two parameter nonholonomic solitonic pp–waves and flows

Finally, we give an explicit example of solutions with two parameter (θ', θ) –metrics (see definition of such frame transform by formulas (81) in Ref. [7] and (69) in Ref. [4]). We begin with the ansatz metric ${}_1\check{\mathbf{g}}_{[2a]}(\theta)$ (47) with the coefficients subjected to constraints (18) for $\lambda \rightarrow 0$ and coordinates parametrized $x^2 = x, x^3 = y, y^4 = p$ and $y^5 = v$. We consider that the solitonic wave ϕ is included as a multiple in η_5 and that $\kappa = \check{\kappa}(x, y)k(p)$ is a pp–wave. This family of vacuum metrics ${}_1\check{\mathbf{g}}_{[2a]}(\theta)$ does not depend on variable v , i.e. it possess a Killing vector $\partial/\partial v$, which allows us to apply a parametric transform as we described in the previous example. The resulting two parameter family of solutions, with redefined polarization functions, is given by the ansatz

$$\begin{aligned} \delta s_{[2a]}^2 &= -e^{\psi(x, y, \theta)} (\bar{\eta}_2(x, y, p, \theta') dx^2 + \bar{\eta}_3(x, y, p, \theta') dy^2) \\ &\quad - 2\check{\kappa}(x, y)k(p) \eta_4(x, y, p, \theta) \bar{\eta}_4(x, y, p, \theta') \delta p^2 \\ &\quad + \frac{\eta_5(x, y, p, \theta) \bar{\eta}_5(x, y, p, \theta')}{8\check{\kappa}(x, y)k(p)} \delta v^2 \\ \delta p &= dp + w_2(x, y, p, \theta) \bar{\eta}_2^4(x, y, p, \theta') dx + w_3(x, y, p, \theta) \bar{\eta}_3^4(x, y, p, \theta') dy, \\ \delta v &= dv + n_2(x, y, \theta) \bar{\eta}_2^5(x, y, p, \theta') dx + n_3(x, y, \theta) \bar{\eta}_3^5(x, y, p, \theta') dy. \end{aligned} \tag{54}$$

The set of multiples in the coefficients are parametrized following the conditions: The value $\check{\kappa}(x, y)$ is just that defining an exact vacuum solution for the primary metric (44) stating the first type of parametric transforms. Then we consider the pp–wave component $k(p)$ and the solitonic wave included in $\eta_5(x, y, p, \theta)$ such way that the functions $\psi, \eta_{4,6}, w_{2,3}$ and $n_{2,3}$ are subjected to the condition to define the class of metrics (47).

The metrics are parametrized both by θ , following solutions of the Geroch equations (see, for instance, the Killing (8) and (9) in Ref. [4]), and by a N-connection splitting with $w_{2,3}$ and $n_{2,3}$, all adapted to the corresponding nonholonomic deformation derived for $g_2(\theta) = g_3(\theta) = e^{\psi(\theta)}$ and $g_4 = 2\check{\kappa}k \eta_4$ and $g_5 = \eta_5/8\check{\kappa}k$ subjected to the conditions (18)). This set of functions also defines a new set of Killing equations, for any metric (47) allowing to find the "overlined" polarizations $\bar{\eta}_i(\theta')$ and $\bar{\eta}_i^a(\theta')$.

For compatible nonholonomic Ricci flows of the h-metric and N-connection coefficients, the class of two parametric vacuum solutions can be extended for a prescribed value λ and new parameter χ ,

$$\begin{aligned} \delta s_{[2a,\chi]}^2 &= -e^{\psi(x,y,\theta,\chi)} (\bar{\eta}_2(x,y,p,\theta',\chi)dx^2 + \bar{\eta}_3(x,y,p,\theta',\chi)dy^2) \\ &\quad - 2\check{\kappa}(x,y)k(p)\eta_4(x,y,p,\theta)\bar{\eta}_4(x,y,p,\theta')\delta p^2 \\ &\quad + \frac{\eta_5(x,y,p,\theta)\bar{\eta}_5(x,y,p,\theta')}{8\check{\kappa}(x,y)k(p)}\delta v^2 \\ \delta p &= dp + w_2(x,y,p,\theta)\bar{\eta}_2^4(x,y,p,\theta')dx \\ &\quad + w_3(x,y,p,\theta)\bar{\eta}_3^4(x,y,p,\theta')dy, \\ \delta v &= dv + n_2(x,y,\theta,\chi)\bar{\eta}_2^5(x,y,p,\theta',\chi)dx \\ &\quad + n_3(x,y,\theta,\chi)\bar{\eta}_3^5(x,y,p,\theta',\chi)dy, \end{aligned} \tag{55}$$

where, for simplicity, we redefine the coefficients in the form

$$\begin{aligned} \tilde{\eta}_2(x,y,p,\theta',\theta,\chi) &= e^{\psi(x,y,\theta,\chi)}\bar{\eta}_2(x,y,p,\theta',\chi) = \\ \tilde{\eta}_3(x,y,p,\theta',\theta,\chi) &= e^{\psi(x,y,\theta,\chi)}\bar{\eta}_3(x,y,p,\theta',\chi), \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_4(x,y,p,\theta',\theta) &= \eta_4(x,y,p,\theta)\bar{\eta}_4(x,y,p,\theta'), \\ \tilde{\eta}_5(x,y,p,\theta',\theta) &= \eta_5(x,y,p,\theta)\bar{\eta}_5(x,y,p,\theta'), \\ \tilde{w}_{2,3}(x,y,p,\theta',\theta) &= w_{2,3}(x,y,p,\theta)\bar{\eta}_{2,3}^4(x,y,p,\theta'), \\ \tilde{n}_{2,3}(x,y,p,\theta',\theta,\chi) &= n_{2,3}(x,y,\theta,\chi)\bar{\eta}_{2,3}^5(x,y,p,\theta',\chi). \end{aligned}$$

Such polarization functions, in general, parametrized by (θ',θ,χ) , allow us to write the conditions (7) for the classes of metrics (55) in a compact form,

$$\frac{\partial}{\partial \chi} \left\{ -\tilde{\eta}_{2,3}(\theta',\theta,\chi) + \frac{\tilde{\eta}_5(\theta',\theta)}{8\check{\kappa}k} [\tilde{n}_{2,3}(\theta',\theta,\chi)]^2 \right\} = 0,$$

where, for simplicity, we emphasized only the parametric dependencies.

For the configurations with Levi Civita connections, we have to consider additional constraints of type (18),

$$\begin{aligned} \tilde{\eta}_2^{\bullet\bullet}(\chi) + \tilde{\eta}_2'(\chi) &= -\lambda \\ h_5^*\phi/h_4h_5 &= \lambda, \\ \tilde{w}_2' - \tilde{w}_3^\bullet + \tilde{w}_3\tilde{w}_2^* - \tilde{w}_2\tilde{w}_3^* &= 0, \\ n_2'(x,y,\theta',\theta,\chi) - n_3^\bullet(x,y,\theta',\theta,\chi) &= 0 \end{aligned}$$

for

$$\begin{aligned}\tilde{h}_4 &= -2\check{\kappa}(x, y)k(p)\tilde{\eta}_4(x, y, p, \theta', \theta), \quad \tilde{h}_5 = \tilde{\eta}_5(x, y, p, \theta', \theta)/8\check{\kappa}(x, y)k(p), \\ \tilde{w}_i &= \partial_i \tilde{\varphi} / \tilde{\varphi}^*, \quad \text{where } \tilde{\varphi} = -\ln \left| \sqrt{|h_4 h_5|} / |h_5^*| \right|.\end{aligned}$$

The classes of vacuum Einstein metrics (54) and their, normalized by a prescribed λ , nonholonomic Ricci flows (55), depend on certain classes of general functions (nonholonomic and parametric transform polarizations and integration functions). It is obvious that they define two parameter (θ', θ) nonlinear superpositions of solitonic waves and pp-waves evolving on parameter χ . From formal point of view, the procedure can be iterated for any finite or infinite number of θ -parameters not depending on coordinates (in principle, such parameters can depend on flow parameter, but we omit such constructions in this work). We can construct an infinite number of nonholonomic vacuum states in gravity, and their possible Ricci flows, constructed from off-diagonal superpositions of nonlinear waves. Such two transforms do not commute and depend on order of successive applications.

The nonholonomic deformations not only mix and relate nonlinearly two different "Killing" classes of solutions but introduce into the formalism the possibility to consider flow evolution configurations and other new very important and crucial properties. For instance, the polarization functions can be chosen such ways that the vacuum solutions will possess noncommutative and/algebroid symmetries even at classical level, or generalized configurations in order to contain contributions of torsion, nonmetricity and/or string fields in various generalized models of Lagrange–Hamilton algebroids, string, brane, gauge, metric–affine and Finsler–Lagrange gravity, see Refs. [10, 9, 8, 39].

5. Ricci Flows and Parametric Nonholonomic Deformations of the Schwarzschild Metric

We construct new classes of exact solutions for Ricci flows and nonholonomic deformations of the Schwarzschild metric. There are analyzed physical effects of parametrized families of generic off-diagonal flows and interactions with solitonic pp-waves.

5.1 Deformations and flows of stationary backgrounds

Following the methods outlined Refs. [4, 7], we nonholonomically deform on angular variable φ the Schwarzschild type solution (20) into a generic off-diagonal stationary metric. For nonholonomic Einstein spaces, we shall use an ansatz of type

$$\begin{aligned}\delta s_{[1]}^2 &= \epsilon_1 d\chi^2 - \eta_2(\xi) d\xi^2 - \eta_3(\xi) r^2(\xi) d\vartheta^2 \\ &\quad - \eta_4(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \delta\varphi^2 + \eta_5(\xi, \vartheta, \varphi) \varpi^2(\xi) \delta t^2, \\ \delta\varphi &= d\varphi + w_2(\xi, \vartheta, \varphi) d\xi + w_3(\xi, \vartheta, \varphi) d\vartheta, \\ \delta t &= dt + n_2(\xi, \vartheta) d\xi + n_3(\xi, \vartheta) d\vartheta,\end{aligned}\tag{56}$$

where we shall use 3D spacial spherical coordinates, $(\xi(r), \vartheta, \varphi)$ or (r, ϑ, φ) . The non-holonomic transform generating this off-diagonal metric are defined by $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$ where $(\check{g}_i, \check{h}_a)$ are given by data (21).

5.1.1 General nonholonomic polarizations

We can construct a class of metrics of type (2) with the coefficients subjected to the conditions (18) (in this case, for the ansatz (56) with coordinates $x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t$). The solution of (13), for $\lambda = 0$, in terms of polarization functions, can be written

$$\sqrt{|\eta_4|} = h_0 \sqrt{\left| \frac{\check{h}_5}{\check{h}_4} \right|} \left(\sqrt{|\eta_5|} \right)^* , \quad (57)$$

where \check{h}_a are coefficients stated by the Schwarzschild solution for the chosen system of coordinates but η_5 can be any function satisfying the condition $\eta_5^* \neq 0$. We shall use certain parametrizations of solutions when

$$\begin{aligned} -h_0^2 (b^*)^2 &= \eta_4(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \\ b^2 &= \eta_5(\xi, \vartheta, \varphi) \varpi^2(\xi) \end{aligned}$$

The polarizations η_2 and η_3 can be taken in a form that $\eta_2 = \eta_3 r^2 = e^{\psi(\xi, \vartheta, \chi)}$,

$$\psi^{\bullet\bullet} + \psi'' = 0,$$

defining solutions of (12) for $\lambda = 0$. The solutions of (14) and (15) for vacuum configurations of the Levi Civita connection are constructed

$$w_2 = \partial_\xi(\sqrt{|\eta_5|} \varpi) / \left(\sqrt{|\eta_5|} \right)^* \varpi, \quad w_3 = \partial_\vartheta(\sqrt{|\eta_5|}) / \left(\sqrt{|\eta_5|} \right)^*$$

and any $n_{2,3}(\xi, \vartheta)$ for which $n'_2(\chi) - n'_3(\chi) = 0$. For any function $\eta_5 \sim a_1(\xi, \vartheta) a_2(\varphi)$, the integrability conditions (18) and (19).

We conclude that the stationary nonholonomic deformations of the Schwarzschild metric are defined by the off-diagonal ansatz

$$\begin{aligned} \delta s_{[1]}^2 &= \epsilon_1 d\chi^2 - e^\psi (d\xi^2 + d\vartheta^2) \\ &\quad - h_0^2 \varpi^2 \left[\left(\sqrt{|\eta_5|} \right)^* \right]^2 \delta\varphi^2 + \eta_5 \varpi^2 \delta t^2, \\ \delta\varphi &= d\varphi + \frac{\partial_\xi(\sqrt{|\eta_5|} \varpi)}{\left(\sqrt{|\eta_5|} \right)^* \varpi} d\xi + \frac{\partial_\vartheta(\sqrt{|\eta_5|})}{\left(\sqrt{|\eta_5|} \right)^*} d\vartheta, \\ \delta t &= dt + n_2 d\xi + n_3 d\vartheta, \end{aligned} \quad (58)$$

where the coefficients do not depend on Ricci flow parameter λ . Such vacuum solutions were constructed mapping a static black hole solution into Einstein spaces with locally anisotropic backgrounds (on coordinate φ) defined by an arbitrary function $\eta_5(\xi, \vartheta, \varphi)$ with $\partial_\varphi \eta_5 \neq 0$, an arbitrary $\psi(\xi, \vartheta)$ solving the 2D Laplace equation and certain integration functions $n_{2,3}(\xi, \vartheta)$ and integration constant h_0^2 . In general, the solutions from the

target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole. We can also consider a prescribed physical situation when, for instance, η_5 mimics 3D, or 2D, solitonic polarizations on coordinates ξ, ϑ, φ , or on ξ, φ .

For a family of metrics (58), we can consider the "nearest" extension to flows of N-connection coefficients $w_{2,3} \rightarrow w_{2,3}(\chi)$ and $n_{2,3} \rightarrow n_{2,3}(\chi)$, when for $\lambda = 0$, and $R_{\alpha\beta} = 0$, the equation (7) is satisfied if

$$h_0^2 \left[\left(\sqrt{|\eta_5|} \right)^* \right]^2 \frac{\partial (w_{2,3})^2}{\partial \chi} = \eta_5 \frac{\partial (n_{2,3})^2}{\partial \chi}. \tag{59}$$

The metric coefficients for such Ricci flows are the same as for the exact vacuum non-holonomic deformation but with respect to evolving N-adapted dual basis

$$\begin{aligned} \delta\varphi(\chi) &= d\varphi + w_2(\xi, \vartheta, \varphi, \chi)d\xi + w_3(\xi, \vartheta, \varphi, \chi)d\vartheta, \\ \delta t &= dt + n_2(\xi, \varphi, \chi)d\xi + n_3(\xi, \vartheta, \chi)d\vartheta, \end{aligned} \tag{60}$$

with the coefficients being defined by any solution of (59) and (18) and (19) for $\lambda = 0$.

5.1.2 Solutions with small nonholonomic polarizations

In a more special case, in order to select physically valuable configurations, it is better to consider decompositions on a small parameter $0 < \varepsilon < 1$ in (58), when

$$\begin{aligned} \sqrt{|\eta_4|} &= q_4^{\hat{0}}(\xi, \varphi, \vartheta) + \varepsilon q_4^{\hat{1}}(\xi, \varphi, \vartheta) + \varepsilon^2 q_4^{\hat{2}}(\xi, \varphi, \vartheta) \dots, \\ \sqrt{|\eta_5|} &= 1 + \varepsilon q_5^{\hat{1}}(\xi, \varphi, \vartheta) + \varepsilon^2 q_5^{\hat{2}}(\xi, \varphi, \vartheta) \dots, \end{aligned}$$

where the "hat" indices label the coefficients multiplied to $\varepsilon, \varepsilon^2, \dots$ ⁸ The conditions (57) are expressed in the form

$$\varepsilon h_0 \sqrt{\left| \frac{\check{h}_5}{\check{h}_4} \right|} \left(q_5^{\hat{1}} \right)^* = q_4^{\hat{0}}, \quad \varepsilon^2 h_0 \sqrt{\left| \frac{\check{h}_5}{\check{h}_4} \right|} \left(q_5^{\hat{2}} \right)^* = \varepsilon q_4^{\hat{1}}, \dots$$

This system can be solved in a form compatible with small decompositions if we take the integration constant, for instance, to satisfy the condition $\varepsilon h_0 = 1$ (choosing a corresponding system of coordinates). For this class of small deformations, we can prescribe a function $q_4^{\hat{0}}$ and define $q_5^{\hat{1}}$, integrating on φ (or inversely, prescribing $q_5^{\hat{1}}$, then taking the partial derivative ∂_φ , to compute $q_4^{\hat{0}}$). In a similar form, there are related the coefficients $q_4^{\hat{1}}$ and $q_5^{\hat{2}}$. A very important physical situation is to select the conditions when such small nonholonomic deformations define rotoid configurations. This is possible, for instance, if

$$2q_5^{\hat{1}} = \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0) - \frac{1}{r^2}, \tag{61}$$

⁸ Of course, this way we construct not an exact solution, but extract from a class of exact ones (with less clear physical meaning) certain subclasses of solutions decomposed (deformed) on a small parameter being related to the Schwarzschild metric.

where ω_0 and φ_0 are constants and the function $q_0(r)$ has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before δt^2 is approximated in the form

$$\eta_5 \varpi^2 = 1 - \frac{2\mu}{r} + \varepsilon \left(\frac{1}{r^2} + 2q_5^{\hat{1}} \right).$$

This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration) given by

$$r_+ \simeq \frac{2\mu}{1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0)}.$$

Such solutions with ellipsoid symmetry seem to define static black ellipsoids (they were investigated in details in Refs. [40, 41]). The ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. This class of vacuum metrics violates the conditions of black hole uniqueness theorems [30] because the "surface" gravity is not constant for stationary black ellipsoid deformations. So, we can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distortions related to generic off-diagonal metric terms. Putting $\varphi_0 = 0$, in the limit $\omega_0 \rightarrow 0$, we get $q_5^{\hat{1}} \rightarrow 0$ in (61). This allows to state the limits $q_4^{\hat{0}} \rightarrow 1$ for $\varepsilon \rightarrow 0$ in order to have a smooth limit to the Schwarzschild solution for $\varepsilon \rightarrow 0$. Here, one must be emphasized that to extract the spherical static black hole solution is possible if we parametrize, for instance,

$$\delta\varphi = d\varphi + \varepsilon \frac{\partial_\xi(\sqrt{|\eta_5|}\varpi)}{\left(\sqrt{|\eta_5|}\right)^* \varpi} d\xi + \varepsilon \frac{\partial_\vartheta(\sqrt{|\eta_5|})}{\left(\sqrt{|\eta_5|}\right)^*} d\vartheta$$

and

$$\delta t = dt + \varepsilon n_2(\xi, \vartheta) d\xi + \varepsilon n_3(\xi, \vartheta) d\vartheta.$$

One can be defined certain more special cases when $q_5^{\hat{2}}$ and $q_4^{\hat{1}}$ (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

For Ricci flows on N-connection coefficients, such stationary rotoid configurations evolve with respect to small deformations of co-frames (60), $\delta\varphi(\chi)$ and $\delta t(\chi)$, with the coefficients proportional to ε .

5.1.3 Parametric nonholonomic transforms of the Schwarzschild solution and their flows

The ansatz (58) does not depend on time variable and possesses a Killing vector $\partial/\partial t$. We can apply parametric transforms and generate families of new solutions depending on a

parameter θ . Following the same steps as for generating (51), we construct

$$\begin{aligned} \delta s_{[1]}^2 &= -e^\psi (\tilde{\eta}_2(\theta)d\xi^2 + \tilde{\eta}_3(\theta)d\vartheta^2) \\ &\quad - h_0^2 \varpi^2 \left[\left(\sqrt{|\eta_5|} \right)^* \right]^2 \tilde{\eta}_4(\theta)\delta\varphi^2 + \eta_5 \varpi^2 \tilde{\eta}_5(\theta)\delta t^2, \\ \delta\varphi &= d\varphi + \tilde{\eta}_2^4(\theta) \frac{\partial_\xi(\sqrt{|\eta_5|\varpi})}{\left(\sqrt{|\eta_5|\varpi}\right)^* \varpi} d\xi + \tilde{\eta}_3^4(\theta) \frac{\partial_\vartheta(\sqrt{|\eta_5|\varpi})}{\left(\sqrt{|\eta_5|\varpi}\right)^*} d\vartheta, \\ \delta t &= dt + \tilde{\eta}_2^5(\theta)n_2(\xi, \vartheta)d\xi + \tilde{\eta}_3^5(\theta)n_3(\xi, \vartheta)d\vartheta, \end{aligned} \tag{62}$$

where polarizations $\tilde{\eta}_{\tilde{\alpha}}(\xi, \vartheta, \varphi, \theta)$ and $\tilde{\eta}_{\tilde{i}}^a(\xi, \vartheta, \varphi, \theta)$ are defined by solutions of the Geroch equations for Killing symmetries of the vacuum metric (58). Even this class of metrics does not satisfy the vacuum equations for a typical anholonomic ansatz, they define vacuum exact solutions and we can apply the formalism on decomposition on a small parameter ε like we described in previous section 5.1.2 (one generates not exact solutions, but like in quantum field theory it can be more easy to formulate a physical interpretation). For instance, we consider a vacuum background consisting from solitonic wave polarizations, with components mixed by the parametric transform, and then to compute nonholonomic deformations of a Schwarzschild black hole self-consistently imbedded in a such nonperturbative background.

Nonholonomic Ricci flows induced by the N-connection coefficients are given by flow equations of type (60), $\delta\varphi(\chi)$ and $\delta t(\chi)$, with the coefficients depending additionally on χ , for instance,

$$\begin{aligned} w_{2,3}(\xi, \vartheta, \varphi, \theta, \chi) &= \tilde{\eta}_{2,3}^4(\xi, \vartheta, \varphi, \theta, \chi) \frac{\partial_\xi(\sqrt{|\eta_5|\varpi})}{\left(\sqrt{|\eta_5|\varpi}\right)^* \varpi}, \\ n_{2,3}(\xi, \vartheta, \theta, \chi) &= \tilde{\eta}_2^5(\xi, \vartheta, \theta, \chi)n_2(\xi, \vartheta). \end{aligned} \tag{63}$$

Of course, in order to get Ricci flows with the Levi Civita connection, the coefficients of (62) and evolving N-connection coefficients (63) have to be additionally constrained by conditions of type (18) and (19) for $\lambda = 0$.

5.2 Anisotropic polarizations on extra dimension coordinate

On can be constructed certain classes of exact off-diagonal solutions when the extra dimension effectively polarizes the metric coefficients and interaction constants. We take as a primary metric the ansatz (22) (see the parametrization for coordinates for that quadratic element, with $x^1 = \varphi$, $x^2 = \check{\vartheta}$, $x^3 = \check{\xi}$, $y^4 = \varkappa$, $y^5 = t$) and consider the off-diagonal target metric

$$\begin{aligned} \delta s_{[5\varkappa]}^2 &= -r_g^2 d\varphi^2 - r_g^2 \eta_2(\xi, \check{\vartheta})d\check{\vartheta}^2 + \eta_3(\xi, \check{\vartheta})\check{g}_3(\check{\vartheta}) d\check{\xi}^2 \\ &\quad + \epsilon_4 \eta_4(\xi, \check{\vartheta}, \varkappa)\delta\varkappa^2 + \eta_5(\xi, \check{\vartheta}, \varkappa) \check{h}_5(\xi, \check{\vartheta}) \delta t^2 \\ \delta\varkappa &= d\varkappa + w_2(\xi, \check{\vartheta}, \varkappa)d\xi + w_3(\xi, \check{\vartheta}, \varkappa)d\check{\vartheta}, \\ \delta t &= dt + n_2(\xi, \check{\vartheta}, \varkappa)d\xi + n_3(\xi, \check{\vartheta}, \varkappa)d\check{\vartheta}. \end{aligned} \tag{64}$$

The coefficients of this ansatz,

$$\begin{aligned} g_1 &= -r_g^2, g_2 = -r_g^2 \eta_2(\xi, \check{\vartheta}), g_3 = \eta_3(\xi, \check{\vartheta}) \check{g}_3(\check{\vartheta}), \\ h_4 &= \epsilon_1 \eta_4(\xi, \check{\vartheta}, \varkappa), h_5 = \eta_5(\xi, \check{\vartheta}, \varkappa) \check{h}_5(\xi, \check{\vartheta}) \end{aligned}$$

are subjected to the condition to solve the system of equations (12)–(15) with a nontrivial cosmological constant defined, for instance, from string gravity by a corresponding ansatz for H -fields with $\lambda = -\lambda_H^2/2$, or other type cosmological constants, see details on such nonholonomic configurations in Refs. [4, 7].

The general solution is given by the data

$$-r_g^2 \eta_2 = \eta_3 \check{g}_3 = \exp 2\psi(\xi, \check{\vartheta}), \quad (65)$$

where ψ is the solution of

$$\begin{aligned} \psi^{\bullet\bullet} + \psi'' &= \lambda, \\ \eta_4 &= h_0^2(\xi, \check{\vartheta}) [f^*(\xi, \check{\vartheta}, \varkappa)]^2 |\varsigma(\xi, \check{\vartheta}, \varkappa)| \\ \eta_5 \check{h}_5 &= [f(\xi, \check{\vartheta}, \varkappa) - f_0(\xi, \check{\vartheta})]^2, \end{aligned} \quad (66)$$

where

$$\begin{aligned} \varsigma(\xi, \check{\vartheta}, \varkappa) &= \varsigma_{[0]}(\xi, \check{\vartheta}) \\ &+ \frac{\epsilon_4}{16} h_0^2(\xi, \check{\vartheta}) \lambda_H^2 \int f^*(\xi, \check{\vartheta}, \varkappa) [f(\xi, \check{\vartheta}, \varkappa) - f_0(\xi, \check{\vartheta})] d\varkappa. \end{aligned}$$

The N-connection coefficients $N_i^4 = w_i(\xi, \check{\vartheta}, \varkappa)$, $N_i^5 = n_i(\xi, \check{\vartheta}, \varkappa)$ are computed following the formulas

$$w_{\hat{i}} = -\frac{\partial_{\hat{i}} \varsigma(\xi, \check{\vartheta}, \varkappa)}{\varsigma^*(\xi, \check{\vartheta}, \varkappa)} \quad (67)$$

$$n_{\hat{k}} = n_{\hat{k}[1]}(\xi, \check{\vartheta}) + n_{\hat{k}[2]}(\xi, \check{\vartheta}) \int \frac{[f^*(\xi, \check{\vartheta}, \varkappa)]^2}{[f(\xi, \check{\vartheta}, \varkappa) - f_0(\xi, \check{\vartheta})]^3} \varsigma(\xi, \check{\vartheta}, \varkappa) d\varkappa. \quad (68)$$

The solutions depend on arbitrary nontrivial functions $f(\xi, \check{\vartheta}, \varkappa)$ (with $f^* \neq 0$), $f_0(\xi, \check{\vartheta})$, $h_0^2(\xi, \check{\vartheta})$, $\varsigma_{[0]}(\xi, \check{\vartheta})$, $n_{k[1]}(\xi, \check{\vartheta})$ and $n_{k[2]}(\xi, \check{\vartheta})$, and value of cosmological constant λ . These values have to be defined by certain boundary conditions and physical considerations. In the sourceless case, $\varsigma_{[0]} \rightarrow 1$. For the Levi Civita connection, we have to consider $h_0^2(\xi, \check{\vartheta}) \rightarrow const$ and have to prescribe the integration functions of type $n_{\hat{k}[2]} = 0$ and $n_{\hat{k}[1]}$ solving the equation $\partial_{\check{\vartheta}} n_{2[1]} = \partial_{\xi} n_{3[1]}$, in order to satisfy some conditions of type (18) and (19).

The class of solutions (64) define self-consistent nonholonomic maps of the Schwarzschild solution into a 5D backgrounds with nontrivial sources, depending on a general function $f(\xi, \check{\vartheta}, \varkappa)$ and mentioned integration functions and constants. Fixing $f(\xi, \check{\vartheta}, \varkappa)$ to be a 3D soliton (we can consider also solitonic pp-waves as in previous sections) running on extra dimension \varkappa , we describe self-consisted embedding of the Schwarzschild solutions

into nonlinear wave 5D curved spaces. In general, it is not clear if any target solutions preserve the black hole character of primary solution. It is necessary a very rigorous analysis of geodesic configurations on such spacetimes, definition of horizons, singularities and so on. Nevertheless, for small nonholonomic deformations (by introducing a small parameter ε , like in the section 5.1.2), we can select classes of "slightly" deformed solutions preserving the primary black hole character. In 5D, such solutions are not subjected to the conditions of black hole uniqueness theorems, see [30] and references therein.

The ansatz (64) posses two Killing vector symmetries, on $\partial/\partial t$ and $\partial/\partial\varphi$. In the sourceless case, we can apply a parametric transform and generate new families depending on a parameter θ' . The constructions are similar to those generating (62) (we omit here such details). Here we emphasize that we can not apply a parametric transform to the primary metric (22) (it is not a vacuum solution) in order to generate families of parametric solutions with the aim to subject them to further anholonomic transforms.

For nontrivial cosmological constant (normalization), the metric (64) can be generalized for nonholonomic Ricci flows of type

$$\begin{aligned} \delta s_{[5\mathcal{X}\chi]}^2 &= -r_g^2 d\varphi^2 - r_g^2 \eta_2(\xi, \check{\vartheta}, \chi) d\check{\vartheta}^2 + \eta_3(\xi, \check{\vartheta}, \chi) \check{g}_3(\check{\vartheta}) d\check{\xi}^2 \\ &\quad + \epsilon_4 \eta_4(\xi, \check{\vartheta}, \mathcal{X}) \delta \mathcal{X}^2 + \eta_5(\xi, \check{\vartheta}, \mathcal{X}) \check{h}_5(\xi, \check{\vartheta}) \delta t^2 \\ \delta \mathcal{X} &= d\mathcal{X} + w_2(\xi, \check{\vartheta}, \mathcal{X}) d\xi + w_3(\xi, \check{\vartheta}, \mathcal{X}) d\check{\vartheta}, \\ \delta t(\chi) &= dt + n_2(\xi, \check{\vartheta}, \chi) d\xi + n_3(\xi, \check{\vartheta}, \mathcal{X}, \chi) d\check{\vartheta}. \end{aligned} \tag{69}$$

where the equation (7) imposes constraints of type (11)

$$\frac{\partial}{\partial \chi} [g_{2,3}(\xi, \check{\vartheta}, \chi) + h_5(\xi, \check{\vartheta}, \mathcal{X}) (n_{2,3}(\xi, \check{\vartheta}, \mathcal{X}, \chi))^2] = 0,$$

with is very different from constraints of type (59), for

$$\begin{aligned} g_2 &= -r_g^2 \eta_2(\xi, \check{\vartheta}, \chi), g_3 = \eta_3(\xi, \check{\vartheta}, \chi) \check{g}_3(\check{\vartheta}), h_5 = \eta_5(\xi, \check{\vartheta}, \mathcal{X}) \check{h}_5(\xi, \check{\vartheta}), \\ n_{\check{k}}(\chi) &= n_{\check{k}}^{[1]}(\xi, \check{\vartheta}, \chi) + n_{\check{k}}^{[2]}(\xi, \check{\vartheta}, \chi) \int \frac{[f^*(\xi, \check{\vartheta}, \mathcal{X})]^2}{[f(\xi, \check{\vartheta}, \mathcal{X}) - f_0(\xi, \check{\vartheta})]^3} \varsigma(\xi, \check{\vartheta}, \mathcal{X}) d\mathcal{X}. \end{aligned}$$

For holonomic Ricci flows with the Levi Civita connection, we have to consider additional constraints

$$\begin{aligned} \psi^{\bullet\bullet}(\chi) + \psi''(\chi) &= \lambda \\ h_5^* \phi / h_4 h_5 &= \lambda, \\ w_2' - w_3^\bullet + w_3 w_2^* - w_2 w_3^* &= 0, \\ n_2'(\chi) - n_3^\bullet(\chi) &= 0. \end{aligned} \tag{70}$$

for

$$\begin{aligned} g_2(\xi, \check{\vartheta}, \chi) &= g_3(\xi, \check{\vartheta}, \chi) = e^{2\psi(\xi, \check{\vartheta}, \chi)}, h_4 = \epsilon_4 \eta_4(\xi, \check{\vartheta}, \mathcal{X}), \\ w_{\check{i}} &= \partial_{\check{i}} \phi / \phi^*, \text{ where } \varphi = -\ln \left| \sqrt{|h_4 h_5|} / |h_5^*| \right|, \\ n_{2,3}(\chi) &= n_{2,3}^{[1]}(\xi, \check{\vartheta}, \chi). \end{aligned}$$

This class of Ricci flows, defined by the family of solutions (69) describes deformed Schwarzschild metrics, running on extra dimension coordinate \varkappa with mutually compatible evolution of the h–component of metric and the n –coefficients of the N–connection.

5.3 5D solutions with nonholonomic time like coordinate

We use the primary metric (24) (which is not a vacuum solution and does not admit parametric transforms but can be nonholonomically deformed) resulting in a target off–diagonal ansatz,

$$\begin{aligned} \delta s_{[3t]}^2 &= -r_g^2 d\varphi^2 - r_g^2 \eta_2(\xi, \check{\vartheta}) d\check{\vartheta}^2 + \eta_3(\xi, \check{\vartheta}) \check{g}_3(\check{\vartheta}) d\check{\xi}^2 \\ &\quad + \eta_4(\xi, \check{\vartheta}, t) \check{h}_4(\xi, \check{\vartheta}) \delta t^2 + \epsilon_5 \eta_5(\xi, \check{\vartheta}, t) \delta \varkappa^2, \\ \delta t &= dt + w_2(\xi, \check{\vartheta}, t) d\xi + w_3(\xi, \check{\vartheta}, t) d\check{\vartheta}, \\ \delta \varkappa &= d\varkappa + n_2(\xi, \check{\vartheta}, t) d\xi + n_3(\xi, \check{\vartheta}, t) d\check{\vartheta}, \end{aligned} \quad (71)$$

where the local coordinates are established $x^1 = \varphi$, $x^2 = \check{\vartheta}$, $x^3 = \check{\xi}$, $y^4 = t$, $y^5 = \varkappa$ and the polarization functions and coefficients of the N–connection are chosen to solve the system of equations (12)–(15). Such solutions are generic 5D and emphasize the anisotropic dependence on time like coordinate t . The coefficients can be computed by the same formulas (65) and (66) as in the previous section, for the ansatz (64), by changing the coordinate t into \varkappa and, inversely, \varkappa into t . This class of solutions depends on a function $f(\xi, \check{\vartheta}, t)$, with $\partial_t f \neq 0$, and on integration functions (depending on ξ and $\check{\vartheta}$) and constants. We can consider more particular physical situations when $f(\xi, \check{\vartheta}, t)$ defines a 3D solitonic wave, or a pp–wave, or their superpositions, and analyze configurations when a Schwarzschild black hole is self–consistently embedded into a dynamical 5D background. We analyzed certain similar physical situations in Ref. [12] when an extra dimension soliton ”running” away a 4D black hole.

The set of 5D solutions (71) posses two Killing vector symmetry, $\partial/\partial t$ and $\partial/\partial \varkappa$, like in the previous section, but with another types of vectors. For the vacuum configurations, it is possible to perform a parametric transform and generate parametric (on θ') 5D solutions (labelling, for instance, packages of nonlinear waves).

For nontrivial cosmological constant (normalization), the metric (71) also can be generalized to describe nonholonomic Ricci flows

$$\begin{aligned} \delta s_{[3t,\chi]}^2 &= -r_g^2 d\varphi^2 - r_g^2 \eta_2(\xi, \check{\vartheta}, \chi) d\check{\vartheta}^2 + \eta_3(\xi, \check{\vartheta}, \chi) \check{g}_3(\check{\vartheta}) d\check{\xi}^2 \\ &\quad + \eta_4(\xi, \check{\vartheta}, t) \check{h}_4(\xi, \check{\vartheta}) \delta t^2 + \epsilon_5 \eta_5(\xi, \check{\vartheta}, t) \delta \varkappa^2, \\ \delta t &= dt + w_2(\xi, \check{\vartheta}, t) d\xi + w_3(\xi, \check{\vartheta}, t) d\check{\vartheta}, \\ \delta \varkappa(\chi) &= d\varkappa + n_2(\xi, \check{\vartheta}, t, \chi) d\xi + n_3(\xi, \check{\vartheta}, t, \chi) d\check{\vartheta}, \end{aligned} \quad (72)$$

where the equation (7) imposes constraints of type (11)

$$\frac{\partial}{\partial \chi} [g_{2,3}(\xi, \check{\vartheta}, \chi) + h_5(\xi, \check{\vartheta}, \varkappa) (n_{2,3}(\xi, \check{\vartheta}, \varkappa, \chi))^2] = 0,$$

for

$$g_2 = -r_g^2 \eta_2(\xi, \check{\nu}, \chi), g_3 = \eta_3(\xi, \check{\nu}, \chi) \check{g}_3(\check{\nu}), h_5 = \epsilon_5 \eta_5(\xi, \check{\nu}, t),$$

$$n_{\check{k}}(\chi) = n_{\check{k}}^{[1]}(\xi, \check{\nu}, \chi) + n_{\check{k}}^{[2]}(\xi, \check{\nu}, \chi) \int \frac{[f^*(\xi, \check{\nu}, t)]^2}{[f(\xi, \check{\nu}, t) - f_0(\xi, \check{\nu})]^3} \varsigma(\xi, \check{\nu}, t) dt.$$

For holonomic Ricci flows with the Levi Civita connection, we have to consider additional constraints of type (70) with re-defined coefficients and coordinates, when

$$g_2(\xi, \check{\nu}, \chi) = g_3(\xi, \check{\nu}, \chi) = e^{2\psi(\xi, \check{\nu}, \chi)}, h_4 = \eta_4(\xi, \check{\nu}, t) \check{h}_4(\xi, \check{\nu}),$$

$$w_{\check{i}} = \partial_{\check{i}} \phi / \phi^*, \text{ where } \varphi = -\ln \left| \sqrt{|h_4 h_5|} / |h_5^*| \right|,$$

$$n_{2,3}(\chi) = n_{2,3}^{[1]}(\xi, \check{\nu}, \chi).$$

This class of Ricci flows, defined by the family of solutions (72) describes deformed Schwarzschild metrics, running on time like coordinate t with mutually compatible evolution of the h-component of metric and the n -coefficients of the N-connection.

6. Discussion

We constructed exact solutions in gravity and Ricci flow theory following superpositions of the parametric and anholonomic frame transforms. A geometric method previously elaborated in our partner works [4, 7] was applied to generalizations of valuable physical solutions (like solitonic waves, pp-waves and Schwarzschild metrics) in vacuum gravity. In this work, our investigations were restricted to nonholonomic Ricci flows of the mentioned type solutions modelled with respect certain classes of compatible metric and associated nonlinear connection (N-connection) coefficients when the solutions of evolution/ field equations can be obtained in general form.

The first advance is the possibility to generalize vacuum metrics by allowing realistic string gravity or matter field sources which can be encoded as an effective (in general, nonhomogeneous) cosmological constant on nonholonomic (pseudo) Riemannian spaces of dimensions four and five (4D and 5D) and deriving nonlinear solitonic and pp-wave interactions and their Ricci flows.

The second kind of progress is the proof of existence of multi-parametric transforms, associated to certain Killing symmetries, like the Geroch equations [28, 29], mapping certain target metrics (in our case of physical importance) into different classes of generic off-diagonal exact solutions admitting different scenarios of Ricci flows depending on the type of nonholonomic frame constraints.

The outcome of the first advance is rather satisfactory: we can in a similar way consider parametric deformations of metrics and flows of geometric and physical objects by obtaining, for instance, static rotoid configurations, solitonic and pp-wave propagation of black holes on time like and extra dimension coordinates.

However, the outcome of the second kind of progress raises as many problems as it solves: we should provide a physical motivation for the multi-parameter dependence and

'hidden' Killing symmetry under nonholonomic deformations. If one of the parameters is identified with the Ricci flow parameter, it may be considered to describe a corresponding evolution. In general, this may be associated to chains of Ricci multi-flows but not obligatory. We have to speculate additionally on physical meaning of such parametric solutions both for vacuum gravitational and Einstein spaces and in Ricci flow theory when metrics and connections are subjected to nonholonomic constraints on coefficients and associated frames.

Whereas most previous work on Ricci flow theory and applications has concentrated on some approximate methods and simplest classes of solutions, the present paper aims to elaboration of general geometric methods of constructing solutions and deriving their physically important symmetries. There were stated exact principles how the physically important solutions in gravity theories can be deformed in multi-parametric ways to describe off-diagonal nonlinear gravitational and matter field interactions and the evolution of physical and geometric objects. The first step was to derive exact solutions in the most possible general form preserving dependence not only on transform and flow parameters but on classes of generating and integration functions and constants. Further work would be needed to analyze more rigorously certain important physical effects with exactly defined boundary and initial conditions when the integration functions and constants are defined in explicit form.

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A Cosmological Constants and Strings

The simplest way to perform a local covariant calculus by applying d-connections is to use N-adapted differential forms and to introduce the d-connection 1-form $\Gamma^\alpha_\beta = \Gamma^\alpha_{\beta\gamma} \mathbf{e}^\gamma$, when the N-adapted components of d-connection $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$ are computed following formulas

$$\Gamma^\gamma_{\alpha\beta}(u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma, \quad (\text{A.1})$$

where " \rfloor " denotes the interior product. This allows us to define in N-adapted torsion $\mathbf{T} = \{\mathcal{T}^\alpha\}$,

$$\mathcal{T}^\alpha \doteq \mathbf{D} \mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma^\alpha_\beta \wedge \mathbf{e}^\beta, \quad (\text{A.2})$$

and curvature $\mathbf{R} = \{\mathcal{R}^\alpha_\beta\}$,

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D} \Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\gamma_\beta \wedge \Gamma^\alpha_\gamma.$$

In string gravity, the nontrivial torsion components and string corrections to matter sources in the Einstein equations can be related to certain effective interactions with the strength (torsion)

$$H_{\mu\nu\rho} = \mathbf{e}_\mu B_{\nu\rho} + \mathbf{e}_\rho B_{\mu\nu} + \mathbf{e}_\nu B_{\rho\mu}$$

of an antisymmetric field $B_{\nu\rho}$, when

$$R_{\mu\nu} = -\frac{1}{4}H_{\mu}{}^{\nu\rho}H_{\nu\lambda\rho} \quad (\text{A.3})$$

and

$$D_{\lambda}H^{\lambda\mu\nu} = 0, \quad (\text{A.4})$$

see details on string gravity, for instance, in Refs. [5, 6]. The conditions (A.3) and (A.4) are satisfied by the ansatz

$$H_{\mu\nu\rho} = \widehat{Z}_{\mu\nu\rho} + \widehat{H}_{\mu\nu\rho} = \lambda_{[H]}\sqrt{|g_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} \quad (\text{A.5})$$

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and the distortion (from the Levi–Civita connection) and

$$\widehat{Z}_{\mu\alpha\beta}\mathbf{c}^{\mu} = \mathbf{e}_{\beta}\rfloor\mathcal{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\mathcal{T}_{\beta} + \frac{1}{2}(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\mathcal{T}_{\gamma})\mathbf{c}^{\gamma}$$

is defined by the torsion tensor (A.2), which for the canonical d–connection is induced by the coefficients of N–connection, see details in [10, 8, 1, 2]. We emphasize that our H –field ansatz is different from those formally used in string gravity when $\widehat{H}_{\mu\nu\rho} = \lambda_{[H]}\sqrt{|g_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho}$. In our approach, we define $H_{\mu\nu\rho}$ and $\widehat{Z}_{\mu\nu\rho}$ from the respective ansatz for the H –field and nonholonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the ‘deformed’ H–field as $\widehat{H}_{\mu\nu\rho} = \lambda_{[H]}\sqrt{|g_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} - \widehat{Z}_{\mu\nu\rho}$.

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