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The Motion of A Test Particle in the Gravitational Field of A Collapsing Shell

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Abstract: We use the Israel formalism to describe the motion of a test particle in the gravitational field of a collapsing shell. The formalism is developed in both of Schwarzchild and Kruskal coordinates.

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1. Introduction

The dynamics of thin shells of matter in general relativity has been discussed by many authors. Our approach is similar to that used by Israel [1] and Kuchar [2] for studying the collapse of spherical shells. Israel [1] found invariant boundary conditions connecting the extrinsic curvature of a shell in space-times on both sides of it with the matter of this shell. In this paper we study the motion of a test particle in the gravitational field of a collapsing shell. In Section 2 we give the general formalism. In Section 3 this formalism is applied to the thin shell in Schwarzschild space-time and test particle. The equations of motion for shell and particle in Kruskal coordinates are given in Section 4, and the equations of motion of a shell and particle in Minkowski space are given in Section 5. Finally Section 6 is devoted to the numerical results and discussion. Also adopt the units such that $c = G = 1$.

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2. Formalism

The junction conditions for arbitrary surfaces and the equation for a shell have been well understood since the works of Israel [1], Barrabes [3], de la Cruz and Israel [4], Kuchar [2], Chase [5], Balbinot and Poisson [6], and Lake [7].

The time-like hypersurfaceΣ , which divides the Riemannian space-time M into two regions, M $^-$and M^+, represents the history of a thin spherical shell of matter in general. The regions M^- and M^+ are covered by the mutually independent coordinate systems X^α_- and X^α_+. The hypersurface Σ represents the boundary of M^- and M^+ respectively; consequently the intrinsic geometry of Σ induced by the metrics of M^- and M^+ must be the same. Let Σ be parameterized by intrinsic coordinates ζ^a,

$$X^\alpha_\pm = X^\alpha_\pm(\zeta^\alpha) \tag{1}$$

(Greek letters refer to 4-dimensional indices, Latin letters refer to 3-dimensional indices on Σ. The signature of the metric is +2, and the Newtonian gravitational constant and light velocity are equal to unity). The basic vectors $e_a = \frac{\partial}{\partial \zeta^a}$ tangent to Σ have the components

$$e^\alpha_{a\pm} = \frac{\partial X^a_\pm}{\partial \zeta^a} \tag{2}$$

with respect to the two four-dimensional coordinate systems in M^- and M^+. Their scalar products define the metric induced on the hypersurface Σ,

$$g^\pm_{ab} = g^\pm_{\mu\nu} e^\mu_{a\pm} e^\nu_{b\pm} \tag{3}$$

The metric induced by the metrics of both regions M^- and M^+ must be identical, $g^+_{ab}(\zeta) = g^-_{ab}(\zeta) = g_{ab}(\zeta)$. The condition is stated independently of the coordinate systems in $M^-$and M^+. The unit normal vector n to Σ:

$$n.n \mid_\pm = 1 \tag{4}$$

is directed from M^- and M^+. The manner in which Σ is bent in space M^- and M^+ is characterized by the three-dimensional extrinsic curvature tensor

$$K^\pm_{ab} = -n^\pm_\alpha \frac{De^\alpha_{a\pm}}{\partial \zeta^b} = e_{a\alpha\pm} \frac{Dn^{\alpha\pm}}{\partial \zeta^b} \tag{5}$$

where $\frac{D}{\partial \zeta^b}$ represents the absolute derivative with respect to ζ^b. The surface energy-momentum tensor t_{ab} is determined by the jump $[K_{ab}] = K^+_{ab} - K^-_{ab}$. Σ represents the history of a surface layer (a singular hypersurface of order one) if $K^+_{ab} \neq K^-_{ab}$. The Einstein equation determines the relations between the extrinsic curvature $K^\pm_{ab}$ and the three-dimensional intrinsic energy-momentum tensor ($t_{ab} = t_{\alpha\beta} e^\alpha_a e^\beta_b$) is given by the Lanczos equation

$$[K_{ab}] = -8\pi(t_{ab} - \frac{1}{2}tg_{ab}) \tag{6}$$

where $t = t^a_a$. We can write this relation in the form

$$t_{ab} = \frac{-1}{8\pi}([K_{ab}] - g_{ab}[K] \tag{7}$$

where $[K] = g^{ab}[K_{ab}]$. These are the field equations for the shell

3. The Motion of Shell and Test Particle

3.1 The Motion of One Shell in The Schwarzschild Space

The shell is spherically symmetric. Therefore, the space-time outside the shell can be described by the line element,

$$ds^2 = -f dt_+^2 + f^{-1}dr_+^2 + r_+^2 d\Omega^2 \tag{8}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the line element on the unit sphere and

$$f = 1 - \frac{2m}{r}$$

where m is the gravitational mass in the exterior space. Inside the shell space-time is flat, i.e., $f = 1$. As exterior and interior coordinates, we use $X_+^\alpha = (t_+, r_+, \theta, \phi)$ and $X_-^\alpha = (t_-, r_-, \theta, \phi)$ respectively. The intrinsic coordinates on Σ are the proper time τ measured by the co-moving observer on the shell and the spherical angles $\theta, \phi : \xi^a = (\tau, \theta, \phi)$. Let us suppose that Σ is determined by

$$r_\pm = R(\tau),\ t_\pm = t_\pm(\tau),\ \theta_\pm = \theta,\ \phi_\pm = \phi$$

(condition (3) implies the continuity of r on the shell). We get from (2) and (4)

$$\begin{aligned} e^\alpha_{\tau\pm} &= (\dot{t}_\pm, \dot{R}, 0, 0), \\ e^\alpha_{\theta\pm} &= (0, 0, 1, 0), \\ e^\alpha_{\phi\pm} &= (0, 0, 0, 1), \end{aligned} \tag{9.a}$$

and

$$n_{\alpha\pm} = (-\dot{R}, \dot{t}_\pm, 0, 0) \tag{9.b}$$

where the dot represents the derivative with respect to proper time, and R is the radius of the shell. Thus the three-dimensional metric tensor on Σ is

$$g_{ab} = (-1, R^2, R^2 \sin^2 \theta)$$

From (5) we get the extrinsic curvature $K^{\pm}_{ab}$ in M^- and M^+,

$$K^-_{\tau\tau} = \dot{R}\ddot{t_-} - \ddot{R}t_-,$$

$$K^-_{\theta\theta} = R\dot{t_-},$$
$$K^-_{\phi\phi} = R\dot{t_-} \sin^2 \theta, \tag{10.a}$$

and

$$K^+_{\tau\tau} = -\dot{t_+}\ddot{R} + \dot{R}\ddot{t_+} - \frac{1}{2}\dot{t_+}[\dot{t_+}^2 f\acute{f} - 3f^{-1}\acute{f}\dot{R}^2],$$
$$K^+_{\theta\theta} = fR\dot{t_+},$$
$$K^+_{\phi\phi} = fR\dot{t_+} \sin^2 \theta \tag{10.b}$$

We suppose that the form of the three-dimensional energy-momentum tensor is

$$t_{ab} = Pg_{ab} + (P + \sigma)U_a U_b$$

where σ is the surface density and P is the surface pressure. Therefore the components of t_{ab} are

$$t_{\tau\tau} = \frac{-1}{4\pi R}(\dot{t_+}f - \dot{t_-}) \tag{11.a}$$

$$t_{\theta\theta} = \frac{-R^2}{8\pi}[H + f\dot{t}_+(\frac{3\dot{R}^2}{2f} - \frac{1}{2}f\dot{t_+}^2) + \frac{1}{R}(\dot{t_-} - f\dot{t_+})] \tag{11.b}$$

$$t_{\phi\phi} = t_{\theta\theta} \sin^2 \theta \tag{11.c}$$

where

$$H = \ddot{R}(\dot{t_-} - \dot{t_+}) + \dot{R}(\ddot{t_+} - \ddot{t_-})$$

Because τ is the proper time on the shell the conditions

$$\dot{t}_-^2 = 1 + \dot{R}^2 \tag{12.a}$$

$$(f\dot{t}_+)^2 = f + \dot{R}^2 \tag{12.b}$$

must be fulfilled. Supposing $t_{\tau\tau} = \sigma$ and $t_{\theta\theta} = t_{\phi\phi} = P$, we get from (12) and (11) ,

$$4\pi R\sigma = \sqrt{1 + \dot{R}^2} \mp \sqrt{f + \dot{R}^2} \tag{13.a}$$

$$8\pi R^2 P = \frac{1}{\sqrt{1 + \dot{R}^2}\sqrt{f + \dot{R}^2}}[m\sqrt{1 + \dot{R}^2} + L] \tag{13.b}$$

where

$$L = R(R\ddot{R} - \sqrt{1+\dot{R}^2}\sqrt{f+\dot{R}^2})(\sqrt{1+\dot{R}^2} \mp \sqrt{f+\dot{R}^2})$$

All relations $(8-13)$ are valid both above and under the horizon. Above the horizon $\dot{t}_+$ and $f>0$, so that $f\dot{t}_+ = \sqrt{f+\dot{R}^2}$ and the upper sign is valid in $(13a)$. Under the horizon $f<0$ and $\dot{t}_+$ can be either positive or negative. Consequently, if $\dot{t}_+ > 0$ the sign $(+)$ must be taken in (13). The sign changes for $R = \frac{M^2}{2m}$ where $M = 4\pi R^2\sigma$. We put $(13a)$ into $(13b)$ to get the relation between the surface pressure and the surface density in the form

$$P = \frac{m}{8\pi R^2\sqrt{f+\dot{R}^2}} + (\frac{R\ddot{R} - \sqrt{1+\dot{R}^2}\sqrt{f+\dot{R}^2}}{2\sqrt{1+\dot{R}^2}\sqrt{f+\dot{R}^2}})\sigma \tag{14}$$

In the case of dust $P=0$ and $U^a = (1,0,0)$; then equation (13a) where $4\pi R^2\sigma = M = const.$,

$$\sqrt{1+\dot{R}^2} \mp \sqrt{f+\dot{R}^2} = \frac{M}{R} \tag{15}$$

is an integral of motion of (14), and it fully determines the motion of the shell.

3.2 The Motion of a Test Particle

The motion of a test particle is given by the geodesic equation. The velocity of the particle is

$$\dot{r} = \pm\sqrt{c_1 - f_p} \tag{16}$$

where $f_p = 1 - \frac{2m}{r}$ and c_1 is determined according to our choice of the initial value. Therefore, the equations of motion of the test particle in Kruskal coordinates are

$$\ddot{r} = -\frac{m}{r^2} \tag{17.a}$$

$$\ddot{u}_p = A_p\dot{v}_p[r\ddot{r} + (1+\frac{r}{2m})\dot{r}^2 + \frac{r}{4m}] + v_p[\frac{\dot{r}}{8m}(1+\frac{r}{2m})] \tag{17.b}$$

$$\ddot{v}_p = A_p\dot{u}_p[r\ddot{r} + (1+\frac{r}{2m})\dot{r}^2 + \frac{r}{4m}] + u_p[\frac{\dot{r}}{8m}(1+\frac{r}{2m})] \tag{17.c}$$

where

$$A_p = \frac{-1}{8m^2(v_p\dot{u}_p - u_p\dot{v}_p)}e^{\frac{r}{2m}}$$

Equation $(17.a)$ represents the acceleration of the particle if it is outside the shell and if r is far from the centre of the shell. If the particle is inside the shell, in flat-space, its acceleration is zero and its velocity is the same as we obtained during passage through the surface of the shell.

3.3 The Intersection of The Shell and Particle

Let us take a particle moving in the gravitational field of a collapsing thin shell. We suppose that the particle interacts with the shell only gravitationally. In this case it must be fulfilled that the projection of the four-velocity of the particle before the intersection U^α_+ on n^+_α is equal to the projection of the four-velocity of the particle after the intersection U^α_- on n^-_α

$$U^\alpha_+ n_{\alpha+} \mid_{(before)} = U^\alpha_- n_{\alpha-} \mid_{(after)} .$$

Also, the projection of the four-velocities of the particle before and after crossing on the tangent e^α_a are equal,

$$e^\alpha_{a+} U_{\alpha+} \mid_{(before)} = e^\alpha_{a-} U_{\alpha-} \mid_{(after)} .$$

From these relations we get the velocity of the particle after crossing the shell from outside

$$\dot{r}_- \equiv \dot{r}_{p-} = \psi^2 \dot{R}_- [\dot{v}\dot{v}_p - \dot{u}\dot{u}_p] + \dot{t}_- [\dot{v}\dot{u}_p - \dot{u}\dot{v}_p] \tag{18}$$

Similarly, if the particle crosses the shell from inside,

$$\dot{u}_p = \dot{r}_{p-} [\dot{t}_- \dot{v} - \dot{R}_- \dot{u}] + \dot{t}_{p-} [\dot{t}_- \dot{u} - \dot{R}_- \dot{v}] \tag{19}$$

Because τ_p is a proper time of the test particle, the conditions

$$\begin{aligned} \dot{t}_{p-} &= \sqrt{1 + \dot{r}^2_{p-}}, \\ \dot{t}_{p+} &= f_p^{-1} \sqrt{f_p + \dot{r}^2_{p+}}, \\ \dot{v}_p &= \sqrt{\dot{u}^2_p + \psi_p^{-2}}, \\ \dot{t}_- &= \sqrt{1 + \dot{R}^2_-} \end{aligned} \tag{20}$$

must be fulfilled. At the point of intersection $R = r = R_0$, where R_0 is the radius of the shell at the point of intersection. The suffix '+' means a value of the quantity in the region to which the normal vector is directed at the points of the shell and '-' means the quantity on the other side of the shell.

3.4 The Motion of a Shell and Particle in a Flat Space

The radius of the shell changes smoothly on the surface of the shell, $R_+(\tau) = R_-(\tau)$, therefore the equation of the motion of the shell in Minkowski space is

$$\dot{R}_- = -\sqrt{(\frac{M}{2R} + \frac{m}{M})^2 - 1}$$

and

$$\dot{t}_- = \sqrt{1 + \dot{R}_-^2}.$$

The acceleration of the particle inside the shell will be $\ddot{r}_- = 0$ and $\ddot{t}_- = 0$. Therefore, the velocity is $\dot{r}_- = const. \equiv \dot{r}_{p-}$ and $\dot{t}_- = const. \equiv \dot{t}_{p-}$, where $\dot{r}_{p-}$ and $\dot{t}_{p-}$ are determined from (18) at the point of intersection. This represents

$$r_- = \dot{r}_{p-}\tau_p + R_0,$$

and

$$t_- = \dot{t}_{p-}\tau_p + t_{01}$$

where t_{01} is the Minkowskian time and R_0 is the radius of the shell at the point of intersection.

To find the space-time point at which the particle's four- velocity intersects the hypersurface representing the shell we use the Kruskal coordinate v as the independent variable.

4. The Motion of A Shell and Particle in Kruskal Coordinates

The motion of the shell can be described by

$$R' = G[-Rve^{\frac{R}{2m}} \pm \frac{u}{q}\sqrt{q^2R^2e^{\frac{R}{m}} - 2mR\phi e^{\frac{R}{2m}}}] \tag{21.a}$$

$$u' = \frac{1}{u}(v + \frac{RR'}{8m^2}e^{\frac{R}{2m}}) \tag{21.b}$$

where

$$G = \frac{8m^2q^2\psi^2}{(64m^4u^2 + R^2q^2\psi^2e^{\frac{R}{m}})},$$

$$q = -\sqrt{(\frac{M}{2R} + \frac{m}{M})^2 - 1},$$

$$\phi \equiv v^2 - u^2 = (1 - \frac{R}{2m})e^{\frac{R}{2m}}.$$

The motion of the test particle is described by

$$r' = G_p[-rv_pe^{\frac{r}{2m}} \pm \frac{u_p}{q_p}\sqrt{q_p^2r^2e^{\frac{r}{m}} - 2mr\phi_pe^{\frac{r}{2m}}}] \tag{22.a}$$

$$u_p' = \frac{1}{u_p}(v_p + \frac{rr'}{8m^2}e^{\frac{r}{2m}}) \tag{22.b}$$

where

$$G_p = \frac{8m^2q_p^2\psi_p^2}{(64m^4u_p^2 + r^2q_p^2\psi_p^2e^{\frac{r}{m}})},$$

$$u_p = \sqrt{v_p^2 + (\frac{r}{2m} - 1)e^{\frac{r}{2m}}},$$

$$\phi_p \equiv v_p^2 - u_p^2 = (1 - \frac{r}{2m})e^{\frac{r}{2m}},$$

$$q_p = -\sqrt{\frac{2m}{r} - 1 + c_1}.$$

The constant c_1 is determined by the initial value. We shall study the shell with $m = M$ (the shell which starts to collapse from infinity with vanishing velocity R'). We integrate its equation of motion, starting from some initial radius R_0. For the particle we assume an initial radius r_0 and the value of the constant c_1 is determined by the condition $r' = R'$ in the initial moment. Therefore,

$$c_1 = 1 - \frac{2m}{r_0} + \frac{m}{R_0} + (\frac{m}{2R_0})^2 \tag{23}$$

as the consequence of (15) and (16). The relation between Minkowskian time, Schwarzschild time and independent variable v are

$$t'_- = (\frac{M}{2R} + \frac{m}{M})\sqrt{\psi^2(1 - u'^2)} \tag{24.a}$$

and

$$t' = \frac{4m(u - vu')}{u^2 - v^2} \tag{24.b}$$

where $R' = \frac{dR}{dv}$ and $\dot{v} = \frac{dv}{d\tau}$.

5. The Motion of A Shell and Particle in Minkowski Space

Since $R_- = R_+$ on the surface of the shell and $v = v_a$ at the point of intersection is the independent variable, therefore the motion of the shell is

$$\widetilde{R}_- = q\sqrt{\psi^2(1 - u'^2)} \tag{25.a}$$

where

$$q = -\sqrt{(\frac{M}{2R} + \frac{m}{M})^2 - 1}$$

with

$$\widetilde{t}_- = (\frac{M}{2R} + \frac{m}{M})\sqrt{\psi^2(1 - u'^2)} \tag{25.b}$$

where $\widetilde{R} = \frac{dR}{dv_a}$ and $\widetilde{t} = \frac{dt}{dv_a}$. The motion of the particle is

$$\widetilde{r}_- = r'_{p-}\widetilde{t}_- \tag{25.c}$$

where r'_{p-} is the velocity of the particle after crossing the shell from outside determined from (18) at the point of intersection and given by

$$r'_{p-} = \frac{Q}{\sqrt{1+Q^2}} \tag{25.d}$$

where

$$Q = H[(\frac{M}{2R} + \frac{m}{M})(u'_p - u') + \frac{\widetilde{R}_-(1-u'u'_p)}{\sqrt{\psi^2(1-u'^2)}}]$$

and

$$H = \frac{1}{\sqrt{(1-u'^2)(1-{u'_p}^2)}}.$$

The particle can intersect the shell again and come out to the exterior Schwarzschild space. From (19) we can determine the velocity u'_{p+} of the particle after crossing,

$$u'_{p+} = \frac{Z\sqrt{\psi^2}}{\sqrt{1+Z^2\psi^2}} \tag{26}$$

where

$$Z = \frac{(r'_{p-} - \widetilde{R}_-) + u'(1 - r'_{p-}\widetilde{R}_-)}{\sqrt{(1-\widetilde{R}^2_-)(1-{r'_{p-}}^2)(\psi^2(1-u'^2))}}.$$

In this case q_p in equation $(22.b)$ will change to q_{p1} given by

$$q_{p1} = -\sqrt{\frac{2m}{r} - 1 + c_2} \tag{27.a}$$

Therefore,

$$r' = q_{p1}\sqrt{\psi_p^2(1-{u'_p}^2)} \tag{27.b}$$

Determine c_2 from

$$c_2 = f_b + \frac{2me^{\frac{-r_b}{2m}}(\omega u_p - v_p)^2}{r_b(1-\omega^2)}$$

where

$$\omega = \frac{1}{u_p}(v_p + \frac{r_b r'}{8m^2}e^{\frac{r_b}{2m}})$$

where $R_- = r_- = r_b$ and $f_b = 1 - \frac{2m}{r_b}$. Inserting c_2 into (27) and solving differential equations (21) and (22) we get the motion of the shell and of the particle.

6. Numerical Solution and Discussion

Let us solve differential equations (21), (22) and (24) with these initial values

$$r_g = 2m = 200, \qquad R_0 = 5r_g, \qquad r_0 = 1.001R_0$$

$$t_0 = t_{0-} = 0, \qquad v = 0$$

From (23) we get the value of c_1 and u_0, u_{p0} from

$$u_0 = \sqrt{v^2 + (\frac{R_0}{2m} - 1)e^{\frac{R_0}{2m}}}$$
$$u_{p0} = \sqrt{v^2 + (\frac{r_0}{2m} - 1)e^{\frac{r_0}{2m}}}.$$

Since the velocity of the particle is greater than the velocity of the shell, the particle crosses the shell. In the flat space inside the shell the particle moves at a constant velocity, while the shell is accelerated. Consequently, they will intersect again, and so on; i.e., the particle oscillates around the shell. We found many intersections before they collapse to singularity. This is shown in Figure 1.

During the collapse, the lapses of the proper time for the shell and the particle ($\Delta\tau_{s(i)} = \tau_{s(i)} - \tau_{s(i-1)}$, $\Delta\tau_{p(i)} = \tau_{p(i)} - \tau_{p(i-1)}$) between the points of intersection are diminishing. The rate of change of the proper time ($\frac{\Delta\tau_s}{\Delta\tau_p}$) between the points of intersection is diminishing, too. The shell goes to singularity faster than the test particle from the point of view of the co-moving observers.

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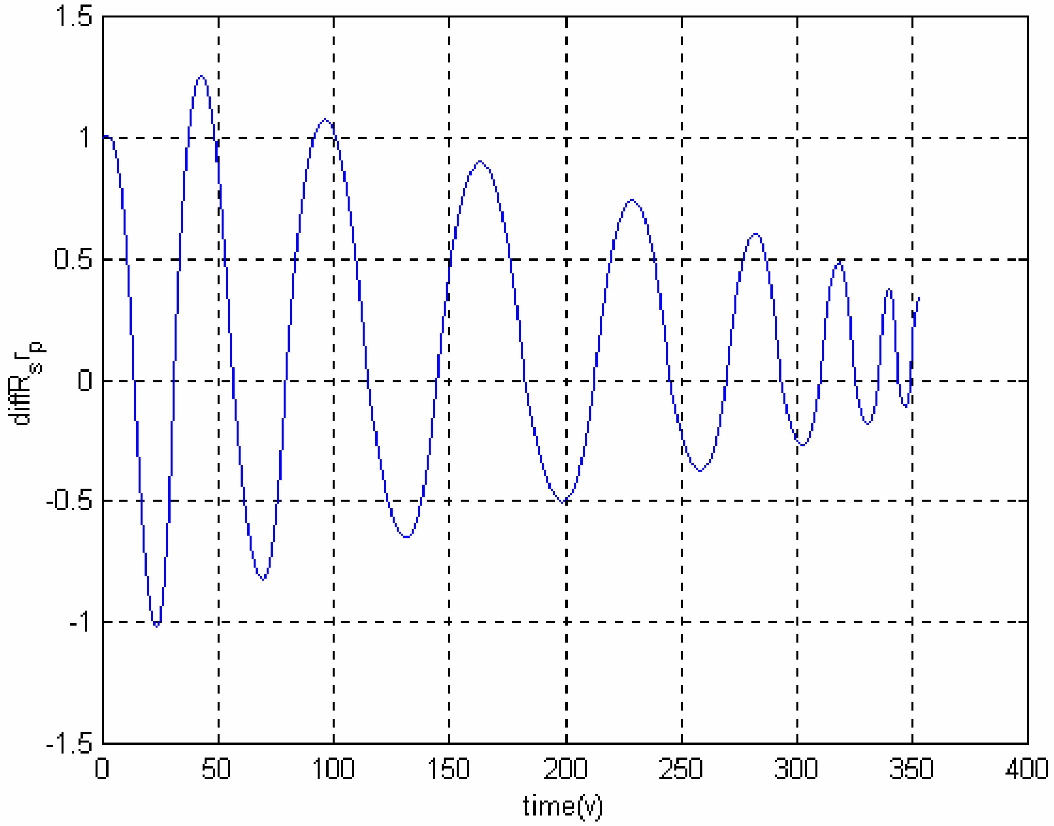

Fig. 1 The x-axis is the time v between the shell and particle, and the y-axis is the distance between the shell and particle. Conditions of collapse: the mass of the shell is 100 solar masses; the starting radius of the shell is 10 times the mass; the starting radius of the particle is 1.001 x the starting radius of the shell; the starting velocity of the particle is greater than the shell.