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Some LRS Bianchi Type VI_0 Cosmological Models with Special Free Gravitational Fields

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Abstract: The properties of the free gravitational fields and their invariant characterizations are discussed and also obtained LRS Bianchi type VI_0 cosmological models imposing different conditions over the free gravitational fields. Models thus formed are then discussed in detail with respect to their physical and kinematical parameters in the last section of the paper.

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1. Introduction

Homogeneous cosmological models filled with perfect fluid have already been widely studied in the framework of general relativity. The perfect fluid, however, is supposed to be only a portion of the total constituents of the universe. Cosmological models representing matter and radiation are of particular interest as they depict early stages of the universe when neutrino and photon decoupling take place and parts of these behave like unidirectional streams moving with the velocity of light. Bianchi type V universes are studied by Roy and Singh [1,2] for such material configurations. These universes are a generalization of the open FRW models which eventually isotropize. The standard cosmological models of Friedmann satisfactorily describe the universe at least since the epoch of the last scattering. However a number of questions regarding the early stages of the evolution and the observed inhomogeneities and anisotropies on a smaller scale can not be explained

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within this framework. It is, therefore, one looks for the

possibility of obtaining models of a less restrictive nature displaying anisotropy and inhomogeneity. Considerable work has been done in obtaining various Bianchi type models and their inhomogeneous generalizations. Barrow [3] has pointed out that Bianchi VI_0 universes give a better explanation of some of the cosmological problems like primordial helium abundance and they also isotropize in a special sense. Looking to the importance of Bianchi type VI_0 universes, Tikekar and Patel [4], Bali et al. [5,6], Pradhan and Bali [7] have investigated Bianchi type VI_0 string cosmological models in presence and absence of magnetic field. Bianchi type VI universes representing different types of material distributions have been studied by a number of authors namely Wainwright et al. [8], Dunn and Tupper [9,10], Collins and Hawking [11], Hughston and Jacob [12], Ellis and MacCallum [13], Collins [14], Ruban [15], Tupper [16], MacCallum [17], Uggla and Rosquist [18], Coley and Dunn [19], Wainwright and Anderson [20], Lorentz-Petzold [21,22,23] and De Rop [24].

The role of the free gravitational field in the determination of the flow of matter in the cosmological models has been emphasized by Ellis [25]. It is well known that near the singularities, the curvature of spacetime becomes dominating so that the evolution of the universe in its early stages is largely affected by the character of the free gravitational field. The study of models with specific types of the free gravitational field has also become necessary due to the speculative existence of copious amount of gravitational radiation following the big-bang. Models for which the free gravitational field is of the 'electric type' are considered to have their Newtonian analogues while a 'magnetic type' free gravitational field is purely general relativistic. Since there is an intimate relationship between the flow patterns of the fluid and the free gravitational field, it is tempted to impose adhoc conditions on the latter. Bianchi type VI_0 magnetic type solutions in presence of a viscous fluid and an incident magnetic field were obtained by Roy and Singh [26]. Bianchi VI electric type solutions are given for different material distributions by Roy and Banerjee [27,28].

In the present paper we have redefined the two specific types of free gravitational fields which we term as the 'electric type', the 'magnetic type' and also recall the idea of the 'gravitational wrench' of unit 'pitch'. The concept of the 'gravitational wrench' of unit 'pitch' was first explored by Roy and Banerjee [29]. We receive this idea here again and give a brief discussion about the properties of these free gravitational fields and their invariant characterizations.

In classical electromagnetic theory, the electromagnetic field has two independent invariants $F_{ij}F^{ij}$ and $^*F_{ij}F^{ij}$. The classification of the field is characterized by the property of the scalar $k^2 = (F_{ij}F^{ij})^2 + (^*F_{ij}F^{ij})^2$. When k = 0 the field is said to be null and for any observer | E | = | H | and E. H = 0 where E and H are the electric and magnetic vectors respectively. When k $\neq$ 0, the field is non-null and there exists an observer for which mE = nH, m and n being scalars. It is easy to see that if $^*F_{ij}F^{ij} = 0, F_{ij}F^{ij} \neq 0$ then either $E = 0$ or $H = 0$. These we call the magnetic and electric fields, respectively. If $F_{ij}F^{ij} = 0, ^*F_{ij}F^{ij} \neq 0$ then $E = \pm H$. This we call the 'electromagnetic wrench' with unit 'pitch'. In this case Maxwell's equations lead to an empty electromagnetic field with constant electric and magnetic intensities. In the case of gravitational field, the number of independent scalar invariants of the second order is fourteen. The independent scalar invariants formed from the conformal curvature tensor are four in number. In the case of Petrov type D spacetimes, the number of independent scalar invariants are only two, viz. $C_{hijk}C^{hijk}$ and $^{*}C_{hijk}C^{hijk}$. Analogous to the electromagnetic case, the electric and magnetic parts of free gravitational field for an observer with velocity v^i are defined as $E_{\alpha\beta} = C_{\alpha j\beta i}v^i v^j$ and $H_{\alpha\beta} = {}^{*}C_{\alpha j\beta i}v^i v^j$ (Ellis [25]). It is clear from the canonical form of the conformal curvature for a general type D spacetime that there exists an observer for which $E_{\alpha\beta} = (n/m)H_{\alpha\beta}$, where m,n being integers and m $\neq$ 0. The field is said to be purely magnetic type for n = 0, m $\neq$ 0. In this case we have $E_{\alpha\beta} = 0$ and $H_{\alpha\beta} \neq 0$. The physical significance for the gravitational field of being magnetic type is that the matter particles do not experience the tidal force. When m $\neq$ 0 and also n $\neq$ 0, we call that there is a 'gravitational wrench' of unit 'pitch' $\mid$ n/m $\mid$ in the free gravitational field [29]. If 'pitch' is unity then we have $E_{\alpha\beta} = \pm H_{\alpha\beta}$.

The spacetime having a symmetry property is invariant under a continuous group of transformations. The transformation equations for such a group of order r is given by

$$X^i = f^i(x^1, ..., x^r, a^1, ..., a^r) \tag{1}$$

which satisfy the differential equations

$$\frac{\partial X^i}{\partial x^\alpha} = \xi^i_{(\beta)}(X)A^\beta_\alpha(a), (\alpha, \beta = 1, ..., r) \tag{2}$$

where $a^1, \ldots, a^r$ are r essential parameters. The vectors $\xi^i_{(\alpha)}$ are the Killing vectors for the group G_r of isometry satisfying the Killing's equation

$$\xi_{(\alpha)i;j} + \xi_{(\alpha)j;i} = 0. \tag{3}$$

A subspace of spacetime is said to be the surface of transitivity of the group if any point of this space can be transformed to another point of it by the action of this group. A spacetime is said to be spatially homogeneous if it admits a group G_r of isometry which is transitive on three dimensional space-like hypersurfaces. The group G_3 of isometry was first considered by Bianchi [30] who obtained nine different types of isometry group known as the Bianchi types. The space-time which admits G_4 group of isometry is known as locally rotationally symmetric (LRS) which always has a G_3 as its subgroup belonging to one of the Bianchi types provided this G_3 is simply transitive on the three dimensional hypersurface t = constant.

Here we have considered an LRS Bianchi type VI_0 spacetime and obtained models with free gravitational field of purely 'magnetic type' and also in the presence of 'gravitational wrench' of unit 'pitch' in the free gravitational field. It is found that the 'magnetic' part of the free gravitational field induces shear in the fluid flow, which is zero in the case of a 'electric' type free gravitational field representing an unrealistic distribution in this case. In section 2, solutions representing LRS Bianchi type VI_0 cosmological models with perfect fluid imposing different conditions on the free gravitational field are obtained. In the last section, we discuss the models so formed with respect to their physical and kinematical parameters.

2. Formation of the Line Element

We consider an LRS Bianchi type VI_0 universe for which

$$ds^2 = \eta_{ab}\,\theta^a\theta^b \tag{4}$$

where $\theta^1 = Adx, \theta^2 = Bexp^x dy, \theta^3 = Bexp^{-x}dz, \theta^4 = dt$;A and B being functions of t alone. The non-vanishing physical components of C_{abcd} for the line element (1) are given by

$$C_{2323} = \frac{1}{6}\left[\frac{2\ddot{A}}{A} - \frac{2\ddot{B}}{B} - \frac{2\dot{A}\dot{B}}{AB} + \frac{2\dot{B}^2}{B^2} + \frac{4}{A^2}\right]$$

$$= -\frac{1}{2}C_{3131} = C_{1212} = -C_{1414} = \frac{1}{2}C_{2424} = -C_{3434} \tag{5}$$

$$C_{2314} = -\frac{1}{A}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right] = -\frac{1}{2}C_{3124} = C_{1234}. \tag{6}$$

The Einstein field equations

$$R^a_b - \frac{1}{2}R\delta^b_a + \Lambda\,\delta^b_a = -8\pi T^b_a \tag{7}$$

for the line-element (4), it leads to

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{A^2} + \Lambda = -8\pi T^1_1, \tag{8}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} + \Lambda = -8\pi T^2_2 = -8\pi T^3_3, \tag{9}$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{1}{A^2} + \Lambda = -8\pi T^4_4 \tag{10}$$

and

$$0 = -8\pi T^b_a \quad (a \neq b). \tag{11}$$

The matter in general will represent an anisotropy fluid with time-lines as the flow-lines of the fluid. The kinematical parameters of the fluid flow are given by

$$\theta = \frac{\dot{A}}{A} + \frac{2\dot{B}}{B}, \tag{12}$$

$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right). \tag{13}$$

We now solve the equations (8) – (10) under the following two alternative possibilities:

Case − I: Free gravitational field is Magnetic type (m $\neq 0$, n = 0 i.e. $H_{\alpha\beta} \neq 0$, and $E_{\alpha\beta}$=0). From (5), we have

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{2}{A^2} = 0. \tag{14}$$

We assume the matter to be a perfect fluid with comoving flow vector v^a;

$$T_a^b = (\in +p)v_a v^b + p\delta_a^b, \tag{15}$$

with $v^a = \delta_0^a$, $\in$ and p being respectively, the density and the thermodynamic pressure of the fluid. We have

$$T_1^1 = T_2^2 = T_3^3 = p, T_4^4 = - \in . \tag{16}$$

The field equation (8) together with equation (9) gives

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) - \frac{2}{A^2} = 0. \tag{17}$$

From equations (14) and (17), we further have two independent equations

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = 0, \tag{18}$$

$$\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) - \frac{2}{A^2} = 0. \tag{19}$$

If we write A/B as U, respectively, we get from equations (18) and (19)

$$UU_{\xi\xi} + \dot{A}UU_{\xi} - 2U_{\xi}^2 = 0 \tag{20}$$

and

$$U_{\xi}^2 - \dot{A}UU_{\xi} + 2U^2 = 0. \tag{21}$$

where ξ is defined by

$$\frac{d}{dt} = \frac{1}{A}\frac{d}{d\xi}. \tag{22}$$

Equations (20) and (21) lead to a second order differential equation

$$UU_{\xi\xi} - U_{\xi}^2 + 2U^2 = 0. \tag{23}$$

On substituting U = e^{λ}, the above equation reduces to the form

$$\lambda_{\xi\xi} + 2 = 0, \tag{24}$$

which yields,

$$\lambda = c_1\xi - \xi^2 + c_2, \tag{25}$$

where c_1 and c_2 are arbitrary constants. Thus

$$U = c_3\, e^{-\tau(\tau - c_1)}, \tag{26}$$

where τ stands for ξ.

Equations (21), (22) and (26) together give

$$A = \frac{c_4}{(c_1 - 2\tau)}\, e^{-\tau(\tau - c_1)}, \tag{27}$$

$$B = \frac{c_5}{c_1 - 2\tau}\,, \tag{28}$$

where τ is given by

$$\frac{d\tau}{dt} = \frac{1}{c_4}\,(c_1 - 2\tau)\, e^{\tau(\tau - c_1)}, \tag{29}$$

c_3, c_4 and c_5 being arbitrary constants. The metric (4) then reduces to the form

$$ds^2 = \frac{c_4^2}{(c_1 - 2\tau)^2}\, e^{-2\tau(\tau - c_1)}[-d\tau^2 + dx^2 + e^{2\tau(\tau - c_1)}\{e^{2x}dy^2 + e^{-2x}dz^2\}]. \tag{30}$$

Expressions for $\in$ and p are respectively given by

$$8\pi p = -\frac{3}{c_4^2}\,[4 - (c_1 - 2\tau)^2]\, e^{2\tau(\tau - c_1)} - \Lambda \tag{31}$$

and

$$8\,\pi\, \in = \frac{3}{c_4^2}\,[4 + (c_1 - 2\tau)^2]\, e^{2\tau(\tau - c_1)} + \Lambda. \tag{32}$$

The kinematical parameters given by equation (12) and equation (13) take the form

$$\theta = \frac{1}{c_4}\,\{6 + (c_1 - 2\tau)^2\}\, e^{\tau(\tau - c_1)}, \tag{33}$$

$$\sigma = \frac{1}{c_4\sqrt{3}}\,(c_1 - 2\tau)^2\, e^{\tau(\tau - c_1)}. \tag{34}$$

Case II: There is a 'gravitational wrench' of unit 'pitch' in the free gravitational field i.e. $E_{\alpha\beta} = \pm H_{\alpha\beta}$.

In this case, we have

$$E_{\alpha\beta} = eH_{\alpha\beta}, \;\; e^2 = 1. \tag{35}$$

Equation (35) together with equations (5) and (6) gives

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \frac{2}{A^2} = -\frac{3e}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right). \tag{36}$$

For a perfect fluid medium, with the help of the equations (17) and (36), we have

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = -\frac{3e}{2A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right), \tag{37}$$

$$\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)-\frac{2}{A^2}=\frac{3e}{2A}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right). \tag{38}$$

On assuming A/B = U, the equations (37) and (38) respectively reduce to the forms

$$U_{\xi\xi}+\dot{A}U_\xi=-\frac{3e}{2}U_\xi \tag{39}$$

and

$$\dot{A}UU_\xi-U_\xi^2-2U^2=\frac{3e}{2}UU_\xi \tag{40}$$

where ξ is defined by (22). Equations (39) and (40) lead to a second order differential equation

$$UU_{\xi\xi}+3eUU_\xi+U_\xi^2+2U^2=0. \tag{41}$$

On substituting U = exp(λ), equation (41) take the form

$$\lambda_{\xi\xi}+2\lambda_\xi^2+3e\lambda_\xi+2=0 \tag{42}$$

which yields,

$$U^4=(2\tau^2+3e\tau+2)^{-1}\,exp\left(\frac{6e}{\sqrt{7}}\,tan^{-1}\frac{4\tau+3e}{\sqrt{7}}\right) \tag{43}$$

where, $\tau = \frac{d\lambda}{d\xi}$. From (40), we have

$$\dot{A}=\frac{(2\tau^2+3e\tau+4)}{2\tau}. \tag{44}$$

Equations (22) and (42) together lead to

$$A^2=\frac{(2\tau^2+3e\tau+2)^{1/2}}{\tau^2}\,exp\left(\frac{3e}{\sqrt{7}}\,tan^{-1}\frac{4\tau+3e}{\sqrt{7}}\right), \tag{45}$$

$$B^2=\frac{(2\tau^2+3e\tau+2)}{\tau^2} \tag{46}$$

where, τ is given by

$$\frac{d\tau}{dt}=\frac{-(2\tau^2+3e\tau+2)}{A}. \tag{47}$$

Metric (4) then takes the form

$$ds^2=\frac{(2\tau^2+3e\tau+2)^{1/2}}{\tau^2}\,exp\left(\frac{3e}{\sqrt{7}}\,tan^{-1}\frac{(4\tau+3e)}{\sqrt{7}}\right)\Bigg[-\frac{d\tau^2}{(2\tau^2+3e\tau+2)^2}+dx^2$$
$$+(2\tau^2+3e\tau+2)^{1/2}\,exp\left(-\frac{3e}{\sqrt{7}}\,tan^{-1}\frac{(4\tau+3e)}{\sqrt{7}}\right)\{e^{2x}dy^2+e^{-2x}dz^2\}\Bigg] \tag{48}$$

Expressions for $\in$, p, θ and σ for the above metric are

$$8\pi\in=\frac{(12e\tau^3+39\tau^2+72e\tau+48)}{4(2\tau^2+3e\tau+2)^{1/2}}\,exp\left(\frac{-3e}{\sqrt{7}}\,tan^{-1}\frac{(4\tau+3e}{\sqrt{7}}\right)+\Lambda, \tag{49}$$

$$8\pi p = \frac{(12e\tau^3 + 29\tau^2 + 72e\tau - 48)}{4(2\tau^2 + 3e\tau + 2)^{1/2}} exp\left(\frac{-3e}{\sqrt{7}} tan^{-1}\frac{(4\tau + 3e)}{\sqrt{7}}\right) - \Lambda, \quad (50)$$

$$\theta = \frac{(2\tau^2 + 9e\tau + 12)}{2(2\tau^2 + 3e\tau + 2)^{1/4}} exp\left(\frac{-3e}{2\sqrt{7}} tan^{-1}\frac{(4\tau + 3e)}{\sqrt{7}}\right), \quad (51)$$

$$\sigma = \frac{\tau^2}{\sqrt{3}\,(2\tau^2 + 3e\tau + 2)^{1/4}} exp\left(\frac{-3e}{2\sqrt{7}} tan^{-1}\frac{(4\tau + 3e)}{\sqrt{7}}\right). \quad ... \quad (52)$$

3. Discussion

The model (30) starts expansion with a big-bang singularity from τ=-∞ and it goes on expanding till $\tau = \frac{c_1}{2}$ where $\tau = -\infty$ and $\tau = \frac{c_1}{2}$ respectively correspond to the cosmic time t = 0 and t = ∞. The model is found to be realistic everywhere in this time interval for $\Lambda > \frac{-12}{c_4^2} \exp\left(\frac{-c_1^2}{2}\right)$. The model behaves like a steady-state de-Sitter type universe at late times where the physical and kinematical parameters $\in$, p and θ tend to a finite value however shear vanishes there. The model has a non-zero anisotropy at the initial stage of its expansion which goes on decreasing with time and asymptotically it dies out completely. The model has a point singularity at t = 0. The weak and strong energy conditions are also satisfied identically within this time span.

The model (48) is realistic for e = +1. It starts expansion with a single shot explosion from a point singularity state $\tau = \infty$ and goes on expanding till $\tau = 0$ where $\tau = \infty$ and $\tau = 0$ respectively correspond to the cosmic time t = 0 and t = ∞. At t = 0 the model has a point singularity. The energy conditions $\in \ > 0$ and $\in \ \geq$ p are satisfied throughout the time span for $6\sqrt{2}\exp\left(\frac{-3}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}\right) + \Lambda > 0$. At t = ∞, the model enters a steady-state de-Sitter type phase where the physical and kinematical parameters θ, $\in$ and p approach a finite value while the shear vanishes. Initially at t = 0, the model has non-zero anisotropy which goes on decreasing with time and finally the model gets isotropized at large times where $\frac{\sigma}{\theta}$ vanishes. The necessary covariant condition for the existence of gravitational waves in cosmology explained by Maartens et al. [31], is $D^b E_{ab} = 0 \neq \text{curl}\, E_{ab}$, $D^b H_{ab} = 0 \neq \text{curl}\, H_{ab}$. In the present model (48), the above two conditions are not satisfied which indicates as a result no presence of the gravitational wave.

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