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Representation of su(1,1) Algebra and Hall Effect

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Abstract: In this paper we consider the Schwinger and Heisenberg representation of su(1,1) algebra under Hall effect. In presence of magnetic field, we obtain the generators of su(1,1) algebra in terms of ladder operators, and magnetic field for the one and two bosons system. Also the Casimir operator for both systems are obtained by ladder operators.

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1. Introduction

When a magnetic field placed transverse to the current of particles we have the Hall effect in system. On the other hand the Hall effect is correspond to the non - commutative geometry. The non - commutative geometries have become the subject literature on the q-deformed groups $E_q(2)$ and $SU_q(2)$ which are the automorphism groups of the quantum plane $zz^\star = qz^\star z$ and the quantum sphere respectively[1,2,3], these lead us to show effect of non-commutativity in $su(1,1)$ algebra. In another word we would like to consider bosons moving in two dimensional non-commutative space when uniform external magnetic field is present. In the usual case this system leads to Hall effect. Indeed, we know that in representation algebra one can obtains Hall effect in terms of an effective magnetic field [4]. Two bosons system is the same as non - relativistic linear singular harmonic oscillator. It realizes the unitary irreducible representation $D^+(l+1)$ of $SU(1,1)$ group and the $su(1,1)$ algebra in non - commutative space, which is corresponding to the Hall effect.

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First, we are going to introduce the $su(1,1) \approx sp(2,R) \approx so(2,1)$ algebra which is defined by the following commutation relations,

$$\begin{aligned}
[K_1, K_2] &= -iK_0, \\
[K_0, K_1] &= iK_2, \\
[K_2, K_0] &= iK_1.
\end{aligned} \tag{1}$$

So we can choose generators of $su(1,1)$ algebra as K_+, K_- and K_0, and we have,

$$K_\pm = (K_1 \pm iK_2), \tag{2}$$

which satisfy with following relations,

$$\begin{aligned}
[K_+, K_-] &= -2K_0, \\
[K_0, K_\pm] &= \pm K_\pm,
\end{aligned} \tag{3}$$

and the Casimir operator is given by,

$$C = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \frac{1}{4}(N+1)(N-1). \tag{4}$$

If the eigenvalues of the operator C is denoted by,

$$C = l(l+1), \tag{5}$$

the irreducible of $su(1,1)$ algebra are characterized by a total boson number,

$$N = -(2l+1), \tag{6}$$

this operator commutes with all of the K's. Since the $SU(1,1)$ group is non-compact, all its unitary irreducible representations are infinite dimensional.
In Section 2, we review the Schwinger representation of $su(1,1)$ algebra for one and two bosons system [5,6]. In order to construct a spectrum generating algebra for system with the finite number of bound states we introduce a set of boson creation and annihilation operators [7], and then we consider Hall effect and obtain generators of algebra and Casimir operator for both systems.
In section 3, we repeat our formalism in non-commutative geometries as Heisenberg representation of $su(1,1)$ algebra [8].

2. Schwinger Representation of $su(1,1)$ Algebra

Here, we review single and two bosons system and see the generators of $su(1,1)$ algebra in terms of ladder operators. Finally, we consider Hall effect and find generators of $su(1,1)$ algebra in terms of ladder operators and B field which $su(1,1)$ algebra lives invariant. In

Schwinger representation of $su(1,1)$ algebra for one boson system, we have generators of $su(1,1)$ algebra in terms of creation and annihilation operators,

$$\begin{aligned} K_1 &= \frac{1}{4}(a^\dagger a^\dagger + aa), \\ K_2 &= \frac{1}{4i}(a^\dagger a^\dagger - aa), \end{aligned} \tag{7}$$

and

$$\begin{aligned} K_+ &= \frac{1}{2}a^\dagger a^\dagger, \\ K_- &= \frac{1}{2}aa, \\ K_0 &= \frac{1}{4}(2a^\dagger a + 1) = \frac{1}{4}(aa^\dagger + a^\dagger a), \end{aligned} \tag{8}$$

with

$$[a, a^\dagger] = 1, \tag{9}$$

equations (7) and (8) satisfy commutative relations (1) and (3). In this case the Casimir operator reduces identically to,

$$C = -\frac{3}{16}. \tag{10}$$

Now we introduce two bosons system with operators a_1 and a_2 which obey the usual commutation relations,

$$\begin{aligned} \left[a_1, a_1^\dagger\right] &= \left[a_2, a_2^\dagger\right] = 1, \\ \left[a_1, a_2^\dagger\right] &= \left[a_2, a_1^\dagger\right] = 0, \\ \left[a_1^\dagger, a_1^\dagger\right] &= \left[a_1^\dagger, a_2^\dagger\right] = 0, \\ [a_1, a_1] &= [a_1, a_2] = 0. \end{aligned} \tag{11}$$

The bilinear combinations $a_1^\dagger a_1$, $a_1 a_2$, $a_1^\dagger a_2^\dagger$ and $a_2^\dagger a_2$ generate the $SU(1,1)$ group. So, we have,

$$\begin{aligned} K_1 &= \frac{1}{2}(a_1^\dagger a_2^\dagger + a_2 a_1), \\ K_2 &= \frac{1}{2i}(a_1^\dagger a_2^\dagger - a_2 a_1), \end{aligned} \tag{12}$$

and,

$$\begin{aligned} K_+ &= a_1^\dagger a_2^\dagger, \\ K_- &= a_2 a_1, \\ K_0 &= \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1). \end{aligned} \tag{13}$$

The total boson number operator of this algebra is given by,

$$N = a_1^\dagger a_1 - a_2^\dagger a_2, \tag{14}$$

in this case the Casimir operator is,

$$C = \frac{1}{4}(a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}. \tag{15}$$

In the next step, we introduce Hall effect and see that how changes the above operations. The ladder operators and generators of algebra in presence of magnetic field will be changed but the commutation relation (1) and (3) are not changed. In that case for single boson one can find,

$$\begin{aligned} a &= -2\frac{\partial}{\partial Z} + \frac{B}{2}\bar{Z}, \\ a^\dagger &= 2\frac{\partial}{\partial \bar{Z}} + \frac{B}{2}Z, \end{aligned} \tag{16}$$

where Z and $\bar{Z}$ are complex coordinates in poincaré space and B is magnetic field. Now equation (9) become,

$$[a, a^\dagger] = -2B. \tag{17}$$

Therefore we find the generators algebra as follow,

$$\begin{aligned} K_1 &= \frac{1}{8B}(a^\dagger a^\dagger + aa), \\ K_2 &= -\frac{i}{8B}(a^\dagger a^\dagger - aa), \end{aligned} \tag{18}$$

and

$$\begin{aligned} K_+ &= \frac{1}{4B}a^\dagger a^\dagger, \\ K_- &= \frac{1}{4B}aa, \\ K_0 &= \frac{1}{4B}(B - a^\dagger a). \end{aligned} \tag{19}$$

One can obtain the equation (19) in terms of complex coordinate in poincaré space by the following expressions,

$$\begin{aligned} K_+ &= \frac{1}{B}\partial_{\bar{Z}}^2 + \frac{1}{2}Z\partial_{\bar{Z}} + \frac{B}{16}Z^2, \\ K_- &= \frac{1}{B}\partial_{Z}^2 - \frac{1}{2}\bar{Z}\partial_{Z} + \frac{B}{16}\bar{Z}^2, \\ K_0 &= \frac{1}{B}\partial_{\bar{Z}}\partial_Z, + \frac{1}{4}\left(Z\partial_Z - \bar{Z}\partial_{\bar{Z}}\right) - \frac{B}{16}Z\bar{Z}. \end{aligned} \tag{20}$$

We see that the Casimir operator is same as equation (10).
Now we consider two bosons system, so creation and annihilation operators are,

$$\begin{aligned} a_i &= -2\partial_{Z_i} + \frac{B}{2}\bar{Z}_i, \\ a_i^\dagger &= 2\partial_{\bar{Z}_i} + \frac{B}{2}Z_i, \end{aligned} \tag{21}$$

therefore the commutation relations will be as,

$$\begin{aligned}\left[a_i, a_i^\dagger\right] &= -2B,\\ \left[a_i, a_j^\dagger\right] &= 0,\\ [a_i, a_i] &= [a_i, a_j] = 0,\end{aligned} \tag{22}$$

where i=1,2.
In order to preserve the $su(1,1)$ algebra, the generators can be written by,

$$\begin{aligned}K_1 &= \frac{1}{4B}(a_1^\dagger a_2^\dagger + a_2 a_1),\\ K_2 &= -\frac{i}{4B}(a_1^\dagger a_2^\dagger - a_2 a_1),\end{aligned} \tag{23}$$

and

$$\begin{aligned}K_+ &= \frac{1}{2B}{a_1}^\dagger a_2^\dagger,\\ K_- &= \frac{1}{2B}a_2 a_1,\\ K_0 &= \frac{1}{4B}(2B - a_1^\dagger a_1 - a_2^\dagger a_2).\end{aligned} \tag{24}$$

and the number operators is,

$$N = \frac{1}{2B}(a_1^\dagger a_1 - a_2^\dagger a_2). \tag{25}$$

The same as one boson case, we can write generators in terms of complex coordinates,

$$\begin{aligned}K_+ &= \frac{2}{B}\partial_{\bar{Z}_1}\partial_{\bar{Z}_2} + \frac{B}{8}Z_1 Z_2 + \frac{1}{2}(Z_1\partial_{\bar{Z}_2} + Z_2\partial_{\bar{Z}_1}),\\ K_- &= \frac{2}{B}\partial_{Z_2}\partial_{Z_1} + \frac{B}{8}\bar{Z}_1\bar{Z}_2 - \frac{1}{2}(\bar{Z}_1\partial_{Z_2} + \bar{Z}_2\partial_{Z_1}),\\ K_0 &= \frac{1}{B}(\partial_{\bar{Z}_1}\partial_{Z_1} + \partial_{\bar{Z}_2}\partial_{Z_2}) - \frac{B}{16}(Z_1\bar{Z}_1 + Z_2\bar{Z}_2)\\ &+ \frac{1}{4}(Z_1\partial_{Z_1} + Z_2\partial_{Z_2} - \bar{Z}_1\partial_{\bar{Z}_1} - \bar{Z}_2\partial_{\bar{Z}_2}).\end{aligned} \tag{26}$$

But, we see that the Casimir operator is depend to B, so we have,

$$C = \frac{1}{16B^2}(a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}, \tag{27}$$

this is maybe interesting result that Casimir operator in the case of one boson system in presence magnetic field and without magnetic field have same form as $C = -\frac{3}{16}$. But in the case of two bosons system Casimir operator in presence of magnetic field (Hall effect) have a different form with respect to without magnetic field B.
In the next section, we work in Heisenberg representation of $su(1,1)$ algebra.

3. Heisenberg Representation of $su(1,1)$ Algebra

In this section first we express Heisenberg representation of $su(1,1)$ algebra, and then introduce Hall effect. The $SU(1,1)$ group is considered as the automorphism group of the Heisenberg algebra H. The commutative relations between generators in this case are like equation (1) and (3), but its generators and creation and annihilation operators are different. So, in this representation we have Z^* and Z, so creation and annihilation operators are respectively,

$$
\begin{aligned}
Z^* &= \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x}), \\
Z &= \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x}),
\end{aligned} \tag{28}
$$

and their commutation relation is,

$$
[Z, Z^*] = 1, \tag{29}
$$

where Z, $Z*$ and 1 are basic elements of one dimensional Heisenberg algebra in the 3 - dimensional vector space. Now we like to turn on magnetic field and find creation and annihilation operators,

$$
\begin{aligned}
Z^* &= \frac{1}{\sqrt{2}}(2Bx - \frac{\partial}{\partial x}), \\
Z &= \frac{1}{\sqrt{2}}(2Bx + \frac{\partial}{\partial x}),
\end{aligned} \tag{30}
$$

and their commutation relation is,

$$
[Z, Z^*] = 2B. \tag{31}
$$

The generators of $su(1,1)$ algebra satisfy equations (1) and (3) can be written by,

$$
\begin{aligned}
K_1 &= \frac{1}{8B}(ZZ + Z^*Z^*), \\
K_2 &= \frac{i}{8B}(Z^*Z^* - ZZ),
\end{aligned} \tag{32}
$$

and

$$
\begin{aligned}
K_+ &= \frac{ZZ}{4B}, \\
K_- &= \frac{Z^*Z^*}{4B}, \\
K_0 &= \frac{1}{4B}(B - ZZ^*).
\end{aligned} \tag{33}
$$

We obtain generators of $su(1,1)$ algebra in Heisenberg representation in terms of magnetic field B.

Conclusion

In this study we discuss about Hall effect in $su(1,1)$ algebra for Schwinger and Heisenberg representation. In presence of magnetic field we know that algebra is invariant, therefore we obtain generators of algebra in terms of ladder operators and magnetic field B. Earth-shaking result found in this paper is that magnetic field is not affect on Casimir operator in the case of single boson system for Schwinger representation of $su(1,1)$ algebra , it means that Hall effect for Casimir operator vanish in that system. But for two bosons system it depends to B field and not invariant under Hall effect.

References

[1] N. Ya. Vilenkin and A. U. Klimyk, Representations of Lie Groups and Special Functions, vol **3**, Kluwer Academic Press, The Netherland (1991).

[2] S.L. Woronowicz, comm. Math. Phys. **144** , 417(1992); Comm. Math. Phys. **149** , 637(1992); Lett. Math. Phys. **23** , 251(1991).

[3] L.L. Vaksman and L.I. Korogodski, Dokl. Akad. Nauk SSSR. **304** , 1036 (1989).

[4] Omer F. Dayi and Ahmed Jellal, J. Math. Phys. **43** 4592 (2002).

[5] Marcel Novaes, Revista Brasilieira de Ensimo de Fisica, **26**, NO 4, p. 351 (2004).

[6] Ramazan Koc, Hayriye Tütüncüler and Mehmet Koca, Nuovo Cimento Vol. **119** B, N. 10 (2004).

[7] Alhassid Y, Gürsey F and Iachello F, ann. Phys. **148** (1983) 346.

[8] H.Ahmedov and I.H.Duru, J. Phys. A: Math. Gen. **34** (2001)227.