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# Partial Swapping, Unitarity and No-signalling

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**Abstract:** It is a well known fact that a quantum state $|\psi(\theta,\phi)\rangle$ is represented by a point on the Bloch sphere, characterized by two parameters $\theta$ and $\phi$. In a recent work we already proved that it is impossible to partially swap these quantum parameters. Here in this work we will show that this impossibility theorem is consistent with principles like unitarity of quantum mechanics and no signalling principle.
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## 1. Introduction

In quantum information theory understanding the limits of fidelity of different operations has become an important area of research. Noticing these kind of operations which are feasible in classical world but have a much restricted domain in quantum information theory started with the famous 'no-cloning' theorem [1]. The theorem states that one cannot make a perfect replica of a single quantum state.Later it was also shown by Pati and Braunstein that we cannot delete either of the two quantum states when we are provided with two identical quantum states at our input port [2]. In spite of these two famous 'no-cloning' [1] and 'no-deletion' [2] theorem there are many other 'no-go' theorems like 'no-self replication' [3] , 'no-partial erasure' [4], 'no-splitting' [5] and many more which have come up.
Recently in ref [6], we introduce a new no-go theorem, which we refer as 'no partial

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swapping' of quantum information. Since we know that the information content in a qubit is dependent on the angles azimuthal and phase angles $\theta$ and $\phi$, then the partial swapping of quantum parameters $\theta$ and $\phi$ is given by,

$$|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta_1},\bar{\phi_1})\rangle \longrightarrow |A(\theta_1,\bar{\phi_1})\rangle|\bar{A}(\bar{\theta_1},\phi_1)\rangle \tag{1}$$

$$|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta_1},\bar{\phi_1})\rangle \longrightarrow |A(\bar{\theta_1},\phi_1)\rangle|\bar{A}(\theta_1,\bar{\phi_1})\rangle \tag{2}$$

However in ref [6] we showed that this operation is impossible in the quantum domain from the linear structure of quantum theory.

In this work we once again claim this impossibility from two different principles namely [i] unitarity of quantum mechanics [ii] no signalling principle. The organization of the work is as follows: In the first section we will prove this impossibility from the unitarity of quantum mechanics. In the second section we will do the same from the principle of no signalling. Then the conclusion follows.

## 2. Partial Swapping: Unitarity of Quantum Mechanics

Let us consider a set S consisting of two non orthogonal states $S = \{|A(\theta_1,\phi_1)\rangle, |B(\theta_2,\phi_2)\rangle\}$ Let us assume that hypothetically it is possible to partially swap the parameters of these two states $|A(\theta_1,\phi_1)\rangle, |B(\theta_2,\phi_2)\rangle$.

First of all we will assume that swapping of phase angles of two quantum states are possible, keeping the azimuthal angles fixed. Therefore the transformation describing such an action is given by,

$$\begin{aligned}
|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta_1},\bar{\phi_1})\rangle &\longrightarrow |A(\theta_1,\bar{\phi_1})\rangle|\bar{A}(\bar{\theta_1},\phi_1)\rangle \\
|A(\theta_2,\phi_2)\rangle|\bar{A}(\bar{\theta_2},\bar{\phi_2})\rangle &\longrightarrow |A(\theta_2,\bar{\phi_2})\rangle|\bar{A}(\bar{\theta_2},\phi_2)\rangle
\end{aligned} \tag{3}$$

To preserve the unitarity of the above transformation,will preserve the inner product.

$$\langle A(\theta_1,\phi_1)|A(\theta_2,\phi_2)\rangle\langle\bar{A}(\bar{\theta_1},\bar{\phi_1})|\bar{A}(\bar{\theta_2},\bar{\phi_2})\rangle = \langle A(\theta_1,\bar{\phi_1})|A(\theta_2,\bar{\phi_2})\rangle\langle\bar{A}(\bar{\theta_1},\phi_1)|\bar{A}(\bar{\theta_2},\phi_2)\rangle \tag{4}$$

The above equality will not hold for all values of $(\theta,\phi)$. The equality will hold if $i)\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \tan\frac{\bar{\theta_1}}{2}\tan\frac{\bar{\theta_2}}{2}$ or $ii)(\phi_2-\phi_1) = (\bar{\phi_2}-\bar{\phi_1}) \pm 2k\pi$, where $k$ is an integer. These two conditions characterizes the set of states on the Bloch sphere for which the partial swapping of the phase angles are possible. However in general this is not possible for all possible values of $\theta$, $\phi$.

Let us now assume that partial swapping of azimuthal angles are possible, without altering the phase angles of the quantum states.

$$\begin{aligned}
|A(\theta_1,\phi_1)\rangle|\bar{A}(\bar{\theta_1},\bar{\phi_1})\rangle &\longrightarrow |A(\bar{\theta_1},\phi_1)\rangle|\bar{A}(\theta_1,\bar{\phi_1})\rangle \\
|A(\theta_2,\phi_2)\rangle|\bar{A}(\bar{\theta_2},\bar{\phi_2})\rangle &\longrightarrow |A(\bar{\theta_2},\phi_2)\rangle|\bar{A}(\theta_2,\bar{\phi_2})\rangle
\end{aligned} \tag{5}$$

Now once again, in order to preserve the unitarity of such a transformation we arrive at the same conditions, (i) and (ii). This clearly indicates the fact that there are certain class of states on the bloch sphere for which partial swapping of phase angles and azimuthal angles are possible. However in this context we cannot say that this is true for all such values of $\theta$ and $\phi$ on the bloch sphere. Therefore it is evident that the unitarity of quantum mechanics, doesn't allow partial swapping of quantum parameters for all such pairs of non orthogonal states on the bloch sphere.

## 3. Partial Swapping: Principle of No signalling

Suppose we have two identical singlet states $|\chi\rangle$ shared by two distant parties Alice and Bob. Since the singlet states are invariant under local unitary operations, it remains same in all basis. The states are given by

$$\begin{aligned}|\chi\rangle|\chi\rangle &= \frac{1}{2}(|\psi_1(\theta_1,\phi_1)\rangle|\bar{\psi_1}(\bar{\theta_1},\bar{\phi_1})\rangle - |\bar{\psi_1}(\bar{\theta_1},\bar{\phi_1})\rangle|\psi_1(\theta_1,\phi_1)\rangle)\\ &(|\psi_1(\theta_1,\phi_1)\rangle|\bar{\psi_1}(\bar{\theta_1},\bar{\phi_1})\rangle - |\bar{\psi_1}(\bar{\theta_1},\bar{\phi_1})\rangle|\psi_1(\theta_1,\phi_1)\rangle)\\ &= \frac{1}{2}(|\psi_2(\theta_2,\phi_2)\rangle|\bar{\psi_2}(\bar{\theta_2},\bar{\phi_2})\rangle - |\bar{\psi_2}(\bar{\theta_2},\bar{\phi_2})\rangle|\psi_2(\theta_2,\phi_2)\rangle)\\ &(|\psi_2(\theta_2,\phi_2)\rangle|\bar{\psi_2}(\bar{\theta_2},\bar{\phi_2})\rangle - |\bar{\psi_2}(\bar{\theta_2},\bar{\phi_2})\rangle|\psi_2(\theta_2,\phi_2)\rangle)\end{aligned} \tag{6}$$

where $\{|\psi_1\rangle, |\bar{\psi_1}\rangle\}$ and $\{|\psi_2\rangle, |\bar{\psi_2}\rangle\}$ are two sets of mutually orthogonal spin states (qubit basis). Alice possesses the first particle while Bob possesses the second particle. Alice can choose to measure the spin in any one of the qubit basis namely $\{|\psi_1\rangle, |\bar{\psi_1}\rangle\}$, $\{|\psi_2\rangle, |\bar{\psi_2}\rangle\}$.

The theorem of no signalling tells us that the measurement outcome of one of the two parties are invariant under local unitary transformation done by other party on his or her qubit.The density matrix $\rho_B = tr\rho_{AB} = tr[(U_A \otimes I_B)\rho_{AB}(U_A \otimes I_B)^\dagger]$ is invariant under local unitary operation by the other party. Hence the first party cannot distinguish two mixtures due to the unitary operation done at remote place.
At this point one may ask if Alice(Bob) partially swap the quantum parameters of her(his) particle and if Bob(Alice) measure his(her) particle in either of the two basis then is there any possibility that Alice(Bob) know the basis in which Bob(Alice) measures his(her) qubit or in other words, is there any way by which Alice(Bob) using a perfect partial swapping machine can distinguish the statistical mixture in her(his) subsystem resulting from the measurement done by Bob(Alice). If Alice(Bob) can do this then signalling will take place, which is impossible. Note that whatever measurement Bob(Alice) does, Alice(Bob) does not learn the results and her(his) description will remain as that of a completely random mixture , i.e., $\rho_{A(B)} = \frac{I}{2} \otimes \frac{I}{2}$. In other words we can say that the local operations performed on his(her) subspace has no effect on Alice's(Bob's) description of her(his) states.
Let us consider a situation where Alice is in possession of a hypothetical machine which can partially swap quantum parameters $\theta$ and $\phi$.
Let us first of all consider the case where with the help of the machine we can partially swap the phase angles keeping the azimuthal angles of the states fixed. The action of

such a machine is given by,

$$|\psi_i(\theta_i,\phi_i)\rangle|\bar{\psi}_i(\bar{\theta}_i,\bar{\phi}_i)\rangle \longrightarrow |\psi_i(\theta_i,\bar{\phi}_i)\rangle|\bar{\psi}_i(\bar{\theta}_i,\phi_i)\rangle \tag{7}$$

where $(i = 1, 2)$. Now if after the action of such a transformation on Alice's qubit,the entangled state initially shared between these two parties takes the form,

$$\begin{aligned}|\chi\rangle_{PS}|\chi\rangle_{PS} = &\frac{1}{2}[(|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle)_A(|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_B + \\ &(|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_A(|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle)_B - \\ &(|\psi_1(\theta_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\phi_1)\rangle)_A(|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\psi_1(\theta_1,\phi_1)\rangle)_B - \\ &(|\bar{\psi}_1(\bar{\theta}_1,\phi_1)\psi_1(\theta_1,\bar{\phi}_1)\rangle)_A(|\psi_1(\theta_1,\phi_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_B] \\ = &\frac{1}{2}[(|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle)_A(|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_B + \\ &(|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_A(|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle)_B - \\ &(|\psi_2(\theta_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\phi_2)\rangle)_A(|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\psi_2(\theta_2,\phi_2)\rangle)_B - \\ &(|\bar{\psi}_2(\bar{\theta}_2,\phi_2)\psi_2(\theta_2,\bar{\phi}_2)\rangle)_A(|\psi_2(\theta_2,\phi_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_B]\end{aligned} \tag{8}$$

where A, B denotes the particles in Alice's and Bob's possession respectively.
Now, if Bob does his measurement on $\{|\psi_1\rangle, |\bar{\psi}_1\rangle\}$ qubit basis, then the reduced density matrix describing Alice's subsystem is given by,

$$\begin{aligned}\rho_A = &\frac{1}{4}[|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle\langle\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)| + \\ &|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle\langle\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)| + \\ &|\psi_1(\theta_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\phi_1)\rangle\langle\psi_1(\theta_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\phi_1)| + \\ &|\bar{\psi}_1(\bar{\theta}_1,\phi_1)\psi_1(\theta_1,\bar{\phi}_1)\rangle\langle\bar{\psi}_1(\bar{\theta}_1,\phi_1)\psi_1(\theta_1,\bar{\phi}_1)|]\end{aligned} \tag{9}$$

Interestingly if Bob does his measurement in $\{|\psi_2\rangle, |\bar{\psi}_2\rangle\}$ qubit basis, then the density matrix representing Alice's subsystem is given by,

$$\begin{aligned}\rho_A = &\frac{1}{4}[|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle\langle\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)| + \\ &|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle\langle\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)| + \\ &|\psi_2(\theta_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\phi_2)\rangle\langle\psi_2(\theta_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\phi_2)| + \\ &|\bar{\psi}_2(\bar{\theta}_2,\phi_2)\psi_2(\theta_2,\bar{\phi}_2)\rangle\langle\bar{\psi}_2(\bar{\theta}_2,\phi_2)\psi_2(\theta_2,\bar{\phi}_2)|]\end{aligned} \tag{10}$$

It is clearly evident that the equations (9) and (10) are not identical in any respect and henceforth we will conclude that Alice can distinguish the basis in which Bob has performed the measurement, This is impossible in principle as this will violate the causality. Hence we arrive at a contradiction with the assumption that the partial swapping of phase angle is possible.
Next we see that whether the partial swapping of azimuthal angles is consistent with the principle of no signalling or not.If we assume that the partial swapping of phase angles is possible keeping the azimuthal angles fixed, then its action is given by,

$$|\psi_i(\theta_i,\phi_i)\rangle|\bar{\psi}_i(\bar{\theta}_i,\bar{\phi}_i)\rangle \longrightarrow |\psi_i(\bar{\theta}_i,\phi_i)\rangle|\bar{\psi}_i(\theta_i,\phi_i)\rangle \tag{11}$$

Let us assume that this hypothetical machine is in possession of Alice, and she applies the transformation (11) on her particles as a result of which the entangled state (6) takes the form,

$$\begin{aligned}
|\chi\rangle_{PS}|\chi\rangle_{PS} = & \frac{1}{2}[(|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle)_A(|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_B + \\
& (|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_A(|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle)_B - \\
& (|\psi_1(\bar{\theta}_1,\phi_1)\bar{\psi}_1(\theta_1,\phi_1)\rangle)_A(|\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\psi_1(\theta_1,\phi_1)\rangle)_B - \\
& (|\bar{\psi}_1(\theta_1,\phi_1)\psi_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_A(|\psi_1(\theta_1,\phi_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle)_B] \\
& = \frac{1}{2}[(|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle)_A(|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_B + \\
& (|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_A(|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle)_B - \\
& (|\psi_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\theta_2,\phi_2)\rangle)_A(|\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\psi_2(\theta_2,\phi_2)\rangle)_B - \\
& (|\bar{\psi}_2(\theta_2,\phi_2)\psi_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_A(|\psi_2(\theta_2,\phi_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle)_B] \qquad (12)
\end{aligned}$$

If Bob does his measurement in any one of the two basis $\{|\psi_1\rangle, |\bar{\psi}_1\rangle\}$ and $\{|\psi_2\rangle, |\bar{\psi}_2\rangle\}$, then the respective density matrix representing Alice's subsystem is given as,

$$\begin{aligned}
\rho_A = & \frac{1}{4}[|\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)\rangle\langle\psi_1(\theta_1,\phi_1)\psi_1(\theta_1,\phi_1)| + \\
& |\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\rangle\langle\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)\bar{\psi}_1(\bar{\theta}_1,\bar{\phi}_1)| + \\
& |\psi_1(\bar{\theta}_1,\phi_1)\bar{\psi}_1(\theta_1,\bar{\phi}_1)\rangle\langle\psi_1(\bar{\theta}_1,\phi_1)\bar{\psi}_1(\theta_1,\bar{\phi}_1)| + \\
& |\bar{\psi}_1(\theta_1,\bar{\phi}_1)\psi_1(\bar{\theta}_1,\phi_1)\rangle\langle\bar{\psi}_1(\theta_1,\bar{\phi}_1)\psi_1(\bar{\theta}_1,\phi_1)|] \\
& = \frac{1}{4}[|\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)\rangle\langle\psi_2(\theta_2,\phi_2)\psi_2(\theta_2,\phi_2)| + \\
& |\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\rangle\langle\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)\bar{\psi}_2(\bar{\theta}_2,\bar{\phi}_2)| + \\
& |\psi_2(\bar{\theta}_2,\phi_2)\bar{\psi}_2(\theta_2,\bar{\phi}_2)\rangle\langle\psi_2(\bar{\theta}_2,\phi_2)\bar{\psi}_2(\theta_2,\bar{\phi}_2)| + \\
& |\bar{\psi}_2(\theta_2,\bar{\phi}_2)\psi_2(\bar{\theta}_2,\phi_2)\rangle\langle\bar{\psi}_2(\theta_2,\bar{\phi}_2)\psi_2(\bar{\theta}_2,\phi_2)|] \qquad (13)
\end{aligned}$$

Alice can easily distinguish two statistical mixtures and as a consequence of which she can easily understand in which basis Bob has performed his measurement. This is not possible in principle as this will violate causality. Hence we conclude that the partial swapping of azimuthal angles is not possible.

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