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Editorial

‘Physical Laws Should have Mathematical Beauty’

*Paul Dirac* (August 1902-October 1984)

The scope of the Electronic Journal of Theoretical Physics (EJTP) covers several high quality aspects of theoretical and mathematical physics. It is aimed to support the scientific collaboration between all theoretical physicists.

Recently, the **Majorana Prize** 2008 was awarded to:

- **Geoffrey F. Chew** For his fundamental contributions in thinking the whole Particle Physics following a philosophy which is giving new impulses to the most recent Physics’ areas and for his latest contributions on Quantum Cosmology.
- **Giuseppe Vitiello** for his paper entitled ‘Topological Defects, Fractals and The Structure of Quantum Field Theory’ which will be published in our project on the Quantum Field Theory, Springer, 2009.

The twenty first issue of the Electronic Journal of Theoretical Physics (EJTP) contains a collection of high quality research papers in different topics of mathematical and theoretical physics.
Licata and Sakaji reproduced E. Recami’s original paper entitled ‘The Tolman-Regge Antitelephone Paradox: Its Solution by Tachyon Mechanics’, Lett. Nuovo Cimento, 44, 587 (1985), in this esteemed paper E. Recami discusses the Tolman Paradox and explains how to solve the old Paradox via Tachyon Mechanics. The reason for reproducing it here is that superluminal motions have been actually met, starting with 1992, in a series of experiments reported in a number of papers and researches, in addition to the recent interest in this area. Michail Zak, from the Jet Propulsion Laboratory California Institute of Technology, reports in his remarkable paper, a new approach of Monte-Carlo simulations based on the coupling of dynamical equations and the corresponding Liouville equation without using the random number generator. In her esteemed paper entitled ‘First Passage Random Walk of Coupled Detector-System Pairs and Quantum Measurement’, Fariel Shafee, from Princeton University, discusses an important topic on the quantum measurement and presents a new model for the measurement of a characteristic of a microscopic quantum state by a large system that selects stochastically the different eigenstates with appropriate quantum weights. Peter Enders, from Germany, discusses extensively the gauge freedom in the electromagnetic potentials and explains the underdeterminacy in Maxwell’s theory.

Sidharth from the International Institute for Applicable Mathematics and Information Sciences, discusses some aspects on quantum gravity using non commutative geometry at the Compton scale techniques. In his seminal paper Sergiu I. Vacaru, from The Fields Institute for Research in Mathematical Science, Canada, addresses the nonholonomic Ricci flows of metrics and geometric objects subjected to nonintegrable constraints especially the Deformations of the Solitonic pp-Waves and Schwarzschild Solutions. In his remarkable paper on Quantum Entanglement Singh, from India, examines the behavior of the maximally entangled Bell state of two spin 1/2 massive particles under relativistic transformations. In their seminal paper, Chakrabarty and Choudhury, from India, discuss the impossibility theorem of partially swap and it is consistency to the unitarity principles of quantum mechanics.

Dibakar Ghosh, from India, addresses the chaotic oscillators and the time scale synchronization between two different time- delay systems. Khan, Iqbal and Farooq Ahmad, from India, discuss the phase transitions occurring in the gravitational clustering of galaxies on the basis of thermodynamic fluctuation theory. Bipin Singh Koranga, from India, reports the Neutrino Oscillation Probability from Tri-Bimaximality due to Planck Scale Effects. Sadeghi and Pourhassan, from Mazandaran University, consider the Schwinger and Heisenberg representation of su(1,1) algebra under Hall effect. Bali, Banerjee and Banerjee, from India, address Some LRS Bianchi Type VI0 Cosmological Models with Special Free Gravitational Fields. Eid, and Hamza, from Egypt, presents The Motion of a Test Particle in the Gravitational Field of a Collapsing Shell.
All the papers have gone through EJTP standard peer review process, the hard work of editors and referees are extremely important to ensure the quality of this issue.

I am grateful to the people that have supported me, the distinguished authors, the referees for their careful and timely job, and the EJTP Editors for their reviewing and fruitful suggestions. In particular, Beny Neta, José Luis Lopez-Bonilla, Leonardo Chiatti, and Tepper L. Gill. Special thanks to Ignazio Licata for his valuable suggestions and Editorial help.


I hope that this issue will make more physicists aware of the current and hot research that concern Theoretical Physics.

*Ammar Sakaji, EJTP Editor*
The Tolman-Regge Antitelephone Paradox: Its Solution by Tachyon Mechanics∗†

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Abstract: The possibility of solving (at least “in microphysics”) all the ordinary causal paradoxes devised for tachyons is not yet widely recognized; on the contrary, the effectiveness of the Stuckelberg-Feynman switching principle is often misunderstood. We want, therefore, to show in detail and rigorously how to solve the oldest causal paradox, originally proposed by Tolman, which is the kernel of so many further tachyon paradoxes. The key to the solution is a careful application of tachyon kinematics, which can be unambiguously derived from special relativity. A systematic, thorough analysis of all tachyon paradoxes is going to appear elsewhere.


Keywords: Special Relativity
PACS (2008): 03.30

Preamble

At the beginning of the seventies, the research group leaded by the author (E. Recami, in collaboration especially with R. Mignani, et al.) extended the theory of Special Relativity, SR, for describing also (antimatter and) superluminal motions, on the basis of

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the standard postulates of SR. Such an Extended, or rather non-restricted, Relativity, ER, does not imply, consequently, any “violations”: In particular, the so-called causal paradoxes, seemingly associated with tachyons, were solved long ago; and reviews about such solutions appeared in 1986 (Riv. Nuovo Cimento 9(6), 1-178) and especially in 1987 (Found. Phys. 17, 239-296). The first paper in this direction is here re-published. One reason is that superluminal motions have been actually met, starting with 1992, in a series of experiments reported in a number of papers (e.g., Phys. Rev. Lett. 1997, and 2000): cf., for instance, our book Localized Waves (J. Wiley; 2008), and particularly its introductory Chapters 1 and 2. Thus, many scientists at present are trying to reconstruct something like ER, often with poor results and sometimes even with a few mistakes, being they unaware of the existing literature which in due time appeared only on paper and not in electronic form. This, as we were saying, is the principal reason for the present reproduction.

P.S.: further information can be found, e.g., in the home-page www.unibg.it/recami

1. Introduction

It has been claimed since long [1] that all the ordinary causal paradoxes proposed for tachyons can be solved (at least “in microphysics”) [2] on the basis of the switching procedure (SWP) by STÜCKELBERG[3], FEYNMAN[3] and SUDARSHAN[1], also known as the reinterpretation principle: a principle which has been given the status of a fundamental postulate[4] of special relativity (SR), both for bradyons[5] and for tachyons. Recently, SCHWARTZ gave the SWP a formalization in which it becomes automatic[6]. Most of the authors, however, seem still to ignore the effectiveness of those solutions; sometimes showing to be aware only of some older —and, therefore, preliminary and incomplete— papers, while being unaware of the more recent and complete literature. Some authors appear, moreover, to misunderstand even the literature known to them: for a late, remarkable example see, e.g., GIRARD and MARCHILDON. We want, therefore, to show here (in detail and rigorously) how to solve the oldest paradox, i.e. the antitelephone one, originally proposed by TOLMAN[8] and then re-proposed by many authors. We shall refer to its most recent formulation, by REGGE[9], and spend some care in solving it, since it is the kernel of many other paradoxes. Let us stress that: i) any careful solution of the tachyon causal paradoxes must have recourse to explicit calculations based on the mechanics of tachyons; ii) such tachyon mechanics can be unambiguously and univocally derived from SR, by referring the spacelike objects to the class of the ordinary, subluminal observers only (i.e., without any need of introducing Superluminal reference frames); iii) the reader will find a lot of help, moreover, by referring himself, first, to the (subluminal, ordinary) SR based on the whole proper Lorentz
group $L_+ \equiv L^1_+ \cup L^1_-$, rather than on its orthochronous subgroup $L^1_+$ only (see ref.[5], and references therein). A systematic, thorough analysis of the tachyon causal problems can be found in refs.[2,10], which appeared elsewhere. Before going on, let us refer the reader —for a modern approach to the classical theory of tachyons—to refs.[2,10,11].

2. Tachyon Kinematics

In ref.[12] it can be found exploited the basic tachyon kinematics related to the following processes: a) the proper (or “intrinsic”) emission of a tachyon $T$ by an ordinary body $A$; b) the “intrinsic” absorption of a tachyon $T$ by an ordinary body $A$; c) the exchange of a tachyon $T$ between two ordinary bodies $A$ and $B$. The word “intrinsic” refers to the fact that those processes (emission, absorption) are describes in the rest-frame of the body $A$; while particle $T$ can represent both a tachyon and an antitachyon. Let us recall only the following results.

First, let us consider a tachyon moving with velocity $V$ in the frame $s_0$. If we go on to a second frame $s'$, endowed with velocity $u$ with respect to (w.r.t.) frame $s_0$, then the new observer $s'$ will see —instead of the initial tachyon $T$— an antitachyon $\bar{T}$ travelling the opposite way in space (due to the SWP), if and only if

$$u \cdot V > c^2.$$  

(1)

Remember in particular that, if $u \cdot V < 0$, the “switching” does never come into play.

Now, let us explore some of the unusual and unexpected consequences of the mere fact that in the case of tachyons it is

$$|E| = +\sqrt{p^2 - m_0^2} \quad (m_0 \text{ real}; \ V^2 > 1),$$  

(2)

where we chose units so that, numerically, $c = 1$.

Let us, e.g., describe the phenomenon of “intrinsic emission” of a tachyon, as seen in the rest frame of the emitting body: Namely, let us consider in its rest frame an ordinary body $A$, with initial rest mass $M$, which emits a tachyon (or antitachyon) $T$ endowed with (real) rest mass $m \equiv m_0$, four-momentum $p^\mu \equiv (E_T, p)$, and speed $V$ along the $x$-axis. Let $M'$ be the final rest mass of body $A$. The four-momentum conservation requires

$$M = \sqrt{p^2 - m^2} + \sqrt{p^2 + M'^2} \quad \text{(rest frame)}$$  

(3)

that is to say:

$$2M|p| = [(m^2 + \Delta)^2 + 4m^2M^2]^{1/2}; \quad V = [1 + 4m^2M^2/(m + \Delta)^2]^{1/2},$$  

(4)

where [we put $E_T \equiv +\sqrt{p^2 - m^2}$]:

$$\Delta \equiv M'^2 - M^2 = -m^2 - 2ME_T \quad \text{(emission)}$$  

(5)
so that

\[-M^2 < \Delta \leq -|p|^2 \leq -m^2 \]  \hspace{1cm} \text{(emission)} \hspace{1cm} (6)

It is essential to notice that \( \Delta \) is, of course, an \textit{invariant} quantity, that in a generic frame \( s \) writes

\[ \Delta = -m^2 - 2p_\mu P^\mu , \]  \hspace{1cm} (7)

where \( P^\mu \) is the initial four-momentum of body A w.r.t. frame \( s \).

Notice that in the generic frame \( s \) the process of (intrinsic) emission can appear both as a \( T \) emission and as a \( \bar{T} \) absorption (due to a possible “switching”) by body A. It holds, however, the theorem[2,10,12]:

\textit{Theorem 1: }<< necessary and sufficient condition for a process to be a tachyon emission in the A rest-frame (i.e., to be an \textit{intrinsic emission}) is that during the process the body A lowers its rest-mass (invariant statement!) in such a way that \(-M^2 < \Delta \leq -m^2 >>.\)

Let us now describe the process of “intrinsic absorption” of a tachyon by body A; i.e., let us consider an ordinary body A to absorb \textit{in its rest} frame a tachyon (or antitachyon) \( T \), travelling again with speed \( V \) along the \( x \)-direction. The four-momentum conservation now requires

\[ M + \sqrt{p^2 - m^2} = \sqrt{p'^2 + M'^2} , \]  \hspace{1cm} (rest frame) \hspace{1cm} (8)

which corresponds to

\[ \Delta \equiv M'^2 - M^2 = -m^2 + 2ME_T , \]  \hspace{1cm} (absorption) \hspace{1cm} (9)

so that

\[-m^2 \leq \Delta \leq +\infty . \]  \hspace{1cm} (absorption) \hspace{1cm} (10)

In a generic frame \( s \), the quantity \( \Delta \) takes on the invariant form

\[ \Delta = -m^2 + 2p_\mu P^\mu . \]  \hspace{1cm} (11)

It follows the theorem[2,10,12]:


Theorem 2: << necessary and sufficient condition for a process (observed either as the emission or as the absorption of a tachyon $T$ by an ordinary body $A$) to be a tachyon absorption in the $A$-rest-frame —i.e., to be an intrinsic absorption— is that $\Delta \geq -m^2 >>$.

We have now to describe the tachyon exchange between two ordinary bodies $A$ and $B$. We have to consider the four-momentum conservation at $A$ and at $B$; we need to choose a (single) frame wherefrom to describe the whole interaction; let us choose the rest-frame of $A$. Let us explicitly remark, however, that —when bodies $A$ and $B$ exchange one tachyon $T$— the tachyon kinematics is such that the “intrinsic descriptions” of the processes at $A$ and at $B$ can a priori correspond to one of the following four cases[12]:

\[
\begin{align*}
1) & \text{ emission—absorption} , \\
2) & \text{ absorption—emission} , \\
3) & \text{ emission—emission} , \\
4) & \text{ absorption—absorption} .
\end{align*}
\]

Case 3) can happen, of course, only when the tachyon exchange takes place in the receding phase (i.e., while $A$, $B$ are receding from each other); case 4) can happen, on the contrary, only in the approaching phase.

Let us consider, here, only the particular tachyon exchanges in which we have an “intrinsic emission” at $A$, and moreover the velocities $u$ of $B$ and $V$ of $T$ w.r.t. body $A$ are such that $u \cdot V > 1$. Due to the last condition and the consequent “switching” (cf. Eq.(1)), in the rest-frame of $B$ it will then be observed an antitachyon $\overline{T}$ emitted by $B$ and absorbed by $A$ (necessary condition for this to happen, let us recall, being that $A$, $B$ be receding from each other).

More in general, the kinematical conditions for a tachyon to be exchangeable between $A$ and $B$ can be shown [12] to be the following:

A) Case of “intrinsic emission” at $A$:

\[
\begin{align*}
\text{if } u \cdot V < 1 , \text{ then } \Delta_B > -m^2 \quad (\rightarrow \text{ intrinsic absorption at } B); \\
\text{if } u \cdot V > 1 , \text{ then } \Delta_B < -m^2 \quad (\rightarrow \text{ intrinsic emission at } B).
\end{align*}
\]
B) Case of “intrinsic absorption” at A:

\[
\begin{align*}
\text{if } \mathbf{u} \cdot \mathbf{V} < 1, \quad &\text{then } \Delta_B < -m^2 \quad (\rightarrow \text{intrinsic emission at B}); \\
\text{if } \mathbf{u} \cdot \mathbf{V} > 1, \quad &\text{then } \Delta_B > -m^2 \quad (\rightarrow \text{intrinsic absorption at B}).
\end{align*}
\]

(14)

Now, let us finally pass to examine the Tolman-Regge paradox.

3. The Paradox

In Figs.1,2 the axes \( t \) and \( t' \) are the world-lines of two devices A and B, respectively, able to exchange tachyons and moving with constant relative speed \( u \), \([u^2 < 1]\), along the \( x \)-axis. According to the terms of the paradox (Fig.1), body A sends tachyon 1 to B (in other words, tachyon 1 is supposed to move forward in time w.r.t. device A). The apparatus B is constructed in such a way to send back tachyon 2 to A as soon as it receives a tachyon 1 from A. If B has to emit (in its rest-frame) tachyon 2, then 2 must move forward in time w.r.t. body B; that is to say, the world-line \( BA_2 \) must have a slope lower than the slope \( BA'_1 \) of the \( x' \)-axis (where \( BA'//x' \)): this means that \( A_2 \) must stay above \( A' \). If the speed of tachyon 2 is such that \( A_2 \) falls between \( A' \) and \( A_1 \), it seems that 2 goes back to A (event \( A_2 \)) before the emission of 1 (event \( A_1 \)). This seems to realize an anti-telephone.

![Fig. 1 The apparent chain of events, according to the terms of the paradox.](image)

4. The Solution

First of all, since tachyon 2 moves backwards in time w.r.t. body A, the event \( A_2 \) will appear to A as the emission of an antitachyon \( \overline{2} \). The observer “\( t \)” will see his own
apparatus A (able to exchange tachyons) emit successively towards B the antitachyon $\overline{2}$ and the tachyon 1.

At this point, some supporters of the paradox (overlooking tachyon kinematics, as well as relations (12)) would say that, well, the description forwarded by the observer “t” can be orthodox, but then the device B is no longer working according to the stated premises, because B is no longer emitting a tachyon 2 on receipt of tachyon 1. Such a claim, however, would be wrong, since the fact that “t” sees an “intrinsic emission” at A$_2$ does not mean that “t’” will see an “intrinsic absorption” at B! On the contrary, we are just in the case considered above, between eqs. (12) and (13): an intrinsic emission by A, at A$_2$, with $u \cdot V_2 > c^2$, where u and V$_2$ are the velocities of B and 2 w.r.t. body A, respectively; so that both A and B suffer an intrinsic emission (of tachyon 2 or of antitachyon $\overline{2}$) in their own rest frame.

But the terms of the “paradox” were cheating us even more, and ab initio. In fact Fig.1 makes it clear that, if $u \cdot V_2 > c^2$, then for tachyon 1 a fortiori $u \cdot V_1 > c^2$, where u and V$_1$ are the velocities of B and 1 w.r.t. body A. Therefore, due to the above-seen tachyon kinematics, observer “t’” will see B intrinsically emit also tachyon 1 (or, rather, antitachyon $\overline{1}$). In conclusion, the proposed chain of events does not include any tachyon absorption by B (in its rest frame).

For body B to absorb tachyon 1 (in its own rest frame), the world-line of 1 ought to have a slope higher than the slope of the x’-axis (see Fig.2). Moreover, for body B to emit (“intrinsically”) tachyon 2, the slope of 2 should be lower than the x’-axis’. In other words, when the body B, programmed to emit 2 as soon as it receives 1, does actually do so, the event A$_2$ does regularly happen after A$_1$ (cf. Fig.2).

5. The Moral

The moral of the story is twofold: i) one should never mix together the descriptions (of one phenomenon) yielded by different observers; otherwise —even in ordinary physics—
one would immediately meet contradictions: in Fig. 1, e.g., the motion direction of 1 is assigned by A and the motion direction of 2 is assigned by B; this is “illegal”; ii) when proposing a problem about tachyons, one must comply [1] with the rules of tachyon mechanics [12]: just as when formulating the text of an ordinary problem one must comply with laws of ordinary physics (otherwise the problem in itself is “wrong”).

Most of the paradoxes proposed in the literature suffered the above shortcomings (cf. e.g. [7]).

Notice that, in the case of Fig. 1, neither A nor B regard event A$_1$ as the cause of event A$_2$ (or vice-versa). In the case of Fig. 2, on the contrary, both A and B consider event A$_1$ to be the cause of event A$_2$; but in this case A$_1$ does chronologically precede A$_2$ according to both observers, in agreement with the relativistic covariance of the law of retarded causality. For a systematic, thorough analysis of the tachyon causal problems we refer once more the interested reader to refs.[2,10].

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The author gladly acknowledges the constant encouragement and interest of A. GIGLI BERZOLARI, as well as, for the present reproduction, I. LICATA and A. SAKAJI.

References


[5] See e.g. M.PAVSIC and E.RECAMI: *Lett. Nuovo Cimento*, 34, 357 (1982); 35, 354 (1982); E.RECAMI and W.A.RODRIGUES Jr.: *Found. Phys.*, 12, 709 (1982) [in connection with the last paper, let us take advantage of the present opportunity for correcting some important misprints: at p.741, correct (nonnegative energy objects exist) into (no negative-energy objects exist); at p.742, correct $PT = +1$ into $PT = -1$, and $A^+\updownarrow$ into $A^+\downarrow$].


New Physical Principle for Monte-Carlo simulations

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Abstract: New physical principle for Monte-Carlo simulations has been introduced. It is based upon coupling of dynamical equations and the corresponding Liouville equation. The proposed approach does not require a random number generator since randomness is generated by instability of dynamics triggered and controlled by the feedback from the Liouville equation. Direct simulation of evolutionary partial differential equations have been proposed, discussed, and illustrated.

Keywords: Monte-Carlo Simulations; Dynamical Equations; Liouville Equation; Numerical Methods

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1. Introduction

Monte-Carlo is one of the most powerful computational tools for solving high-dimensional problems in physics, chemistry, economics, and information processing. It is especially effective for problems in which the computational complexity grows exponentially with the dimensionality (PDE, NP-complete problems). The main advantage of the Monte-Carlo simulations over other computational techniques is independence of the computational resources upon the problem dimensionality. There are many modifications of this method such as "classical" Monte Carlo, (samples are drawn from a probability distribution), "quantum" Monte Carlo, (random walks are used to compute quantum-mechanical energies and wavefunctions), "path-integral" quantum Monte Carlo, (quantum statistical mechanical integrals are computed to obtain thermodynamic properties), "simulation" Monte Carlo, (stochastic algorithms are used to generate initial conditions for quasiclassical trajectory simulations), etc. However, success of this approach depends upon effi-
cient implementations of multi-dimensional stochastic processes with prescribed density
distribution, and that necessitates a fast and effective way to generate random num-
bers uniformly distributed on the interval [0,1]. It should be noticed that often-used
computer-generated numbers are not really random, since computers are deterministic.
In particular, if a random number seed is used more than once, one will get identical ran-
dom numbers every time. Therefore, for multiple trials, different random number seeds
must be applied. The proposed modification of the Monte-Carlo method is free of such
limitation since it is based upon randomness that is generated by instability of dynamics
triggered and controlled by the feedback from the Liouville equation. The approach is
motivated by the mathematical formalism of coupled Liouville-Langevin equations being
applied to modeling active systems,[1], as well as by the Madelung version of quantum
mechanics, [2]. Our strategy exploits different feedbacks from the Liouville equation to
the corresponding dynamical ODE. These feedbacks may be different from those used
for modeling active or quantum systems and their choice is based upon computational
efficiency rather than upon a physical meaning of the underlying models.

2. Destabilizing Liouville Feedback

For mathematical clarity, we will start here with a one-dimensional motion of a unit mass
under action of a force \( f \) depending upon the velocity \( x \) and time \( t \)

\[
\dot{x} = f(x,t),
\]

If initial conditions are not deterministic, and their probability density is given in the
form

\[
\rho_0 = \rho_0(X), \quad \text{where } \rho \geq 0, \quad \text{and } \int_{-\infty}^{\infty} \rho dX = 1
\]

while \( \rho \) is a single- valued function, then the evolution of this density is expressed by the
corresponding Liouville equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X}(\rho f) = 0
\]

The solution of this equation subject to initial conditions and normalization constraints
(2) determines probability density as a function of \( X \) and \( t \)

\[
\rho = \rho(X,t)
\]

Let us now specify the force \( f \) as a feedback from the Liouville equation

\[
f(x,t) = \varphi[\rho(x,t)]
\]

and analyze the motion after substituting the force (5) in to Eq.(1)

\[
\dot{x} = \varphi[\rho(x,t)],
\]
This is a fundamental step in our approach. Although theory of ODE does not impose any restrictions upon the force as a function of space coordinates, the Newtonian physics does: equations of motion are never coupled with the corresponding Liouville equation. Moreover, as shown in [3], such a coupling leads to non-Newtonian properties of the underlying model. Indeed, substituting the force \( f \) from Eq. (5) into Eq. (3), one arrives at the nonlinear equation for evolution of the probability density

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X}\{\rho \varphi[\rho(X,t)]\} = 0
\] (7)

Regardless of a physical meaning of the underlying model, we will exploit the mathematical property of this equation for simulation of a prescribed probability density.

Now we will demonstrate the destabilizing effect of the feedback (6). For that purpose, it should be noted that the derivative \( \frac{\partial \rho}{\partial x} \) must change its sign, at least once, within the interval \(-\infty < x < \infty\), in order to satisfy the normalization constraint (2). But since

\[
\text{Sign } \frac{\partial \dot{x}}{\partial x} = \text{Sign } \frac{d\varphi}{d\rho} \text{Sign } \frac{\partial \rho}{\partial x}
\] (8)

there will be regions of \( x \) where the motion is unstable, and this instability generates randomness with the probability distribution guided by the Liouville equation (7).

3. Simulation of Stochastic Processes with Prescribed Probability Distributions

Most of the Monte-Carlo simulations require generation of stochastic processes having prescribed probability density distribution. In this section we propose to generate such processes using controlled instability of the ODE driven by a feedback from the corresponding Liouville equation.

Let us consider Eqs. (1) and (7) defining \( f \) as the following function of probability density

\[
f = \frac{1}{\rho(x,t)} \int_{-\infty}^{x} [\rho(\zeta,t) - \rho^*(\zeta)]d\zeta
\] (9)

Then these equations take the form, respectively

\[
\dot{x} = \frac{1}{\rho(x,t)} \int_{-\infty}^{x} [\rho(\zeta,t) - \rho^*(\zeta)]d\zeta
\] (10)

\[\frac{d\rho}{dt} + \rho(t) - \rho^* = 0\] (11)

The last equation has the analytical solution

\[
\rho = (\rho_0 - \rho^*)e^{-t} + \rho^*
\] (12)
Subject to the initial condition
\[ \rho(t = 0) = \rho_0 \] (13)
this solution converges to a prescribed stationary distribution \( \rho^*(x) \).

Substituting the solution (12) into Eq. (10), one arrives at the ODE that simulates the stochastic process with the probability distribution (12),
\[ \dot{x} = \frac{e^{-t}}{[\rho_0(x) - \rho^*(x)]e^{-t} + \rho^*(x)} \int_{-\infty}^{x} [\rho_0(\zeta) - \rho^*(\zeta)]d\zeta \] (14)

As noticed above, the randomness of the solution to Eq. (14) is caused by instability that is controlled by the corresponding Liouville equation. It should be emphasized that in order to run the stochastic process started with the initial distribution \( \rho_0 \) and approaching a stationary process with the distribution \( \rho^* \), one should substitute into Eq. (14) analytical expressions for these functions.

It is reasonable to assume that the solution (12) starts with sharp initial condition
\[ \rho_0(X) = \delta(X) \] (15)
As a result of that assumption, all the randomness is supposed to be generated only by the controlled instability of Eq. (14). Substitution of Eq. (15) in Eq. (14) leads to two different domains, [4]
\[ \int_{-\infty}^{x} \rho^*(\zeta)d\zeta = \frac{C_1}{e^{-t} - 1}, \quad x \neq 0 \] (16)
\[ x = 0 \] (17)
Eq. (17) represents a singular solution, while Eq. (16) is a regular solution that includes arbitrary constant \( C_1 \). The regular solution is unstable at \( t=0 \), \( |x| \to 0 \) where the Lipschitz condition is violated
\[ \frac{d\dot{x}}{dx} \to \infty \text{ at } t \to 0, \quad |x| \to 0 \] (18)
and therefore, an initial error always grows generating randomness. This type of non-Lipschitz instability has been introduced and analyzed in [3]. The solutions (15) and (16) describe an irreversible motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of Lipschitz condition at \( t=0 \), while the backward motion obtained by replacement of \( t \) with \((-t)\) leads to imaginary values of velocities. As follows from Eq. (14),
\[ \dot{x} \to 0 \text{ at } t \to \infty \text{ for } -\infty < x < \infty \] (19)
If Eq. (14) is run for fixed initial conditions, i.e. for fixed values of the arbitrary constant \( C_1 \), the solution (16) produces a sample of the corresponding stochastic process. If the same equation (14) is run at different values of \( C_1 \), the solution produces the whole...
ensemble characterizing the underlying stochastic process. Due to the condition (19), eventually the process becomes stationary as it prescribed by the solution of Eq. (12).

The approach is generalized to n-dimensional case simply by replacing $x$ with a vector

$$x = x_1, x_2, ..., x_n$$  \hspace{1cm} (20)

Application of the results described in this section to finding global maximum of an integrable (but not necessarily differentiable) function has been introduced in [2].

Remark 1. It has been assumed that both probability distributions, $\rho^*$ and $\rho_0$ satisfy the normalization constraints

$$\int_{-\infty}^{\infty} \rho^*(X) dX = 1, \quad \int_{-\infty}^{\infty} \rho_0(X) dX = 1$$  \hspace{1cm} (21)

Then, integrating Eq. (11) over $X$, with reference to Eqs. (19), one obtains

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho(X) dX = 0,$$  \hspace{1cm} (22)

Therefore, if the probability distribution satisfies the normalization constraint at $t = 0$, it will satisfy the same constraint for $t > 0$.

Remark 2. In order to control the speed of convergence to the stationary distribution, one can modify Eq. (9) by introducing a control parameter $a$

$$f = \frac{a}{\rho(X, t)} \int_{-\infty}^{X} [\rho(\zeta, t) - \rho^*(\zeta)] d\zeta$$  \hspace{1cm} (23)

4. Example

Turning to the solution (16), let us choose the final distributions $\rho^*$ as following

$$\rho^*(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}}$$  \hspace{1cm} (24)

Substituting the expressions (21) into Eq. (16) at $X = x$, one obtains

$$x = erf^{-1}(\frac{C_1}{e^{-t} - 1} - 1)$$  \hspace{1cm} (25)

5. Direct Simulation of Parabolic PDE

Computational complexity of deterministic numerical methods for solving PDE grows exponentially with the growth of the problem dimensionality, and that make such problems intractable. On the other hand, the Monte-Carlo methods, which are based upon simulation of a random walk, are free of this major limitation. However, simulating of a
random work with prescribed probability distribution has its own limitations. (Actually, the approach proposed above can be applied for that purpose). In this section we will demonstrate direct simulations of the solutions to parabolic PDE using a feedback from the Liouville equation. The main idea of the proposed approach is to consider a given parabolic PDE (to be solved) as a Liouville equation for some dynamical system described by random ODE, and thereby, to replace solution of PDE by a system of random ODE. The first candidate for the demonstration of the proposed approach is a multi-dimensional Fokker-Planck equation. The computational complexity of integrating this equation is on the order of \((1/\varepsilon)^\ell\) — that is, the reciprocal of the error threshold raised to a power equal to the number of variables that is exponential in \(\ell\). In contradistinction to that, the resources for simulations by Monte-Carlo (MC) method is on the order of \((1/\varepsilon^2)\), i.e., they do not depend upon the dimensionality of the problem.

There is another advantage of MC-simulations of the Fokker-Planck equation: suppose that we are interested in behavior of the solution in a local region of the variables \(\{x\}\); then, in case of computing, one has to find the global solution first, and only after that the local solution can be extracted, while the last procedure requires some additional integrations in order to enforce the normalization constraints. On the other hand, in case of MC simulations, one can project all the simulations onto a desired sub-space \(j_\alpha \otimes j_\beta\) of the total space \(j_1 \otimes \ldots j_\ell\) and directly obtain the local solution just disregarding the rest of the space.

All these advantages of MC method are preserved in the proposed approach that is based upon new physical principle of generating randomness.

For the purpose of mathematical clarity, we will start with a simplest case of the one-dimensional Fokker-Planck equation

\[
\frac{\partial \rho}{\partial t} = \sigma^2 \frac{\partial^2 \rho}{\partial X^2} \tag{26}
\]

The solution of Eq. (26) subject to the sharp initial condition is

\[
\rho = \frac{1}{2\sigma\sqrt{\pi t}} \exp\left(-\frac{X^2}{4\sigma^2 t}\right) \tag{27}
\]

Let us now find a dynamical equation for which Eq. (26) plays the role of the corresponding Liouville equation. It can be verified that such an equation is the following

\[
\dot{x} = -\sigma^2 \frac{\partial}{\partial X} \ln \rho, \tag{28}
\]

Substituting the solution into Eq. (28) at \(X = x\) one arrives at the differential equation with respect to \(x(t)\)

\[
\dot{x} = \frac{x}{2t} \tag{29}
\]

and therefore,

\[
x = C\sqrt{t} \tag{30}
\]

where \(C\) is an arbitrary constant. Since \(x = 0\) at \(t = 0\) for any value of \(C\), the solution (30) is consistent with the sharp initial condition for the solution (27) of the corresponding
Liouville equation (26). As the solution (15) and (16), the solution (30) describes the simplest irreversible motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of Lipschitz condition at \( t = 0 \), Fig. 1), while the backward motion obtained by replacement of \( t \) with \((−t)\) leads to imaginary values of velocities. One can notice that the probability density (27) possesses the same properties.

For a fixed \( C \), the solution (29) is unstable since

\[
\frac{d\dot{x}}{dx} = \frac{1}{2t} > 0
\]

and therefore, an initial error always grows generating randomness, (compare to Eq.(25)). Initially, at \( t = 0 \), this growth is of infinite rate since the Lipschitz condition at this point is violated

\[
\frac{d\dot{x}}{dx} \to \infty \quad \text{at} \quad t \to 0
\]

Considering first Eq. (30) at fixed \( C \) as a sample of the underlying stochastic process (28), and then varying \( C \), one arrives at the whole ensemble characterizing that process, (see Fig. 1). One can verify that, as follows from Eq. (27), the expectation and the variance of this process are, respectively

\[
M_x = 0, \quad D_x = 2\sigma^2 t
\]

The same results follow from the ensemble (30) at \(-\infty \leq C \leq \infty\). Indeed, the first equality in (33) results from symmetry of the ensemble with respect to \( x = 0 \); the second one follows from the fact that

\[
D_x \propto x^2 \propto t
\]

It is interesting to notice that the stochastic process (30) is an alternative to the following Langevin equation, [5]

\[
\dot{x} = \Gamma(t), \quad M\Gamma = 0, \quad D\Gamma = \sigma
\]

that corresponds to the same Fokker-Planck equation (26).

Here \( \Gamma(t) \) is the Langevin (random) force with zero mean and constant variance \( \sigma \). Thus, the solution to Eq. (28) has a well-organized structure: as follows from Eq. (30) (Fig 1), the initial “explosion” of instability driven by the violation of the Lipschitz condition at \( t = 0 \) distributes the motion over the family of smooth trajectories with the probability expressed by Eq. (27). Therefore, the solution (27) can be reconstructed from the solution (30) by taking cross-section over the ensemble at fixed \( t \), and then computing the probability density for each \( t \). It should be emphasized that in this particular case, we have already known the solution (see Eq. (27)), and this solution was used to obtain the solution (30) in the analytical form. However, in the next example we will show that actually we need to know only the initial value of the solution (27).

Let us turn now to the Fokker-Planck equation with a variable diffusion coefficient

\[
\frac{\partial \rho}{\partial t} + \frac{\partial^2}{\partial X^2}[a(X)\rho] = 0
\]
to be solved subject to the following initial conditions

$$\rho(t = 0) = \rho_*(X)$$

First of all, we have to find such an ODE for which Eq. (35) plays the role of the corresponding Liouville equation. Although such an ODE is not unique, we will select the simplest one

$$\dot{x} = \frac{1}{\rho} \frac{\partial}{\partial x} [a(x) \rho(x)]$$

Substituting the initial value of the probability density (36) into Eq. (37), one finds an approximate solution to Eq. (37) within a small initial time interval $\Delta t$

$$x = x_0 + \frac{1}{\rho_*(x_0)} \left\{ \frac{\partial}{\partial x} [a(x) \rho_*(x)] \right\}_{x=x_0} \Delta t, \quad 0 < t < \Delta t$$

where $x_0$ is the initial value of $x$.

Varying this value as $-\infty < x_0 < \infty$, one obtains the whole ensemble of solutions during the initial time interval $\Delta t$. Then computing the probability density $\rho_*(x_1)$ at $t = \Delta t$ one arrives at the next approximation cycle that starts with the “initial” density $\rho_*(x_1)$, where

$$x_1 = x_0 + \frac{1}{\rho_*(x_0)} \left\{ \frac{\partial}{\partial x} [a(x) \rho_*(x)] \right\}_{x=x_0} \Delta t$$

etc. It should be emphasized that the solution is obtained without exploiting the original PDE (35). Nothing would change in the proposed methodology if

$$a = a(x, t, \rho, \frac{\partial \rho}{\partial x}...)$$

This would include a nonlinear version of the Fokker-Planck equation, the Burgers equation, the Korteveg-de-Vries equation, etc. A generalization to multi-dimensional case

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\nabla (a \rho)] = 0, \quad a = a(\{X\}, t, \rho, ...)$$

where $\{X\} = X_1, X_2, ...X_n$ is n-dimensional vector, would require a system of n ODE to replace Eq. (37)

$$\{\dot{x}\} = \frac{1}{\rho} \nabla (a \rho), \quad a = a(\{x\}, t, \rho, ...)$$

For this system, Eq.(41) plays the role of the Liouville equation, and the proposed methodology described above can be applied without significant modification.

Remark 3. It should be noticed that as in other modifications of the MC simulations, the normalization constraints imposed upon the probability density are implemented “automatically”, and this is another significant advantage of simulations over computations. However, if the variable of a PDE to be solved is not a probability density, the normalization constraint can be applicable only in special cases of mass or heat conservation, while in case of sinks or sources this constraint has to be modified.
6. Direct Simulation of Hyperbolic PDE

Let us modify Eq. (44) as following

\[ \dot{x} = -\frac{1}{\rho} \frac{\partial (\sigma^2 \psi)}{\partial x}, \quad \sigma = \sigma(x), \quad \dot{\psi} = \rho \] \hspace{1cm} (43)

Then the corresponding Liouville equation is

\[ \frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial X^2} (\sigma^2 \psi) \] \hspace{1cm} (44)

or, after differentiation with respect to time,

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial X^2} (\sigma^2 \rho) \] \hspace{1cm} (45)

This is a hyperbolic equation describing elastic waves in strings, as well as longitudinal waves in bars, and its solution is simulated by the dynamical system (43). A generalization to multi-dimensional case is straightforward

\[ \frac{\partial^2 \rho}{\partial t^2} = \nabla^2 (\sigma^2 \rho), \quad \sigma = \sigma(\{X\}) \] \hspace{1cm} (46)

where \( \{X\} = X_1, X_2, \ldots X_n \) is n-dimensional vector, would require a system of n ODE to replace Eq. (43)

\[ \{\dot{x}\} = \frac{1}{\rho} \nabla (\sigma^2 \psi), \quad \dot{\psi} = \rho \] \hspace{1cm} (47)

The waves with dispersion that is represented by the Klein-Gordon equation of quantum theory

\[ \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial X} (\sigma^2 \frac{\partial \rho}{\partial X}) - \beta^2 \rho \] \hspace{1cm} (48)

are simulated by the following dynamical system

\[ \dot{x} = -\frac{\sigma^2}{\rho} \frac{\partial \psi}{\partial x} + \frac{\beta^2}{\rho} \int_{-\infty}^{x} \rho dx, \quad \sigma = \sigma(x), \quad \dot{\psi} = \rho, \quad \beta = \text{const} \] \hspace{1cm} (49)

The same approach can be applied to non-hyperbolic wave equations. For instance, the equation for flexural vibration of a beam

\[ \frac{\partial^2 \rho}{\partial t^2} = -\gamma^2 \frac{\partial^4 \rho}{\partial X^4} \] \hspace{1cm} (50)

is simulated by the dynamical system

\[ \dot{x} = -\frac{\gamma^2}{\rho} \frac{\partial^3 \psi}{\partial x^3}, \quad \dot{\psi} = \rho \] \hspace{1cm} (51)

However, the simulation strategy of these wave equations is different from the parabolic PDE described above since solutions of the hyperbolic equations do not have transitions
from a delta-function to distributed functions as the parabolic equations do. Therefore, such a transition should be introduced prior to simulation. We will illustrate the proposed strategy based upon Eqs. (43) and (44). Suppose that

$$\rho(t = 0) = \rho_0(x), \quad \psi(t = 0) = \psi_0(x)$$

(52)

and introduce an auxiliary dynamical system of the type (10)

$$\dot{x} = \frac{1}{\rho(x, t)} \int_{-\infty}^{\infty} \left[ \rho(\zeta, t) - \rho^*(\zeta) \right] d\zeta, \quad \rho^*(x) = \frac{1}{\rho_0} \frac{d\psi_0}{dx}$$

(53)

that has the corresponding Liouville equation of the type (11)

$$\frac{\partial \rho}{\partial t} + \rho(t) - \rho^* = 0$$

(54)

The simulation starts with the system (53) which approaches the stationary distribution of the probability $$\rho^*(X)$$ after the time period (see Eq. (17))

$$\tau \approx \ln \frac{1 - \rho^*}{\varepsilon \rho^*}$$

(55)

Using this solution as random initial conditions for Eqs. (43), one can start the simulation of Eqs. (43).

7. Direct Simulation of Elliptic PDE

Elliptic equations describe stationary processes, and in many cases they can be considered as a limit of the solutions of the corresponding parabolic equations. We will demonstrate the proposed strategy of the simulations using the solution of the Laplace equation

$$\nabla^2 \rho = 0$$

(56)

as a limit of the solution to the corresponding Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \sigma^2 \nabla^2 \rho \quad \text{at} \quad t \to \infty$$

(57)

taking advantage of the fact that the solution to Eq. (57) always tends to a stationary limit regardless of boundary conditions imposed upon the solution, [5].

Therefore, the dynamical system simulating the Laplace equation (56) can be presented as following

$$\{\dot{x}\} = \frac{1}{\rho} \nabla(\sigma^2 \rho), \quad \sigma = \sigma(\{x\}, t, \rho, ...), \quad t \to \infty$$

(58)

The speed of convergence of the simulations (58) to a stationary solution can be control by a choice of the factor $$\sigma$$.
Conclusion

New physical principle for Monte-Carlo simulations has been introduced. It is based upon coupling of dynamical equations and the corresponding Liouville equation. The proposed approach does not require a random number generator since randomness is generated by instability of dynamics triggered and controlled by the feedback from the Liouville equation. Simulations of random processes with prescribed probability density as well as direct simulation of evolutionary partial differential equations have been proposed, discussed, and illustrated.

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References

First Passage Random Walk of Coupled Detector-System Pairs and Quantum Measurement

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Abstract: We propose a new model for a measurement of a characteristic of a microscopic quantum state by a large system that selects stochastically the different eigenstates with appropriate quantum weights. Unlike previous works which formulate a modified Schrödinger equation or an explicit modified Hamiltonian, or more complicated mechanisms for reduction and decoherence to introduce transition to classical stochasticity, we propose the novel use of couplings to the environment, and random walks in the product Hilbert space of the combined system, with first passage stopping rules, which seem intuitively simple, as quantum weights and related stochasticity is a commonality that must be preserved under the widest range of applications, independent of the measured quantity and the specific properties of the measuring device.

Keywords: Quantum Indeterminism, Measurement, Random Walk, First Passage
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APS code: 81P15, 81P20

1. Introduction

Quantum measurement remains one of the most puzzling physical processes. Although the laws of the quantum world and those of the classical world remain experimentally verifiable, the process of gaining information about the quantum world in the macroscopically perceived world remains unsolved.

Quantum mechanically an object may have various states associated with different values of a property superposed together, and interference patterns support the existence of such superposed quantum waves [1]. On the other hand in the classical world, superposed states do not exist, and interference patterns are not observed with of classical
matter constituting of many quantum particles.

In an ensemble, the probability of finding an object with a certain well-defined quantum property is equal to the square of the amplitude of the respective component of the wave function. Hence, in an ensemble of quantum particles such that each has two possible spin states, up and down, with wave amplitudes \( a \) and \( b \) superposed together in a quantum world, after a measurement is made in the classical world each particle comes either in the state up or in the state down. The probability of finding an up particle in such an ensemble is \( |a|^2 \) and the probability of finding a down particle is \( |b|^2 \). So, in each measurement process only one of the possible quantum states is retrieved and the rest of the superposed states are lost. In the measurement process, therefore, interference patterns are lost with the loss of coherence among quantum states, and only one of many states is obtained in the classical world, so that information about that quantum system is also partially lost.

In recent years there has been renewed interest (review in [14]) in the indeterminism in the measurement process in quantum mechanics, which has remained an enigmatic probabilistic characteristic of quantum theory since Bohr and the Copenhagen interpretation [2, 3]. Some people have even seen in the confusion a fuzzy mixing of ontology and epistemology, and the role of consciousness, which continues to be debated [8]. Intriguing many-world interpretations with bifurcating realities have also been suggested [19].

In [23], it was proposed that it is the many degrees of freedom of the environment that causes the stochastic collapse of the wave function. Each quantum state finds an environmental state partner, which it prefers, and these pairs get separated, and hence decohered, because of the orthogonality of the environmental states due to many degrees of freedom of the environment. However, the sudden loss of information and coherence remains confusing.

The existence of these measurer-microsystem entangled systems, and the reduction by itself, raises the question of where these lost component states are disappearing and how the environment can have so many orthogonal states corresponding to each quantum state, and how or why the macroscopic states are getting bound to these small quantum states.

In [9] an interesting scheme with quantized detector networks was also suggested.

In this paper, we propose a new model where stochasticity is derived in a dynamic fashion and in steps. Decoherence occurs simultaneously with reduction. The initial pairing is among quantum and corresponding mesoscopic states. One of these pairs gets amplified in the detector and locks the detector in an energetically favorable state indicating the existence of that one amplified quantum state. The rest of the bound pairs are dissipated in the environment or are rotated, so that information about the other quantum states are not expressed in the environment. The many degrees of freedom associated with the environment causes the dissipated states to be lost for ever in small packets each carrying parts of the information. The couplings of the detector causes the one expressed state to be amplified to a scale where information can be retrieved macroscopically. So, this new approach introduces dynamics in the amplification and
loss of states together with the loss of coherence.

Our model uses random walks to amplify the preferred state within the detector. Random walks with hierarchical constraints have also been presented recently [10] in a different picture to explain reduction to a pure eigenstate. In classical systems there is no intrinsic indeterminism, and random walks are simply an algorithmic approximation to simulate extremely complicated dynamics resulting from a system with a large number of coupled degrees of freedom. In the quantum context too one might expect a similar evolution replicable in the large by a probabilistic interpretation, when a mesoscopic detector interacts with a microscopic system, despite a deterministic set of rules of dynamics. However, there is as yet no universally acceptable theory, and all existent models have their advantages and weaknesses needing further development. Our objective is to produce a picture that is more comprehensible and plausible than the others in certain respects.

In the following subsection we shall discuss briefly the mechanism of reduction proposed recently by Omnés, and thereafter another interesting method proposed by Sewell [15], before we present details of our work, where we give a general method of picturing and explaining the transition from mixed quantum states to pure eigenstates as first passage walks in Hilbert space. This model also includes a preparation stage, where the system and the detector form a virtual superposed quantum bound state before the walk begins.

2. Comparison with Recent Models

2.1 Current Models of Decoherence

Theories of quantum collapse are usually based on decoherence or reduction [23]. In decoherence, a quantum state interacts with the environment, which contains a collection of all possible observable states, and due to this interaction the cross terms in the density matrix (|i⟩⟨j|, j ≠ i) gradually vanish, leaving the arena to the diagonal terms only (i = j), but that mixture too finally reduces to one single eigenstate, which corresponds to the result of the measurement.

The quantum system is initially in the state

|Ψ⟩ = ∑ |i⟩⟨i|Ψ⟩

when expressed in terms of the |i⟩’s which form the basis selected or preferred by the environment. If the system is brought into contact with the environment, |E⟩, a joint state results in the form ∑ |i⟩|E⟩⟨i|Ψ⟩ This compounded system is allowed to interact, so that either the system is lost to the environment,

|i⟩|E⟩ → |E_i⟩

or, the environment is modified, so that the joint system evolves to

∑ |i, E_i⟩⟨i|Ψ⟩
In both cases, the selection of one of many possible quantum states is achieved by assuming the “orthogonality” of the environment states posed by the many degrees of freedom in the environment $<E_i|E_j> \delta_{ij}$

However, the main criticisms of decoherence remain:

1. The “einsteinas” (environment-selected eigenstates) are inserted in an *ad hoc* manner, with no explanation.

2. The splitting of the macro-system into a relevant system and the environment, by means of a set of projection operators, is also done in an *ad hoc* manner, without a credible strong procedural explanation. Nevertheless, it is this drastic classification that leads to the loss of coherence.

3. The diagonalization of the reduced density matrix of the relevant system does not offer any insight into the dynamics of the mechanism, and also does not completely eliminate small probabilities. The problems with diagonalization was pointed out in [10].

4. The orthogonality of the environment states is mentioned. However, what these orthogonal environment states could be physically, and how they exist independent of the quantum system, or what happens to the remaining environment states after one is selected, is not clear. The fate of the unselected quantum states is also unclear in regard to where they disappear.

2.2 The Omnés Paradigm of Reduction

Some of the ambiguities posed in the process proposed by [23] was resolved by Omnés. In order to account for the dynamics of decoherence, in a recent work, Omnés [10] combined reduction with decoherence. His model suggested the adjustment of the coefficients of the projection operators for different eigenstates so that an infinitesimal reduction in the j’th state changes the projection operator $P_j$ to $(1 + \epsilon_j) P_j$, in turn changing the probability amplitude of the j’th state. The random adjustment of the weights was carried out by using homogenous, isotropic Brownian motion.

In his paper, Omnés proposed combining two possible explanations of quantum measurement, namely decoherence and also reduction. The addition of reduction adds a timescale, so that the sudden diagonalization problem is solved, and the problems with small probabilities existing in a decohered system is also solved when the reduction of states in each vertex is introduced.

However, although this method shed some light into possible dynamics leading to decoherence, the main criticisms of decoherence, regarding the *ad hoc* nature of the eigenbasis and projection vectors in the environment, and the physical basis of projection vectors readjusting themselves remain unanswered. The projections were taken to be possible states of the detector indicating different quantum eigenvalues. The role of the larger environment is not considered.

In the Omnés approach reduction occurs as the last step of decoherence. In his paper, it was shown that if decoherence already did not take place, the projection operators
were inconsistent with field theoretic formulations. This necessitates the reduction of already existing multiple states into one state. Where the reduced decohered states vanish remains unanswered.

2.3 The Sewell Paradigm

Sewell proposes using many-body Schrödinger equations for the large number of degrees of freedom for the composite of the microsystem and the macroscopic measuring system. This finite closed system with conservative dynamics with no dissipation is claimed to be sufficient to bring about the collapse of the superposed state to an eigenstate in number of steps.

This method avoids the need of the decoherence process to end eventually in ‘consciousness’, as envisaged in the steps proposed by von Neumann [20] and Wigner [21]. It gives a robust one-to-one correspondence between the microstates of the measured system and the macrostates of the instrument, irrespective of the initial quantum state.

Sewell obtains conditions on the measuring devices imposed by the requirement of obtaining quantum measurement as probabilistic observation. However, one property signifies that the micro-macro coupling removes the interference between the different components of the pure state and thus represents a complete decoherence effect.

2.4 Our Approach

In [16] a draft model was proposed to take into account dynamic reduction of a coupled quantum-meso pair within a detector. In a manner somewhat similar to [10], which was proposed a few months earlier than our draft, the model made use of stochastic reduction. First passage random walks are used for the coupled pair to eventually reduce to one of the possible observables. However, instead of using projection operators, and stochastically changing the projection operators, our model relied on a coupled detector-system pair. The coupling of the quantum system with an image present in the detector and the reduction of the entire coupled pair introduces the square of the quantum wave amplitudes, giving rise to the squared probabilities expressed stochastically in the classical world in an elegant manner.

In [17], a possible mechanism for generating an image of the quantum system within the detector was proposed. The image was created within the detector in a manner similar to inducing charges in a conductor.

In this paper, we review, polish and extend the previous draft [16] and present it in a paper form. We compare our model with existing models and add some explanations regarding certain mechanisms.

We propose a more physically comprehensible approach, which circumvents some of the mathematical abstractions posed in the previous works. Detailed arguments and steps are inserted to reduce the ambiguities in existing models. Our approach also treats the elimination of cross terms simultaneously with reduction, so that multiple deco-
hered states do not exist entangled independently with possible orthonormal environment states, to be eliminated one by one, with intrinsic problems of normalization. We also clarify what these environment states may be and how one state emerges macroscopically while the others are eliminated. Our approach does not necessitate the existence of an ultimate undefinable detector (consciousness) to measure a state. The approach also does not make it necessary to introduce parallel universes or multiple universes to explain the disappearance of any unmeasured quantum state.

The formation of the coupled detector-system pair [17] is also explained using an energy landscape. Couplings between the microscopic world and mesostates are obtained by using interactions, and the coupling constants are taken to be proportional to the amplitudes of the superposed waves. The initial couplings cause the quantum system to become further connected because of the couplings within the measuring device. In the presence of stochastic interactions of the detector subsystems with the environment, the entire system is reduced by means of first passage random walk. The cross terms also reduce with the reduction of multiple states.

In the next part of this paper, we present a method of concurrent decoherence and reduction of quantum information within a coupled detector by using first passage random walks and the formation of images. In the last part, we discuss some possible methods of such image formation and reduction given the necessity to preserve unitarity. Some related dynamical mechanisms and philosophical questions are addressed, and the relevance of the model proposed by us, given the degree of ambiguity and incompleteness still existing in the process of quantum measurement, is discussed.

3. Formation of Images

Though physical laws are all expected to be based on quantum principles, the full quantum picture is clear and usable only for small systems, such as the interaction among a small number of particles. When a large number of particles are involved, approximations become inevitable. Even in quantum field theory one often has to resort to approximate effective interactions to simulate contributions of the sum of large diagrams involving many propagators and loops, losing in the process some vital components of the quantum theory such as unitarity, and making perturbative convergence suspect.

In a measuring device there will in general be a large number of subsystems which can directly or indirectly couple to the attribute of the small system which is to be measured and they will in turn couple to large macroscopic recording devices whose states indicating the different values of the measured quantity are macroscopically so different with such huge energy barriers in the transition paths that it is not possible to tunnel from one such state to another in a realistic time limit, so that we do not expect quantum superpositions of the states of the recording device after measurement is completed. However, during the process of measurement the microsystem components of the device (D) coupling to the measured microsystem (S) may be in states of superposition of eigenstates. We shall now assume that D and S couple in a way to form a virtual bound pair, the D state being
an image of the S state in the following sense:

\[ |\psi\rangle_S = \sum_i a_i |i\rangle_S \]  \hspace{1cm} (3)

has the

\[ |\psi\rangle_D = \sum_i a_i^* |i^*\rangle_D \]  \hspace{1cm} (4)

We can here refer to a comparison with the creation of a conjugate image charge in a grounded neutral conductor, which has a sea of charges of both kinds available. When a free charge approaches it, there is an induction of the image charge, which may be a manifestation of a re-arranged charge distribution on the conductor, and not of any particular real charge on the conductor. So \( |\psi\rangle_D \) may be the effective state resulting from the combination of a large number of micro subsystem components of the device. Hence, the image states are not exact clones of the micro-system’s states, and they correspond on a class-to-one basis, each image class containing a superposition of a large number of quantum states, not discriminable on a coarse-grained macro or mesoscale, and the “no-cloning theorem” \[22\] is not applicable here. Elsewhere \[17\] we have studied and demonstrated a possible mechanism of the formation of such conjugate images.

The formation of the image states and the virtual bound pairs can also be briefly justified as below (the example can be generalized): 1. Consider detection of spin \( s_z \) along the preferred direction of the environment of the detector. Let the mesoscopic corresponding variable in D be \( S_z \). The interaction energy is \( \pm ks_z S_z \). So the lowest energies correspond to \( s_z = -1/2, S_z = +S \), and \( s_z = -1/2, S_z = -S \), where S may be a little fuzzy when obtained by coarse graining. If the incoming state is polarized along some other direction initially, it will be expressible as a linear superposition of the preferred eigenbasis as, say,

\[ |in\rangle = a|+1/2\rangle + b|-1/2\rangle \]  \hspace{1cm} (5)

2. The interaction energy between the system and the detector for the two states would be proportional to \( a \) and \( b \) respectively.

3.1 Detector Image Coefficients and Quantum Wave Coefficients

In the model proposed above, the detector forms an image corresponding to the quantum wave and the states of the detector share the same (conjugate) coefficients as those of the quantum states. However, the detector itself is a large system comprising of many subsystems. Hence, the image “wave” in the detector must comprise of an ensemble of coupled subsystems. This evoked detector wave is coupled to the quantum wave by means of interaction, and the detector image itself is created by the interactions among the coupled subsystems within the detector. This is evoked by the coupling with the quantum system and is dependent on the coefficient of the quantum wave.
The number of subsystems corresponding to each quantum state (the detector subsystem (meso-system) binding with the corresponding quantum system in an energetically favorable interaction) would reflect the coefficient of the corresponding detector image component. The cross terms derive from the interactions from energetically unfavorable pairs of mismatched states. If all the detector subsystems correspond to a single quantum eigenstate, and are coupled together to form a favorable energy minimum, the detector state is locked to indicate that certain quantum state, and since all the states correspond to a single quantum state, the cross-terms also disappear.

The use of an energy landscape with regard to interactions makes it possible to allow detector segments to get locked into a state, indicating a certain eigenvalue or observable. We also assume these subsystems in the mesoscopic detector system are coupled to the macroscopic recording part of the device (R) in such a way that when they act in unison they can change the state of R to indicate one of the eigenvalues of S. The image state of D and the recording state of R may be degenerate states and superpositions of the corresponding multi-component meso or macro systems may correspond to unique eigenvalues of S. We do not need to make use of any decoherence arguments in the usual sense.

4. Loss of Reduced States

In forming the virtual S-D bound system, if conservation rules demand, excess values of the measured attribute and intrinsically related quantities may pass on to other subsystems of R, or to the environment, in the same way as the excess charge from a grounded conductor passes to the ground after the formation of the image. We can state this as a theorem:

**Theorem:** It is not possible in general for a closed quantum system comprising only the detected system and the detector to perform a complete measurement.

**Proof:** Let us consider a microsystem with spin-1/2 given in Eq. 1. This can be generalized to other types of measurement. Let the detector have a preferred basis (i.e. z-direction) different from that used in Eq. 1. After interaction leading to measurement, the micro-system assumes an eigenstate corresponding to the eigenbasis of D, and hence the composite system now has no component of the spin in any direction normal to the eigenbasis direction, even though the original system did have some, in general. Since angular momentum must be conserved, this is not possible. In other words, the component of spin of the microsystem orthogonal to the direction of the eigenbasis of D must escape from the (S-D) quantum system to the environment.

Though in QM we always have the same left-hand side in the completeness relation of a basis

\[ 1 = \sum |i\rangle \langle i| \]  \hspace{1cm} (6)

the basis we choose on the right-hand side must be relevant to the context. In a
measurement problem the operative basis set corresponds to the environment in which the detector is placed, and not the original orientation of the quantization axis of the microsystem which is detected. When a two-body problem is reduced to a one-body problem using reduced mass, the origin remains closer to the heavier body. In the quantum measurement problem too, the mesoscale D system consisting of a very large number of microsystems comparable to the detected system S. Though quantum states are normalized to unity, independent of the size of the system, they can also be represented as rays in Hilbert space for many purposes, where absolute probability is not needed. The energies of alignment are quite different and cannot be treated in terms of a size-independent parameter, as in the case of probability. The energy exchanges involved in rotating the alignment axis of S, with respect to the environment (e.g. an external magnetic field) to bring it in line with that of S will be many orders of magnitudes larger than the energy required to for S to align to D. Hence, in a sense, the detector D has a higher quantum “inertia” and fixes the frame of the basis set for the measurement process.

The S-D coupled system may be represented by

\[
|\psi\rangle_{SD} = \sum_i |a_i|^2 |i\rangle_S |i^*\rangle_D + \sum_{i \neq j} a_i a_j^* |i\rangle_S |j^*\rangle_D \tag{7}
\]

5. **First Passage Transition to Eigenstates**

We now propose that the complicated quantum interactions of the macroscopic D-R complex and S can be represented by a random walk in Hilbert space of the (SD) coupled system with active transfers among the microstate-image pair diagonal terms in Eq. 3, with the off-diagonal terms, required by unitarity, which are uncoupled free spectator components adjusting their coefficients passively to conserve unitarity. When the coupled system finally reaches a particular eigenstate pair, all such off-diagonal terms vanish. We do not intend to construct a model of explicit unitary transformations which would lead to such a scenario. But it is possible to show easily that a sequence of simple unitary rotations in Hilbert space through a uniform stepping angle of random sign cannot achieve the right probabilities even in the case of a qubit, i.e. a spin-1/2 object. It is probably unrealistic to expect that the micro-transitions leading eventually to a complete collapse to an eigenstate is expressible in terms of a calculable sequence of operations. Indeed each successive operation may be drastically different from its predecessor. Brownian motion [11, 12, 7, 4] may be the most simplified model for such a sequence.

We approach the transition as a first passage problem [13], where, on reaching an eigenstate, which forms a bounding wall in the Hilbert space, the evolution of the system stops, the recorder shows the eigenvalue, and that eigenstate of S continues until further measurement. In game theory language it can also be described as a winner-takes-all game with the players (the competing eigenstates) betting against one another in pairs by random turns, with an equal small 1 : 1 stake (the interaction) at each turn, and players eliminated one by one on going bankrupt till the eventual winner emerges. An
incomplete game would indicate a quantum mixed final state, which may be the case when the recorder’s state is coupled to such a mixture.

For the measurement of a qubit

\[ |S_{in}\rangle = a_0 |0\rangle + a_1 |1\rangle \tag{8} \]

our model yields the compound (SD) state (omitting passive cross-terms):

\[ |SD_{in}\rangle = |a_0|^2 |00\rangle_{SD} + |a_1|^2 |11\rangle_{SD} + \text{off-diagonal cross-terms} \tag{9} \]

We locate this initial state as the point \((x_0, y_0) = (|a_0|^2, |a_1|^2)\) in a two-dimensional Hilbert space. We may anticipate \(x + y = 1\) here, and eliminate one co-ordinate, but later we shall keep the co-ordinates free till the last stage of the calculation.

So we have a random diffusion of the probability concentration from \(x_0\) to \(x = 0\) (pure eigenstate \(|1\rangle\)) and to \(x = 1\) (pure eigenstate \(|0\rangle\)).

With the diffusion equation

\[ \frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \tag{10} \]

where the diffusion constant \(D\) is related to an effective strength of interaction or a length of the walk, and \(t\) is a continuous variable representing the sequence index of small discrete operations, which may be proportional to real time. In Fig. 1 we give a diagram of the process. We should, however, not interpret it as a Feynman-type graph, and should also remember that the cross-terms also change with every step passively to maintain unitarity. The situation is to some extent analogous to the quantization of spin \(s_z\) by an interacting external magnetic field \(H_z\), while the magnitude of the resultant of the other components add up vectorially to maintain the constrained value of the overall magnitude of the spin \(s\).

Since we shall have to sum over all possible step numbers, and equivalently integrate over all \(t\), it is more convenient to work with the Laplace transform of \(c\), with the transformed equation

\[ \frac{d^2 \tilde{c}(x, s)}{dx^2} - \frac{(s/D)}{\tilde{c}(x, s)} = -c(x, t = 0)/D \tag{11} \]

\(s\) being the Laplace conjugate of \(t\).

The initial \(c(x, 0)\) is the delta function \(\delta(x - x_0)\) when the diffusion (walk) begins. We also have the boundary conditions \(\tilde{c} = 0\) at the two absorbing walls \(x = 0, 1\), where the first passage walks stop. Hence the solution is the normalized Green’s function

\[ \tilde{c}(x, s) = \frac{\sinh \left( \sqrt{(s/D)}x_< \right) \sinh \left( \sqrt{(s/D)}(1 - x_>) \right)}{\sqrt{(sD)} \sinh \left( \sqrt{(s/D)} \right)} \tag{12} \]

with \(x_< = \min(x, x_0)\) and \(x_> = \max(x, x_0)\).

The probability of passage to the walls (the eigenstates) is given by the space derivatives with \(s \rightarrow 0\).
adjusting cross-terms

\[ (x_0 - dx)|00^*\rangle, \quad (y_0 + dx)|11^*\rangle \]

\[ y = 0 \]

\[ |0^*D\rangle \]

\[ |D\rangle \]

\[ |1^*\rangle \]

\[ |a_1\rangle \]

\[ |x_0\rangle \]

\[ |y_0\rangle \]

\[ |00^*\rangle_{SD} + |a_1|^2|11^*\rangle_{SD} + |a_2|^2|22^*\rangle_{SD} + \ldots \] (14)

\[ p(x = 0) = D \left. \frac{\partial \tilde{c}}{\partial x} \right|_{s \to 0, x = 0} = 1 - x_0 = |a_1|^2 \]

\[ p(x = 1) = -D \left. \frac{\partial \tilde{c}}{\partial x} \right|_{s \to 0, x = 1} = x_0 = |a_0|^2 \] (13)

in conformity with quantum mechanics.

6. Walks in Higher Dimensional Hilbert Spaces

It is also possible to arrive at the above results using the two variables \( x \) and \( y \) at all stages independently, and then finally constraining them by the normalization condition \( x + y = 1 \). This provides a general procedure for an \( n \)-eigenvalue situation, for arbitrary \( n \). We shall illustrate the method for \( n = 3 \), with the initial (SD) state

\[ |\psi\rangle_{SD} = |a_0|^2|00^*\rangle_{SD} + |a_1|^2|11^*\rangle_{SD} + |a_2|^2|22^*\rangle_{SD} + \ldots \] (14)

where we have labeled the three eigenvectors in an arbitrary sequence, and as before we indicate the position of the initial vector by \((x_0, y_0, z_0)\), and calculate the probabilities of transitions to \( x = 1 \), \( y = 1 \) and \( z = 1 \).

Fig. 2 shows the triangle in which the co-ordinates are constrained, but for a symmetric calculation in all three co-ordinates we shall impose the normalization relation at the end. The diffusion equation is now

\[ \nabla^2 \tilde{c}(x, s) - \left( s/D \right) \tilde{c}(x; s) = -c(x, t = 0)/D \] (15)
Fig. 2 Three-dimensional Hilbert space with (SD) eigenstate coefficients along the three axes. Walks begin at \((x_0, y_0, z_0)\) and proceed on the triangle to any of the vertices representing a pure eigenstate

with the boundary condition

\[
\frac{\partial c}{\partial x_i} \bigg|_{x_i=0} = 0 \tag{16}
\]

which indicates zero diffusion out of the sides of the triangle of Fig. 2. In game theory terms reaching \(x_i = 0\) eliminates \(i\) from the rest of the game.

Instead of using a Dirac delta function, we can normalize more simply and with full symmetry in the co-ordinates by demanding that at the source

\[
\sum_i (-D/2) \left( \frac{\partial \tilde{c}}{\partial x_i} \bigg|_{x_i=x_0,+e,s-0} + \frac{\partial \tilde{c}}{\partial x_i} \bigg|_{x_i=x_0,-e,s-0} \right) = 1 \tag{17}
\]

With symmetry among the co-ordinates, and hence the same velocity of the walk (denoted by the parameter \(k\) below) we finally get

\[
\tilde{c}(x, s) = A \prod_i \left[ \cosh(kx_i<) \right] \cosh[k(2 - \sum_i x_i>)] \tag{18}
\]

with \(A\) given by normalization

\[
A = \prod_i \left[ \cosh(kx_{i0}) \right] \sum_i \frac{\sinh[k(1 - x_{i0})]}{\cosh(kx_{i0})} \tag{19}
\]

and
\[ k = \sqrt{\left( \frac{s}{3D} \right)} \]  

This gives

\[ p_i = -D \frac{\partial \tilde{c}}{\partial x_i} \bigg|_{s \rightarrow 0, x_i = 1} = x_i = |a_i|^2 \]  

which is a postulate in quantum theory.

For higher dimensions the procedure seems to be easily extensible, with higher dimensional complexes in Hilbert space providing the arena for the first passage diffusion. In each case we would need the same boundary constraint for zero diffusion when a coordinate goes to zero. The walk then proceeds in a lower dimensional complex, till a vertex is reached.

7. Interpretational Implications

7.1 Measuring Devices

A measuring device records a particular state of an observable. Hence, by quantum mechanical axiom, a measuring device is a macroscopic system that can indicate any one of the superposed quantum states and eliminate the rest, although the detailed mechanism of the process is not defined. The manifestation of a highly complex quantum state in the classical world of the device is a macroscopically averaged truncation, that requires reorganization of a large number of smaller segments in favor of one effective quantum state to be expressed. The elimination of the remaining superposed components from being recorded in any measuring device is denoted by a “collapse” or a cancelation. This implies the impossibility of the remainder of the states of being coupled to another detector subsequently, in a manner as to be recordable. Because of the coupling of the measuring device with the environment, the recorded system’s surviving information is expressed within the S-D system. The eliminated states are not expressed within S-D, but their characteristics disperse into the inactive components of the device that do not couple to the recorder, or into the external environment coupled to the device in a random manner.

7.2 Coarse-graining and Macrostates

Decoherence has been related to coarse graining in an attempt to explain probabilistic histories [5]. We extend that idea to claim that macroscopic expression of a state comes from its expression in a manner coherent enough to be expressible as reasonably sharp fuzzy sum (of states degenerate with respect to the characteristic measured, but with other attributes not necessarily agreeing unless constrained) at a certain scale. In order
for a state to be expressible on a classical scale on a macroscopic recorder, a certain cluster of subsystems need to be expressed in a correlated manner within the system. The transformation of expressed subclusters into a correlated cluster expressing a single macrostate allows the entire cluster to influence other coupled clusters of the same scale. Just as the discrete microscopic quantum spin “up” and “down” states exist, the macroscopically organized “block spin” states may be thought to come in discrete assemblies as well that are in reality fuzzy sets of well-defined separated limits. For example, in biology, a hemoglobin can exist in one of two possible states depending on where or not an oxygen molecule is bound to it [6].

However, in the quantum domain, states can coexist as superpositions. The expression of a quantum component within the quantum scale (a small system) includes objects with clean wave functions with a minimal number of variables. The wavelike property of a small object can be observed when small (quantum scale) objects are allowed to pass through slits, and hence interfere. The interference reflects several possible states existing in a superposed manner within the same microsystem’s state function. In large ensembles, such wave-like properties are not usually coherently extended far enough to form quantum interactions with neighbors.

The macroscopic world is a large ensemble of interacting microsystems that does not display such wave properties. The presence of a large number of interacting overlapping neighbors effectively causes the decoherence of the individual microsystems’ wave functions so that they become localized particle-like objects, quantum mechanically as well as in terms of classical mechanics. Such a set of interactions also eliminates bizarre combinations of dead and live cat wave functions, though valid paths exist for live cat states and dead cat states to migrate from one state to the other, quantum mechanically and classically, given sufficient time and energy. Identity of the states of the macroscopic world are recognized not by quantum numbers or states of microscopic states, but rather by the correlation (and hence organization) of the subsystems. A dead cat may also dissipate into dust after some time, signifying that a possible ensemble state in the scale of a macroscopic cat may dissipate into smaller ensembles and disperse into the uncorrelated large degrees of freedom within the environment in a manner such that the initial dead cat state cannot be retrieved from within the environment without an improbable conspiracy. Hence, the cat (a quantum mechanically fuzzy set) transforms from the live cat state to a dead cat state from its association with a detector which tangos with a superposed microsystem to either a dead cat associated with a fully decayed radioactive or a still live cat associated with an intact harmless nucleus. We anticipate that the state of the detector is half-way between a fully quantized microsystem’s and a completely decohered classical macroscopic recorder system, so that it has states pairing coherently with the microsystem on the one hand, and can also couple classically with other mesoscale systems in the recorder.

Hence, in our model, the entire universe does not get split into multiple energetically separate bands and multiple universes. The critical interactions involved in the “collapse” are carried out at the interface of the quantum and classical domains, introducing an
intermediate meso-state. The expression of a quantum state in the classical world is by means of a one-to-one correspondence between a quantum state and a macroscopic state, which might be a certain organization of the entire detector-detector ensemble. As the entire detector is aligned in a certain direction, corresponding to possible quantum states, information about the reduced components is lost to the rest of the recorder or the environment.

7.3 Unitarity and Loss of Information

In quantum computing, the operations are carried out by unitary operators, which are linear. These operations preserve all information, so that it is possible to return to an initial state by means of an inverse function. Hence, the unitary operations map each quantum state into another state on a one-to-one basis. An important theorem in quantum computing states that \[22\], it is not possible to clone an unknown state into a given state by means of unitary operations. The proof is elegant and simple. If two arbitrary initial states can be mapped to a single cloned states by means of a unitary operator, the inverse of that operator must yield both the initial states from the final cloned state, which violates the linearity clause.

However, in the case of quantum measurement, one of the possible superposed initial eigenstates is expressed in the detector, and the probability of choice of the state is dependent on the amplitude. Hence, the quantum measurement process preserves partial information of the state, and also can feel the wave amplitude.

The partial loss of information from the detector makes it clear that the stepwise linear unitary operators connecting the detector and the microsystem alone cannot cause measurement. The following alternate approaches might be able to explain quantum wave functions, so that the classical appearance of the world requires the two stages mentioned above:

a. Approximation of Waves The perceived macroscopically identified universe may be taken to be the result of an extremely large number of random phase superpositions. The components of the detector contains systems with a semilocal truncation near the detector that causes loss of information.

The addition of a new superposed wave function in the large wave system causes a new approximation, which again makes some of the information in the original microscopic and isolated quantum wave function to be lost. This approximation process may be summarized by taking a series of complicated unitary operators, and truncating the series, so that an effective non-linear operator arises, which is representative of the classical operation.

b. Energy Landscape We propose an alternative explanation using interaction energy landscapes. In this scenario, an observer does not introduce a unitary operator with a known operation to an arbitrary state and expect it to evolve to a known (cloned) state. Rather, quantum (many particle) mesoscale pairs exist in the detector, so that introducing a new incident quantum microsystem within the interaction range of an-
other mesoscale system causes the two to interact and get coupled. This process creates (virtually) bound pairs. This explanation allows for cross terms to exist during the entire decoherence-reduction process. The active process involves interactions between two corresponding/similar states forming the matched active pairs involving the micro and mesoscale levels, and the elimination of one state from the scenario automatically diminishes all the cross-terms passively.

This picture does not need the existence of macroscopic environment states that are orthogonal and correspond to definite quantum states. The initial (prior to reduction) existence of the entire set of orthogonal macro-states and the disappearance of all but one, would pose the problem of creating/annihilating large system states within the reduction time scale. Also, the mechanism by which a microscopic quantum wave function is able to cause such a phenomenal situation is hard to explain.

Rather, in the energy landscape picture, pairs are formed at the lowest hierarchy level on a relatively small scale within the detector: sized between meso-subsystems and quantum microsystems. Internal couplings and interactions would then cause some microsystems within the detector to make a random walk to one quantum eigenstate together with the incident system, while the alternative small states are either rotated to favorable configurations, gaining the adaptation energy from the environment, or are dissipated within the environment by random phase cancelation, as there is no coherence glue to add up their contributions to the measured value, so that the level of expression of these “lost states” are not coherently strong enough to indicate the “non-collapsed” quantum states at a macroscopic level. This conversion can then trigger other similar coupled meso-systems within the detector-recorder complex, effectively amplifying the coupled walk at the lowest level.

In this approach, the entire “universe” is broken down into quantum systems, mesosystems, detector components, detector and the rest of the environment, which is large, with many degrees of freedom, allowing some loss of information at each level of the scale hierarchy in the form of dissipation or leakage to external coupled components. The initial interactions are local, but the dissipation of information allows for certain states to be coherently expressed within the highly coupled detector, while the attributes of the other states are dissipated within the large degrees of freedom of the environment.

### 7.4 Choice of Basis

While the outcome of the state in the detector is dependent on the quantum superposed waves interacting with the detector, the choice of basis is determined by the detector and the environment preceding it. For example, in the Stern-Gerlach experiment, the z-axis is determined when the electrons enter the apparatus. The collapse of up states and down states take place along the z-axis.

This determination of the axis may be seen as innately built within the detector-environment complex. A certain detector, by design, may have mesostates or subcomponents that bind with the quantum states only along that axis. Hence, the detector may
be seen as a system that is able to generate energetically favorable subsystems that can couple along only a certain axis. There might be an ensemble of such states available, coupled with one another, which initially make the detector neutral. But the introduction of the quantum state function causes the degenerate states to split and express themselves.

Conclusions

We believe the picture presented above is a simple, effective one for understanding the process of collapse of superposed states to eigenstates. One can also calculate the average time required for the completion of the process trivially from the first passage equations, and it would depend on the parameter $D$ used as the diffusion coefficient, which would vary according to the process steps, as it would involve the details of the coupling between the measuring device and the measured microsystem.

The question of nonlocality for the measurement of entangled states of spatially separated systems is an interesting one. In this picture, and most others, the spatial separation has to be ignored when entangled systems are considered, and the states can then be processed in Hilbert space together with the corresponding measuring devices. In [18] we have shown recently that spatial separation and local measurement of entangled systems are not inconsistent with the established rules of quantum mechanics.

The randomness used here is not an inherent property of nature cited in conventional quantum theory, but is simply the apparently unpredictable outcome of each of the steps representing the interaction between an extremely large number of quantum components in the device itself, and also of possible hidden parameters carried from the source [18].

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References


Underdeterminacy and Redundance in Maxwell’s Equations. I. The Origin of Gauge Freedom

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Abstract: The gauge freedom in the electromagnetic potentials indicates an underdeterminacy in Maxwell’s theory. This underdeterminacy will be found in Maxwell’s (1864) original set of equations by means of Helmholtz’s (1858) decomposition theorem. Since it concerns only the longitudinal electric field, it is intimately related to charge conservation, on the one hand, and to the transversality of free electromagnetic waves, on the other hand (as will be discussed in Pt. II). Exploiting the concept of Newtonian and Laplacian vector fields, the role of the static longitudinal component of the vector potential being not determined by Maxwell’s equations, but important in quantum mechanics (Aharonov-Bohm effect) is elucidated. These results will be exploited in Pt.III for formulating a manifest gauge invariant canonical formulation of Maxwell’s theory as input for developing Dirac’s (1949) approach to special-relativistic dynamics.

Keywords: Electromagnetic Fields; Maxwell Equations; Helmholtz Decomposition; Gauge Theory; Aharonov-Bohm Effect

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1. Introduction

Traditionally, there are two main approaches to classical electromagnetism, viz,

(1) the experimental one going from the phenomena to the rationalized Maxwell equations (eg, Maxwell 1873, Mie 1941, Jackson 1999, Feynman, Leighton & Sands 2001);
(2) the deductive one deriving the phenomena from the rationalized Maxwell equations (eg, Hertz 1889, Lorentz 1909, Sommerfeld 2001).

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"Rationalized Maxwell equations" (Poynting 1884, Heaviside 1892) means the macroscopic Gauss' laws for the magnetic and electric field and the induction and flux laws (SI units).

\[
\begin{align*}
\nabla \cdot \vec{B}(\vec{r}, t) &= 0 \\
\frac{\partial}{\partial t} \vec{B}(\vec{r}, t) &= -\nabla \times \vec{E}(\vec{r}, t) \\
\nabla \cdot \vec{D}(\vec{r}, t) &= \rho(\vec{r}, t) \\
\frac{\partial}{\partial t} \vec{D}(\vec{r}, t) &= \nabla \times \vec{H}(\vec{r}, t) - \vec{j}(\vec{r}, t)
\end{align*}
\]

For moving charges in vacuo, they can be simplified via \( \vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E}(\vec{r}, t), \vec{B}(\vec{r}, t) = \mu_0 \vec{H}(\vec{r}, t) \) to the microscopic Maxwell equations (Lorentz 1892).

\[
\begin{align*}
\nabla \cdot \vec{B}(\vec{r}, t) &= 0 \\
\frac{\partial}{\partial t} \vec{B}(\vec{r}, t) &= -\nabla \times \vec{E}(\vec{r}, t) \\
\nabla \cdot \vec{E}(\vec{r}, t) &= \frac{1}{\varepsilon_0} \rho(\vec{r}, t) \\
\frac{\partial}{\partial t} \vec{E}(\vec{r}, t) &= \frac{1}{\varepsilon_0 \mu_0} \nabla \times \vec{B}(\vec{r}, t) - \frac{1}{\varepsilon_0} \vec{j}(\vec{r}, t)
\end{align*}
\]

For both sets, two fundamental problems have to be clarified, viz,

1. the origin of the gauge freedom in the potentials, and
2. the origin of the transversality of free (unbounded) electromagnetic waves.

For this, I will – following the recommendation by Boltzmann (2001) – return to Maxwell’s (1864) original set of equations. Using the Helmholtz (1858) decomposition of 3D vector fields into transverse and longitudinal components, I will show that this set is both underdetermined and redundant (but not inconsistent). Remarkably enough, both deficiencies are related to longitudinal vector components.

Thus, this Pt. I of a series of papers starts with an exposition of the Helmholtz decomposition. Special attention is paid to the various kinds of fields occurring in electromagnetism, notably to fields like the possible static longitudinal component of the vector potential, \( \vec{A}_L(\vec{r}) \), which is not accounted for in any variant of Maxwell equations, but is determined by boundary conditions. In Section 3, Maxwell’s (1864) original set of equations is rewritten in terms of the transverse and longitudinal components of all fields, and a revised set being free of underdeterminacy and redundancy is proposed. Section 4 considers the role of \( \vec{A}_L(\vec{r}, t) \) for the gauge freedom both in electromagnetism and in Schrödinger’s (1926) wave mechanics, where the latter provides a short-cut to a gauge invariant Hamiltonian. The results are summarized and discussed in Section 5.

2. Helmholtz Decomposition of 3D Vector Fields

In order to apply Helmholtz’s decomposition theorem appropriately, one has carefully to discriminate between certain types of vector fields, viz, Newtonian, Laplacian and vector fields in multiply connected domains.
2.1 Newtonian Vector Fields

Newtonian vector fields are vector fields in unbounded domains with a given distribution of sources and vortices (Schwab 2002). The classical example is Newton’s force of gravity. They are the actual subject of

**Helmholtz’s decomposition theorem:** Any sufficiently well-behaving 3D vector field, \( \vec{f}(\vec{r}) \), can uniquely be decomposed into a transverse or solenoidal, \( \vec{f}_T(\vec{r}) \), a longitudinal or irrotational, \( \vec{f}_L(\vec{r}) \), and a constant components (which I will omit in what follows).

\[
\vec{f}(\vec{r}) = \int_{V'} \vec{f}(\vec{r}') \delta(\vec{r} - \vec{r}') dV' \quad \vec{r} \in V \setminus \partial V
\]

\[
= -\frac{1}{4\pi} \int_{V} \int_{V} \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} dV' + \frac{1}{4\pi} \nabla \times \int_{V} \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} dV'
\]

\[
= \vec{f}_T(\vec{r}) + \vec{f}_L(\vec{r})
\] (9)

(These notions of longitudinal and transverse fields should not be confused with the notions of longitudinal and transverse waves in waveguides!)

It is thus most useful to introduce scalar, \( \phi_f(\vec{r}) \), and vector potentials, \( \vec{a}_f(\vec{r}) \), as

\[
\vec{f}_T(\vec{r}) = \nabla \times \vec{a}_f(\vec{r}); \quad \vec{f}_L(\vec{r}) = -\nabla \phi_f(\vec{r})
\] (10)

The minus sign is chosen to follow the definitions of the mechanical potential energy and the scalar potential in the electric field strength. \( \vec{a}_f \) is *sourceless*; otherwise, one would increases the number of independent field variables.

As a consequence, each such vector field is uniquely determined by its sources, \( \phi_f \), and sourceless vertices, \( \vec{j}_f \).

\[
\nabla \times \vec{f}(\vec{r}) = \nabla \times \vec{f}_T(\vec{r}) = \nabla \times \nabla \times \vec{a}_f(\vec{r}) = -\Delta \vec{a}_f(\vec{r}) = \vec{j}_f(\vec{r})
\] (11)

\[
\nabla \cdot \vec{f}(\vec{r}) = \nabla \cdot \vec{f}_L(\vec{r}) = -\Delta \phi_f(\vec{r}) = \rho_f(\vec{r})
\] (12)

Including the surface terms (Oughstun 2006, Appendix A), the potentials follow as

\[
\phi_f(\vec{r}) = \frac{1}{4\pi} \nabla \cdot \int_V \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} dV'
\]

\[
= \frac{1}{4\pi} \int_V \int_V \frac{\rho_f(\vec{r}')}{|\vec{r}' - \vec{r}|} dV' - \frac{1}{4\pi} \int_V \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} \cdot d\vec{σ}'
\] (13)

\[
\vec{a}_f(\vec{r}) = \frac{1}{4\pi} \nabla \times \int_V \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} dV'
\]

\[
= \frac{1}{4\pi} \int_V \int_V \frac{\vec{j}_f(\vec{r}')}{|\vec{r}' - \vec{r}|} dV' + \frac{1}{4\pi} \int_V \frac{\vec{f}(\vec{r}')}{|\vec{r}' - \vec{r}|} \times d\vec{σ}'
\] (14)
For the balance equations I will also need the
**Orthogonality theorem:** Integrals over mixed scalar products vanishes,
\[
\int\int\int_V \vec{f}_T \cdot \vec{g}_L \, dV = - \int\int\int_V \nabla \times \vec{a}_f \cdot \nabla \phi g \, dV
\]
\[
= - \int\int\int_V \nabla \left( \phi g \nabla \times \vec{a}_f \right) \, dV = - \int\int\int_V \left( \vec{a}_f \times \nabla \phi g \right) \, dV
\]
\[
= - \oint_{\partial V} \phi g \nabla \times \vec{a}_f \cdot d\vec{\sigma} = - \oint_{\partial V} \vec{a}_f \times \nabla \phi g \cdot d\vec{\sigma} = 0 \quad (15)
\]
if the surface, \( \partial V \), lies infinitely away from the sources of the fields (cf. Stewart 2008), or if the fields obey appropriate periodic boundary conditions on \( \partial V \) (Heitler 1954, I.6.3).

If the Orthogonality theorem holds true, the integrals over the scalar products of two vectors separates as
\[
\int\int\int_V \vec{f}(\vec{r}) \cdot \vec{g}(\vec{r}) \, dV = \int\int\int_V \vec{f}_T(\vec{r}) \cdot \vec{g}_T(\vec{r}) \, dV + \int\int\int_V \vec{f}_L(\vec{r}) \cdot \vec{g}_L(\vec{r}) \, dV \quad (16)
\]
In particular, both the Joule power and the electric field energy decompose into the contributions of the transverse and longitudinal components of the (di)electric field vectors.

The validity of this theorem will be assumed throughout this series of papers.

Notice that the electromagnetic vector potential, \( \vec{A} \), is a vector potential in the sense of Helmholtz’s theorem only w.r.t. the magnetic induction, \( \vec{B} \), not, however, w.r.t. the electric field strength, \( \vec{E} \). As a consequence, both its transverse and longitudinal components are physically significant. The clue is thus to Helmholtz-decompose \( \vec{A} \), too.

It should also be noted that the longitudinal and transverse components of a *localized* vector field are spread over the whole volume of definition. For instance, for a point-like body of charge \( q \) moving along the trajectory \( \vec{r}(t) \),
\[
\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}(t)); \quad \vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}(t)) \quad (17)
\]
one has \((\vec{r}_t \equiv \vec{r}(t)); \) here, \( t \) is merely a parameter
\[
\rho_j = \nabla \vec{j} = q \vec{v} \cdot \nabla \delta(\vec{r} - \vec{r}(t)) = - \frac{\partial \rho}{\partial t} \quad (18a)
\]
\[
\vec{a}_j = \nabla \times \vec{j} = q \vec{v} \delta(\vec{r} - \vec{r}_t) \times \vec{v} \quad (18b)
\]
\[
\phi_j = \frac{q}{4\pi} \nabla \cdot \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = - \frac{q}{4\pi} \frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3} \quad (19a)
\]
\[
\vec{a}_j = \frac{q}{4\pi} \nabla \times \frac{\vec{v}}{|\vec{r} - \vec{r}_t|} = - \frac{q}{4\pi} \frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3} \quad (19b)
\]
\[
\vec{j}_L = \frac{q}{4\pi} \nabla \frac{\vec{v} \cdot (\vec{r} - \vec{r}_t)}{|\vec{r} - \vec{r}_t|^3}; \quad \vec{j}_T = - \frac{q}{4\pi} \nabla \times \frac{(\vec{r} - \vec{r}_t) \times \vec{v}}{|\vec{r} - \vec{r}_t|^3} \quad (20)
\]
2.2 Laplacean Vector Fields

Laplacean vector fields are vector fields outside any sources and vortices, they (or their potentials) satisfy the Laplace equation and are essentially determined by the (inhomogeneous) boundary conditions (Schwab 2002). A typical example is the electric field between electrodes.

Since both their divergence and curl vanish identically, Helmholtz’s theorem is not really useful for them.

2.3 Vector Fields in Multiply Connected Domains

Vector fields in multiply connected domains assume an 'androgyne' position in that they (or their potentials) satisfy the Laplace equation in certain, bounded domains, but not globally. A well known example is the magnetic field strength, $\vec{H}$, of a constant current, $I$, through an infinite straight conductor in vacuo. The 'magnetic ring voltage', $\oint \vec{H} \cdot d\vec{s}$, vanishes identically, as long as the path of integration lies entirely outside the conductor, so that no current flows through the area bounded by it. But it equals

$$n \int \int_{\sigma} (\nabla \times \vec{H}) \cdot d\vec{\sigma} = n \int \int_{\sigma} \vec{j} \cdot d\vec{\sigma} = nI$$

(21)

if the path surrounds the conductor $n$ times ($n$ integer). That means, that inside the conductor, $\vec{H}$ is a vortex field: $\nabla \times \vec{H} = \vec{j} \neq \vec{0}$, while outside the conductor, $\vec{H}$ is a gradient field: $\nabla \times \vec{H} = \vec{0}$. Obviously, Helmholtz’s theorem is only conditionally applicable, since the integral rather than the differential form of Ampère’s flux law is appropriate.

An analogous example is the vector potential in the Aharonov-Bohm (1959) setup. A constant current through an ideal straight infinite coil in vacuo with no spacing between its windings creates a magnetic field strength and induction being constant inside and vanishing outside the coil. However, by virtue of its continuity, the vector potential does not vanish outside the coil, but represents a gradient field there. I will return to this issue in Section 4.

3. Maxwell’s (1864) Original Equations Revisited

"He [Maxwell] would not have been so often misunderstood, if one would have started the study not with the treatise, while the specific Maxwellian method occurs much more clearly in his earlier essays." (Boltzmann 2001; cf also Sommerfeld 2001, §1) For this, let us return to Maxwell’s (1864) original set of ”20 equations for the 20 variables” $(F,G,H) = \vec{A}$, $(\alpha, \beta, \gamma) = \vec{H}$, $(P,Q,R) = \vec{E}$, $(p,q,r) = \vec{j}$, $(f,g,h) = \vec{D}$, $(p',q',r') = \vec{J}$, $e = \rho$, $\psi = \Phi$. I will rewrite them in modern notation (the r.h.s. of the foregoing relations), SI units and together with their Helmholtz decomposition. For easier reference, Maxwell’s equation numbering is applied. In place of his eqs. (D) for moving conductors his eqs. (35) for conductors at rest is used. The signs in his eqs. (30) and (33) are changed according to the nowadays use.
3.1 Helmholtz Decomposition

A) The total current density, $\vec{J}$, is the sum of electric (conduction, convection) current density, $\vec{j}$, and displacement (‘total polarization’) current density, $\partial \vec{D}/\partial t$.

$$\vec{J}(\vec{r}, t) = \vec{j}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{D}(\vec{r}, t)$$  \hspace{1cm} (22)

This is Maxwell’s famous and crucial step to generalize Ampère’s flux law to open circuits and to convective currents. The time derivative is a precondition to obtain wave equations for the field variables.

The Helmholtz decomposition of this equation is obvious.

$$\vec{J}_{T,L}(\vec{r}, t) = \vec{j}_{T,L}(\vec{r}, t) + \frac{\partial}{\partial t} \vec{D}_{T,L}(\vec{r}, t)$$  \hspace{1cm} (23)

B) The ”magnetic force” (induction, flux density), $\mu \vec{H}$, is the vortex of the vector potential, $\vec{A}$: $\mu \vec{H} = \nabla \times \vec{A}$. Hence, it has got no longitudinal component.

$$\mu \vec{H}_T(\vec{r}, t) = \nabla \times \vec{A}_T(\vec{r}, t)$$  \hspace{1cm} (24a)

$$\mu \vec{H}_L(\vec{r}, t) \equiv \vec{0}$$  \hspace{1cm} (24b)

C) The total current density, $\vec{J}$, is the vortex of the magnetic field strength, $\vec{H}$.

$$\nabla \times \vec{H}(\vec{r}, t) = \vec{J}(\vec{r}, t)$$  \hspace{1cm} (C)

Hence, it has got no longitudinal component, too.

$$\vec{J}_T(\vec{r}, t) = \nabla \times \vec{H}_T(\vec{r}, t)$$  \hspace{1cm} (25a)

$$\vec{J}_L(\vec{r}, t) = \vec{0}$$  \hspace{1cm} (25b)

D) The ”electromotive force” (electric field strength) equals

$$\vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{A}(\vec{r}, t) - \nabla \Phi(\vec{r}, t)$$  \hspace{1cm} (26)

Therefore,

$$\vec{E}_T(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{A}_T(\vec{r}, t)$$  \hspace{1cm} (27a)

$$\vec{E}_L(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{A}_L(\vec{r}, t) - \nabla \Phi(\vec{r}, t) = -\nabla \phi_E(\vec{r}, t)$$  \hspace{1cm} (27b)

The longitudinal component consists of two terms, for which, however, there is no other equation. This makes the whole set to be underdetermined and is the origin of the gauge freedom in the potentials $\vec{A}$ and $\Phi$. Due to the redundancy in some equations below, it is not inconsistent, however.

This underdeterminacy is overcome, if one can work solely with $\phi_E(\vec{r}, t)$, the 'total scalar potential of $\vec{E}(\vec{r}, t)'$, or if one finds an additional equation for $\vec{A}_L$ and $\Phi$, respectively. An example is the boundary conditions in the Aharonov-Bohm (1959) setup, which determine $\vec{A}_L$ outside the coil.
**E)** Electric field strength and dielectric displacement are related through the "equation of electric elasticity".

\[ \vec{E}(\vec{r}, t) = \frac{1}{\varepsilon} \vec{D}(\vec{r}, t) \]  

Thus,

\[ \vec{E}_{T,L}(\vec{r}, t) = \frac{1}{\varepsilon} \vec{D}_{T,L}(\vec{r}, t) \]  

if \( \varepsilon \) is a scalar constant.

**F)** Electric field strength and electric current density are related through the "equation of electric resistance" (\( \sigma \) being the specific conductivity).

\[ \vec{E}(\vec{r}, t) = \frac{1}{\sigma} \vec{j}(\vec{r}, t) \]  

Thus,

\[ \vec{E}_{T,L}(\vec{r}, t) = \frac{1}{\sigma} \vec{j}_{T,L}(\vec{r}, t) \]  

if \( \sigma \) is a scalar constant.

For \( N \) point-like charges \( \{q_a\} \) in vacuo (\( \sigma = 0 \)), eq. (30) is to be replaced with

\[ \sum_{a=1}^{N} q_a \vec{v}_a(t) \delta(\vec{r} - \vec{r}_a(t)) = \vec{j}(\vec{r}, t) \]  

**G)** The "free" charge density is related to the dielectric displacement through the "equation of free electricity".

\[ \rho(\vec{r}, t) - \nabla \vec{D}(\vec{r}, t) = 0 \]  

Obviously, it concerns the longitudinal component of \( \vec{D} \) only.

\[ \rho(\vec{r}, t) - \nabla \vec{D}_L(\vec{r}, t) = 0 \]  

**H)** In a conductor, there is – in analogy to hydrodynamics – "another condition", the "equation of continuity".

\[ \frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla \cdot \vec{j}(\vec{r}, t) = 0 \]  

It concerns the longitudinal component of \( \vec{j} \) only.

\[ \frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla \cdot \vec{j}_L(\vec{r}, t) = 0 \]  

At once, by virtue of eq.(25b), it is merely a consequence of eq.(34). Here is the redundancy mentioned above.

With

\[ \vec{j} = \vec{j}_T + \vec{j}_L = \nabla \times \vec{a}_j - \nabla \phi_j \]  

one obtains the continuity equation in the form

\[ -\Delta \phi_j(\vec{r}, t) + \frac{\partial}{\partial t} \rho(\vec{r}, t) = 0 \]  

It has the advantage of being a single equation relating two scalar quantities one to another rather than four, as in its usual form (35).
3.2 Elimination of Underdeterminacy and Redundance

The underdeterminacy and redundancy in Maxwell’s original set can be eliminated through removing \((\mu \vec{H})_L\), \(\Phi\) and \(\vec{A}_L\) from the set of field variables, but retaining \(\phi_E = -\partial \phi \vec{A}/\partial t + \Phi\). I also remove the total current in view of its merely historical relevance. Then, it remains 18 equations for the 18 variables \((\mu \vec{H})_T = \vec{B}, \vec{H}, \vec{A}_T, \vec{D}, \vec{E}, \phi_E, \vec{j}\) and \(\rho\).

B') The magnetic induction (flux density), \(\mu \vec{H}\), is solenoidal, since it is the vortex of the transverse component of the vector potential.

\[
\mu \vec{H} = (\mu \vec{H})_T = \nabla \times \vec{A}_T
\]  

(39)

C') The transverse components of the conduction/convection and displacement current densities build the vortex of the transverse component, \(\vec{H}_T\), of the magnetic field strength, \(\vec{H}\).

\[
\nabla \times \vec{H}_T = \vec{j}_T + \frac{\partial}{\partial t} \vec{D}_T
\]  

(40)

D') The electric field strength equals (and Helmholtz decomposes as)

\[
\vec{E} = -\frac{\partial}{\partial t} \vec{A}_T - \nabla \phi_E
\]  

(41)

E) Electric field strength and dielectric displacement are related through the “equation of electric elasticity” (28).

F) Electric field strength and electric current density are related through the “equation of electric resistance” (30).

G') The “free” charge density is related to the longitudinal component of the dielectric displacement through the “equation of free electricity”.

\[
\rho(\vec{r}, t) - \nabla \vec{D}_L(\vec{r}, t) = 0
\]  

(42)

H') The conservation of charge is expressed through the equation of continuity.

\[
\frac{\partial}{\partial t} \rho(\vec{r}, t) + \nabla \vec{j}_L(\vec{r}, t) = 0
\]  

(43)

Therefore, the redundancy is removed in the flux law rather than eliminating the continuity equation from the set of basic equations. The continuity equation is retained, because it is a direct consequence of the fact, that - within this approach – the charge of a point-like body is a given, invariant property of it (like its mass). This also allows for an immediate explanation of the transversality of free electromagnetic waves, as will be shown in Pt.II.

4. Gauge Freedom and the Role of \(\vec{A}_L\)

4.1 Classical Gauge Freedom

As mentioned after eq.(27b) above, there is only one equation for the two fields \(\partial \vec{A}_L/\partial t\) and \(\Phi\). Hence, any change of the scalar and vector potentials such, that the expression
\(-\partial \phi_{\vec{A}}/\partial t + \Phi\) = \(\phi_{\vec{E}}\) remains unchanged, is without any physical effect within Maxwell’s theory.

In fact, the Helmholtz components and potentials of vector potential, \(\vec{A}\), and electrical field strength, \(\vec{E}\),

\[
\vec{A} = \vec{A}_T + \vec{A}_L = \nabla \times \vec{a}_A - \nabla \phi_A
\]

\[
\vec{E} = \vec{E}_T + \vec{E}_L = \nabla \times \vec{a}_E - \nabla \phi_E
\]

are known to be interrelated as

\[
\vec{E}_T = -\frac{\partial}{\partial t} \vec{A}_T; \quad \vec{a}_E = -\frac{\partial}{\partial t} \vec{a}_A
\]

\[
\vec{E}_L = -\frac{\partial}{\partial t} \vec{A}_L - \nabla \Phi; \quad \phi_E = -\frac{\partial}{\partial t} \phi_A + \Phi
\]

Hence, the gauge transformation,

\[
\vec{A} = \vec{A}' - \nabla \chi; \quad \Phi = \Phi' + \frac{\partial \chi}{\partial t}
\]

actually concerns only the scalar potential, \(\phi_A\), of \(\vec{A}\) as

\[
\phi_{\vec{A}} = \phi_A' + \chi
\]

but not the vector potential, \(\vec{a}_A\), of \(\vec{A}\).

In the Lorenz (1867) gauge used in Lorentz covariant formulations of the theory, one has

\[
\nabla \vec{A} = -\Delta \phi_A = -\frac{\partial \Phi}{\partial t}
\]

while in the Coulomb (transverse, radiation) gauge being popular in quantum electrodynamics,

\[
\nabla \vec{A} = -\Delta \phi_A = 0
\]

This all suggests to avoid the gauge indeterminacy at all through working solely with \(\vec{A}_T\) and \(\phi_E\). If necessary, \(\vec{A}_L\) can be determined as boundary value problem.

4.2 Quantum Gauge Freedom (Schrödinger Theory)

Although this series of papers deals with classical electromagnetism, it is enlightening and pedagogically useful to sidestep for looking at gauge freedom within Schrödinger wave mechanics.

In order to be independent of the interpretation of the quantum mechanical formalism, let me proceed a follows (Enders 2006, 2008a,b).

\(|\psi|^2\) and \(<\psi|H|\psi>\) are ‘Newtonian state functions’ of a non-relativistic quantum system as they are time-independent in stationary states and as their time-dependence is governed by solely the time-dependent part of the Hamiltonian. This suggests to extend Helmholtz’s (1847, 1911) explorations about the relationships between forces and energies to the question, which ‘external influences’ leave \(|\psi|^2\) and \(<\psi|H|\psi>\) unchanged?
Obviously, $|\psi|^2$ is unchanged, if an external influence, $w$, affects only the phase, $\varphi$, of $\psi$. (Dirac 1931 required the phase to be independent of the state.)

$$\psi_w = \psi_0 e^{i\varphi(w)}, \quad \varphi(0) = 0$$ (51)

Then, if $\psi_0(\vec{r}, t)$ obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_0 = \frac{\hat{p}^2}{2m} \psi_0 + V \psi_0$$ (52)

$\psi_w(\vec{r}, t)$ obeys the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi_w = \hat{H}_w \psi_w = \frac{1}{2m} \left( \hat{p} - h \nabla \varphi \right)^2 \psi_w + \left( V - h \frac{\partial \varphi}{\partial t} \right) \psi_0$$ (53)

Consequently, in stationary states, $<\psi_w|\hat{H}_w|\psi_w>$ is independent of $w$, because $i\hbar \frac{\partial}{\partial t} \psi_w = E \psi_w$, where – by the very definition of $w - E$ is independent of $w$. This is essentially the gauge invariance of the Schrödinger (Pauli 1926) and Dirac equations (Fock 1929) (see also Weyl 1929, 1931).

For influences caused by external electromagnetic fields, this quite general arguing leads to the following important observation, which will be exploited in Pt.III.

The common quasi-classical Schrödinger equation for a point-like charge, $q$, in an electromagnetic field reads

$$i\hbar \frac{\partial}{\partial t} \psi_{\vec{A}, \Phi}(\vec{r}, t) = \left[ \frac{1}{2m} \left( \hat{p} - q \vec{A}(\vec{r}, t) \right)^2 + q \Phi(\vec{r}, t) \right] \psi_{\vec{A}, \Phi}(\vec{r}, t)$$ (54)

Thus, the wave function

$$\psi_{\vec{A}_T, \phi_E}(\vec{r}, t) = \psi_{\vec{A}, \Phi}(\vec{r}, t) e^{i\phi_{\vec{E}}(\vec{r}, t)}$$ (55)

obeys a Schrödinger equation with a *manifest gauge invariant* Hamiltonian.

$$i\hbar \frac{\partial}{\partial t} \psi_{\vec{A}_T, \phi_E}(\vec{r}, t) = \left[ \frac{1}{2m} \left( \hat{p} - q \vec{A}_T(\vec{r}, t) \right)^2 + q \phi_E(\vec{r}, t) \right] \psi_{\vec{A}_T, \phi_E}(\vec{r}, t)$$ (56)

This suggests that manifest gauge invariant theories can be obtained through replacing $\vec{A}$ with $\vec{A}_T$ and $\Phi$ with $\phi_E$.

It is noteworthy, that in *multiply* connected domains, notably outside an infinite coil, where the $\vec{B}$-field vanishes, $\phi_{\vec{A}}$ is not globally integrable. The phase of the wave function can acquire physical significance, as in the Aharonov-Bohm (1959) effect. This underpins the physical significance of the Helmholtz decomposition of the field variables.

Thus, the longitudinal component of a static vector potential, $\vec{A}_L(\vec{r})$, is classically not observable, because it does not contribute to the Maxwell-Lorentz force (Maxwell 1864, Lorentz 1892),

$$q \vec{E} + q \vec{v} \times \vec{B} = q \left( -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi + \vec{v} \times \nabla \times \vec{A} \right)$$ (57)
This suggests to remove $\vec{A}_L(\vec{r})$ from the classical theory altogether and to consider it to be a 'quantum potential' being proportional to Planck’s quantum of action, $h$. On the other hand, if one requires – for good reasons – $\vec{A}(\vec{r})$ to be continuous, $\vec{A}_L(\vec{r})$ can be finite even in the classical (limit) case.

Eq. (55) suggests to incorporate other non-dynamical fields not entering the Hamiltonian and being determined by Laplacean boundary-value problems, by means of appropriate phase factors, too.

"As emphasized by Yang [1974] the vector potential is an over complete specification of the physics of a gauge theory but the gauge covariant field strength underspecifies the content of a gauge theory. The Bohm-Aharanov [1959] effect is the most striking example of this, wherein there exist physical effects on charged particles in a region where the field strength vanishes. The complete and minimal set of variables necessary to capture all the physics are the non-integrable phase factors." (Gross 1992, II.4) Because there are no such phase factors within classical electromagnetism, their classical limit is rather unclear. The complete and minimal set of classical variables obtained in Pts.II and III, respectively, is only loosely related to those. It is thus hoped that the gauge-free representation presented in Pt.III will narrow this gap between classical and quantum theory.

5. Summary and Discussion

The gauge freedom in classical electromagnetism roots in an underdeterminacy in Maxwell’s (1864) original set of equations. Maxwell’s representation (26) of the electric field strength in terms of the potentials contains two contributions to its longitudinal component, one from the vector potential, $-\frac{\partial}{\partial t}\vec{A}_L$, and one from the scalar potential, $-\nabla\Phi$. For these two, no other equation is established. Consequently, Maxwell’s (1864) set is actually not "20 equations for 20 variables" (§70), but only 19 equations for 20 variables. It is not inconsistent, however, because the equations being related to charge conservation are redundant.

In the rationalized and microscopic Maxwell equations, this underdeterminacy is hidden, because only $\vec{E}(\vec{r}, t)$ occurs. It returned in the canonical theory, as will be discussed in Pt. III of this series, where manifest gauge invariant Lagrangians and Hamiltonians will be proposed. The redundancy of the 1864 set is absent in those equations, because the continuity equation has not been retained. The advantage of this concentration on the field equations is their Lorentz invariance. A disadvantage consists in that the experimentally observed transversality of free electromagnetic waves does not naturally emerges out of the theory. (Within quantum theory, it consists in the artifact of the occurrence of not observable photons.)

Both deficiencies, underdeterminacy and redundancy, are absent in the revised set of equations proposed in this paper. Their ‘rationalization’ leads to separate Poynting theorems for both, the propagating transverse and the non-propagating longitudinal field variables, respectively, and to a complete and minimal set of field variables (see Pts.II and III).
Littlejohn (2008) has stressed correctly, that the gauge transformation changes only the longitudinal component of the vector potential. His conclusion, however, that this is the "nonphysical" part, while the transverse component is the physical one (Sect. 34.8), overlooks its role in the Aharonov-Bohm effect. Such contradictions will be avoided through, (i), working with combinations of $\Phi$ and $\vec{A}$, in which those "nonphysical parts", if present, cancel each another and, (ii), treating this gauge invariant combination separately from the dynamics of the other field components.

It is perhaps no accident that the history of the electromagnetic potentials is even more curvilinear than that of the field strengths. Maxwell (1861, 1862, 1864) saw the vector potential to represent Faraday’s "electrotonic state" and the electromagnetic field momentum, respectively. Later, the potentials were considered to be superfluous or merely mathematical tools for solving the rationalized Maxwell’s equations. This mistake lived for a surprisingly long time, in spite of their appearance in the principle of least action (Schwartzschild 1903), in the Hamiltonian (Pauli 1926, Fock 1929) and, last but not least, in the Aharonov-Bohm (1959) effect. The double role of the vector potential, $\vec{A}$, in the electric field strength, $\vec{E}$, where $\partial \vec{A}/\partial t$ contributes to both the transverse and the longitudinal components, has surely hindered the clarification.

The approach presented here benefits from the methodological advantages of the treatments by Newton, Euler and Helmholtz (see Enders 2006, 2008, 2009). In particular, the subject under investigation (moving charged bodies and the electromagnetic fields created by them and acting back onto them) is defined before the mathematical formalism is developed. This keeps the latter physically clear.

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References


The Origin of Mass, Spin and Interaction

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Abstract: We argue that a non commutative geometry at the Compton scale is at the root of mass, Quantum Mechanical spin and QCD and electromagnetic interactions. It also leads to a reconciliation of linearized General Relativity and Quantum Theory.

Keywords: Quantum Gravity; Noncommutative Geometry; Quantum Fields Interactions; General Relativity

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1. Introduction

Modern Fuzzy Spacetime and Quantum Gravity approaches deal with a non differentiable spacetime manifold. In the latter approach there is a minimum spacetime cut off, which, as shown nearly sixty years ago by Snyder leads to what is nowadays called a non commutative geometry, a feature shared by the Fuzzy Spacetime also [1, 2, 3, 4, 5, 6]. The new geometry is given by

\[
[dx^\mu, dx^\nu] \approx \beta^{\mu\nu} l^2 \neq 0
\]  

(1)

While equation (1) is true for any minimum cut off \( l \), it is most interesting and leads to physically meaningful relations including a rationale for the Dirac equation and the underlying Clifford algebra, when \( l \) is at the Compton scale (Cf.ref.[3]). In any case given (1), the usual invariant line element,

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]  

(2)

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has to be written in terms of the symmetric and nonsymmetric combinations for the product of the coordinate differentials. That is the right side of Equation (2) would become

\[ \frac{1}{2} g_{\mu\nu} \left[ (dx^\mu dx^\nu + dx^\nu dx^\mu) + (dx^\mu dx^\nu - dx^\nu dx^\mu) \right], \]

In effect we would have

\[ g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu} \] (3)

So the noncommutative geometry introduces an extra term, that is the second term on the right side of (3). It has been shown in detail by the author that (1) or (2) lead to a reconciliation of electromagnetism and gravitation and lead to what may be called an extended gauge formulation [7, 8, 9, 10].

The extra term in (3) leads to an energy momentum like tensor but it must be stressed that its origin is in the non commutative geometry (1). All this of course is being considered at the Compton scale of an elementary particle.

2. Compton Scale Considerations

As in the case of General Relativity [11, 12], but this time remembering that neither the coordinates nor the derivatives commute we have

\[ \partial_\lambda \partial^\lambda h^{\mu\nu} - (\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}) \]

\[ -\eta^{\mu\nu} \partial_\lambda \partial^\lambda h + \eta^{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} = -k T^{\mu\nu} \] (4)

It must be reiterated that the non commutativity of the space coordinates has thrown up the analogue of the energy momentum tensor of General Relativity, viz., \( T^{\mu\nu} \). We identify this with the energy momentum tensor.

At this stage, we note that the usual energy momentum tensor is symmetric, this being necessary for the conservation of angular momentum. This condition does not hold in (4), and the circumstance requires some discussion. Let us first consider the usual case with commuting coordinates [13]. Here as is well known, we start with the action integral

\[ S = \int \Lambda \left( q, \frac{\partial q}{\partial x^i} \right) dV dt = \frac{1}{c} \int \Lambda d\Omega, \] (5)

In (5) \( \Lambda \) is a function of the generalized coordinates \( q \) of the system, as also their first derivatives with respect to the space and time coordinates. In our case the \( q \) will represent the four potential \( A^\mu \) (Cf.[13]) as will be seen again in (24). Requiring that (5) should be stationary leads to the usual Euler-Lagrange type equations,

\[ \frac{\partial}{\partial x^i} \frac{\partial \Lambda}{\partial q, i} - \frac{\partial \Lambda}{\partial q} = 0 \] (6)

In (6), the summation convention holds. We also have from first principles

\[ \frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q} \frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial q, k} \frac{\partial q, k}{\partial x^i} \] (7)
At this stage we note that in the usual theory we have in (7)
\[
\left\{ \frac{\partial q_{k}}{\partial x^{i}} - \frac{\partial q_{i}}{\partial x^{k}} \right\} = A_{k,a}^{i} - A_{i,a}^{k} = 0
\] (8)

Using (8) it then follows that conservation of angular momentum requires
\[
T^{jk} = T^{kj}
\] (9)

However in our case the right side of (8) does not vanish due to the non commutativity of the coordinates and the partial derivatives, as will be seen more explicitly in (24). This means that the condition (9) does not hold for non commutative coordinates, and hence (4) does not contradict the conservation of angular momentum. However there is new physics here and this new physics will be seen in equations following from (24): we recover electromagnetism as an effect.

Remembering that $h_{\mu\nu}$ is a small effect, we can use the methods of linearized General Relativity [11, 12], to get from (4),
\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, h_{\mu\nu} = \int \frac{4T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} d^{3}x'
\] (10)

It was shown several years ago in the context of linearized General Relativity, that for distances $|\vec{x} - \vec{x}'|$ much greater than the distance $\vec{x}'$, that is well outside the Compton wavelength in our case, we can recover from (10) the electromagnetic potential (Cf.ref.[14] and references therein). We will briefly return to this point.

In (10) we use the well known expansions [12]
\[
\bar{T}_{\mu\nu}(t - |x - x'|, x') = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{\partial^{n}}{\partial t^{n}} T_{\mu\nu}(t - r, x') \right] (r - |x - x'|)^{n},
\] (11)
\[
r - |x - x'| = x^{j} \left( \frac{x^{j}'}{r} \right) + \frac{1}{2} x^{j} x^{k} \left( \frac{x^{j'} x^{k'} - r'^{2} \delta_{jk}}{r^{2}} \right) + \cdots,
\] (12)
\[
\frac{1}{|x - x'|} = \frac{1}{r} + x^{j} \frac{x^{j'}}{r^{2}} + \frac{1}{2} \frac{x^{j} x^{k} (3x^{j'} x^{k'} - r'^{2} \delta_{jk})}{r^{3}} + \cdots,
\] (13)

where $r \equiv |\vec{x}|$. We note that
\[
r = |\vec{x}| \sim l
\] (14)

where $l$ is of the order of the Compton wavelength. So the expansion of the integral in (10) now gives using (12) and (13),
\[
\frac{T}{r} + T' \cdot \frac{1}{r} (r - |x - x'|) + \frac{1}{2} T'' \frac{r - |x - x'|}{r}
\] (15)

where primes denote the derivatives and we have dropped the superscripts for the moment. Denoting $(r - |x - x'|) \equiv r'$, where $0 \leq r' \leq r$, we can write
\[
\langle r' \rangle \approx \gamma r \text{ where } \gamma \sim 0(1)
\] (16)
Finally the expansion gives on the use of (15) and (16), the expression
\[
\frac{T}{r} + \gamma T' + \frac{\gamma^2}{2} T'' \cdot r 
\]
(17)

That is we have, from (10) and (17),
\[
h_{\mu\nu} = 4 \int \frac{T_{\mu\nu}(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + (\text{terms independent of } \vec{x}) + 2 \int \frac{d^2}{dt^2} T_{\mu\nu}(t, \vec{x}') |\vec{x} - \vec{x}'| d^3x' + 0(\vec{x} - \vec{x}'|^2)
\]
(18)

The first term gives on the use of (17), a Coulombic $\frac{\alpha}{r}$ type interaction except that the coefficient $\alpha$ is of much greater magnitude as compared to the gravitational or electromagnetic case, because in the expansion (12) and (13) all terms are of comparable order in view of (14). The second term on the right side of (18) is of no dynamical significance as it is independent of $\vec{x}$. The third term however is of the form constant $\times r$. That is the potential (18) is exactly of the form of the QCD potential [15]
\[
-\frac{\alpha}{r} + \beta r
\]
(19)

In (19) $\alpha$ is of the order of the mass of the particle as follows from (18) and the fact that $T_{\mu\nu}$ is the energy omentum tensor given by
\[
T_{\mu\nu} = \rho u^\mu u^\nu
\]
(20)

where in (20), remembering that (14) holds, that is we are at the Compton scale, $u^i \sim c$.

We now deduce two relations which can be deduced directly from the theory of the Dirac equation [16]. We do it here to show the continuity of the above theme. Remembering that from (1), we are within a sphere of radius given by the Compton length where the velocities equal that of light, as noted above, we have equations
\[
\left| \frac{du_v}{dt} \right| = |u_v| \omega
\]
(21)
\[
\omega = \frac{|u_v|}{R} = \frac{2mc^2}{\hbar}
\]
(22)

Alternatively as remarked, we can get (21) from the theory of the Dirac equation itself [16], viz.,
\[
\hbar \frac{d}{dt} (u_v) = -2mc^2(u_v),
\]

Using (20), (21) and (22) we get
\[
\frac{d^2}{dt^2} T_{\mu\nu} = 4\rho u^\mu u^\nu \omega^2 = 4\omega^2 T_{\mu\nu}
\]
(23)

(Equation (23) too is obtained in the Dirac theory (loc.cit)). Whence, as can be easily verified, $\alpha$ and $\beta$ in (19) have the correct values required for the QCD potential (Cf. also
[14]). (Alternatively $\beta r$ itself can be obtained, as in the usual theory by a comparison with the Regge angular momentum mass relation: It is in fact the constant string tension like potential which gives quark confinement and its value is as in the usual theory [17]).

Let us return to the considerations which lead via a non commutative geometry to an energy momentum tensor in (4). We can obtain from here the origin of mass and spin itself, for we have as is well known (Cf.ref.[12])

$$m = \int T^{00} d^3x$$

and via

$$S_k = \int \epsilon_{klm} x^l T^{m0} d^3x$$

the equation

$$S_k = e <x^l> \int \rho d^3x.$$

While $m$ above can be immediately and consistently identified with the mass, the last equation gives the Quantum Mechanical spin if we remember that we are working at the Compton scale so that

$$\langle x^l \rangle = \hbar^2 mc.$$ 

Returning to the considerations in (1) to (4) it follows that (Cf.ref.[7])

$$\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \text{goes over to} \frac{\partial}{\partial x^\lambda} \Gamma_{\mu\nu} - \frac{\partial}{\partial x^\mu} \Gamma_{\lambda\nu}$$  \hspace{1cm} (24)

Normally in conventional theory the right side of (24) would vanish. Let us designate this non vanishing part on the right by

$$\frac{e}{c\hbar} \Gamma_{\mu\lambda}$$  \hspace{1cm} (25)

We have shown here that the non commutativity in momentum components leads to an effect that can be identified with electromagnetism and in fact from expression (25) we have

$$A^\mu = \hbar \Gamma_{\mu\nu}$$  \hspace{1cm} (26)

where $A_\mu \equiv q$, which we encountered in (6), as noted can be identified with the electromagnetic four potential and the Coulomb law deduced for $|\vec{x} - \vec{x}'|$ in (10) much greater than $|\vec{x}'|$ that is well outside the Compton scale (Cf.ref.[3] and also ref. [14]). Indeed we have referred to this in the discussion after (4). It must be mentioned that despite non commutativity, we are using as an approximation the usual continuous partial derivatives, though these latter do not commute amongst themselves now. This facilitates the analysis and brings out the physical effects. In any case as can be seen from (1), the effects are of the order $l^2$.

To see this in the light of the usual gauge invariant minimum coupling (Cf.ref.[14]), we start with the effect of an infinitesimal parallel displacement of a vector in this non commutative geometry,

$$\delta a^\sigma = -\Gamma_{\sigma\mu} a^\mu dx^\nu$$  \hspace{1cm} (27)
As is well known, (27) represents the effect due to the curvature and non integrable nature of space - in a flat space, the right side would vanish. Considering the partial derivatives with respect to the $\mu^{th}$ coordinate, this would mean that, due to (27)

$$\frac{\partial a^\sigma}{\partial x^\mu} \rightarrow \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma_{\mu\nu} a^\nu$$

(28)

Letting $a^\mu = \partial^\mu \phi$, we have, from (28)

$$D_{\mu\nu} \equiv \partial_\nu \partial^\mu \rightarrow D'_\mu\nu \equiv \partial_\nu \partial^\mu - \Gamma^\mu_{\lambda\nu} \partial^\lambda$$

$$= D_\mu - \Gamma^\mu_{\lambda\nu} \partial^\lambda$$

(29)

Now we can also write

$$D_{\mu\nu} = (\partial^\mu - \Gamma^\mu_{\lambda\lambda})(\partial_\nu - \Gamma^\lambda_{\lambda\nu}) + \partial^\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\mu_{\lambda\lambda} \partial_\nu$$

So we get

$$D_{\mu\nu} - \Gamma^\mu_{\lambda\lambda} \partial_\nu = (p^\mu)(p_\nu)$$

where

$p^\mu \equiv \partial^\mu - \Gamma^\mu_{\lambda\lambda}$

Or,

$$D_{\mu\mu} - \Gamma^\mu_{\lambda\lambda} \partial_\mu = (p^\mu)(p_\mu)$$

Further we have

$$D'_{\mu\mu} = D_{\mu\mu} - \Gamma^\mu_{\lambda\mu} \partial^\lambda$$

Thus, (29) gives, finally,

$$D'_{\mu\nu} = (p^\mu)(p_\nu)$$

That is we have

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}$$

Comparison with (26) establishes the required identification.

It is quite remarkable that equation (26) is mathematically identical to Weyl's unified formulation, though this was not originally acceptable because of the ad hoc insertion of the electromagnetic potential. Here in our case it is a consequence of the geometry - the noncommutative geometry (Cf.refs.[14] and [18] for a detailed discussion).

It was also described in detail how in the usual commutative spacetime the Dirac spinorial wave functions conceal the noncommutative character \((1) [3]\).

Indeed we can verify all these considerations in a simple way as follows:

First let us consider the usual spacetime, in which the Dirac wave function is given by

$$\psi = \begin{pmatrix} \chi \\ \Theta \end{pmatrix},$$
where $\chi$ and $\Theta$ are two component spinors. It is well known that under reflection while the so called positive energy spinor $\Theta$ behaves normally, on the contrary $\chi \rightarrow -\chi$, $\chi$ being the so called negative energy spinor which comes into play at the Compton scale [19]. That is, space is doubly connected. Because of this property as shown in detail [8], there is now a covariant derivative given by, in units, $\hbar = c = 1$,

$$
\frac{\partial \chi}{\partial x^\mu} \rightarrow \left[ \frac{\partial}{\partial x^\mu} - n A^\mu \right] \chi
$$

where

$$
A^\mu = \Gamma^\mu_{\sigma \sigma} = \frac{\partial}{\partial x^\mu} \log(\sqrt{|g|})
$$

$\Gamma$ denoting the Christoffel symbols.

$A^\mu$ in (31) is now identified with the electromagnetic potential, exactly as in Weyl’s theory except that now, $A^\mu$ arises from the bi spinorial character of the Dirac wave function or the double connectivity of spacetime. In other words, we return to (26) via an alternative route.

What all this means is that the so called ad hoc feature in Weyl’s unification theory is really symptomatic of the underlying noncommutative spacetime geometry (1). Given (1) (or (3)) we get both gravitation and electromagnetism in a unified picture, because both are now the consequence of spacetime geometry. We could think that gravitation arises from the symmetric part of the metric tensor (which indeed is the only term if $0(l^2)$ is neglected) and electromagnetism from the antisymmetric part (which manifests itself as an $0(l^2)$ effect). It is also to be stressed that in this formulation, we are working with noncommutative effects at the Compton scale, this being true for the Weyl like formulation also.

3. Discussion

The Compton scale comes as a Quantum Mechanical effect, within which we have zitterbewegung effects and a breakdown of Causal Physics [16]. Indeed Dirac had noted this aspect in connection with two difficulties with his electron equation. Firstly the speed of the electron turns out to be the velocity of light. Secondly the position coordinates become complex or non Hermitian. His explanation was that in Quantum Theory we cannot go down to arbitrarily small spacetime intervals, for the Heisenberg Uncertainty Principle would then imply arbitrarily large momenta and energies. So Quantum Mechanical measurements are an average over intervals of the order of the Compton scale. Once this is done, we recover meaningful physics. All this has been studied afresh by the author more recently, in the context of a non differentiable spacetime and noncommutative geometry [20].

Weinberg also notices the non physical aspect of the Compton scale [21]. Starting with the usual light cone of Special Relativity and the inversion of the time order of events, he goes on to add, and we quote at a little length and comment upon it, “Although the relativity of temporal order raises no problems for classical physics, it plays a profound
role in quantum theories. The uncertainty principle tells us that when we specify that a particle is at position \(x_1\) at time \(t_1\), we cannot also define its velocity precisely. In consequence there is a certain chance of a particle getting from \(x_1\) to \(x_2\) even if \(x_1 - x_2\) is spacelike, that is, \(|x_1 - x_2| > |x_1^0 - x_2^0|\). To be more precise, the probability of a particle reaching \(x_2\) if it starts at \(x_1\) is nonnegligible as long as

\[
(x_1 - x_2)^2 - (x_1^0 - x_2^0)^2 \leq \frac{\hbar^2}{m^2}
\]

where \(\hbar\) is Planck’s constant (divided by \(2\pi\)) and \(m\) is the particle mass. (Such space-time intervals are very small even for elementary particle masses; for instance, if \(m\) is the mass of a proton then \(\hbar/m = 2 \times 10^{-14}\text{cm}\) or in time units \(6 \times 10^{-25}\text{sec}\). Recall that in our units \(1\text{sec} = 3 \times 10^{10}\text{cm}\).) We are thus faced again with our paradox; if one observer sees a particle emitted at \(x_1\), and absorbed at \(x_2\), and if \((x_1 - x_2)^2 - (x_1^0 - x_2^0)^2\) is positive (but less than or equal \(\hbar^2/m^2\)), then a second observer may see the particle absorbed at \(x_2\) at a time \(t_2\) before the time \(t_1\) it is emitted at \(x_1\).

“There is only one known way out of this paradox. The second observer must see a particle emitted at \(x_2\) and absorbed at \(x_1\). But in general the particle seen by the second observer will then necessarily be different from that seen by the first. For instance, if the first observer sees a proton turn into a neutron and a positive pi-meson at \(x_1\) and then sees the pi-meson and some other neutron turn into a proton at \(x_2\), then the second observer must see the neutron at \(x_2\) turn into a proton and a particle of negative charge, which is then absorbed by a proton at \(x_1\) that turns into a neutron. Since mass is a Lorentz invariant, the mass of the negative particle seen by the second observer will be equal to that of the positive pi-meson seen by the first observer. There is such a particle, called a negative pi-meson, and it does indeed have the same mass as the positive pi-meson. This reasoning leads us to the conclusion that for every type of charged particle there is an oppositely charged particle of equal mass, called its antiparticle. Note that this conclusion does not obtain in nonrelativistic quantum mechanics or in relativistic classical mechanics; it is only in relativistic quantum mechanics that antiparticles are a necessity. And it is the existence of antiparticles that leads to the characteristic feature of relativistic quantum dynamics, that given enough energy we can create arbitrary numbers of particles and their antiparticles.”

We note however that there is a nuance here which distinguishes Weinberg’s explanation from that of Dirac. In Weinberg’s analysis, one observer sees only protons at \(x_1\) and \(x_2\), whereas the other observer sees only neutrons at \(x_1\) and \(x_2\) while in between, the first observer sees a positively charged pion and the second observer a negatively charged pion. We remark that in Weinberg’s explanation which is in the spirit of the Feynman-Stuckeberg diagrams there is no charge conservation, though the Baryon number is conserved. The explanation for this is to be found in the considerations leading from (10) to (19) - within the Compton scale we have the QCD interactions - electromagnetic interaction is outside the Compton scale [22].

Our analysis uses the Compton length (and time) as the fundamental parameter. It may be added that there is a close parallel between the above considerations and the orig-
inal Dirac monopole theory: in the latter it is the nodal singularity that gives rise to magnetism, while in the former, the multiply connected nature of space (or non-commutativity) gives rise to electromagnetism. This has been discussed in [23]. So too, it may be mentioned that the considerations in equations (21), (22) and (23) are connected with Dirac’s membrane (and more recently and generally the p-brane) theory [24] - though Dirac himself approached the membrane problem from a different route. Finally, it may be pointed out that Einstein himself always disliked the energy momentum tensor in his General Relativistic equation [25] as it was mechanical and non geometric! Pleasingly, in the above formulation, this term has a geometric origin - albeit, a non-commutative geometry which also provides a unified description of linearized General Relativity and Quantum Mechanics.

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References


Nonholonomic Ricci Flows and Parametric Deformations of the Solitonic pp–Waves and Schwarzschild Solutions

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Abstract: We study Ricci flows of some classes of physically valuable solutions in Einstein and string gravity. The anholonomic frame method is applied for generic off–diagonal metric ansatz when the field/ evolution equations are transformed into exactly integrable systems of partial differential equations. The integral varieties of such solutions, in four and five dimensional gravity, depend on arbitrary generation and integration functions of one, two and/ or three variables. Certain classes of nonholonomic frame constraints allow us to select vacuum and/or Einstein metrics, to generalize such solutions for nontrivial string (for instance, with antisymmetric torsion fields) and matter field sources. A very important property of this approach (originating from Finsler and Lagrange geometry but re–defined for semi–Riemannian spaces) is that new classes of exact solutions can be generated by nonholonomic deformations depending on parameters associated to some generalized Geroch transforms and Ricci flow evolution. In this paper, we apply the method to construct in explicit form some classes of exact solutions for multi–parameter Einstein spaces and their nonholonomic Ricci flows describing evolutions/interactions of solitonic pp–waves and deformations of the Schwarzschild metric. We explore possible physical consequences and speculate on their importance in modern gravity.

Keywords: Ricci Flows; Exact Solutions; Nonholonomic Frames; Gravitational Solitons; pp–waves; Methods of Finsler and Lagrange Geometry; Nonlinear Connections
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1. Introduction

This is the fifth paper in a series of works on nonholonomic Ricci flows of metrics and geometric objects subjected to nonintegrable (nonholonomic) constraints [1, 2, 3, 4]. It is devoted to explicit applications of new geometric methods in constructing exact solutions in gravity and Ricci flow theory. Specifically, we shall consider a set of particular solutions with solitonic pp–waves and anholonomic deformations of the Schwarzschild metric, defined by generic off–diagonal metrics, describing nonlinear gravitational interactions and gravitational models with effective cosmological constants, for instance, induced by string corrections [5, 6] or effective approximations for matter fields. We shall analyze how such gravitational configurations (some of them generated as exact solutions by geometric methods in Section 4 of Ref. [7]) may evolve under Ricci flows; on previous results and the so–called anholonomic frame method of constructing exact solutions, see works [8, 9, 10, 11, 12, 13] and references therein.

Some of the most interesting directions in modern mathematics are related to the Ricci flow theory [14, 15], see reviews [16, 17, 18]. There were elaborated a set of applications with such geometric flows, following low dimensional or approximative methods to construct solutions of evolution equations, in modern gravity and mathematical physics, for instance, for low dimensional systems and gravity [19, 20, 21, 22] and black holes and cosmology [23, 24]. One of the most important tasks in such directions is to formulate certain general methods of constructing exact solutions for gravitational systems under Ricci flow evolution of fundamental geometric objects.

In Refs. [25, 26, 27], considering the Ricci flow evolution parameter as a time like, or extra dimension, coordinate, we provided the first examples when physically valuable Ricci flow solutions were constructed following the anholonomic frame method. A quite different scheme was considered in Ref. [3] with detailed proofs that the information on general metrics and connection in Riemann–Cartan geometry, and various generalizations to nonholonomic Lagrange–Finsler spaces, can be encoded into bi–Hamilton structure and related solitonic hierarchies. There were formulated certain conditions when nonholonomic solitonic equations can be constrained to extract exact solutions for the Einstein equations and Ricci flow evolution equations.

The previous partner paper [4] was devoted to the geometry of parametrized nonholonomic frame transforms as superpositions of the Geroch transforms (generating exact vacuum gravitational solutions with Killing symmetries, see Refs. [28, 29]) and the anholonomic frame deformations and oriented to carry out a program of generating off–diagonal exact solutions in gravity [7] and Ricci flow theories. The goal of this work is to show how such new classes of parametric nonholonomic solutions, formally constructed for the Einstein and string gravity [7], can be generalized to satisfy certain geometric evolution equations and define Ricci flows of physically valuable metrics and connections.

The structure of the paper is the following: In section 2, we outline the necessary formulas for nonholonomic Einstein spaces and Ricci flows. There are introduced the general...
ansatz for generic off–diagonal metrics (for which, we shall construct evolution/ field exact solutions) and the primary metrics used for parametric nonholonomic deformations to new classes of solutions.

In section 3, we construct Ricci flow solutions of solitonic pp–waves in vacuum Einstein and string gravity.

Section 4 is devoted to a study of parametric nonholonomic tranforms (defined as superpositions of the parametric transforms and nonholonomic frame deformations) in order to generate (multi-) parametric solitonic pp–waves for Ricci flows and in Einstein spaces.

Section 5 generalizes to Ricci flow configurations the exact solutions generated by parametric nonholonomic frame transforms of the Schwarzschild metric. There are analyzed deformations and flows of stationary backgrounds, considered anisotropic polarizations on extra dimension coordinate (possibly induced by Ricci flows and extra dimension interactions) and examined five dimensional solutions with running of parameters on nonholonomic time coordinate and flow parameter.

The paper concludes with a discussion of results in section 6.

The Appendix contains some necessary formulas on effective cosmological constants and nonholonomic configurations induced from string gravity.

2. Preliminaries

We work on five and/or four dimensional, (5D and/ or 4D), nonholonomic Riemannian manifolds \( V \) of necessary smooth class and conventional splitting of dimensions \( \text{dim } V = n + m \) for \( n = 3 \), or \( n = 2 \) and \( m = 2 \), defined by a nonlinear connection (N–connection) structure \( N = \{ N^a_i \} \), such manifolds are also called N–anholonomic [8]. The local coordinates are labelled in the form \( u^a = (x^i, y^a) = (x^1, x^i, y^4 = v, y^5) \), for \( i = 1, 2, 3 \) and \( \hat{i} = 2, 3 \) and \( a, b, ... = 4, 5 \). Any coordinates from a set \( u^a \) can be for a three dimensional (3D) space, time, or extra dimension (5th coordinate). Ricci flows of geometric objects will be parametrized by a real \( \chi \in [0, \chi_0] \). Four dimensional (4D) spaces, when the local coordinates are labelled in the form \( u^a = (x^\tilde{i}, y^a) \), i. e. without coordinate \( x^1 \), are defined as a trivial embedding into 5D ones. In general, we shall follow the conventions and methods stated in Refs. [4, 7] (the reader is recommended to consult those works on main definitions, denotations and geometric constructions).

2.1 Ansatz for the Einstein and Ricci flow equations

A nonholonomic manifold \( V \), provided with a N–connection (equivalently, with a locally fibred) structure and a related preferred system of reference, can be described in equivalent forms by two different linear connections, the Levi Civita \( \nabla \) and the canonical distinguished connection, d–connection \( \hat{D} \), both completely defined by the same metric
structure

\[ g = g_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) \mathbf{e}^a \otimes \mathbf{e}^b, \]

\[ e^i = dx^i, \quad e^a = dy^a + N_i^a(u) dx^i, \]

in metric compatible forms, \( \nabla g = 0 \) and \( \hat{D} g = 0 \). It should be noted that, in general, \( \hat{D} \) contains some nontrivial torsion coefficients induced by \( N_i^a \), see details in Refs. [1, 2, 3, 4, 8, 9, 10, 11, 12, 13]. For simplicity, we shall omit "hats" and indices, or other labels, and write, for instance, \( u = (x, y) \), \( \partial_i = \partial / \partial x^i \) and \( \partial_a = \partial / \partial y^a \), ... if such simplifications will not result in ambiguities.

In order to consider Ricci flows of geometric objects, we shall work with families of ansatz \( \chi g = g(\chi) \), of type (1), parametrized by \( \chi \),

\[ \chi g = g_1 dx^1 \otimes dx^1 + g_2(x^2, x^3, \chi) dx^2 \otimes dx^2 + g_3(x^2, x^3, \chi) dx^3 \otimes dx^3 \]

\[ + h_4(x^k, v, \chi) \chi \delta v \otimes \chi \delta v + h_5(x^k, v, \chi) \chi \delta y \otimes \chi \delta y, \]

\[ \chi \delta v = dv + w_i(x^k, v, \chi) dx^i, \quad \chi \delta y = dy + n_i(x^k, v, \chi) dx^i, \]

for \( g_1 = \pm 1 \), with corresponding flows for \( N \)-adapted bases,

\[ \chi \mathbf{e}_a = \mathbf{e}_a(\chi) = (\chi \mathbf{e}_i = \mathbf{e}_i(\chi) = \partial_i - N_i^a(u, \chi) \partial_a, \mathbf{e}_a), \]

\[ \chi \mathbf{e}^a = \mathbf{e}^a(\chi) = (\mathbf{e}^i, \chi \mathbf{e}^a = \mathbf{e}^a(\chi) = dy^a + N_i^a(u, \chi) dx^i) \]

defined by \( N_i^a(u, \chi) = w_i(x^k, v, \chi) \) and \( N_i^5(u, \chi) = n_i(x^k, v, \chi) \). For any fixed value of \( \chi \), we may omit the Ricci flow parametric dependence.

The frames (4) satisfy certain nonholonomy (equivalently, anholonomy) relations

\[ [\mathbf{e}_a, \mathbf{e}_\beta] = \mathbf{e}_a \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_a = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \]

with anholonomy coefficients

\[ W_{ia}^b = \partial_a N_i^b \] and \( W_{ji}^a = \Omega_i^a_{ij} = \mathbf{e}_j(N_i^a) - \mathbf{e}_j(N_i^a). \]

A local basis is holonomic (for instance, the local coordinate basis) if \( W_{a\beta}^\gamma = 0 \) and integrable, i.e. it defines a fibred structure, if the curvature of \( N \)-connection \( \Omega_i^a_{ij} = 0 \).

We can elaborate on a \( N \)-anholonomic manifold \( \mathbf{V} \) (i.e. on a manifold provided with \( N \)-connection structure) a \( N \)-adapted tensor and differential calculus if we decompose the geometric objects and basic equations with respect to \( N \)-adapted bases (4) and (3) and using the canonical \( d \)-connection \( \hat{D} = \nabla + \mathbf{Z} \), see, for instance, the formulas (A.17) in Ref. [4] for the components of distorsion tensor \( \mathbf{Z} \), which contains certain nontrivial torsion coefficients induced by "off-diagonal" \( N_i^a \). Two different linear connections, \( \nabla \) and \( \hat{D} \), define respectively two different Ricci tensors, \( R_{\alpha\beta} \) and \( \tilde{R}_{\alpha\beta} = [\tilde{R}_{ij}, \tilde{R}_{ia}, \tilde{R}_{bj}, \tilde{S}_{ab}] \) (see (A.12) and related formulas in [1]). Even, in general, \( \hat{D} \neq \nabla \), for certain classes of ansatz and \( N \)-adapted frames, we can obtain, for some nontrivial coefficients, relations of type \( \tilde{R}_{\alpha\beta} = R_{\alpha\beta} \). This allows us to constrain some classes of solutions of the Einstein
and/or Ricci flow equations constructed for a more general linear connection $\hat{D}$ to define also solutions for the Levi Civita connection $\nabla$.

For a 5D space initially provided with a diagonal ansatz (1), when $g_{ij} = \text{diag}[\pm 1, g_2, g_3]$ and $g_{ab} = \text{diag}[g_4, g_5]$, we considered [25, 4] the nonholonomic (normalized) evolution equations (parametrized by an ansatz (2))

$$\frac{\partial}{\partial \chi} g_{ii} = -2 \left[ \hat{R}_{ii} - \lambda g_{ii} \right] - h_{cc} \frac{\partial}{\partial \chi} \left( N_i^c \right)^2,$$

$$\frac{\partial}{\partial \chi} h_{aa} = -2 \left( \hat{R}_{aa} - \lambda h_{aa} \right),$$

$$\hat{R}_{\alpha \beta} = 0 \quad \text{for} \quad \alpha \neq \beta,$$

with the coefficients defined with respect to $N$–adapted frames (3) and (4). This system of constrained (nonholonomic) evolution equations in a particular case is related to families of metrics $\lambda g = \lambda g_{\alpha \beta} e^\alpha \otimes e^\beta$ for nonholonomic Einstein spaces, considered as solutions of

$$\hat{R}_{\alpha \beta} = \lambda g_{\alpha \beta}, \quad (10)$$

with effective cosmological constant $\lambda$ (in more general cases, we can consider effective, locally anisotropically polarized cosmological constants with dependencies on coordinates and $\chi$ : such solutions were constructed in Refs. [10, 8]). For any solution of (10) with nontrival

$$g_{\alpha \beta} = [\pm 1, g_{2,3}(x^2, x^3), h_{4,5}(x^2, x^3, v)], \quad \text{and} \quad w_{2,3}(x^2, x^3, v), n_{2,3}(x^2, x^3, v),$$

we can consider nonholonomic Ricci flows of the horizontal metric components, $g_{2,3}(x^2, x^3) \rightarrow g_{2,3}(x^2, x^3, \chi)$, and of certain $N$–connection coefficients $n_{2,3}(x^2, x^3, v) \rightarrow n_{2,3}(x^2, x^3, v, \chi)$, constrained to satisfy the equation (7), i.e.

$$\frac{\partial}{\partial \chi} \left[ g_{2,3}(x^2, x^3, \chi) + h_5(x^2, x^3, v) \left( n_{2,3}(x^2, x^3, v, \chi) \right)^2 \right] = 0. \quad (11)$$

Having constrained an integral variety of (10) in order to have $\hat{R}_{\alpha \beta} = \lambda R_{\alpha \beta}$ for certain subclasses of solutions, the equations (11) define evolutions of the geometric objects just for a family of Levi Civita connections $\lambda \nabla$.

Computing the components of the Ricci and Einstein tensors for the metric (2) (see details on tensors components’ calculus in Refs. [10, 8]), one proves that the equations

---

2 we note that such equalities are obtained by deformation of the nonholonomic structure on a manifold which change the transformation laws of tensors and different linear connections

3 this imposes certain additional restrictions on $n_{2,3}$ and $g_{2,3}$, see discussions related to formulas (A.16)–(A.20) and explicit examples for Ricci flows stated by constraints (49), or (74) in Ref. [4]; in the next sections, we shall consider explicit examples
transform into a parametric on $\chi$ system of partial differential equations:

\[
\frac{1}{2g_2(\chi)g_3(\chi)}\left[g_2^*(\chi)g_3^*(\chi) + \frac{(g_3^*(\chi))^2}{2g_3(\chi)} - g_3^{**}(\chi) + \frac{g_2^*(\chi)g_3'(\chi)}{2g_3(\chi)} + \frac{(g_2^*(\chi))^2}{2g_2(\chi)} - g_2''(\chi)\right] = -\lambda,
\]

\[
\frac{1}{2h_4(\chi)h_5(\chi)} \times \left[h_5^*(\chi)\left(\ln \sqrt{|h_4(\chi)h_5(\chi)|}\right)^* - h_5^{**}(\chi)\right] = -\lambda,
\]

\[
\frac{-w_i(\chi)}{2h_5(\chi)} [n_i^{**}(\chi) + \gamma(\chi)n_i^*(\chi)] = 0,
\]

where, for $h_{4,5}^* \neq 0$,

\[
\alpha_i = h_5^* \partial_i \phi, \quad \beta = h_5^* \phi^*,
\]

\[
\gamma = \frac{3h_5^*}{2h_5} \frac{h_4^*}{h_4^*} - \frac{h_5^*}{h_4^*}, \quad \phi = \ln \left| \frac{h_5^*}{\sqrt{|h_4h_5|}} \right|,
\]

when the necessary partial derivatives are written in the form $a^* = \partial a/\partial x^2$, $a' = \partial a/\partial x^3$, $a^* = \partial a/\partial v$. In the vacuum case, we shall put $\lambda = 0$. Here we note that the dependence on $\chi$ can be considered both for classes of functions and integrations constants and functions defining some exact solutions of the Einstein equations or even for any general metrics on a Riemann-Cartan manifold (provided with any compatible metric and linear connection structures).

For an ansatz (2) with $g_2(x^2, x^3, \chi) = \epsilon_2 e^{\psi(x^2, x^3, \chi)}$ and $g_2(x^2, x^3, \chi) = \epsilon_3 e^{\psi(x^2, x^3, \chi)}$, we can restrict the solutions of the system (12)–(15) to define Ricci flows solutions with the Levi Civita connection if the coefficients satisfy the conditions

\[
\epsilon_2 \psi^{**}(\chi) + \epsilon_3 \psi''(\chi) = \lambda
\]

\[
\frac{h_5^*}{h_4^*} = \lambda,
\]

\[
w_2' - w_3^* + w_3 w_2^* - w_2 w_3^* = 0,
\]

\[
n_2'(\chi) - n_3^*(\chi) = 0,
\]

for

\[
\frac{\partial_i}{\partial \phi^*}, \quad \text{where} \quad \phi = -\ln \left| \frac{h_4h_5}{|h_5^*|} \right|,
\]

for $\hat{i} = 2, 3$, see formulas (49) in Ref. [4].

### 2.2 Five classes of primary metrics

We introduce a list of 5D quadratic elements, defined by certain primary metrics, which will be subjected to parametrized nonholonomic transforms in order to generate new
classes of exact solutions of the Einstein and Ricci flow equations, i.e. of the system (12)–(15) and (10) with possible additional constraints in order to get geometric evolutions in terms of the Levi Civita connection $\nabla$.

The first type quadratic element is taken
\[ \delta s_1^2 = \epsilon_1 dx^2 - d\xi^2 - r^2(\xi) \, d\vartheta^2 - r^2(\xi) \sin^2 \vartheta \, d\varphi^2 + \varpi^2(\xi) \, dt^2, \]
where the local coordinates and nontrivial metric coefficients are parametrized in the form
\[ x^1 = \kappa, x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t, \]
\[ \tilde{g}_1 = \epsilon_1 = \pm 1, \quad \tilde{g}_2 = -1, \quad \tilde{g}_3 = -r^2(\xi), \quad \tilde{h}_4 = -r^2(\xi) \sin^2 \vartheta, \quad \tilde{h}_5 = \varpi^2(\xi), \]
for
\[ \xi = \int dr \left| 1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2} \right|^{1/2} \quad \text{and} \quad \varpi^2(r) = 1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}. \]
For the constants $\varepsilon \to 0$ and $\mu$ being a point mass, the element (20) defines just a trivial embedding into 5D (with extra dimension coordinate $\kappa$) of the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$.

The second quadratic element is
\[ \delta s_2^2 = -r_g^2 \, d\varphi^2 - r_g^2 \, d\vartheta^2 + \tilde{g}_3(\tilde{\vartheta}) \, d\tilde{\xi}^2 + \epsilon_1 \, d\chi^2 + \tilde{h}_5 (\xi, \tilde{\vartheta}) \, dt^2, \]
where the local coordinates are
\[ x^1 = \varphi, x^2 = \tilde{\vartheta}, x^3 = \tilde{\xi}, y^4 = \chi, y^5 = t, \]
for
\[ d\tilde{\vartheta} = d\vartheta/\sin \vartheta, \quad d\tilde{\xi} = dr/r \sqrt{|1 - 2\mu/r + \varepsilon/r^2|}, \]
and the Schwarzschild radius of a point mass $\mu$ is defined $r_g = 2G_{[4]\mu}/c^2$, where $G_{[4]}$ is the 4D Newton constant and $c$ is the light velocity. The nontrivial metric coefficients in (22) are parametrized
\[ \tilde{g}_1 = -r_g^2, \quad \tilde{g}_2 = -r_g^2, \quad \tilde{g}_3 = -1/\sin^2 \vartheta, \]
\[ \tilde{h}_4 = \epsilon_1, \quad \tilde{h}_5 = \left[ 1 - 2\mu/r + \varepsilon/r^2 \right] / r^2 \sin^2 \vartheta. \]
The quadratic element defined by (22) and (23) is a trivial embedding into 5D of the Schwarzschild quadratic element multiplied to the conformal factor $(r \sin \vartheta/r_g)^2$. We emphasize that this metric is not a solution of the Einstein or Ricci flow equations but it will be used in order to construct parametrized nonholonomic deformations to such solutions.

4 For simplicity, we consider only the case of vacuum solutions, not analyzing a more general possibility when $\varepsilon = e^2$ is related to the electric charge for the Reissner–Nordström metric (see, for example, [30]). In our further considerations, we shall treat $\varepsilon$ as a small parameter, for instance, defining a small deformation of a circle into an ellipse (eccentricity).
We shall use a quadratic element when the time coordinate is considered to be "anisotropic",

$$\delta s^2 = -r_g^2 d\varphi^2 - r_g^2 d\vartheta^2 + \hat{g}_3(\vartheta) d\xi^2 + \hat{h}_4 (\xi, \vartheta) \ d t^2 + \epsilon_1 \ d\varphi^2$$  \hspace{1cm} (24)

where the local coordinates are

$$x^1 = \varphi, \ x^2 = \vartheta, \ x^3 = \xi, \ y^4 = t, \ y^5 = \varphi,$$

and the nontrivial metric coefficients are parametrized

$$\hat{g}_1 = -r_g^2, \ \hat{g}_2 = -r_g^2, \ \hat{g}_3 = -1/\sin^2 \vartheta, \ \hat{h}_4 = \left[1 - 2\mu/r + \epsilon/r^2\right] / r^2 \sin^2 \vartheta, \ \hat{h}_5 = \epsilon_1.$$  \hspace{1cm} (25)

The formulas (24) and (25) are respective reparametrizations of (22) and (23) when the 4th and 5th coordinates are inverted. Such metrics will be used for constructing new classes of exact solutions in 5D with explicit dependence on time like coordinate.

The forth quadratic element is introduced by inverting the 4th and 5th coordinates in (20)

$$\delta s^2 = \epsilon_1 d\varphi^2 - d\xi^2 - r^2(\xi) d\vartheta^2 + \omega^2(\xi) \ dt^2 - r^2(\xi) \sin^2 \vartheta \ d\varphi^2$$  \hspace{1cm} (26)

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$x^1 = \varphi, \ x^2 = \xi, \ x^3 = \vartheta, \ y^4 = t, \ y^5 = \varphi,$$

$$\hat{g}_1 = \epsilon_1 = \pm 1, \ \hat{g}_2 = -1, \ \hat{g}_3 = -r^2(\xi), \ \hat{h}_4 = \omega^2(\xi), \ \hat{h}_5 = -r^2(\xi) \sin^2 \vartheta.$$  \hspace{1cm} (27)

Such metrics can be used for constructing exact solutions in 4D gravity and Ricci flows with anisotropic dependence on time coordinate.

Finally, we consider

$$\delta s^2 = \epsilon_1 d\varphi^2 - d\xi^2 - d\vartheta^2 - 2\kappa(x,y,p) \ dp^2 + \ dv^2/8\kappa(x,y,p),$$  \hspace{1cm} (28)

where the local coordinates are

$$x^1 = \varphi, \ x^2 = x, \ x^3 = y, \ y^4 = p, \ y^5 = v,$$

and the nontrivial metric coefficients are parametrized

$$\hat{g}_1 = \epsilon_1 = \pm 1, \ \hat{g}_2 = -1, \ \hat{g}_3 = -1,$$

$$\hat{h}_4 = -2\kappa(x,y,p), \ \hat{h}_5 = 1/8 \kappa(x,y,p).$$  \hspace{1cm} (29)

The metric (28) is a trivial embedding into 5D of the vacuum solution of the Einstein equation defining pp–waves [38] for any $\kappa(x,y,p)$ solving

$$\kappa_{xx} + \kappa_{yy} = 0,$$
with \( p = z + t \) and \( v = z - t \), where \((x, y, z)\) are usual Cartesian coordinates and \( t \) is the time like coordinates. The simplest explicit examples of such solutions are

\[
\kappa = (x^2 - y^2) \sin p,
\]
defining a plane monochromatic wave, or

\[
\begin{align*}
\kappa &= \frac{xy}{(x^2 + y^2)^2} \exp \left[ p_0^2 - p^2 \right], \text{ for } |p| < p_0; \\
&= 0, \text{ for } |p| \geq p_0,
\end{align*}
\]
defining a wave packet travelling with unit velocity in the negative \( z \) direction.

3. Solitonic pp–Waves and String Torsion

Pp–wave solutions are intensively exploited for elaborating string models with nontrivial backgrounds [31, 32, 33]. A special interest for pp–waves in general relativity is related to the fact that any solution in this theory can be approximated by a pp–wave in vicinity of horizons. Such solutions can be generalized by introducing nonlinear interactions with solitonic waves [12, 34, 35, 36, 37] and nonzero sources with nonhomogeneous cosmological constant induced by an ansatz for the antisymmetric tensor fields of third rank, see Appendix. A very important property of such nonlinear wave solutions is that they possess nontrivial limits defining new classes of generic off–diagonal vacuum Einstein spacetimes and can be generalized for Ricci flows induced by evolutions of N–connections.

In this section, we use an ansatz of type (2),

\[
\begin{align*}
\delta s_{(5)}^2 &= \epsilon_1 \, dx^2 - e^{\psi(x,y,\chi)} \left( dx^2 + dy^2 \right)  \\
& \quad -2\kappa(x, y, p) \eta_4(x, y, p) \delta p^2 + \frac{\eta_5(x, y, p)}{8\kappa(x, y, p)} \delta v^2 \\
\delta p &= dp + w_2(x, y, p) dx + w_3(x, y, p) dy, \\
\delta v &= dv + n_2(x, y, p, \chi) dx + n_3(x, y, p, \chi) dy
\end{align*}
\]

where the local coordinates are

\[
x^1 = \kappa, \ x^2 = x, \ x^3 = y, \ y^4 = p, \ y^5 = v,
\]

and the nontrivial metric coefficients and polarizations are parametrized

\[
\begin{align*}
\tilde{g}_1 &= \epsilon_1 = \pm 1, \ \tilde{g}_2 = -1, \ \tilde{g}_3 = -1, \\
\tilde{h}_4 &= -2\kappa(x, y, p), \ \tilde{h}_5 = 1/8\kappa(x, y, p), \\
\eta_1 &= 1, \eta_\alpha = \eta_\alpha \eta_\alpha.
\end{align*}
\]

For trivial polarizations \( \eta_\alpha = 1 \) and \( w_{2,3} = 0, n_{2,3} = 0 \), the metric (30) is just the pp–wave solution (28).
3.1 Ricci flows of solitonic pp–wave solutions in string gravity

Our aim is to define such nontrivial values of polarization functions when \( \eta_5(x, y, p) \) is defined by a 3D soliton \( \eta(x, y, p) \), for instance, as a solution of solitonic equation

\[
\eta^{\bullet\bullet} + \epsilon(\eta^\prime + 6\eta \eta^\ast + \eta^{\bullet\bullet\ast})^\ast = 0, \ \epsilon = \pm 1,
\]

and \( \eta_2 = \eta_3 = e^{\psi(x,y,\chi)} \) is a family solutions of (12) transformed into

\[
\psi^{\bullet\bullet}(\chi) + \psi^\prime(\chi) = \frac{\lambda H}{2}.
\]

The solitonic deformations of the pp–wave metric will define exact solutions in string gravity with \( H \)-fields, see in Appendix the equations (A.3) and (A.4) for the string torsion ansatz (A.5), when with \( \lambda = \lambda_H \).

Introducing the above stated data for the ansatz (30) into the equation (13), we get two equations relating \( h_4 = \eta_4 \hat{h}_4 \) and \( h_5 = \eta_5 \hat{h}_5 \),

\[
\eta_5 = 8 \kappa(x, y, p) \left[ h_{5[0]}(x, y) + \frac{1}{2\lambda H} e^{2\eta(x,y,p)} \right]
\]

and

\[
|h_4| = \frac{e^{-2\phi(x,y,p)}}{2\kappa^2(x, y, p)} \left[ \left( \sqrt{\eta_5} \right)^\ast \right]^2,
\]

where \( h_{5[0]}(x, y) \) is an integration function. Having defined the coefficients \( h_a \), we can solve the equations (14) and (15) expressing the coefficients (16) and (17) through \( \eta_4 \) and \( \eta_5 \) defined by pp– and solitonic waves as in (34) and (33). The corresponding solutions are

\[
w_1 = 0, w_2 = (\phi^\ast)^{-1} \partial_x \phi, w_3 = (\phi^\ast)^{-1} \partial_x \phi,
\]

for \( \phi^\ast = \partial \phi / \partial p \), see formulas (19) and

\[
n_1 = 0, n_{2,3} = n_{2,3}[0](x, y, \chi) + n_{2,3}[1](x, y, \chi) \int \left| \eta_4 \eta_5^{-3/2} \right| dp,
\]

where \( n_{2,3}[0](x, y, \chi) \) and \( n_{2,3}[1](x, y, \chi) \) are integration functions, restricted to satisfy the conditions (11),

\[
\frac{\partial}{\partial \chi}[-e^{\psi(x,y,\chi)} + \eta_5(x, y, p) \hat{h}_5(x, y, p) (n_{2,3}[0](x, y, \chi) + n_{2,3}[1](x, y, \chi)]
\]

\[
+ |n_{2,3}[1](x, y, \chi) \int |n_4(x, y, p)\eta_5^{-3/2}(x, y, p)| dp|] = 0.
\]

We note that the ansatz (30), without dependence on \( \chi \) and with the coefficients computed following the equations and formulas (32), (34), (33), (35) and (36), defines a class

\[5\] as a matter of principle we can consider that \( \phi \) is a solution of any 3D solitonic, or other, nonlinear wave equation.

\[6\] such solutions can be constructed in general form (see, in details, the formulas (26)–(28) in Ref. [4], for corresponding reparametrizations)
of exact solutions (depending on integration functions) of gravitational field equations in string gravity with $H$–field. For corresponding families of coefficients evolving on $\chi$ and constrained to satisfy the conditions (37) we get solutions of nonholonomic Ricci flow equations (7)–(9) normalized by the effective constant $\lambda_H$ induced from string gravity.

Putting the above stated functions $\psi, k, \phi$ and $\eta_5$ and respective integration functions into the corresponding ansatz, we define a class of evolution and/or gravity field solutions,

$$\delta s^2_{\text{sol2}} = \epsilon_1 \, dx^2 - e^{\psi(x)} \left( dx^2 + dy^2 \right)$$

$$+ \frac{\eta_5}{8\kappa} \delta p^2 - \kappa^{-1} \, e^{-2\phi} \left[ \left( \sqrt{\eta_5} \right)^* \right]^2 \delta \nu^2(\chi),$$

$$\delta p = dp + (\phi^*)^{-1} \partial_x \phi \, dx + (\phi^*)^{-1} \partial_y \phi \, dy, \quad (38)$$

$$\delta v(\chi) = dv + \left\{ n_2^{[0]}(\chi) + \hat{n}_2^{[1]}(\chi) \int k^{-1} e^{2\phi} \left[ \left( |\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} \, dx$$

$$+ \left\{ n_3^{[0]}(\chi) + \hat{n}_3^{[1]}(\chi) \int k^{-1} e^{2\phi} \left[ \left( |\eta_5|^{-1/4} \right)^* \right]^2 dp \right\} \, dy,$$

where some constants and multiples depending on $x$ and $y$ are included into $\hat{n}_2, \hat{n}_3(x, y, \chi)$ and we emphasize the dependence of coefficients on Ricci flow parameter $\chi$. Such families of generic off–diagonal metrics posses induced both nonholonomically and from string gravity torsion coefficients for the canonical $d$–connection (we omit explicit formulas for the nontrivial components which can be computed by introducing the coefficients of our ansatz into (A.2)). This class of solutions describes nonlinear interactions of pp–waves and 3D solutions in string gravity in Ricci flow theory.

The term $\epsilon_1 \, dx^2$ can be eliminated in order to describe only 4D configurations. Nevertheless, in this case, there is not a smooth limit of such 4D solutions for $\lambda_H^2 \to 0$ to those in general relativity, see the second singular term in (33), proportional to $1/\lambda_H^2$.

Finally, note that explicit values for the integration functions and constants can be defined (for a fixed system of reference and coordinates) from certain initial value and boundary conditions. In this work, we shall analyze the properties of the derived classes of solutions and their multi–parametric transforms and geometric flows working with general forms of generation and integration functions.

3.2 Solitonic pp–waves in vacuum Einstein gravity and Ricci flows

In this section, we show how the anholonomic frame method can be used for constructing 4D metrics induced by nonlinear pp–waves and solitonic interactions for vanishing sources and the Levi Civita connection. For an ansatz of type (30), we write

$$\eta_5 = 5k b^2 \quad \text{and} \quad \eta_4 = h_0^2 (b^*)^2 / 2\kappa.$$  

A 3D solitonic solution can be generated if $b$ is subjected to the condition to solve a solitonic equation, for instance, of type (31), or other nonlinear wave configuration. We chose a parametrization when

$$b(x, y, p) = \tilde{b}(x, y) q(p) k(p),$$
for any $\tilde{b}(x, y)$ and any pp–wave $\kappa(x, y, p) = \tilde{\kappa}(x, y)k(p)$ (we can take $\tilde{b} = \tilde{\kappa}$), where $q(p) = 4 \tan^{-1} e^{\pm p}$ is the solution of “one dimensional” solitonic equation

$$q^{**} = \sin q.$$  \hspace{1cm} (39)

In this case,

$$w_2 = [\ln |qk|]^{-1} \partial_x \ln |\tilde{b}| \text{ and } w_3 = [\ln |qk|]^{-1} \partial_y \ln |\tilde{b}|.$$  \hspace{1cm} (40)

The final step in constructing such vacuum Einstein solutions is to chose any two functions $n_{2,3}(x, y)$ satisfying the conditions $n_2^* = n_3^* = 0$ and $n_2 - n_3^*$ = 0 which are necessary for Riemann foliated structures with the Levi Civita connection, see discussion of formulas (42) and (43) in Ref. [4] and conditions (18). This mean that in the integrals of type (36) we shall fix the integration functions $n_2^{[1]}(x, y) = 0$ but take such $n_2^{[0]}(x, y)$ satisfying $(n_2^{[0]})' - (n_3^{[0]}) = 0$.

We can consider a trivial solution of (12), i.e. of (32) with $\lambda = \lambda_H = 0$.

Summarizing the results, we obtain the 4D vacuum metric

$$\delta s_{[sol2a]}^2 = -(dx^2 + dy^2) - h_0^2 \tilde{b}^2((\chi)^2) \delta p^2 + \tilde{b}^2((\chi)k)^2 \delta v^2,$$

$$\delta p = dp + [(\ln |qk|)]^{-1} \partial_x \ln |\tilde{b}| \, dx + [(\ln |qk|)]^{-1} \partial_y \ln |\tilde{b}| \, dy,$$

$$\delta v = dv + n_2^{[0]}(\chi) \, dx + n_3^{[0]}(\chi) \, dy,$$

(41)

defining nonlinear gravitational interactions of a pp–wave $\kappa = \hat{\kappa}k$ and a soliton $q$, depending on certain type of integration functions and constants stated above. Such vacuum Einstein metrics can be generated in a similar form for 3D or 2D solitons but the constructions will be more cumbersome and for non–explicit functions, see a number of similar solutions in Refs. [12, 8].

Now, we generalize the ansatz (41) in a form describing normalized Ricci flows of the mentioned type vacuum solutions extended for a prescribed constant $\lambda$ necessary for normalization. We chose

$$\delta s_{[sol2a]}^2 = -(dx^2 + dy^2) - h_0^2 \tilde{b}^2((\chi)(qk)^2)^2 \delta p^2 + \tilde{b}^2((\chi)(qk)^2) \delta v^2,$$

$$\delta p = dp + [(\ln |qk|)]^{-1} \partial_x \ln |\tilde{b}| \, dx + [(\ln |qk|)]^{-1} \partial_y \ln |\tilde{b}| \, dy,$$

$$\delta v = dv + n_2^{[0]}(\chi) \, dx + n_3^{[0]}(\chi) \, dy,$$

(42)

where we introduced the parametric dependence on $\chi$,

$$b(x, y, p, \chi) = \tilde{b}(x, y, \chi)q(p)k(p)$$

which allows us to use the same formulas (40) for $w_{3,4}$ not depending on $\chi$. The values $\tilde{b}^2(\chi)$ and $n_2^{[0]}(\chi)$ are constrained to be solutions of

$$\frac{\partial}{\partial \chi} \left[ \tilde{b}^2(n_2^{[0]})^2 \right] = -2\lambda \text{ and } \frac{\partial}{\partial \chi} \tilde{b}^2 = 2\lambda \tilde{b}^2$$

(43)
in order to solve, respectively, the equations (7) and (8). As a matter of principle, we can consider a flow dependence as a factor $\psi(\lambda)$ before $(dx^2 + dy^2)$, i.e. flows of the
h–components of metrics which will generalize the ansatz (42) and constraints (43). For simplicity, we have chosen a minimal extension of vacuum Einstein solutions in order to describe nonholonomic flows of the v–components of metrics adapted to the flows of N–connection coefficients $n_{2,3}^0(\chi)$. Such nonholonomic constraints on metric coefficients define Ricci flows of families of vacuum Einstein solutions defined by nonlinear interactions of a 3D soliton and a pp–wave.


There are different possibilities to apply parametric and frame transforms and define Ricci flows and nonholonomic deformations of geometric objects. The first one is to perform a parametric transform of a vacuum solution and then to deform it nonholonomically in order to generate pp–wave solitonic interactions. In the second case, we can subject an already nonholonomically generated solution of type (41) to a one parameter transforms. Finally, in the third case, we can derive two parameter families of nonholonomic soliton pp–wave interactions. For simplicity, Ricci flows will be considered after certain classes of exact solutions of field equations will have been constructed.

4.1 Flows of solitonic pp–waves generated

by parametric transforms

Let us consider the metric

$$\delta s^2_{[5a]} = -dx^2 - dy^2 - 2\kappa(x, y) \, dp^2 + \frac{dv^2}{8\kappa(x, y)}$$

which is a particular 4D case of (28) when $\kappa(x, y, p) \rightarrow \kappa(x, y)$. It is easy to show that the nontrivial Ricci components $R_{\alpha\beta}$ for the Levi Civita connection are proportional to $\kappa^{\bullet\bullet} + \kappa''$ and the non–vanishing components of the curvature tensor $R_{\alpha\beta\gamma\delta}$ are of type $R_{\alpha_1\beta_1} \simeq R_{\alpha_2\beta_2} \simeq \sqrt{(\kappa^{\bullet\bullet})^2 + (\kappa^\bullet)^2}$. So, any function $\tilde{\kappa}$ solving the equation $\tilde{\kappa}^{\bullet\bullet} + \tilde{\kappa}'' = 0$ but with $(\kappa^{\bullet\bullet})^2 + (\kappa^\bullet)^2 \neq 0$ defines a vacuum solution of the Einstein equations. In the simplest case, we can take $\kappa = x^2 - y^2$ or $\kappa = xy/\sqrt{x^2 + y^2}$ like it was suggested in the original work [38], but for the metric (44) we do not consider any multiple $q(p)$ depending on $p$.

Subjecting the metric (44) to a parametric transform, we get an off–diagonal metric of type

$$\delta s^2_{[2p]} = -\eta_2(x, y, \theta)dx^2 + \eta_3(x, y, \theta)dy^2$$

$$-2\tilde{\kappa}(x, y) \eta_4(x, y, \theta, \chi)dp^2 + \frac{\eta_5(x, y, \theta, \chi)}{8\tilde{\kappa}(x, y)} \delta v^2,$$

$$\delta p = dp + w_2(x, y, \theta)dx + w_3(x, y, \theta)dy,$$

$$\delta v = dv + n_2(x, y, \theta, \chi)dx + n_3(x, y, \theta, \chi)dy.$$
which may define Ricci flows, or vacuum solutions of the Einstein equations, if the coefficients are restricted to satisfy the necessary conditions. Such parametric transforms consist a particular case of frame transforms when the coefficients $g_{a\beta}$ are defined by the coefficients of (44) and $\tilde{g}_{a\beta}$ are given by the coefficients (45). The polarizations $\eta_{5}(x, y, \theta, \chi)$ and N–connection coefficients $w_{5}(x, y, \theta)$ and $n_{1}(x, y, \theta, \chi)$ determine the coefficients of the matrix of parametric, or Geroch, transforms (for details on nonholonomic generalizations and Geroch equations, see section 4.2 and Appendix B in Ref. [4] and sections 2.2 and 3 in Ref. [7]; we note that in this work we have an additional to $\theta$ Ricci flow parameter $\chi$).

Considering that $\eta_{2} \neq 0$, we multiply (45) on conformal factor $(\eta_{2})^{-1}$ and redefining the coefficients as $\tilde{\eta}_{3} = \eta_{3}/\eta_{2}$, $\tilde{\eta}_{a} = \eta_{a}/\eta_{2}$, $\tilde{w}_{a} = w_{a}$ and $\tilde{n}_{a} = n_{a}$, for $i = 2, 3$ and $a = 4, 5$, we obtain

$$\delta s_{[2\alpha]}^{2} = -dx^{2} + \tilde{\eta}_{3}(x, y, \theta, \chi)dy^{2}$$

$$-2\kappa(x, y) \tilde{\eta}_{4}(x, y, \theta, \chi)\delta p^{2} + \frac{\tilde{\eta}_{5}(x, y, \theta, \chi)}{8\kappa(x, y)} \delta v^{2}$$

$$\delta p = dp + \tilde{w}_{2}(x, y, \theta)dx + \tilde{w}_{3}(x, y, \theta)dy,$$

$$\delta v = dv + \tilde{n}_{2}(x, y, \theta, \chi)dx + \tilde{n}_{3}(x, y, \theta, \chi)dy,$$

which is not an exact solution but can be nonholonomically deformed into exact vacuum solutions by multiplying on additional polarization parameters. Firstly, we first introduce the polarization $\eta_{2} = \exp\psi(x, y, \theta, \chi)$ when $\eta_{3} = \tilde{\eta}_{3} = -\exp\psi(x, y, \theta, \chi)$ are defined as families of solutions of $\psi^{*}\chi + \psi''(\chi) = \lambda$. Then, secondly, we redefine $\tilde{\eta}_{a} \rightarrow \eta_{a}(x, y, p, \chi)$ (for instance, multiplying on additional multiples) by introducing additional dependencies on "anisotropic" coordinate $p$ such a way when the ansatz (46) transform into

$$\delta s_{[2\alpha]}^{2} = -e^{\psi(x,y,\theta,\chi)} (dx^{2} + dy^{2})$$

$$-2\kappa(x,y)k(p) \eta_{4}(x, y, p, \theta)\delta p^{2} + \frac{\eta_{5}(x, y, p, \theta)}{8\kappa(x, y)k(p)} \delta v^{2}$$

$$\delta p = dp + w_{2}(x, y, p, \theta)dx + w_{3}(x, y, p, \theta)dy,$$

$$\delta v = dv + n_{2}(x, y, \theta, \chi)dx + n_{3}(x, y, \theta, \chi)dy.$$  

The "simplest" Ricci flow solutions induced by flows of the h–metric and N–connection coefficients are

$$w_{1} = 0, w_{2}(\theta) = (\phi^{*})^{-1} \partial_{\phi} \phi, w_{3}(\theta) = (\phi^{*})^{-1} \partial_{\phi} \phi,$$

for $\phi^{*} = \partial \phi/\partial p$, see formulas (19), and

$$n_{2,3} = n^{[0]}_{2,3}(x, y, \theta, \chi) + n^{[1]}_{2,3}(x, y, \theta, \chi) \int |\eta_{4}(\theta)\eta_{5}^{-3/2}(\theta)| dp,$$

where $n^{[0]}_{2,3}(x, y, \theta, \chi)$ and $n^{[1]}_{2,3}(x, y, \theta, \chi)$ are constrained as (11),

$$\frac{\partial}{\partial \chi} [-e^{\psi(x,y,\theta,\chi)} + \eta_{5}(x, y, p, \theta) \tilde{h}_{5}(x, y, p) \eta_{5}^{-[0]}(x, y, \theta, \chi)$$

$$+ n^{[1]}_{2,3}(x, y, \theta, \chi) \int |\eta_{4}(x, y, p, \theta)\eta_{5}^{-3/2}(x, y, p, \theta)| dp]^{2} = 0.$$  

$^7$ $\eta_{2} \rightarrow 1$ and $\eta_{3} \rightarrow 1$ for infinitesimal parametric transforms
The difference of formulas (48), (49) and (50) and respective formulas (35), (36) and (37) is that the set of coefficients defining the nonholonomic Ricci flow of metrics (47) depend on a free parameter \( \theta \) associated to some ‘primary’ Killing symmetries like it was considered by Geroch [28]. The analogy with Geroch’s (parametric) transforms is more complete if we do not consider dependencies on \( \chi \) and take the limit \( \lambda \to 0 \) which generates families, on \( \theta \), of vacuum Einstein solutions, see formula (105) in Ref. [7].

In order to define Ricci flows for the Levi Civita connection, with \( g_4 = -2\bar{\kappa}k\eta_4 \) and \( g_5 = \eta_5/8\bar{\kappa}k \) from (47), the coefficients of this metric must solve the conditions (18), when the coordinates are parametrized \( x^2 = x, x^3 = y, y^4 = p \) and \( y^5 = v \). This describes both parametric nonholonomic transform and Ricci flows of a metric (44) to a family of evolution/field exact solutions depending on parameter \( \theta \) and defining nonlinear superpositions of pp–waves \( \kappa = \bar{\kappa}(x,y)k(p) \).

It is possible to introduce solitonic waves into the metric (47). For instance, we can take \( \eta_5(x,y,p,\theta) \sim q(p) \), where \( q(p) \) is a solution of solitonic equation (39). We obtain nonholonomic Ricci flows of a family of Einstein metrics labelled by parameter \( \theta \) and defining nonlinear interactions of pp–waves and one–dimensional solitons. Such solutions with prescribed \( \psi = 0 \) can be parametrized in a form very similar to the ansatz (41).

4.2 Parametrized transforms and flows of nonholonomic solitonic pp–waves

We begin with the ansatz (41) defining a vacuum off–diagonal solution. That metric does not depend on variable \( v \) and possess a Killing vector \( \partial/\partial v \). It is possible to apply the parametric transform writing the new family of metrics in terms of polarization functions,

\[
\delta s^2_{\text{sol229}} = -\eta_2(\theta') \, dx^2 + \eta_3(\theta') \, dy^2 - \eta_4(\theta') \, h_0^2 \bar{b}^2 [(qk)^*]^2 \delta p^2 \\
+ \eta_5(\theta') \, \bar{b}^2 (qk)^2 \delta v^2,
\]

\[
\delta p = dp + \eta_2^4(\theta') \cdot [(\ln |qk|)^*]^{-1} \, \partial_x \ln |\bar{b}| \, dx \\
+ \eta_3^4(\theta') \cdot [(\ln |qk|)^*]^{-1} \, \partial_y \ln |\bar{b}| \, dy,
\]

\[
\delta v = dv + \eta_2^5(\theta') n_2^0 \, dx + \eta_3^5(\theta') n_3^0 \, dy,
\]

where all polarization functions \( \eta_5(x,y,p,\theta') \) and \( \eta_4^0(x,y,p,\theta') \) depend on anisotropic coordinate \( p \), labelled by a parameter \( \theta' \). The new class of solutions contains the multiples \( q(p) \) and \( k(p) \) defined respectively by solitonic and pp–waves and depends on certain integration functions like \( n_2^0(x,y) \) and integration constant \( h_0^2 \). Such values can defined exactly by stating an explicit coordinate system and for certain boundary and initial conditions.

It should be noted that the metric (51) can not be represented in a form typical for nonholonomic frame vacuum ansatz for the Levi Civita connection (i.e. it can not be represented, for instance, in the form (78) with the coefficients satisfying the conditions (79) in Ref. [7]).
frame method. Nevertheless, such classes of metrics define exact vacuum solutions as a consequence of the Geroch method (for nonholonomic manifolds, we call it also the method of parametric transforms). This is the priority to consider together both methods: we can parametrize different type of transforms by polarization functions in a unified form and in different cases such polarizations will be subjected to corresponding type of constraints, generating anholonomic deformations or parametric transforms.

Nevertheless, we can generate nonholonomic Ricci flows solutions from the very beginning, considering such flows, at the first step of transforms, for the metric (41), prescribing a constant \( \lambda \) necessary for normalization, which result in (42) and at the second step to apply the parametric transforms. After first step, we get an ansatz

\[
\delta s^2_{\text{sol2a,\chi}} = -\left[\eta_2(\chi)dx^2 + \eta_3(\chi)dy^2\right] - h_0^2\tilde{b}^2(\chi) [(qk)^*]^2 \delta p^2 + \tilde{b}^2(\chi)(qk)^2 \delta v^2,
\]

\[\delta p = dp + [(\ln |qk|)^*]^{-1} \partial_x \ln |\tilde{b}(\chi)| \, dx
+ [(\ln |qk|)^*]^{-1} \partial_y \ln |\tilde{b}(\chi)| \, dy,
\]

\[\delta v = dv + n_2^0(\chi)dx + n_3^0(\chi)dy,
\]

which allows us to use the same formulas (40) for \( w_{3,4} \) not depending on \( \chi \). The values \( \tilde{b}^2(\chi) \) and \( n_2^0(\chi) \) are constrained to be solutions of

\[
\frac{\partial}{\partial \chi} \left[ -\eta_{2,3}(x, y, \chi) + \tilde{b}^2(x, y, \chi)(q(p)k(p))^2 n_{2,3}^0(x, y, \chi)^2 \right] = 0,
\]

obtained by introducing (52) into (7). The second step is to introduce polarizations functions \( \eta_3(x, y, p, \theta') \) and \( n_2^0(x, y, p, \theta') \) for a parametric transform, which is possible because we have a Killing symmetry on \( \partial/\partial v \) and (52) is an Einstein metric (we have to suppose that such parametric transforms can be defined as solutions of the Geroch equations [7, 28] for the Einstein spaces, not only for vacuum metrics, at least for small prescribed cosmological constants). Finally, we get a two parameter metric, on \( \theta' \) and \( \chi \),

\[
\delta s^2_{\text{sol2a,\chi}} = -\left[\eta_2(\theta')dx^2 + \eta_3(\theta')dy^2\right] - \eta_4(\theta')h_0^2\tilde{b}^2(\chi) [(qk)^*]^2 \delta p^2 + \eta_5(\theta') \tilde{b}^2(\chi)(qk)^2 \delta v^2,
\]

\[\delta p = dp + \eta_2^0(\theta') [(\ln |qk|)^*]^{-1} \partial_x \ln |\tilde{b}| \, dx
+ \eta_3^0(\theta') [(\ln |qk|)^*]^{-1} \partial_y \ln |\tilde{b}| \, dy,
\]

\[\delta v = dv + \eta_2^0(\theta')n_2^0(\chi)dx + \eta_3^0(\theta')n_3^0(\chi)dy,
\]

with the nonholonomic Ricci flow evolution equations

\[
\frac{\partial}{\partial \chi} \{ -\eta_{2,3}(x, y, p, \chi, \theta') + \eta_3(x, y, p, \theta')\tilde{b}^2(x, y, \chi)(q(p)k(p))^2 \left[ \eta_3^0(x, y, p, \theta')n_{2,3}^0(x, y, \chi) \right]^2 \} = 0,
\]
to which one reduces the equation (7) by introducing (52). Such parametric nonholonomic Ricci flows can be constrained for the Levi Civita connection if we consider coefficients satisfying certain conditions equivalent to (18) and (19), imposed for the coefficients of auxiliary metric (52), when

\[ \eta_2(x, y, \chi) = g_3(x, y, \chi) = -e^{\psi(x,y,\chi)}, \]
\[ h_4 = -h_0^2b^2(\chi)\right[(gk)\right]^2, h_5 = b^2(\chi)(gk)^2, \]
\[ w_{\chi} = \partial_t\phi/\phi^*, \text{ where } \varphi = -\ln \left[|h_4h_5|^{|h_5^*|} \right], \]
\[ n_{2,3}(\chi) = n_{2,3}^{[0]}(x, y, \chi) \]

are constrained to

\[ \psi^{**}(\chi) + \psi''(\chi) = -\lambda \]
\[ h_5^*\phi/h_4h_5 = \lambda, \]
\[ w_2 - w_3^* + w_3w_3^* - w_2w_3^* = 0, \]
\[ n_2^*(\chi) - n_3^*(\chi) = 0. \]

In a more general case, we can model Killing—Ricci flows for the canonical d–connections.

4.3 Two parameter nonholonomic solitonic pp–waves and flows

Finally, we give an explicit example of solutions with two parameter ($\theta', \theta$)–metrics (see definition of such frame transform by formulas (81) in Ref. [7] and (69) in Ref. [4]). We begin with the ansatz metric $\tilde{g}$ (47) with the coefficients subjected to constraints (18) for $\lambda \to 0$ and coordinates parametrized $x^2 = x, x^3 = y, y^4 = p$ and $y^5 = v$. We consider that the solitonic wave $\phi$ is included as a multiple in $\eta_5$ and that $\kappa = \kappa(x,y)k(p)$ is a pp–wave. This family of vacuum metrics $\tilde{g}$ does not depend on variable $v$, i.e. it possess a Killing vector $\partial/\partial v$, which allows us to apply a parametric transform as we described in the previous example. The resulting two parameter family of solutions, with redefined polarization functions, is given by the ansatz

\[ \delta s_{[20]}^2 = -e^{\psi(x,y,\theta)} \left( \eta_2(x, y, p, \theta')dx^2 + \eta_3(x, y, p, \theta')dy^2 \right) \]
\[ -2\kappa(x,y)k(p)\eta_4(x, y, p, \theta)\eta_4(x, y, p, \theta')\delta p^2 \]
\[ + \frac{\eta_5(x, y, p, \theta)\eta_5(x, y, p, \theta')}{8\kappa(x,y)k(p)}\delta v^2 \]
\[ \delta p = dp + w_2(x, y, p, \theta)\eta_3(x, y, p, \theta')dx + w_3(x, y, p, \theta)\eta_3(x, y, p, \theta')dy, \]
\[ \delta v = dv + n_2(x, y, \theta)\eta_3(x, y, \theta)\eta_3(x, y, p, \theta')dx + n_3(x, y, \theta)\eta_3(x, y, p, \theta')dy. \]

The set of multiples in the coefficients are parametrized following the conditions: The value $\kappa(x,y)$ is just that defining an exact vacuum solution for the primary metric (44) stating the first type of parametric transforms. Then we consider the pp–wave component $k(p)$ and the solitonic wave included in $\eta_5(x, y, p, \theta)$ such way that the functions $\psi, \eta_4, w_2, 3$ and $n_2, 3$ are subjected to the condition to define the class of metrics (47).
The metrics are parametrized both by $\theta$, following solutions of the Geroch equations (see, for instance, the Killing (8) and (9) in Ref. [4]), and by a $N$–connection splitting with $w_{2,3}$ and $n_{2,3}$, all adapted to the corresponding nonholonomic deformation derived for $g_2(\theta') = g_3(\theta) = e^{\psi(\theta)}$ and $g_4 = 2\kappa k \eta_4$ and $g_5 = \eta_5$ subjected to the conditions (18)). This set of functions also defines a new set of Killing equations, for any metric (47) allowing to find the “overlined” polarizations $\overline{\eta}_2(\theta')$ and $\overline{\eta}_3(\theta')$.

For compatible nonholonomic Ricci flows of the $h$–metric and $N$–connection coefficients, the class of two parametric vacuum solutions can be extended for a prescribed value $\lambda$ and new parameter $\chi$,

$$\delta s^2_{[2\alpha,\chi]} = -e^{\psi(x,y,\theta,\chi)}(\overline{\eta}_2(x,y,p,\theta',\chi)dx^2 + \overline{\eta}_3(x,y,p,\theta',\chi)dy^2)$$

$$-2\kappa(x,y)k(p)\eta_4(x,y,p,\theta)\overline{\eta}_4(x,y,p,\theta')\delta p^2$$

$$+ \frac{\eta_5(x,y,p,\theta)\overline{\eta}_5(x,y,p,\theta')}{8\kappa(x,y)k(p)}\delta v^2$$

$$\delta p = dp + w_2(x,y,p,\theta)\overline{\eta}_2(x,y,p,\theta')dx$$

$$+ w_3(x,y,p,\theta)\overline{\eta}_3(x,y,p,\theta')dy,$$

$$\delta v = dv + n_2(x,y,\theta,\chi)\overline{\eta}_2(x,y,p,\theta')dx$$

$$+ n_3(x,y,\theta,\chi)\overline{\eta}_3(x,y,p,\theta')dy,$$

where, for simplicity, we redefine the coefficients in the form

$$\overline{\eta}_2(x,y,p,\theta',\theta,\chi) = e^{\psi(x,y,\theta,\chi)}\overline{\eta}_2(x,y,p,\theta',\chi) =$$

$$\overline{\eta}_3(x,y,p,\theta',\theta,\chi) = e^{\psi(x,y,\theta,\chi)}\overline{\eta}_3(x,y,p,\theta',\chi),$$

$$\overline{\eta}_4(x,y,p,\theta',\theta) = \eta_4(x,y,p,\theta)\overline{\eta}_4(x,y,p,\theta'),$$

$$\overline{\eta}_5(x,y,p,\theta',\theta) = \eta_5(x,y,p,\theta)\overline{\eta}_5(x,y,p,\theta'),$$

$$\overline{w}_{2,3}(x,y,p,\theta',\theta) = w_{2,3}(x,y,p,\theta)\overline{\eta}_2(x,y,p,\theta'),$$

$$\overline{n}_{2,3}(x,y,p,\theta',\theta,\chi) = n_{2,3}(x,y,\theta,\chi)\overline{\eta}_2(x,y,p,\theta').$$

Such polarization functions, in general, parametrized by $(\theta', \theta, \chi)$, allow us to write the conditions (7) for the classes of metrics (55) in a compact form,

$$\frac{\partial}{\partial \chi}\{-\overline{\eta}_{2,3}(\theta',\theta,\chi) + \frac{\overline{\eta}_5(\theta',\theta)}{8\kappa k} [\overline{n}_{2,3}(\theta',\theta,\chi)]^2\} = 0,$$

where, for simplicity, we emphasized only the parametric dependencies.

For the configurations with Levi Civita connections, we have to consider additional constraints of type (18),

$$\overline{\eta}_2^\ast(\chi) + \overline{\eta}_2^\prime(\chi) = -\lambda$$

$$h_5^\ast \phi / h_4 h_5 = \lambda,$$

$$\overline{w}_2^\prime - \overline{w}_3^\ast + \overline{w}_3 \overline{w}_2^\prime - \overline{w}_2 \overline{w}_3^\ast = 0,$$

$$n_3^\prime(x,y,\theta',\theta,\chi) = n_3^\prime(x,y,\theta',\theta,\chi) = 0.$$
The classes of vacuum Einstein metrics (54) and their, normalized by a prescribed $\lambda$, nonholonomic Ricci flows (55), depend on certain classes of general functions (nonholonomic and parametric transform polarizations and integration functions). It is obvious that they define two parameter $(\theta', \theta)$ nonlinear superpositions of solitonic waves and pp–waves evolving on parameter $\chi$. From formal point of view, the procedure can be iterated for any finite or infinite number of $\theta$–parameters not depending on coordinates (in principle, such parameters can depend on flow parameter, but we omit such constructions in this work). We can construct an infinite number of nonholonomic vacuum states in gravity, and their possible Ricci flows, constructed from off–diagonal superpositions of nonlinear waves. Such two transforms do not commute and depend on order of successive applications.

The nonholonomic deformations not only mix and relate nonlinearly two different ”Killing” classes of solutions but introduce into the formalism the possibility to consider flow evolution configurations and other new very important and crucial properties. For instance, the polarization functions can be chosen such ways that the vacuum solutions will posses noncommutative and/algebroid symmetries even at classical level, or generalized configurations in order to contain contributions of torsion, nonmetricity and/or string fields in various generalized models of Lagrange–Hamilton algebroids, string, brane, gauge, metric–affine and Finsler–Lagrange gravity, see Refs. [10, 9, 8, 39].

5. Ricci Flows and Parametric Nonholonomic Deformations of the Schwarzschild Metric

We construct new classes of exact solutions for Ricci flows and nonholonomic deformations of the Schwarzschild metric. There are analyzed physical effects of parametrized families of generic off–diagonal flows and interactions with solitonic pp–waves.

5.1 Deformations and flows of stationary backgrounds

Following the methods outlined Refs. [4, 7], we nonholonomically deform on angular variable $\varphi$ the Schwarzschild type solution (20) into a generic off–diagonal stationary metric. For nonholonomic Einstein spaces, we shall use an ansatz of type

$$
\delta s^2_{[1]} = \epsilon_1 d\varphi^2 - \eta_2(\xi) d\xi^2 - \eta_3(\xi) r^2(\xi) d\vartheta^2 - \eta_4(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \, d\varphi^2 + \eta_5(\xi, \vartheta, \varphi) w^2(\xi) \, dt^2,
$$

$$
\delta \varphi = d\varphi + w_2(\xi, \vartheta, \varphi) d\xi + w_3(\xi, \vartheta, \varphi) d\vartheta,
$$

$$
\delta t = dt + n_2(\xi, \vartheta) d\xi + n_3(\xi, \vartheta) d\vartheta,
$$

for

$$
\tilde{h}_4 = -2\kappa(x, y)k(p)\tilde{\eta}_4(x, y, p, \theta', \theta), \quad \tilde{h}_5 = \tilde{\eta}_5(x, y, p, \theta') / 8\kappa(x, y)k(p),
$$

$$
\tilde{w}_i = \partial_i \tilde{\varphi} / \tilde{\varphi}^* \quad \text{where} \quad \tilde{\varphi} = - \ln \left| \sqrt{|\tilde{h}_4|} |\tilde{h}_5| \right|.
$$
where we shall use 3D spacial spherical coordinates, \((\xi, \vartheta, \varphi)\) or \((r, \vartheta, \varphi)\). The nonholonomic transform generating this off–diagonal metric are defined by \(g_i = \eta_i \hat{g}_i\) and \(h_a = \eta_a \hat{h}_a\) where \((\hat{g}_i, \hat{h}_a)\) are given by data (21).

5.1.1 General nonholonomic polarizations

We can construct a class of metrics of type (2) with the coefficients subjected to the conditions (18) (in this case, for the ansatz (56) with coordinates \(x^2 = \xi, x^3 = \vartheta, y^4 = \varphi, y^5 = t\)). The solution of (13), for \(\lambda = 0\), in terms of polarization functions, can be written

\[
\sqrt{|\eta_4|} = h_0 \sqrt{|\hat{h}_5|} \left(\sqrt{|\eta_5|}\right)^* ,
\]

where \(\hat{h}_a\) are coefficients stated by the Schwarzschild solution for the chosen system of coordinates but \(\eta_5\) can be any function satisfying the condition \(\eta_5^* \neq 0\). We shall use certain parametrizations of solutions when

\[
-h_0^2(b^*)^2 = \eta_4(\xi, \vartheta, \varphi)r^2(\xi) \sin^2 \vartheta
\]

\[
b^2 = \eta_5(\xi, \vartheta, \varphi) \varpi^2(\xi)
\]

The polarizations \(\eta_2\) and \(\eta_3\) can be taken in a form that \(\eta_2 = \eta_3 r^2 = e^{\psi(\xi, \vartheta, \chi)}, \psi^{**} + \psi'' = 0\), defining solutions of (12) for \(\lambda = 0\). The solutions of (14) and (15) for vacuum configurations of the Levi Civita connection are constructed

\[
w_2 = \partial_\xi(\sqrt{|\eta_5|} \varpi) / (\sqrt{|\eta_5|})^* \varpi, \ w_3 = \partial_\vartheta(\sqrt{|\eta_5|}) / (\sqrt{|\eta_5|})^* \varpi
\]

and any \(n_{2,3}(\xi, \vartheta)\) for which \(n_2^*(\chi) - n_3^*(\chi) = 0\). For any function \(\eta_5 \sim a_1(\xi, \vartheta)a_2(\varphi)\), the integrability conditions (18) and (19).

We conclude that the stationary nonholonomic deformations of the Schwarzschild metric are defined by the off–diagonal ansatz

\[
\delta s^2_{[1]} = \epsilon_1 d\chi^2 - e^\psi (d\xi^2 + d\vartheta^2)
\]

\[
-h_0^2 \varpi^2 \left(\sqrt{|\eta_5|}\right)^* \varpi \delta \varphi^2 + \eta_5 \varpi^2 \delta t^2 ,
\]

\[
\delta \varphi = d\varphi + \partial_\xi(\sqrt{|\eta_5|} \varpi)d\xi + \partial_\vartheta(\sqrt{|\eta_5|})^* d\vartheta ,
\]

\[
\delta t = dt + n_2 d\xi + n_3 d\vartheta,
\]

where the coefficients do not depend on Ricci flow parameter \(\lambda\). Such vacuum solutions were constructed mapping a static black hole solution into Einstein spaces with locally anistoropic backgrounds (on coordinate \(\varphi\)) defined by an arbitrary function \(\eta_5(\xi, \vartheta, \varphi)\) with \(\partial_\varphi \eta_5 \neq 0\), an arbitrary \(\psi(\xi, \vartheta)\) solving the 2D Laplace equation and certain integration functions \(n_{2,3}(\xi, \vartheta)\) and integration constant \(h_0^2\). In general, the solutions from the
target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient $\varpi^2$ vanishing on the horizon of a Schwarzschild black hole. We can also consider a prescribed physical situation when, for instance, $\eta_5$ mimics 3D, or 2D, solitonic polarizations on coordinates $\xi, \vartheta, \varphi$, or on $\xi, \varphi$.

For a family of metrics (58), we can consider the "nearest" extension to flows of N–connection coefficients $w_{2,3} \to w_{2,3}(\chi)$ and $n_{2,3} \to n_{2,3}(\chi)$, when for $\lambda = 0$, and $R_{\alpha\beta} = 0$, the equation (7) is satisfied if

$$h_0^2 \left[ \left( \sqrt{|\eta_5|} \right)^2 \frac{\partial (w_{2,3})^2}{\partial \chi} \right] = \eta_5 \frac{\partial (n_{2,3})^2}{\partial \chi}. \tag{59}$$

The metric coefficients for such Ricci flows are the same as for the exact vacuum nonholonomic deformation but with respect to evolving N–adapted dual basis

$$\delta \varphi(\chi) = d\varphi + w_2(\xi, \vartheta, \varphi, \chi) d\xi + w_3(\xi, \vartheta, \varphi, \chi) d\vartheta, \tag{60}$$

$$\delta t = dt + n_2(\xi, \varphi, \chi) d\xi + n_3(\xi, \varphi, \chi) d\vartheta,$$

with the coefficients being defined by any solution of (59) and (18) and (19) for $\lambda = 0$.

5.1.2 Solutions with small nonholonomic polarizations

In a more special case, in order to select physically valuable configurations, it is better to consider decompositions on a small parameter $0 < \varepsilon < 1$ in (58), when

$$\sqrt{|\eta_4|} = q_4^0(\xi, \varphi, \vartheta) + \varepsilon q_4^1(\xi, \varphi, \vartheta) + \varepsilon^2 q_4^2(\xi, \varphi, \vartheta),$$

$$\sqrt{|\eta_5|} = 1 + \varepsilon q_5^1(\xi, \varphi, \vartheta) + \varepsilon^2 q_5^2(\xi, \varphi, \vartheta),$$

where the "hat" indices label the coefficients multiplied to $\varepsilon, \varepsilon^2, ...$ The conditions (57) are expressed in the form

$$\varepsilon h_0 \sqrt{\frac{|h_5|}{h_4}} \left( q_5^1 \right)^* = q_4^0, \quad \varepsilon^2 h_0 \sqrt{\frac{|h_5|}{h_4}} \left( q_5^2 \right)^* = q_4^1, \ldots$$

This system can be solved in a form compatible with small decompositions if we take the integration constant, for instance, to satisfy the condition $\varepsilon h_0 = 1$ (choosing a corresponding system of coordinates). For this class of small deformations, we can prescribe a function $q_4^0$ and define $q_4^1$, integrating on $\varphi$ (or inversely, prescribing $q_4^1$, then taking the partial derivative $\partial_\varphi$, to compute $q_4^0$). In a similar form, there are related the coefficients $q_4^1$ and $q_5^2$. A very important physical situation is to select the conditions when such small nonholonomic deformations define rotoid configurations. This is possible, for instance, if

$$2q_5^1 = \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0) - \frac{1}{r^2}. \tag{61}$$

Of course, this way we construct not an exact solution, but extract from a class of exact ones (with less clear physical meaning) certain subclasses of solutions decomposed (deformed) on a small parameter being related to the Schwarzschild metric.
where \( \omega_0 \) and \( \varphi_0 \) are constants and the function \( q_0(r) \) has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before \( \delta t^2 \) is approximated in the form

\[
\eta_5 \varpi^2 = 1 - \frac{2\mu}{r} + \varepsilon\left(\frac{1}{r^2} + 2q_5^i\right).
\]

This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into a ellipsoidal one (rotoid configuration) given by

\[
r_+ \simeq \frac{2\mu}{1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0)}.
\]

Such solutions with ellipsoid symmetry seem to define static black ellipsoids (they were investigated in details in Refs. \([40, 41]\)). The ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. This class of vacuum metrics violates the conditions of black hole uniqueness theorems \([30]\) because the "surface" gravity is not constant for stationary black ellipsoid deformations. So, we can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distortions related to generic off–diagonal metric terms. Putting \( \varphi_0 = 0 \), in the limit \( \omega_0 \to 0 \), we get \( q_5^i \to 0 \) in (61). This allows to state the limits \( q_4^0 \to 1 \) for \( \varepsilon \to 0 \) in order to have a smooth limit to the Schwarzschild solution for \( \varepsilon \to 0 \). Here, one must be emphasized that to extract the spherical static black hole solution is possible if we parametrize, for instance,

\[
\delta \varphi = d\varphi + \varepsilon \frac{\partial \xi}{\sqrt{|\eta_5|}} \frac{d\xi}{\varpi} + \varepsilon \frac{\partial \vartheta}{\sqrt{|\eta_5|}} d\vartheta
\]

and

\[
\delta t = dt + \varepsilon n_2(\xi, \vartheta) d\xi + \varepsilon n_3(\xi, \vartheta) d\vartheta.
\]

One can be defined certain more special cases when \( q_5^2 \) and \( q_4^1 \) (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

For Ricci flows on N–connection coefficients, such stationary rotoid configurations evolve with respect to small deformations of co–frames (60), \( \delta \varphi(\chi) \) and \( \delta t(\chi) \), with the coefficients proportional to \( \varepsilon \).

5.1.3 Parametric nonholonomic transforms of the Schwarzschild solution and their flows

The ansatz (58) does not depend on time variable and posses a Killing vector \( \partial / \partial t \). We can apply parametric transforms and generate families of new solutions depending on a
parameter \( \theta \). Following the same steps as for generating (51), we construct

\[
\delta s_{[1]}^2 = -e^{\psi} (\tilde{\eta}_2(\theta) d\xi^2 + \tilde{\eta}_3(\theta) d\bar{\theta}^2) - h_0 \omega^2 \left[ \left( \sqrt{|\eta_5|} \right)^2 \tilde{\eta}_4(\theta) \delta \varphi^2 + \eta_5 \omega^2 \tilde{\eta}_5(\theta) \delta t^2, \right.
\]

\[
\delta \varphi = d\varphi + \tilde{\eta}_2^i(\theta) \frac{\partial_\xi (\sqrt{|\eta_5|} \omega)}{\sqrt{|\eta_5|} \omega} d\xi + \tilde{\eta}_3^i(\theta) \frac{\partial_\theta (\sqrt{|\eta_5|} \omega)}{\sqrt{|\eta_5|} \omega} d\vartheta,
\]

\[
\delta t = dt + \tilde{\eta}_2^5(\theta) n_2(\xi, \theta) d\xi + \tilde{\eta}_3^5(\theta) n_3(\xi, \theta) d\vartheta,
\]

where polarizations \( \tilde{\eta}_i(\xi, \vartheta, \varphi, \theta) \) and \( \tilde{\eta}^i_2(\xi, \vartheta, \varphi, \theta) \) are defined by solutions of the Geroch equations for Killing symmetries of the vacuum metric (58). Even this class of metrics does not satisfy the vacuum equations for a typical anholonomic ansatz, they define vacuum exact solutions and we can apply the formalism on decomposition on a small parameter \( \varepsilon \) like we described in previous section 5.1.2 (one generates not exact solutions, but like in quantum field theory it can be more easy to formulate a physical interpretation). For instance, we consider a vacuum background consisting from solitonic wave polarizations, with components mixed by the parametric transform, and then to compute nonholonomic deformations of a Schwarzschild black hole self–consistently imbedded in a such nonperturbative background.

Nonholonomic Ricci flows induced by the N–connection coefficients are given by flow equations of type (60), \( \delta \varphi(\chi) \) and \( \delta t(\chi) \), with the coefficients depending additionally on \( \chi \), for instance,

\[
w_{2,3}(\xi, \vartheta, \varphi, \theta, \chi) = \tilde{\eta}^4_{2,3}(\xi, \vartheta, \varphi, \theta, \chi) \frac{\partial_\xi (\sqrt{|\eta_5|} \omega)}{\sqrt{|\eta_5|} \omega},
\]

\[
n_{2,3}(\xi, \vartheta, \vartheta, \chi) = \tilde{\eta}^5_{2}(\xi, \vartheta, \vartheta, \chi) n_2(\xi, \vartheta).
\]

Of course, in order to get Ricci flows with the Levi Civita connection, the coefficients of (62) and evolving N–connection coefficients (63) have to be additionally constrained by conditions of type (18) and (19) for \( \lambda = 0 \).

5.2 Anisotropic polarizations on extra dimension coordinate

On can be constructed certain classes of exact off–diagonal solutions when the extra dimension effectively polarizes the metric coefficients and interaction constants. We take as a primary metric the ansatz (22) (see the parametrization for coordinates for that quadratic element, with \( x^1 = \varphi, x^2 = \bar{\vartheta}, x^3 = \bar{\xi}, y^4 = \varpi, y^5 = t \)) and consider the off–diagonal target metric

\[
\delta s_{[5,\varpi]}^2 = -r_g^2 \; d\varphi^2 + r_g^2 \; \eta_4(\xi, \bar{\vartheta}) d\bar{\vartheta}^2 + \eta_3(\xi, \bar{\vartheta}) \bar{\varrho}_3(\bar{\vartheta}) \; d\bar{\xi}^2
\]

\[
+ \epsilon_4 \; \eta_4(\xi, \bar{\vartheta}, \varpi) \delta \varpi^2 + \eta_5(\xi, \bar{\vartheta}, \varpi) \; \tilde{h}_5(\xi, \bar{\vartheta}) \; \delta t^2
\]

\[
\delta \varpi = d\varpi + w_2(\xi, \bar{\vartheta}, \varpi) d\xi + w_3(\xi, \bar{\vartheta}, \varpi) d\bar{\vartheta},
\]

\[
\delta t = dt + n_2(\xi, \bar{\vartheta}, \varpi) d\xi + n_3(\xi, \bar{\vartheta}, \varpi) d\bar{\vartheta}.
\]
The coefficients of this ansatz,
\begin{align*}
g_1 &= -r_g^2, \quad g_2 = -r_g^2 \eta_2(\xi, \tilde{\vartheta}), \quad g_3 = \eta_3(\xi, \tilde{\vartheta}) \hat{g}_3(\tilde{\vartheta}), \\
h_4 &= \epsilon_1 \eta_4(\xi, \tilde{\vartheta}, \varpi), \quad h_5 = \eta_5(\xi, \tilde{\vartheta}, \varpi) \hat{h}_5(\xi, \tilde{\vartheta})
\end{align*}
are subjected to the condition to solve the system of equations (12)–(15) with a nontrivial cosmological constant defined, for instance, from string gravity by a corresponding ansatz for $H$–fields with $\lambda = -\lambda_H^2/2$, or other type cosmological constants, see details on such nonholonomic configurations in Refs. [4, 7].

The general solution is given by the data
\begin{equation}
- r_g^2 \eta_2 = \eta_3 \hat{g}_3 = \exp 2 \psi(\xi, \tilde{\vartheta}),
\end{equation}
where $\psi$ is the solution of
\begin{align*}
\psi^{\bullet\bullet} + \psi'' &= \lambda, \\
\eta_4 &= h_0^2(\xi, \tilde{\vartheta}) \left[ f^\star(\xi, \tilde{\vartheta}, \chi) \right]^2 |\varsigma(\xi, \tilde{\vartheta}, \varpi)|, \\
\eta_5 \hat{h}_5 &= \left[ f(\xi, \tilde{\vartheta}, \varpi) - f_0(\xi, \tilde{\vartheta}) \right]^2,
\end{align*}
where
\begin{equation}
\varsigma(\xi, \tilde{\vartheta}, \chi) = \varsigma_{[0]}(\xi, \tilde{\vartheta}) + \frac{\epsilon_4}{16} h_0^2(\xi, \tilde{\vartheta}) \lambda_H^2 \int f^\star(\xi, \tilde{\vartheta}, \chi) \left[ f(\xi, \tilde{\vartheta}, \chi) - f_0(\xi, \tilde{\vartheta}) \right] d\chi.
\end{equation}
The N–connection coefficients $N_i^4 = w_i(\xi, \tilde{\vartheta}, \varpi)$, $N_i^5 = n_i(\xi, \tilde{\vartheta}, \varpi)$ are computed following the formulas
\begin{align*}
w_i &= - \frac{\partial \varsigma(\xi, \tilde{\vartheta}, \varpi)}{\varsigma(\xi, \tilde{\vartheta}, \varpi)} \\
n_{\tilde{k}} &= n_{\tilde{k}[1]}(\xi, \tilde{\vartheta}) + n_{\tilde{k}[2]}(\xi, \tilde{\vartheta}) \int \frac{f^\star(\xi, \tilde{\vartheta}, \varpi)^2}{\left[ f(\xi, \tilde{\vartheta}, \varpi) - f_0(\xi, \tilde{\vartheta}) \right]^3} \varsigma(\xi, \tilde{\vartheta}, \varpi) d\varpi.
\end{align*}

The solutions depend on arbitrary nontrivial functions $f(\xi, \tilde{\vartheta}, \varpi)$ (with $f^\star \neq 0$), $f_0(\xi, \tilde{\vartheta})$, $h_0^2(\xi, \tilde{\vartheta})$, $\varsigma_{[0]}(\xi, \tilde{\vartheta})$, $n_{\tilde{k}[1]}(\xi, \tilde{\vartheta})$, and $n_{\tilde{k}[2]}(\xi, \tilde{\vartheta})$, and value of cosmological constant $\lambda$. These values have to be defined by certain boundary conditions and physical considerations. In the sourceless case, $\varsigma_{[0]} \to 1$. For the Levi Civita connection, we have to consider $h_0^2(\xi, \tilde{\vartheta}) \to \text{const}$ and have to prescribe the integration functions of type $n_{\tilde{k}[2]} = 0$ and $n_{\tilde{k}[1]}$, solving the equation $\partial_{\tilde{\vartheta}} n_{\tilde{k}[1]} = \partial_\xi n_{\tilde{k}[1]}$, in order to satisfy some conditions of type (18) and (19).

The class of solutions (64) define self-consistent nonholonomic maps of the Schwarzschild solution into a 5D backgrounds with nontrivial sources, depending on a general function $f(\xi, \tilde{\vartheta}, \varpi)$ and mentioned integration functions and constants. Fixing $f(\xi, \tilde{\vartheta}, \varpi)$ to be a 3D soliton (we can consider also solitonic pp–waves as in previous sections) running on extra dimension $\varpi$, we describe self-consisted embedding of the Schwarzschild solutions.
into nonlinear wave 5D curved spaces. In general, it is not clear if any target solutions preserve the black hole character of primary solution. It is necessary a very rigorous analysis of geodesic configurations on such spacetimes, definition of horizons, singularities and so on. Nevertheless, for small nonholonomic deformations (by introducing a small parameter \( \varepsilon \), like in the section 5.1.2), we can select classes of "slightly" deformed solutions preserving the primary black hole character. In 5D, such solutions are not subjected to the conditions of black hole uniqueness theorems, see [30] and references therein.

The ansatz (64) posses two Killing vector symmetries, on \( \partial/\partial t \) and \( \partial/\partial \varphi \). In the sourceless case, we can apply a parametric transform and generate new families depending on a parameter \( \theta \). The constructions are similar to those generating (62) (we omit here such details). Here we emphasize that we can not apply a parametric transform to the primary metric (22) (it is not a vacuum solution) in order to generate families of parametric solutions with the aim to subject them to further anholonomic transforms.

For nontrivial cosmological constant (normalization), the metric (64) can be generalized for nonholonomic Ricci flows of type

\[
\delta s^2_{[5\times]} = -r^2_g \, d\varphi^2 - r^2_g \, \eta_2(\xi, \tilde{\vartheta}, \chi) d\tilde{\vartheta}^2 + \eta_3(\xi, \tilde{\vartheta}, \chi) \tilde{g}_3(\tilde{\vartheta}) \, d\tilde{\xi}^2 + \epsilon_4 \, \eta_4(\xi, \tilde{\vartheta}, \varpi) d\varpi^2 + \eta_5(\xi, \tilde{\vartheta}, \chi) \tilde{h}_5(\xi, \tilde{\vartheta}) \, dt^2
\]

\[
\delta \varpi = dw_2(\xi, \tilde{\vartheta}, \varpi) \, d\xi + w_3(\xi, \tilde{\vartheta}, \varpi) \, d\tilde{\vartheta},
\]

\[
\delta t = dt + n_2(\xi, \tilde{\vartheta}, \chi) \, d\xi + n_3(\xi, \tilde{\vartheta}, \varpi) \, d\tilde{\vartheta}.
\]

where the equation (7) imposes constraints of type (11)

\[
\frac{\partial}{\partial \chi} \left[ g_{2,3}(\xi, \tilde{\vartheta}, \chi) + h_5(\xi, \tilde{\vartheta}, \varpi) \, (n_{2,3}(\xi, \tilde{\vartheta}, \varpi, \chi))^2 \right] = 0,
\]

with is very different from constraints of type (59), for

\[
g_2 = -r^2_g \, \eta_2(\xi, \tilde{\vartheta}, \chi), \quad g_3 = \eta_3(\xi, \tilde{\vartheta}, \chi) \tilde{g}_3(\tilde{\vartheta}), \quad h_5 = \eta_5(\xi, \tilde{\vartheta}, \varpi) \tilde{h}_5(\xi, \tilde{\vartheta}),
\]

\[
n_k(\chi) = n_k^{[1]}(\xi, \tilde{\vartheta}, \chi) + n_k^{[2]}(\xi, \tilde{\vartheta}, \chi) \int \frac{[f^*(\xi, \tilde{\vartheta}, \varpi)]^2}{[f(\xi, \tilde{\vartheta}, \varpi) - f_0(\xi, \tilde{\vartheta})]^2} \zeta(\xi, \tilde{\vartheta}, \varpi) \, d\varpi.
\]

For holonomic Ricci flows with the Levi Civita connection, we have to consider additional constraints

\[
\psi^{**}(\chi) + \psi^{\prime}(\chi) = \lambda
\]

\[
h_5^* \phi / h_4 h_5 = \lambda,
\]

\[
w' - w^*_3 + w_3 w^*_3 = 0,
\]

\[
n' = n^*_3(\chi) = 0.
\]

for

\[
g_2(\xi, \tilde{\vartheta}, \chi) = g_3(\xi, \tilde{\vartheta}, \chi) = e^{2\psi(\xi, \tilde{\vartheta}, \chi)}, \quad h_4 = \epsilon_4 \, \eta_4(\xi, \tilde{\vartheta}, \chi),
\]

\[
w^*_4 = \partial \phi / \phi^*, \quad \text{where} \quad \phi = -\ln \left| \sqrt{|h_4 h_5| / |h_5^*|} \right|,
\]

\[
n_{2,3}(\chi) = n_{2,3}^{[1]}(\xi, \tilde{\vartheta}, \chi).
\]
This class of Ricci flows, defined by the family of solutions (69) describes deformed Schwarzschild metrics, running on extra dimension coordinate $\varpi$ with mutually compatible evolution of the $h$–component of metric and the $n$–coefficients of the N–connection.

5.3 5D solutions with nonholonomic time like coordinate

We use the primary metric (24) (which is not a vacuum solution and does not admit parametric transforms but can be nonholonomically deformed) resulting in a target off–diagonal ansatz,

$$\delta s^2_{[\varpi]} = -r_g^2 d\varphi^2 - r_g^2 \eta_2 (\xi, \tilde{\vartheta}) d\tilde{\vartheta}^2 + \eta_3 (\xi, \tilde{\vartheta}) \tilde{\gamma}_3 (\tilde{\vartheta}) d\xi^2$$

$$+ \eta_4 (\xi, \tilde{\vartheta}, t) \tilde{h}_4 (\xi, \tilde{\vartheta}) d\tilde{t}^2 + \epsilon_5 \eta_5 (\xi, \tilde{\vartheta}, t) \delta \varpi^2,$$

$$\delta t = dt + w_2 (\xi, \tilde{\vartheta}, t) d\xi + w_3 (\xi, \tilde{\vartheta}, t) d\tilde{\vartheta},$$

$$\delta \varpi = d\varpi + n_2 (\xi, \tilde{\vartheta}, t) d\xi + n_3 (\xi, \tilde{\vartheta}, t) d\tilde{\vartheta},$$

(71)

where the local coordinates are established $x^1 = \varphi$, $x^2 = \tilde{\vartheta}$, $x^3 = \xi$, $y^4 = t$, $y^5 = \varpi$ and the polarization functions and coefficients of the N–connection are chosen to solve the system of equations (12)–(15). Such solutions are generic 5D and emphasize the anisotropic dependence on time like coordinate $t$. The coefficients can be computed by the same formulas (65) and (66) as in the previous section, for the ansatz (64), by changing the coordinate $t$ into $\varpi$ and, inversely, $\varpi$ into $t$. This class of solutions depends on a function $f(\xi, \tilde{\vartheta}, t)$, with $\partial_t f \neq 0$, and on integration functions (depending on $\xi$ and $\tilde{\vartheta}$) and constants. We can consider more particular physical situations when $f(\xi, \tilde{\vartheta}, t)$ defines a 3D solitonic wave, or a pp–wave, or their superpositions, and analyze configurations when a Schwarzschild black hole is self–consistently embedded into a dynamical 5D background.

We analyzed certain similar physical situations in Ref. [12] when an extra dimension soliton “running” away a 4D black hole.

The set of 5D solutions (71) posses two Killing vector symmetry, $\partial/\partial t$ and $\partial/\partial \varpi$, like in the previous section, but with another types of vectors. For the vacuum configurations, it is possible to perform a parametric transform and generate parametric (on $\theta'$) 5D solutions (labelling, for instance, packages of nonlinear waves).

For nontrivial cosmological constant (normalization), the metric (71) also can be generalized to describe nonholonomic Ricci flows

$$\delta s^2_{[u, \chi]} = -r_g^2 d\varphi^2 - r_g^2 \eta_2 (\xi, \tilde{\vartheta}, \chi) d\tilde{\vartheta}^2 + \eta_3 (\xi, \tilde{\vartheta}, \chi) \tilde{\gamma}_3 (\tilde{\vartheta}) d\xi^2$$

$$+ \eta_4 (\xi, \tilde{\vartheta}, t) \tilde{h}_4 (\xi, \tilde{\vartheta}) d\tilde{t}^2 + \epsilon_5 \eta_5 (\xi, \tilde{\vartheta}, t) \delta \varpi^2,$$

$$\delta t = dt + w_2 (\xi, \tilde{\vartheta}, t) d\xi + w_3 (\xi, \tilde{\vartheta}, t) d\tilde{\vartheta},$$

(72)

$$\delta \varpi (\chi) = d\varpi + n_2 (\xi, \tilde{\vartheta}, t, \chi) d\xi + n_3 (\xi, \tilde{\vartheta}, t, \chi) d\tilde{\vartheta},$$

where the equation (7) imposes constraints of type (11)

$$\frac{\partial}{\partial \chi} [g_{2,3}(\xi, \tilde{\vartheta}, \chi) + h_5 (\xi, \tilde{\vartheta}, \varpi) (n_{2,3}(\xi, \tilde{\vartheta}, \varpi, \chi))^2] = 0.$$
For holonomic Ricci flows with the Levi Civita connection, we have to consider additional constraints of type (70) with re-defined coefficients and coordinates, when

\[ g_2(\xi, \tilde{\vartheta}, \chi) = g_3(\xi, \tilde{\vartheta}, \chi) = e^{2\psi(\xi, \tilde{\vartheta}, \chi)}, h_4 = h_3(\xi, \tilde{\vartheta}, t), \]

\[ w_i = \partial_i \phi/\phi^* \text{, where } \varphi = -\ln \left| \sqrt{|h_4 h_5|/|h_5^*|} \right|, \]

\[ n_{2,3}(\chi) = n_{2,3}^1(\xi, \tilde{\vartheta}, \chi). \]

This class of Ricci flows, defined by the family of solutions (72) describes deformed Schwarzschild metrics, running on time like coordinate \( t \) with mutually compatible evolution of the h–component of metric and the \( n \)–coefficients of the N–connection.

6. Discussion

We constructed exact solutions in gravity and Ricci flow theory following superpositions of the parametric and anholonomic frame transforms. A geometric method previously elaborated in our partner works [4, 7] was applied to generalizations of valuable physical solutions (like solitonic waves, pp–waves and Schwarzschild metrics) in vacuum gravity. In this work, our investigations were restricted to nonholonomic Ricci flows of the mentioned type solutions modelled with respect certain classes of compatible metric and associated nonlinear connection (N–connection) coefficients when the solutions of evolution/ field equations can be obtained in general form.

The first advance is the possibility to generalize vacuum metrics by allowing realistic string gravity or matter field sources which can be encoded as an effective (in general, nonhomogeneous) cosmological constant on nonholonomic (pseudo) Riemannian spaces of dimensions four and five (4D and 5D) and deriving nonlinear solitonic and pp–wave interactions and their Ricci flows.

The second kind of progress is the proof of existence of multi–parametric transforms, associated to certain Killing symmetries, like the Geroch equations [28, 29], mapping certain target metrics (in our case of physical importance) into different classes of generic off–diagonal exact solutions admitting different scenarios of Ricci flows depending on the type of nonholonomic frame constraints.

The outcome of the first advance is rather satisfactory: we can in a similar way consider parametric deformations of metrics and flows of geometric and physical objects by obtaining, for instance, static rotoid configurations, solitonic and pp–wave propagation of black holes on time like and extra dimension coordinates.

However, the outcome of the second kind of progress raises as many problems as it solves: we should provide a physical motivation for the multi–parameter dependence and
'hidden' Killing symmetry under nonholonomic deformations. If one of the parameters is identified with the Ricci flow parameter, it may be considered to describe a corresponding evolution. In general, this may be associated to chains of Ricci multi-flows but not obligatory. We have to speculate additionally on physical meaning of such parametric solutions both for vacuum gravitational and Einstein spaces and in Ricci flow theory when metrics and connections are subjected to nonholonomic constraints on coefficients and associated frames.

Whereas most previous work on Ricci flow theory and applications has concentrated on some approximate methods and simplest classes of solutions, the present paper aims to elaboration of general geometric methods of constructing solutions and deriving their physically important symmetries. There were stated exact principles how the physically important solutions in gravity theories can be deformed in multi-parametric ways to describe off-diagonal nonlinear gravitational and matter field interactions and the evolution of physical and geometric objects. The first step was to derive exact solutions in the most possible general form preserving dependence not only on transform and flow parameters but on classes of generating and integration functions and constants. Further work would be needed to analyze more rigorously certain important physical effects with exactly defined boundary and initial conditions when the integration functions and constants are defined in explicit form.

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A Cosmological Constants and Strings

The simplest way to perform a local covariant calculus by applying d-connections is to use N-adapted differential forms and to introduce the d-connection 1-form $\Gamma_{\beta}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} e^\gamma$, when the N-adapted components of d-connection $D_a = (e_a, D)$ are computed following formulas

$$\Gamma_{\alpha\beta}^\gamma (u) = (D_a e_b) | e^\gamma,$$

where "\(\cdot\)" denotes the interior product. This allows us to define in N-adapted torsion $\mathcal{T} = \{\mathcal{T}^\alpha\}$,

$$\mathcal{T}^\alpha = D e^\alpha = de^\alpha + \Gamma_{\beta}^{\alpha} \wedge e_{\alpha},$$

and curvature $\mathcal{R} = \{\mathcal{R}^\alpha_{\beta}\}$,

$$\mathcal{R}^\alpha = D \Gamma_{\beta}^{\alpha} - \Gamma_{\beta}^{\gamma} \wedge \Gamma_{\gamma}^{\alpha}.$$}

In string gravity, the nontrivial torsion components and string corrections to matter sources in the Einstein equations can be related to certain effective interactions with the strength (torsion)

$$H_{\mu\nu\rho} = e_\mu B_{\nu\rho} + e_\mu B_{\mu\nu} + e_\mu B_{\rho\mu}.$$
of an antisymmetric field $B_{\nu\rho}$, when
\[
R_{\mu\nu} = -\frac{1}{4} H^\nu_{\mu\rho} H_{\nu\lambda\rho}
\] (A.3)
and
\[
D_{\lambda} H^{\lambda\mu\nu} = 0,
\] (A.4)
see details on string gravity, for instance, in Refs. [5, 6]. The conditions (A.3) and (A.4) are satisfied by the ansatz
\[
H_{\mu\nu\rho} = \hat{Z}_{\mu\nu\rho} + \hat{H}_{\mu\nu\rho} = \lambda[H] \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho}
\] (A.5)
where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and the distorsion (from the Levi–Civita connection) and
\[
\hat{Z}_{\mu\nu\rho} e^\mu = e^\beta T_\alpha - e^\alpha T_\beta + \frac{1}{2} (e_\alpha e_\beta) T_\gamma e^\gamma
\]
is defined by the torsion tensor (A.2), which for the canonical $d$–connection is induced by the coefficients of $N$–connection, see details in [10, 8, 1, 2]. We emphasize that our $H$–field ansatz is different from those formally used in string gravity when $\hat{H}_{\mu\nu\rho} = \lambda[H] \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho}$. In our approach, we define $H_{\mu\nu\rho}$ and $\hat{Z}_{\mu\nu\rho}$ from the respective ansatz for the $H$–field and nonholonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the ‘deformed’ $H$–field as $\hat{H}_{\mu\nu\rho} = \lambda[H] \sqrt{|g_{\alpha\beta}|} \varepsilon_{\nu\lambda\rho} - \hat{Z}_{\mu\nu\rho}$.

References


Relativistic Effects on Quantum Bell States of Massive Spin $\frac{1}{2}$ Particles

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Abstract: We examine the behaviour of the maximally entangled Bell state of two spin $\frac{1}{2}$ massive particles under relativistic transformations. On the basis of explicit calculations of the Wigner rotation and the use of transformation properties of Dirac spinors, we establish that the constituent particles of the Bell state undergo momentum dependent rotation of the spin orientations characterized by the Wigner angle $\phi_W = \tan^{-1} \frac{\sinh \varpi \sinh \tau}{\cosh \varpi + \cosh \tau}$. However, since local unitarity is retained in the process, the corresponding entanglement fidelity is not lost.

Keywords: Wigner Rotation; Bell States; Quantum Entanglement

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1. Introduction

Potential applications of information encoded into the states of quantum systems are finding their way not only into revolutionizing computation but, more importantly, in achieving communication tasks with unprecedented efficiency e.g. quantum teleportation, entanglement enhanced communication, quantum clock synchronization and reference frame alignment, quantum enhanced global positioning etc. These applications warrant long distance signal transmission through quantum mechanical states. As such, in these functional areas of quantum information processing, relativistic effects could significantly contribute to the dynamics.

Even for quantum computing operations like quantum cryptography, error detection etc., it would be desirable to study relativistic implications since they may result in optimal processes and/or superior algorithms.

At a fundamental level, we also need to reconcile quantum information theory with...
general relativity and other contemporary quantum theories like the quantum field theory, loop quantum gravity and string theory to examine where it fits in our overall quest for unification. Some of the literature that report work done in the direction of relativistic quantum information is listed in the references [1-40].

2. Prerequisites [41-43]

2.1 Unitary Representations of the Poincare Group

We start by briefly elucidating the salient features of the theory of unitary representations of the Poincare group as propounded by Wigner in his seminal article [41]. It is well known that the unitary representations of the Lorentz group $\hat{L}(\Lambda)$ and the four dimensional translational group $\hat{T}(a)$ (that together constitute the Poincare group) satisfy

\begin{align}
\hat{T}(a) \hat{T}(b) &= \hat{T}(a + b) \\
\hat{L}(\Lambda) \hat{L}(\Lambda') &= \hat{L}(\Lambda \Lambda') \\
\hat{L}(\Lambda) \hat{T}(a) &= \hat{T}(\Lambda a) \hat{L}(\Lambda)
\end{align}

The above equations constitute the defining equations for obtaining the unitary representations of the Poincare group. Some properties of the representation theory of the Poincare group are summarized below for use in the sequel. Proofs are accessible in any text on the subject e.g.[42,43].

(a) The parameter space of the Poincare group is spanned by a ten parameter basis, four of which relate to the four dimensional translation group $T_4 \equiv \{\hat{T}(a)\}$ (representing translations in the four directions of relativistic spacetime) and the remaining six relate to the Lorentz group $L \equiv \{\hat{L}(\Lambda)\}$ (three of which represent spatial rotations and the other three symbolize “boosts” i.e. spatiotemporal rotations along the three spatial axes).

(b) The parameter space corresponding to $T_4$ is the four dimensional Euclidean space that is simply connected so that irreducible representations (IRRs) of $T_4$ are single valued.

(c) Any element of $L \equiv \{\hat{L}(\Lambda)\}$ can be written as $L(\Lambda)_{\nu}^\sigma = R_{\nu}^\rho B_{\rho}^\sigma (\tau) R_{\sigma}^\rho$ where $\hat{R}$ & $\hat{B}$ are spatial rotations and $B(\tau)$ is a Lorentz boost in the direction of the first spatial axis. Now, the parameter space of $B(\tau)$ is a straight line. It is unidimensional and simply connected so that the IRRs are single valued. On the other hand, the rotation group admits both single valued and double valued IRRs. It follows that the unitary representations of the Poincare group could be either single valued or, at most, double valued depending on the character of the representation of the constituent rotation group.

(d) Infinitesimal Lorentz transformations can be written in terms of the generators as $\hat{L}(\varepsilon, \delta) = \hat{I} + i \varepsilon \hat{J} - i \delta \hat{K}$ where $\hat{J}$ & $\hat{K}$ are operators (generators of spatial rotations and boosts respectively) that satisfy the relations
\[ [\hat{J}_i, \hat{J}_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} \hat{J}_k \] (4)

\[ [\hat{J}_i, \hat{K}_j] = i \sum_{k=1}^{3} \varepsilon_{ijk} \hat{K}_k \] (5)

\[ [\hat{K}_i, \hat{K}_j] = -i \sum_{k=1}^{3} \varepsilon_{ijk} \hat{J}_k \] (6)

For unitary representations, the operators \( \hat{J} \) & \( \hat{K} \) must necessarily be hermitian.

(e) Infinitesimal translations can be written in terms of the generators as \( \hat{T}(a) = \hat{I} + ia^\mu \hat{P}_\mu, \mu = 0, 1, 2, 3 \) where \( \hat{P}_\mu \) are operators (generators of translations) that satisfy the relations

\[ [\hat{P}^0, \hat{J}_n] = 0 \] (7)

\[ [\hat{P}_m, \hat{J}_n] = i \varepsilon^{mnl} \hat{P}_l \] (8)

\[ [\hat{P}_m, \hat{K}_n] = i \delta_{mn} \hat{P}^0 \] (9)

\[ [\hat{P}^0, \hat{K}_n] = i \hat{P}_n \] (10)

Finite translations are, as usual, expressed in terms of the generators by exponentiation as

\[ \hat{T}(a) = e^{i \hat{P}_\mu a^\mu} \] (11)

2.2 Casimir Operators and Classification of IRRs

The operator \( \hat{C}_1 = \hat{P}^\mu \hat{P}_\mu = \hat{P}_0^2 - \hat{P}^2 \) commutes with all the generators of the Lie algebra of the Poincare group and hence constitutes a Casimir operator of the group. Hence, \( \hat{C}_1 = \hat{P}^\mu \hat{P}_\mu = \hat{P}_0^2 - \hat{P}^2 \) is invariant for an IRR. IRRs of the Poincare group can, therefore, be labeled by the eigenvalues of \( \hat{C}_1 = \hat{P}^\mu \hat{P}_\mu = \hat{P}_0^2 - \hat{P}^2 \). Noting that the generators of spatial translations are realized as the “momentum” operators and the generator of time translation as the “energy operator”, we have, in terms of the respective eigenvalues, \( c_1 = p^\mu p_\mu = p_0^2 - p^2 = M^2 \) where \( M \) is the particle mass. Since the eigenvalues \( c_1 \) are not positive definite in the case of the Poincare group, we classify the IRRs as follows:

1. \([L]\) Null vector case corresponding to \( c_1 = 0, p^0 = p = 0 \)
2. \([M_\pm]\) Time like case corresponding \( c_1 > 0 \)
3. \([0_\pm]\) Light like case corresponding to \( c_1 = 0, p \neq 0 \)
4. \([T]\) Space like case corresponding to \( c_1 < 0 \).
We shall restrict ourselves to the time like case corresponding to particles with finite mass since that is relevant to the context. However, the treatment for the other cases follows on similar lines.

In establishing the Casimir nature of $\hat{C}_1$, we have simply used eqs. (7-10). However, eqs. (4-6) and hence, eq. (2) has not been considered. It follows that $c_1$ alone is not sufficient to completely identify the IRR of the Poincare group. Eq. (2) is the defining property of the homogeneous Lorentz group that is known to possess a Casimir operator $\hat{J}^2 - \hat{K}^2$. Therefore, involvement of eq. (2) in the analysis should lead to a second Casimir operator which we symbolize (for the time being) by $\hat{C}_2$ with the eigenvalue $c_2$.

We can completely identify an IRR by these two eigenvalues $c_1, c_2$ and the energy sign $\varepsilon (p_0) = \frac{p_0}{|p_0|}$ since the energy sign also commutes with all the generators of the Poincare group corresponding to the classes of representations $[M \pm]$ and $[0 \pm]$.

The second Casimir operator of the Poincare group is known as the Pauli-Lubanski vector $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ where $\hat{W} \equiv \left( \hat{W}^0, \hat{W} \right) = \left( \hat{P}.\hat{J}, \hat{P}^0.\hat{J} - \hat{P} \times \hat{K} \right)$ that has the following properties:

$$\hat{W}^\lambda \hat{P}_\lambda = 0$$  \hspace{1cm} (12)
$$\left[ \hat{W}^\lambda, \hat{P}^\mu \right] = 0$$  \hspace{1cm} (13)
$$\left[ \hat{W}^\lambda, \hat{j}^{\mu\nu} \right] = i \left( \hat{W}^\mu g^{\lambda\nu} - \hat{W}^\nu g^{\mu\lambda} \right)$$  \hspace{1cm} (14)
$$\left[ \hat{W}^\lambda, \hat{W}^\sigma \right] = i \varepsilon^{\lambda\sigma\mu\nu} \hat{W}_\mu \hat{P}_\nu$$  \hspace{1cm} (15)

The fact that $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is a Casimir operator and commutes with all the generators of the group follows from

(i) Each component of $\hat{W}$ is translationally invariant for $\left[ \hat{W}^\lambda, \hat{P}^\mu \right] = 0$. Hence $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is also translationally invariant;

(ii) Further $\hat{C}_2 = \hat{W}_\mu \hat{W}^\mu$ is the scalar product of a 4-vector with itself. Hence, it is the square of the length of a 4-vector which makes it invariant under homogeneous Lorentz transformations.

Thus, the massive class of representations of the Poincare group has two Casimir operators that are left invariant by group operations plus the energy sign. Hence, we label out the IRRs of the Poincare group by two indices corresponding to the eigenvalues of the two Casimir operators viz. $c_1 \equiv M^2$ and $\frac{1}{p^0} c_2 \equiv \frac{1}{p^0} W^2$ that is identified with the spin $s$ in the case of massive particles ($M > 0$). For a given IRR labeled by the eigenvalues of the two Casimir operators viz. $M^2$ and $\frac{1}{p^0} W^2$, the sign of the energy $\varepsilon (p_0)$ commutes with all the infinitesimal generators of the Poincare group. There are, thus, two IRRs for each combination of values of $M^2$ and $\frac{1}{p^0} W^2$, one for each sign of $\varepsilon (p_0)$.
2.3 Basis Vectors in an IRR Space

We, now, address the issue of the labeling of basis vectors within a given IRR. For the purpose, we note that the four dimensional translation group $T_4$ constitutes an invariant subgroup of the Poincare group. We also know that the basis vectors in a representation of the translation group can be labeled by the eigenvalues of the generators of the translation group. However, the specification of a basis vector using a single index corresponding to the generators of translations is incomplete. For a complete specification of the basis vectors, we need a second index that relates to the generators of “rotations” and is usually taken to be the eigenvalue of $\hat{J}_3$. A complete specification of a basis vector in the representation space $(M, s, \varepsilon (p_0))$ would, thus, consist of four indices, together with the energy sign $\varepsilon (p_0)$, two of them related to the eigenvalues of Casimir operators $\hat{C}_1$ & $\hat{C}_2$ of the Poincare group and the energy sign $\varepsilon (p_0)$ for identifying the representation itself and the other two $(p, m)$ that relate the eigenvalues of $\hat{P}_\mu$ & $\hat{J}_3$ for specifying a basis vector within a representation. Now, all the basis vectors in a given representation space identified by $(M, s, \varepsilon (p_0))$ correspond to the same eigenvalue $c_1 \equiv M^2$. We also have the relativistic mass-energy relation $p_0 = \pm \sqrt{p^2 + M^2}$. It follows that we can as well use the eigenvalues $p$ of 3-momentum in lieu of the eigenvalues $p \equiv (p_0, p)$ of 4-momentum for specification of the basis vectors.

2.4 Little Group Decomposition of the Poincare Group

It can be shown that the independent components of $\hat{W}$ are proportional to the generators of the group $SO (3)$ for, given a basis vector $|p, m\rangle$ in the $(M, s, \varepsilon (p_0))$ representation of the Poincare group, we have (since these basis vectors are eigenstates of generators of translations $P_\mu$), $\hat{P}_\mu |p, m\rangle = p^\mu |p, m\rangle$ where $p^\mu$ is the eigenvalue of $\hat{P}_\mu$ so that $\hat{W}^\lambda |p, m\rangle = \varepsilon^{\lambda \mu \nu \sigma} \hat{J}_\mu \hat{P}_\sigma |p, m\rangle = \varepsilon^{\lambda \mu \nu \sigma} \hat{J}_\mu \hat{P}_\sigma |p, m\rangle$. For the standard vector, $p^0 = M$ and $p = 0$ whence $\hat{W}^0 |0, m\rangle = \varepsilon^{0 \mu \nu \sigma} \hat{J}_\mu \hat{P}_\sigma |0, m\rangle = 0$ and $\hat{W}^i |0, m\rangle = \varepsilon^{i \mu \nu \sigma} \hat{J}_\mu \hat{P}_\sigma |0, m\rangle = \frac{1}{2} \varepsilon^{ijk} \hat{J}_{jk} M |0, m\rangle$ which establishes our result.

Using eqs. (3-11), we obtain

$$e^{i[\hat{L}(\Lambda)\hat{P}_\mu \hat{L}(\Lambda)^{-1}]a^\mu} = \hat{L}(\Lambda)e^{i\hat{P}_\mu a^\mu}\hat{L}(\Lambda)^{-1} = \hat{L}(\Lambda)\hat{T}(a)\hat{L}(\Lambda)^{-1} = \hat{T}(\Lambda a) = e^{i\hat{P}_\mu \Lambda_\mu^a a^\mu}$$

whence we get the transformation rules for the covariant generators of translations as

$$\hat{L}(\Lambda)\hat{P}_\mu \hat{L}(\Lambda)^{-1} = \hat{P}_\mu \Lambda_\mu^a$$

or equivalently

$$\hat{L}(\Lambda)^{-1} \hat{P}_\mu \hat{L}(\Lambda) = \hat{P}_\mu \Lambda_\mu^a$$

(16)

Given a basis vector $|p, m\rangle$ in a representation $(M, s, \varepsilon (p_0))$, we have, on using eq.(16)

$$\hat{P}_\mu \hat{L}(\Lambda) |p, m\rangle = \hat{L}(\Lambda)\hat{P}_\mu \Lambda_\mu^a |p, m\rangle = p_\mu \Lambda_\mu^a \hat{L}(\Lambda) |p, m\rangle$$

thereby showing that $\hat{L}(\Lambda) |p, m\rangle$ is also an eigenvector of $\hat{P}_\mu$ with the eigenvalue $p_\mu \Lambda_\mu^a$. Given two basis vectors, in a representation space $|p, m\rangle, |p', m'\rangle$ of the same energy sign, $\varepsilon (p_0)$, we can define an inner product by
\[ \langle p, m | p', m' \rangle = \delta_{mm'} \omega_p \delta (p - p') \]  

(17)

where \( \omega_p = \sqrt{p^2 + M^2} \). The Lorentz invariance of the integration element \( \omega_k \delta (p - p') \) is well known. The completeness of the space implies

\[ \sum_m \int \frac{dp}{\omega_p} |p, m\rangle \langle p, m| = 1 \]  

(18)

Let us define the operator

\[ \hat{S} (\Lambda) \]  

by \( \hat{S} (\Lambda) |p, m\rangle = |\Lambda p, m\rangle \) \( \hat{S} (\Lambda) \) is unitary because it leaves the inner product (17) invariant for

\[ \langle \Lambda p', m' | \Lambda p, m \rangle = \langle p', m'| \hat{S}^\dagger \hat{S} |p, m\rangle = \delta_{mm'} \omega_{\Lambda p} \delta (\Lambda p - \Lambda p') \]

\[ = \delta_{mm'} \omega_p \delta (p - p') = \langle p', m'| p, m \rangle \]  

(20)

where the penultimate step follows from the Lorentz invariance of \( \omega_k \delta (p - p') \). Further

\[ \hat{S} (\Lambda) \hat{S} (\Lambda') = \hat{S} (\Lambda \Lambda') \]  

(21)

implying that \( \hat{S} (\Lambda) \) is a unitary representation of the Lorentz group. Also being independent of \( m \), it is diagonal with respect to the corresponding generator e.g. \( \hat{J}_3 \).

We also have

\[ \hat{S} (\Lambda) \hat{T} (\Lambda^{-1} a) |p, m\rangle \]

\[ = \hat{S} (\Lambda) e^{i \nu_\mu (\Lambda^{-1} a)\nu} |p, m\rangle = \hat{S} (\Lambda) e^{i \nu_\mu (\Lambda^{-1} a)\nu} |p, m\rangle \]

\[ = e^{i \nu_\mu (\Lambda^{-1} a)\nu} |\Lambda p, m\rangle = e^{i (\Lambda^{-1} \hat{P})_\mu (\Lambda^{-1} a)\nu} |\Lambda p, m\rangle = \hat{T} (a) |\Lambda p, m\rangle = \hat{T} (a) \hat{S} (\Lambda) |p, m\rangle \]

whence, because of the completeness of the set of basis vectors, we can infer that

\[ \hat{T} (\Lambda^{-1} a) \hat{S} (\Lambda^{-1}) = \hat{S} (\Lambda^{-1}) \hat{T} (a) \]  

(22)

From eq. (3), we also infer that \( \hat{L} (\Lambda) \hat{T} (a) = \hat{T} (\Lambda a) \hat{L} (\Lambda) \) so that \( \hat{T} (\Lambda^{-1} a) \hat{L} (\Lambda) \hat{T} (a) \) which, together with eq. (22) gives \( [\hat{L} (\Lambda) \hat{S} (\Lambda)^{-1}, \hat{T} (a)] = 0 \). Since this equation holds for arbitrary \( a \), it follows that \( \hat{L} (\Lambda) \hat{S} (\Lambda)^{-1} \) commutes with the translation operators. This implies that it is diagonal with respect to the generators of translation \( \hat{P}_\mu \). Writing it as \( \hat{Q} (\Lambda, \hat{P}) \), we obtain

\[ \hat{L} (\Lambda) = \hat{Q} (\Lambda, \hat{P}) \hat{S} (\Lambda) \]  

(23)

Further, since both \( \hat{L} (\Lambda), \hat{S} (\Lambda) \) are unitary, it follows that \( \hat{Q} (\Lambda, \hat{P}) \) is also unitary.

Now

\[ \hat{Q} (\Lambda \Lambda', \hat{P}) \hat{S} (\Lambda \Lambda') |p, m\rangle = \hat{L} (\Lambda \Lambda') |p, m\rangle = \hat{L} (\Lambda) \hat{L} (\Lambda') |p, m\rangle \]

\[ = \hat{Q} (\Lambda, \hat{P}) \hat{S} (\Lambda) \hat{Q} (\Lambda', \hat{P}) \hat{S} (\Lambda') |p, m\rangle = \hat{Q} (\Lambda, \hat{P}) \hat{S} (\Lambda) \hat{Q} (\Lambda', \hat{P}) |\Lambda' p, m\rangle \]

\[ = \hat{Q} (\Lambda, \hat{P}) \hat{S} (\Lambda) \hat{Q} (\Lambda', \Lambda' p) |\Lambda' p, m\rangle = \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda', \Lambda' p) \hat{S} (\Lambda) |\Lambda' p, m\rangle \]
\[ \begin{align*}
&= \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda', \Lambda' p) |\Lambda \Lambda' p, m\rangle = \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda', \Lambda^{-1} \hat{P}) |\Lambda \Lambda' p, m\rangle \\
&= \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda', \Lambda^{-1} \hat{P}) \hat{S} (\Lambda \Lambda') |p, m\rangle \\
\text{whence} \\
\hat{Q} (\Lambda \Lambda', \hat{P}) &= \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda', \Lambda^{-1} \hat{P}) \\
\text{(24)} \\
\text{Using eq. (24), we obtain} \\
|p, m\rangle &= \hat{L} (I) |p, m\rangle = \hat{L} (\Lambda \Lambda^{-1}) |p, m\rangle = \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda^{-1}, \Lambda^{-1} \hat{P}) \hat{S} (\Lambda \Lambda^{-1}) |p, m\rangle \\
&= \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda^{-1}, \Lambda^{-1} \hat{P}) |p, m\rangle = \hat{Q} (\Lambda, \hat{P}) \hat{Q} (\Lambda, \Lambda^{-1} \hat{P})^{-1} |p, m\rangle \\
\text{so that} \\
\hat{Q} (\Lambda, \hat{P})^{-1} &= \hat{Q} (\Lambda^{-1}, \Lambda^{-1} \hat{P}) \\
\text{(25)} \\
\text{Let us assume that there exists an operator } \hat{U} (\hat{P}) \text{ in the irreducible representation} \\
\text{space that satisfies (i) unitarity, (ii) is diagonal with respect to } \hat{P}_\mu \text{, (iii) is single valued} \\
\text{in } \hat{P}_\mu \text{ (so that it preserves the single valuedness of the state operator } |p, m\rangle) \text{ with} \\
\text{respect to } \hat{P}_\mu \text{. Under the action of } \hat{U} (\hat{P}) \text{ (i) the translation operator } \hat{T} (a) \text{ is not} \\
\text{affected since } [\hat{T} (a), \hat{U} (\hat{P})] = 0, \text{ (ii) the Lorentz transformation } \hat{L} (\Lambda) \text{ is transformed as} \\
\hat{U} (\hat{P}) \hat{L} (\Lambda) \hat{U} (\hat{P})^{-1}. \text{ We have} \\
\hat{U} (\hat{P}) \hat{L} (\Lambda) \hat{U} (\hat{P})^{-1} |p, m\rangle &= \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{S} (\Lambda) \hat{U} (\hat{P})^{-1} |p, m\rangle \\
&= \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{U} (p)^{-1} \hat{S} (\Lambda) |p, m\rangle \\
&= \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{U} (p)^{-1} |\Lambda p, m\rangle \\
&= \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{U} (\Lambda^{-1} \hat{P})^{-1} |\Lambda p, m\rangle \\
&= \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{U} (\Lambda^{-1} \hat{P})^{-1} \hat{S} (\Lambda) |p, m\rangle \\
\text{(26)} \\
\text{It follows from eq. (26) that } \hat{Q} (\Lambda, \hat{P}) \& \hat{U} (\hat{P}) \hat{Q} (\Lambda, \hat{P}) \hat{U} (\Lambda^{-1} \hat{P})^{-1} \text{ provide equivalent representations for given representations of } \hat{T} (a) \& \hat{S} (\Lambda). \\
\text{To proceed further with the representation theory of the Poincare group, it is necessary} \\
\text{at this point to introduce the concept of “little group”. As mentioned above, the IRRs} \\
\text{of the Poincare group are classified into timelike, null, lightlike and spacelike on the} \\
\text{basis of the nature of the eigenvalues of the Casimir operator } \hat{F}^2 \text{ and then by the sign} \\
\text{of the energy eigenvalue. Thus, given an IRR, the corresponding identifying eigenvalue}
of $\hat{P}_2$ would be classified into one of the classes on the basis of being timelike, null, lightlike or spacelike with a sub classification within this on the basis of the energy sign. In other words, for a given IRR, the operator $\hat{P}_2$ and its eigenvalue $c_1 \equiv M^2$ would fall into one of \([M], [0], [L] \text{ or } [T]\) whereas the operator $\hat{P}_\mu$ and its associated eigenvalue $p_\mu$ would be classified into one of \([M_+], [M_-], [0_+], [0_-], [L] \text{ or } [T]\). In view of $\hat{P}_\mu \hat{L}(\Lambda) |p, m\rangle = p_\nu \Lambda^\nu_\mu \hat{L}(\Lambda) |p, m\rangle$, the eigenvalues $p_\mu$ within a class are connected with each other by a suitable Lorentz transformation. It is to be noted that in the case of class $[T]$ of representations we cannot subclassify the representations on the basis of the energy sign because the energy eigenvalue $p_0$ can change sign within the class itself by an appropriate Lorentz transformation. Physically, the class $[T]$ corresponds to imaginary mass $p^2 \equiv M^2 < 0$ which implies that the energy $p^2_0 = p^2 + M^2$ corresponding to such a representation could be made arbitrarily large negatively by an appropriate Lorentz transformation. Additionally, this representation also does not have a well defined and reasonable nonrelativistic limit. The class \([L]\) corresponds to the case of zero momentum and energy.

Let us denote by \(\{p\}\), the set of eigenvalues $p_\mu$ of a particular class. Let us identify an element $q_\mu$ arbitrarily from the set \(\{p\}\). As mentioned above, the various eigenvalues in \(\{p\}\) are related to each other through suitable Lorentz transformations. Let \(\{\eta\}\) be the set of Lorentz transformations that leave $q_\mu$ invariant i.e.

$$\eta_\nu^\mu q_\nu' = q_\mu \text{ or } \eta q = q \quad (27)$$

\(\{\eta\}\) is a subset of the Lorentz group and is called a “little group” of the Lorentz group. Now, since the elements of \(\{p\}\) are connected inter se by Lorentz transformation, given a $p_\mu \in \{p\}$, there would exist a Lorentz transformation $\kappa_p$ such that

$$p = \kappa_p q \quad (28)$$

Corresponding to an arbitrary element $q_\mu \in \{p\}$, the Lorentz transformation $\eta_p = \kappa_p^{-1} \Lambda \kappa_{A-1} \eta$ is an element of the little group $\{\eta\}$ for, we have

$$\eta_p q = \kappa_p^{-1} \Lambda \kappa_{A-1} \eta q = \kappa_p^{-1} \Lambda \Lambda^{-1} p = \kappa_p^{-1} p = q \quad (29)$$

where we have used eq. (28). Using eqs. (24) & (29), we obtain

$$\hat{Q} \big( \Lambda, \hat{P} \big) = \hat{Q} \big( k_p \eta_p \kappa_{A-1}^{-1}, \hat{P} \big) = \hat{Q} \big( \kappa_p, \hat{P} \big) \hat{Q} \big( \eta_p \kappa_{A-1}^{-1}, \kappa_p^{-1} \hat{P} \big)$$

$$= \hat{Q} \big( \kappa_p, \hat{P} \big) \hat{Q} \big( \eta_p, \kappa_p^{-1} \hat{P} \big) \hat{Q} \big( \kappa_{A-1}^{-1}, \eta_p^{-1} \kappa_p^{-1} \hat{P} \big) \quad (30)$$

Now, operating the right hand side of eq. (29) on the state $|p, m\rangle$, we obtain

$$\hat{Q} \big( \kappa_p, \hat{P} \big) \hat{Q} \big( \eta_p, \kappa_p^{-1} \hat{P} \big) \hat{Q} \big( \kappa_{A-1}^{-1}, \eta_p^{-1} \kappa_p^{-1} \hat{P} \big) |p, m\rangle$$

$$= \hat{Q} \big( \kappa_p, \hat{P} \big) \hat{Q} \big( \eta_p, \kappa_p^{-1} \hat{P} \big) \hat{Q} \big( \kappa_{A-1}^{-1}, \eta_p^{-1} \kappa_p^{-1} \hat{P} \big) |p, m\rangle$$

$$= \hat{Q} \big( \kappa_p, \hat{P} \big) \hat{Q} \big( \kappa_{A-1}^{-1}, \eta_p^{-1} \kappa_p^{-1} \hat{P} \big) \hat{Q} \big( \eta_p, \kappa_p^{-1} \hat{P} \big) |p, m\rangle$$
\[ \hat{Q} \left( \kappa_p, \hat{P} \right) \hat{Q} \left( \kappa_{-1}^{-1} \lambda, \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) |p, m\rangle = \hat{Q} \left( \kappa_p, \hat{P} \right) \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) |p, m\rangle \]

whence

\[ \hat{Q} \left( \Lambda, \hat{P} \right) = \hat{Q} \left( \kappa_p, \hat{P} \right) \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{Q} \left( \kappa_{-1}^{-1} \lambda, \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) |p, m\rangle \]  \hspace{1cm} \text{(31)}

Eq. (31) is identical in form to the expression \( \hat{U} \left( \hat{P} \right)^{-1} \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{U} \left( \Lambda^{-1} \hat{P} \right) \) if we identify \( \hat{U} \left( \hat{P} \right)^{-1} = \hat{Q} \left( \kappa_p, \hat{P} \right) \) or \( \hat{U} \left( \hat{P} \right) = \hat{Q} \left( \kappa_p, \hat{P} \right)^{-1} = \hat{Q} \left( \kappa_{-1}^{-1} \lambda, \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \) and \( \hat{U} \left( \Lambda^{-1} \hat{P} \right) = \hat{Q} \left( \kappa_{-1}^{-1} \lambda, \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \). Since \( \hat{U} \left( \hat{P} \right)^{-1} \) is unitary, it follows that \( \hat{U} \left( \hat{P} \right) \) is unitary as well. It, therefore, follows from eq. (31) that \( \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \) generates a representation of the Poincaré group that is unitarily equivalent to \( \hat{Q} \left( \Lambda, \hat{P} \right) \). Hence, in lieu of eq. (23), we can write the equivalent expression \( \hat{L} \left( \Lambda \right) = \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{S} \left( \Lambda \right) \) without any loss of generality. Further, using eq. (25), we get

\[ \hat{Q} \left( \kappa_p, \hat{P} \right)^{-1} = \hat{Q} \left( \kappa_{p}^{-1}, \kappa_{-1}^{-1} \rho \right) = \hat{Q} \left( \kappa_{-1}^{-1} \rho \right) \]  \hspace{1cm} \text{(32)}

Using eq. (24), we can also write \( \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) = \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \) where \( \eta_p = \kappa_{-1}^{-1} \Lambda \kappa_{-1}^{-1} \rho \) and \( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \). This shows that \( \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \) is a unitary representation of the relevant little group \( \hat{Q} \left( \Lambda, \hat{P} \right) \).

The operator \( \hat{Q} \left( \eta_{-1}^{-1} \kappa_{-1}^{-1} \rho \right) \) is called the Wigner rotation.

### 2.5 Little Group and Wigner Rotation Corresponding to Massive Particles

Now, for the study of the representations corresponding to the massive \( M_{\pm} \) and the identification of the corresponding “little group” let us define the “standard vector” as \( q = (\varepsilon \left( p_0 \right), M, 0, 0, 0) \). We have identified the standard vector in the “rest frame” so that all the spatial components of momentum are zero. The factor group of the Poincaré group with respect to the subgroup of translations in four dimensions, \( T_4 \), (which is abelian) is the homogeneous Lorentz group. The maximal subgroup of the homogeneous Lorentz group that leaves \( q = (\varepsilon \left( p_0 \right), M, 0, 0, 0) \) invariant is the group of three dimensional rotations, identifiable with \( SO \left( 3 \right) \) which, thus, constitutes the “little group” for this class of representations.

This may also be seen by invoking the fact that the generators of the little group of the standard vector e.g. \( q = (\varepsilon \left( p_0 \right), M, 0, 0, 0) \) are the independent components of the corresponding Pauli Lubanski vector \( \hat{W} \equiv (\hat{W}^0, \hat{W}) = \frac{1}{2} \varepsilon^{\mu \nu \sigma \rho} \hat{J}_{\mu \nu} \hat{P}_\sigma = (\hat{J}, \hat{P}^0 \hat{J} - \hat{P} \times \hat{K}) \) corresponding to the standard vector \( q = (\varepsilon \left( p_0 \right), M, 0, 0, 0) \). The second Casimir operator
for massive particles is, therefore, $\hat{W}^\mu \hat{W}_\mu = W, W = \frac{1}{M^2} \hat{J}^2$. The corresponding Lie algebra takes the form $[\hat{W}^i, \hat{W}^j] = i \varepsilon^{ijk} \hat{W}_k \hat{P}_0$ or equivalently $[\hat{J}^i, \hat{J}^j] = i \varepsilon^{ijk} \hat{J}_k$ which is the Lie algebra of the group of rotations in three dimensional space thereby confirming that the little group corresponding to massive representations is $SO(3)$.

As is common in the literature on relativity, we can segregate Lorentz transformation in two types with distinctly identifiable features viz. (i) the $\theta$ transformations that constitute spatial rotations about the three spatial axes respectively and do not involve any mixing of spatial and temporal coordinates and (ii) the $\tau$ transformations that involve Lorentz boosts along the three spatial axes that are in the nature of spatiotemporal rotations. The coordinate transformation equations under these transformations can be summarized as

$$\begin{align*}
\mathbf{x} &\rightarrow \mathbf{x} + \mathbf{x} \times \theta \\
\quad x^0 &\rightarrow x^0 \quad \text{($\theta$ transformation)} \tag{33}
\end{align*}$$

$$\begin{align*}
\mathbf{x} &\rightarrow \mathbf{x} - \tau x^0 \\
\quad x^0 &\rightarrow x^0 - \tau \mathbf{x} \quad \text{($\tau$ transformation)} \tag{34}
\end{align*}$$

We shall, now, obtain explicit expressions for the Wigner rotation for each of these transformations. Corresponding to $q = (\varepsilon(p_0) M, 0, 0, 0)$, the Lorentz transformation that satisfies the equation $p = \kappa_p q$ takes the explicit form

$$\kappa_p = \begin{pmatrix}
\omega_p / M & \pm p_1 / M & \pm p_2 / M & \pm p_3 / M \\
\pm p_1 / M & 1 + \varpi_p p_1 p_2 & \varpi_p p_1 p_3 & \varpi_p p_1 p_2 \\
\pm p_2 / M & \varpi_p p_2 p_1 & 1 + \varpi_p p_2 p_3 & \varpi_p p_2 p_3 \\
\pm p_3 / M & \varpi_p p_3 p_1 & \varpi_p p_3 p_2 & 1 + \varpi_p p_3^2
\end{pmatrix} \tag{35}
$$

where $\varpi_p = \frac{1}{p^2} (\frac{\varpi_p}{M} - 1)$. In terms of the rapidity, $\zeta = \cosh^{-1} \frac{\omega_p}{M} = \sinh^{-1} \frac{|p|}{M}$, eq. (35) may be written as $(\kappa_p)_j^i = \delta_j^i + \frac{p_j p_i}{p^2} (\cosh \zeta - 1)$, $(\kappa_p)_0^i = (\kappa_p)_i^0 = \frac{p_i}{p^2} \sinh \zeta$ and $(\kappa_p)_0^0 = \cosh \zeta$.

Using this expression we can calculate the Wigner rotation $\eta_p = \kappa_p^{-1} \Lambda_{\kappa_p}^{-1} p$. Now, it is obvious from eq. (29) that $\eta_p = \kappa_p^{-1} \Lambda_{\kappa_p}^{-1} p$ is a member of the “little group” which in this case $(M_{\pm})$ constitutes the group of three dimensional rotations. The action of $\eta_p = \kappa_p^{-1} \Lambda_{\kappa_p}^{-1} p$ on a four vector will, therefore, leave the time component invariant and we can write for $x \equiv (x^0, x)$, $\eta_p x = \eta_p (x^0, x) = (x^0, x + x \times \zeta)$

where $\zeta$ is the rotation induced by the element $\eta_p = \kappa_p^{-1} \Lambda_{\kappa_p}^{-1} p \in SO(3)$. For a $\theta$ transformation $\zeta = \theta$ and for a $\tau$ transformation $\zeta = \pm \frac{\kappa_p}{M + \omega_p}$ if we consider only first order terms in the angles.
2.6 Transformation rules for the Momentum - Spin Eigenstate Basis

The Hilbert space $H$ of a massive particle can be represented as a direct sum of the eigenspaces $H_p$ of the momentum operator i.e. $H = \bigoplus_{p \in \mathbb{R}} H_p$. The subspace that relates to the rest frame of the massive particle has the following properties:

(i) it is invariant with respect to rotations for $\hat{P} \left( \hat{I} - i\hat{J}\theta \right) |0\rangle = \hat{P} |0\rangle - i\hat{P} \hat{J}\theta |0\rangle = 0$ showing that the rotated vector has also zero momentum and hence belongs to $H_0$. In arriving at this result, we have used the commutation relation $[\hat{P}_m, \hat{J}_n] = i\varepsilon^{mnl} \hat{P}_l$.

(ii) If we define the relativistic spin operator for positive energy massive particles as

$$\hat{S} = \frac{1}{M} \hat{P}^0 \hat{j} - \frac{1}{M} \hat{P} \times \hat{K} - \frac{1}{\hat{P}_0 + M} \hat{P} \left( \hat{P}.\hat{j} \right)$$

then for $H_0, \hat{S} \equiv \hat{J}$ for $\hat{S}_3|0\rangle = \left[ \frac{1}{M} \hat{P}^0 \hat{j} - \frac{1}{M} \left( \hat{P}_1 \hat{K}_2 - \hat{P}_2 \hat{K}_1 \right) - \frac{1}{\hat{P}_0 + M} \hat{P}_3 \left( \hat{P}.\hat{j} \right) \right] |0\rangle = \frac{1}{M} \hat{j}_3 \hat{P}^0 |0\rangle = \hat{j}_3 |0\rangle$$

where we have used the commutators $[\hat{P}_0, \hat{J}_n] = 0, \left[ \hat{P}_m, \hat{K}_n \right] = i\delta_{mn} \hat{P}^0$ and the fact that for a zero momentum positive energy particle $p_0 = M$.

(iii) A basis can be constructed in $H_0$ consisting of the eigenvectors of $\hat{S}_3$ and labeled by the eigenvalue of $\hat{S}_3$ e.g. $|0,\sigma\rangle$ where $\hat{S}_3|0,\sigma\rangle = \sigma |0,\sigma\rangle$. The action of the rotation operator on these basis vectors is

$$e^{-i\hat{j}\phi}|0,\sigma\rangle = e^{-i\hat{S}\phi}|0,\sigma\rangle = \sum_{\sigma' = -s}^s D^*_{\sigma'\sigma}(\phi) |0,\sigma'\rangle$$

where the $D^*$ are $(2s + 1) \times (2s + 1)$ rotation matrices.

Our next step is to construct the basis for the remaining subspaces $H_p$ with $p \neq 0$. The basis vectors $|0,\sigma\rangle$ of $H_0$ can be transformed by pure boost transformations to generate basis vectors $|p,\sigma\rangle$ of $H_p$. The appropriate boost transformation that does the trick is $\lambda_p \equiv e^{-i\hat{K}\theta_p} \approx \left( \hat{I} - i\hat{K}\theta_p \right)$ where $\theta_p = \frac{p}{|p|} \sin^{-1} \frac{|p|}{M}$ so that, as a first order approximation $\theta_p \approx \frac{p}{|p|}$. Hence, $|p,\sigma\rangle = N(p) \lambda_p |0,\sigma\rangle = N(p) e^{-i\hat{K}\theta_p} |0,\sigma\rangle \approx N(p) \left( \hat{I} - i\hat{K}\theta_p \right) |0,\sigma\rangle$ with $N(p)$ being a normalization factor. We then have,

$$\hat{P}|p,\sigma\rangle = N(p) \hat{P} e^{-i\hat{K}\theta_p} |0,\sigma\rangle = N(p) \left( e^{-i\hat{K}\theta_p} e^{i\hat{K}\theta_p} \hat{P} e^{-i\hat{K}\theta_p} \right) |0,\sigma\rangle \approx N(p) \left( \hat{I} - i\hat{K}\theta_p \right) \left( \hat{I} - i\hat{K}\theta_p \hat{P} \right) |0,\sigma\rangle \approx N(p) \left( \hat{I} - i\hat{K}\theta_p \right) \left( \hat{P} + \hat{P}^0 \hat{P} \right) |0,\sigma\rangle = N(p) \left( \hat{I} - i\hat{K}\theta_p \right) M \theta_p |0,\sigma\rangle \approx N(p) \left( \hat{I} - i\hat{K}\theta_p \right) p |0,\sigma\rangle = p N(p) \left( \hat{I} - i\hat{K}\theta_p \right) |0,\sigma\rangle = p |p,\sigma\rangle$$

(39)
confirming that \(|p, \sigma\rangle\) is an eigenstate of the momentum operator with momentum \(p\).

Now, consider

\[
\hat{S}_3|p, \sigma\rangle = N(p) \hat{S}_3 e^{-i\hat{K} \hat{\theta}_p} |0, \sigma\rangle = N(p) e^{-i\hat{K} \hat{\theta}_p} e^{+i\hat{K} \hat{\theta}_p} \hat{S}_3 e^{-i\hat{K} \hat{\theta}_p} |0, \sigma\rangle
\]

\[
\approx N(p) \left( \hat{I} - i\hat{K} \hat{\theta}_p \right) \left( \left[ \hat{I} + i\hat{K} \hat{\theta}_p \right] \hat{S}_3 \left( \hat{I} - i\hat{K} \hat{\theta}_p \right) \right) |0, \sigma\rangle
\]

\[
\approx N(p) \left( \hat{I} - i\hat{K} \hat{\theta}_p \right) \left( \hat{S}_3 + i \left[ \hat{K} \hat{\theta}_p, \hat{S}_3 \right] \right) |0, \sigma\rangle
\]

We have \(\hat{S}_3|0, \sigma\rangle = |0, \sigma\rangle\) by definition of the basis vector, so it is labeled by the eigenvalue of \(\hat{S}_3\). To determine \(\left[ \hat{K} \hat{\theta}_p, \hat{S}_3 \right] |0, \sigma\rangle\) we make use of the explicit representation of the spin operator as

\[
\hat{S}_3 = \frac{1}{M} \hat{P}_0 \hat{J}_3 - \frac{1}{M} \left( \hat{P}_1 \hat{K}_2 - \hat{P}_2 \hat{K}_1 \right) - \frac{1}{(F_0 + M)^2} \hat{P}_3 \left( \hat{P} . \hat{J} \right).
\]

We have,

\[
\left[ \hat{K}_1 \hat{\theta}_p, \hat{P}_0 \hat{J}_3 \right] |0, \sigma\rangle = \left\{ \hat{P}_0 \left[ \hat{K}_1 \hat{\theta}_p, \hat{J}_3 \right] + \hat{K}_1 \hat{\theta}_p, \hat{P}_0 \hat{J}_3 \right\} |0, \sigma\rangle = \left( -i \hat{P}_0 \hat{K}_2 \hat{\theta}_p^1 - i \hat{P}_1 \hat{\theta}_p^1 \hat{J}_3 \right) |0, \sigma\rangle
\]

\[
\left[ \hat{K}_2 \hat{\theta}_p, \hat{P}_1 \hat{K}_2 \right] |0, \sigma\rangle = \left\{ \hat{P}_1 \left[ \hat{K}_2 \hat{\theta}_p, \hat{K}_2 \right] + \hat{K}_2 \hat{\theta}_p, \hat{P}_1 \hat{K}_2 \right\} |0, \sigma\rangle = \left( -i (i \hat{P}_0 + \hat{K}_2 \hat{\theta}_p) \hat{\theta}_p^1 \right) |0, \sigma\rangle
\]

so that \(\left[ \hat{K} \hat{\theta}_p, \hat{P}_0 \hat{J}_3 \right] |0, \sigma\rangle = \hat{I} |0, \sigma\rangle\).

Putting all these pieces together, we get

\[
\hat{S}_3|p, \sigma\rangle = \sigma N(p) \left( \hat{I} - i\hat{K} \hat{\theta}_p \right) |0, \sigma\rangle = |p, \sigma\rangle \quad (40)
\]

showing that \(|p, \sigma\rangle\) is an eigenstate of \(\hat{S}_3\) with eigenvalue \(\sigma\).

The effect of various transformations constituting the Poincare group on the basis vectors is summarized thus:

(a) Translations - We have

\[
e^{-iP^a |p, \sigma\rangle = e^{-iP^a |p, \sigma\rangle, e^{iHx^a |p, \sigma\rangle = e^{iwp^a} |p, \sigma\rangle \quad (41)}
\]

with \(\omega^2 = p^2 + M^2\).

(b) Spatial Rotations - We have \(e^{-iJ^a |p, \sigma\rangle = N(p) e^{-iJ^a \hat{K} \hat{\theta}_p} |0, \sigma\rangle\)

\[
= N(p) e^{-iJ^a \hat{K} \hat{\theta}_p e^{iJ^a \hat{K} \hat{\theta}_p} |0, \sigma\rangle = N(p) e^{-i(R(\phi)^{-1} \hat{K}) \hat{\theta}_p \sum_{\sigma' = -s} D_{\sigma, \sigma'}(\phi) |0, \sigma'}
\]

\]}
the spin indices for brevity, we have

\[ I \varpropto \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma} (\phi) |0, \sigma'| \]

where we have used \( \hat{U} (R) \hat{P}_i |p| = \hat{U} (R) \hat{P}_i \hat{U} (R)^{-1} \hat{U} (R) |p| = \hat{U} (R) \hat{P}_i \hat{U} (R)^{-1} |p' \)

\[ \hat{U} (R) \hat{P}_i \hat{U} (R)^{-1} = \sum_{j} (R^{-1})^j_i \hat{P}_j = \sum_{j} (R^T)^j_i \hat{P}_j. \]

(c) Lorentz Boosts - Let us apply a Lorentz boost \( \Lambda \) to a basis vector \(|p, \sigma\rangle\) to obtain \( \Lambda|p, \sigma\rangle = \Lambda N (p) \lambda_p |0, \sigma\rangle \). Now, the right hand side, being the product of two boosts, is also a Lorentz transformation and hence, can be represented by the product of a spatial rotation followed by a boost i.e. \( \Lambda|p, \sigma\rangle = N (p) \Lambda \lambda_p |0, \sigma\rangle \)

or equivalently \( \lambda_{p'}^{-1} \Lambda \lambda_p |0, \sigma\rangle = R (\phi_W (p, \Lambda)) |0, \sigma\rangle \). Now, we have shown above that a rotation keeps invariant the subspace of zero momentum. It follows that the sequence of boosts on the left hand side must return each vector of zero momentum to the subspace of zero momentum. Now, \( \lambda_p \) transforms a vector of zero momentum to a vector of momentum \( p \). Subsequent application of the boost \( \Lambda \) would transform this vector’s momentum to \( \Lambda p \).

It follows that the boost \( \lambda_{p'} \) will transform the vector with momentum \( \Lambda p \) back to zero i.e. \( \lambda_{p'} = \lambda_{\Lambda p} \).

Therefore, \( e^{-iK_{\theta}} |p, \sigma\rangle = N (p) \Lambda \lambda_p |0, \sigma\rangle = N (p) \lambda_p \Lambda R (\phi_W (p, \Lambda)) |0, \sigma\rangle \)

\[ = N (p) \lambda_p \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma} (\phi_W (p, \Lambda)) |0, \sigma'\rangle = \frac{N (p)}{N (\Lambda p)} \sum_{\sigma'=-s}^{s} D_{\sigma'\sigma} (\phi_W (p, \Lambda)) |\Lambda p, \sigma'\rangle \]

where \( D_{\sigma'\sigma} (\phi_W (p, \Lambda)) \) are the \((2s + 1) \times (2s + 1)\) unitary rotation matrix representations of the Wigner rotations.

To determine the normalization factor \( N (p) \), we invoke the requirement of the invariance of identity under Lorentz boosts i.e. \( I = e^{-iK_{\theta}} I e^{iK_{\theta}} \). Introducing the spectral resolution of the identity in the momentum representation \( I = \int dp |p\rangle \langle p| \) and omitting the spin indices for brevity, we have

\[ I = e^{-iK_{\theta}} \left( \int dp |p\rangle \langle p| \right) e^{iK_{\theta}} = \int dp \frac{N (p)}{N (\Lambda p)} |\Lambda p\rangle \langle \Lambda p| = \int (\Lambda p) det | \frac{d(p)}{d(\Lambda p)} | \frac{N (p)}{N (\Lambda p)} |\Lambda p\rangle \langle \Lambda p| \]

Since the Jacobian of the transformation \( p \rightarrow \Lambda p \) viz. \( \det | \frac{dp}{d(\Lambda p)} | \) should not depend on the direction of the boost, we choose a boost along the direction of the \( x^3 \) axis so that \( p_1 = (\Lambda p)_1, p_2 = (\Lambda p)_2, p_3 = (\Lambda p)_3 \cos \theta - \omega_{\Lambda p} \sin \theta, \)

\( \omega_p = \sqrt{M^2 + (\Lambda p)_2^2 + (\Lambda p)_2^2 [(\Lambda p)_3 \cos \theta - \omega_{\Lambda p} \sin \theta]^2} \)

\( = \omega_{\Lambda p} \cos \theta - (\Lambda p)_3 \sin \theta. \)

This gives
with the normalization
\[ \langle \psi, \psi \rangle = \langle (\Lambda p) \rangle = \delta_{pp'} \]
and ket vectors have the same energy sign. Under a Lorentz transformation \( \hat{\Lambda} \), we obtain
\[ \psi' = \hat{\Lambda} \, \psi \]
Two different eigenstates of the momentum operator, being different eigenstates of a hermitian operator must necessarily be orthogonal. i.e. \( \langle p | p' \rangle = 0 \) if \( p \neq p' \). Had the spectrum of eigenvalues \( \text{Spec} \{ p \} \) being discrete, we would have the simple normalization \( \langle p | p' \rangle = \delta_{pp'} \). However, such a normalization is obviously not possible for continuous spectra. To obviate this problem and obtain normalizable vectors, we introduce momentum space wavefunctions \( \psi(p) \) and write an arbitrary state vector as \( |\Psi\rangle = \int dp \psi(p) |p\rangle \)
with the normalization \( \langle p' | p \rangle = \delta (p - p') \) so that
\[ \langle p | \Psi \rangle = \langle p | \int dp' \psi(p') |p'\rangle = \int dp' \psi(p') \langle p | p' \rangle = \int dp' \psi(p') \delta (p - p') = \psi(p) \]
where we have, for the time being omitted the spin index and identified the eigenstates only by the momentum eigenvalue.

The transformation law for the momentum space wavefunction \( \psi^m(p) = \langle p, m | \Psi \rangle \)
under a Lorentz transformation \( \hat{\Lambda} = \hat{Q}(\eta_p, q) \hat{S}(\Lambda) \)
can be written as
\[ \psi'^m(p) = \langle p, m | \Psi' \rangle = \sum_m \int \frac{dp'}{\omega_p'} \langle p, m | \hat{Q}(\eta_p, q) |p', m'\rangle \langle p, m | \hat{S}(\Lambda) |\Psi\rangle \] (44)
where we have used the completeness property of the state vectors and where the bra and ket vectors have the same energy sign.

Using eq. (20) and the fact that \( \hat{Q}(\eta_p, q) \) is diagonal with respect to the operator \( \hat{P}_\mu \), we obtain
\[ \langle p, m | \hat{Q}(\eta_p, q) |p', m'\rangle = \omega_p D^s \left[ \hat{Q}(\eta_p, q) \right]_{mm'}^m \delta (p - p') \] (45)
Furthermore, since \( \hat{S}(\Lambda) \) is unitary, we have \( \hat{S}(\Lambda)\dagger = \hat{S}(\Lambda)^{-1} = \hat{S}(\Lambda^{-1}) \) whence
\[ \langle p, m | \hat{S}(\Lambda) |\Psi\rangle = \langle \Lambda^{-1}p, m | \Psi \rangle = \psi^m(\Lambda^{-1}p) \] (46)
Eq. (44) then becomes
\[
\psi'^m(p) = \sum_{mm'} \int \frac{dp'}{\omega_p'} \left\{ \omega_p D^s \left[ \hat{Q}(\eta_p, q) \right]_{mm'}^m \delta (p - p') \right\} \langle \Lambda^{-1}p', m' | \Psi \rangle = \sum_{mm'} D^s \left[ \hat{Q}(\eta_p, q) \right]_{mm'}^m \psi^m(\Lambda^{-1}p) \] (47)
The inner product of two state vectors with same energy signs takes the form

\[ \langle p_1, m_1 \mid p_2, m_2 \rangle = \sum_m \int \frac{dp}{\omega_p} \psi^m_{\ast}(p_1) \psi^m(p_2) \]  

(48)

For the case of infinitesimal rotations induced by \( \hat{Q} (\eta_p, q) \) (which is a unitary representation of the three dimensional rotation operator \( \eta_p = \kappa^{-1}_p \Lambda \kappa^{-1} \in SO(3) \) inducing a rotation by an angle \( \zeta \)), we have

\[ \psi^m(p) = \sum_{m'} (I + i \hat{S} \zeta)^m_{m'} \psi^m(p - p \times \zeta) = \sum_{m'} \left[ I + i \zeta \left( \frac{1}{i} p \times \nabla_p + \hat{S} \right) \right] m' \psi^m(p) \]  

(49)

where we have made used of the differential representation of the translation operator. Defining the generators of spatial rotations, \( \hat{J} \), and Lorentz boosts, \( \hat{K} \) by

\[ \hat{J} = -i \left( p \times \nabla_p \right) + \hat{S} \]  

(50)

and

\[ \hat{K} = p_0 \left( i \nabla_p + \frac{p \times \hat{S}}{\omega_p (M + \omega_p)} \right) \]  

(51)

we can write the transformation laws for the \( \theta \) transformation and \( \tau \) transformation as

\[ \psi^m(p) = \begin{cases} 
\sum_{m'} (I + i \hat{J} \theta)^m_{m'} \psi^{m'}(p) \quad (\theta \text{ transformation}) \\
\sum_{m'} (I - i \hat{K} \tau)^m_{m'} \psi^{m'}(p) \quad (\tau \text{ transformation}) 
\end{cases} \]  

(52)

In deriving the above expressions, we have used e.g.

\[ \hat{K}_1 \psi(p) = i \lim_{\theta \to 0} \frac{d}{d\theta} \left[ e^{-i \hat{K}_1 \theta} \psi(p, \sigma) \right] = i \lim_{\theta \to 0} \frac{d}{d\theta} \left[ \langle p, \sigma \mid e^{-i \hat{K}_1 \theta} \mid \Psi \rangle \right] \]

\[ = i \lim_{\theta \to 0} \frac{d}{d\theta} \left[ \langle p, \sigma \mid e^{i \hat{K}_1 \theta} \right] \mid \Psi \rangle \]  

\[ = i \lim_{\theta \to 0} \frac{d}{d\theta} \left[ \langle p, \sigma \mid e^{i \hat{K}_1 \theta} \right] \mid \Psi \rangle \]  

\[ = i \lim_{\theta \to 0} \frac{d}{d\theta} \left[ \langle p, \sigma \mid e^{i \hat{K}_1 \theta} \right] \mid \Psi \rangle \]  

\[ = i \lim_{\theta \to 0} \sum_{\sigma' = -s}^s D_{\sigma' \sigma} (-\phi_W(p, -\Lambda)) \frac{d}{d\theta} \left[ \sqrt{\frac{\omega_{\Lambda^{-1}}}{\omega_p}} \psi(\Lambda^{-1} p, \sigma) \right] \]

\[ = i \lim_{\theta \to 0} \sum_{\sigma' = -s}^s D_{\sigma' \sigma} (-\phi_W(p, -\Lambda)) \frac{d}{d\theta} \left[ \sqrt{\frac{\omega_{\Lambda^{-1}}}{\omega_p}} \right] \psi(p_1 \cosh \theta - \sqrt{p_1^2 + M^2} \sinh \theta, p_2, p_3, \sigma) \]
\( = -i \sum_{\sigma' = -s}^{s} D_{\sigma' \sigma} (-\phi_W (p, -\Lambda)) \left[ \frac{d}{dp_1} + \frac{p_1}{2\omega_p} \right] \psi(p, \sigma) \)  

(53)

3. Lorentz Transformation of Bell States

We write the state vector corresponding to two massive spin \( \frac{1}{2} \) particles in the momentum representation as

\[ \left| \Psi_{AB} \right\rangle = \sum_{\sigma_1, \sigma_2} \int \int \frac{dp_1 dp_2}{(2\omega_{p_1})(2\omega_{p_2})} \psi_{\sigma_1, \sigma_2} (p_1, p_2) \left| p_1, \sigma_1 \right\rangle \left| p_2, \sigma_2 \right\rangle_B \]  

(54)

where the two particle wavefunctions satisfy the condition

\[ \sum_{\sigma_1, \sigma_2} \int \int \frac{dp_1 dp_2}{(2\omega_{p_1})(2\omega_{p_2})} |\psi_{\sigma_1, \sigma_2}(p_1, p_2)|^2 = 1 \]  

(55)

Since, a multiparticle state transforms as the direct product of single particle states, the relativistic spin entangled Bell state \( \left| \phi \right\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \right) \) can be expressed in terms of Dirac spinors as:

\[ \left| \Phi \right\rangle = \frac{1}{\sqrt{2}} \left( u_A \left( p, \frac{1}{2} \right) \otimes u_B \left( -p, \frac{1}{2} \right) + u_A \left( p, -\frac{1}{2} \right) \otimes u_B \left( -p, -\frac{1}{2} \right) \right) \]  

(56)

To study the effect of relativistic transformations on entangled states, we take the Bell state as a prototype. Further, since quantum multiparticle states transform as direct products of single particle states, it is sufficient for us to study the behaviour of the Dirac spinors \( u(p, s) \) under such transformations.

The Dirac spinors \( u_{A(B)} (\pm p, \pm \frac{1}{2}) \) representing particles in motion with momenta \( \pm p \) can be obtained from the corresponding rest frame spinors \( u_{A(B)} (0, \frac{1}{2}) = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T \) and \( u_{A(B)} (0, -\frac{1}{2}) = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T \) by imparting a suitable boost transformation as explained below.

Consider the motion of a wave particle traveling with a velocity \( v \equiv \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \) in an arbitrary direction represented by the unit vector \( n \equiv \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \) i.e. \( \text{i.e.} \ v = |v| n = |v| \begin{pmatrix} \cos \alpha & \cos \beta & \cos \gamma \end{pmatrix} \). The generators of unit Lorentz transformations along the direction of the unit vector \( n \equiv \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \) are given by:-
\[
(I_n)^\nu_\mu = \begin{pmatrix}
0 & -\cos \alpha - \cos \beta - \cos \gamma \\
-\cos \alpha & 0 & 0 & 0 \\
-\cos \beta & 0 & 0 & 0 \\
-\cos \gamma & 0 & 0 & 0
\end{pmatrix}
\] (57)

and the infinitesimal Lorentz transformations along \( n \equiv (n_1, n_2, n_3) \) are

\[
x^{\nu} = a^{\nu}_{\mu} x^{\mu} = x^{\nu} + \Delta \partial^{\nu}_{\mu} x^{\mu} = x^{\nu} + \Delta \partial (I_n)^{\nu}_{\mu} x^{\mu}
\] (58)

The spinor transformation operator \( \hat{S} \), corresponding to the above Lorentz transformation is:-

\[
\hat{S} = \exp \left[ -\frac{i}{4} \partial \tilde{\sigma}_{\mu \nu} (I_n)^{\nu \mu} \right] = \exp \left\{ -\frac{i\vartheta}{4} \left[ (\hat{\sigma}_{01} I_n^{01} + \hat{\sigma}_{02} I_n^{02} + \hat{\sigma}_{03} I_n^{03}) + (\hat{\sigma}_{10} I_n^{10} + \hat{\sigma}_{20} I_n^{20} + \hat{\sigma}_{30} I_n^{30}) \right] \right\}
\]

because all the other elements of \( I_n \) vanish. Now, since \( \sigma_{\mu \nu} = -\sigma_{\nu \mu} \) and \( I_n^{10} = g^{00} (I_n)^{1}_0 = (I_n)^{1}_0 = (I_n)^0_1 = -g^{11} (I_n)^0_1 = -I_n^{01} \) etc., we have

\[
\hat{S} = \exp \left[ -\frac{i\vartheta}{2} (\hat{\sigma}_{01} \cos \alpha + \hat{\sigma}_{02} \cos \beta + \hat{\sigma}_{03} \cos \gamma) \right]
\] (60)

where we have used \( I_n^{01} = g^{11} (I_n)^0_1 = -(I_n)^0_1 = \cos \alpha \) etc. This gives

\[
\hat{S} = \exp \left( -\frac{\vartheta}{2} \alpha . n \right) = \exp \left( -\frac{\vartheta}{2} \frac{\alpha v}{|v|} \right)
\] (61)

since \( \sigma_{0i} = \frac{1}{2} [\gamma_0, \gamma_i] = i\gamma_0 \gamma_i = i\gamma^0 \gamma^i g_{ii} = -i\gamma^0 \gamma^i = -i\alpha_i \).

Now,

\[
(\alpha . v)^2 = (\alpha^i v_i) (\alpha^j v_j) = \alpha^i \alpha^j v_i v_j = \frac{1}{2} (\alpha^i \alpha^j v_i v_j + \alpha^j \alpha^i v_i v_j) = \frac{1}{2} (\alpha^i \alpha^j + \alpha^j \alpha^i) v_i v_j = \delta^{ij} v_i v_j = v^2
\] (62)

Therefore, \( \frac{(\alpha . v)^2}{|v|^2} = 1 \) and on expanding the exponential of eq. (61) as a power series in \( \left(-\frac{\vartheta}{2} \frac{\alpha . v}{|v|}\right) \) and collecting even and odd powers in \( \frac{\vartheta}{2} \) we obtain

\[
\hat{S} = \exp \left( -\frac{\vartheta}{2} \frac{\alpha . v}{|v|} \right) = I \left[ 1 + \left( -\frac{1}{2!} \frac{\vartheta}{|v|} \right) (\alpha . v) + \frac{1}{2!} \left( -\frac{1}{2!} \frac{\vartheta}{|v|} \right)^2 (\alpha . v)^2 + ... \right]
\]

\[
= I \left[ 1 + \frac{1}{2!} \left( \frac{1}{2} \frac{\vartheta}{|v|} \right)^2 + ... \right] - \left( \frac{\alpha . v}{|v|} \right) \left[ \left( \frac{1}{2} \frac{\vartheta}{|v|} \right) + \frac{1}{3!} \left( \frac{1}{2} \frac{\vartheta}{|v|} \right)^3 + ... \right] = I \cosh \left( \frac{\vartheta}{2} \right) - \left( \frac{\alpha . v}{|v|} \right) \sinh \left( \frac{\vartheta}{2} \right)
\]
\[ \alpha.v = \sum_{k=1}^{3} \alpha^k v_k = \sum_{k=1}^{3} \left( \begin{array}{cc} 0 & \sigma^k v_k \\ \sigma^k v_k & 0 \end{array} \right), \quad v_k = \frac{p_k}{|p|}, \quad \cosh \left( \frac{\vartheta}{2} \right) = \sqrt{\frac{p_0 + M}{2M}}, \quad \sinh \left( \frac{\vartheta}{2} \right) = -\sqrt{\frac{p_0 - M}{2M}} \]

so that

\[
\hat{S} = \frac{1}{\sqrt{2M}} \begin{pmatrix} (p_0 + M)^{1/2} & 0 & \frac{p_3(p_0 - M)^{1/2}}{|p|} \\ 0 & (p_0 + M)^{1/2} & \frac{(p_1 + ip_2)(p_0 - M)^{1/2}}{|p|} \\ \frac{p_3(p_0 - M)^{1/2}}{|p|} & \frac{(p_1 + ip_2)(p_0 - M)^{1/2}}{|p|} & (p_0 + M)^{1/2} \end{pmatrix}
\]

and

\[
\sinh \left( \frac{\vartheta}{2} \right) = \frac{p_3}{|p|} \sinh \left( \frac{\vartheta}{2} \right)
\]

Our objective here is to examine the impact of a Lorentz transformation on the Bell state given by eq. (56).
Let us, now, impart a boost transformation,

\[
\Lambda = \begin{pmatrix}
\cosh \tau & \sinh \tau & 0 & 0 \\
\sinh \tau & \cosh \tau & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
directed along the $x^1$ axis and identified by the rapidity $\tau = -\tanh v_1$ on the spinor $u(p, s)$. Under this transformation, the components of the momentum vector transform as $p'^\mu = (\Lambda p)^\mu = \Lambda^\mu_\nu p^\nu$ so that

$$p' \equiv (p_0 \cosh \tau + p_1 \sinh \tau, p_0 \sinh \tau + p_1 \cosh \tau, p_2, p_3)$$ (66)

and

$$p'' = \Lambda^{-1} p = (p_0 \cosh \tau - p_1 \sinh \tau, -p_0 \sinh \tau + p_1 \cosh \tau, p_2, p_3)$$ (67)

Under this transformation, the spinors $u(p, \sigma)$ will transform as

$$\begin{pmatrix} u'(p, \frac{1}{2}) \\ u'(p, -\frac{1}{2}) \end{pmatrix} = D(\phi_W(p, \Lambda)) \begin{pmatrix} u(\Lambda^{-1} p, \frac{1}{2}) \\ u'(\Lambda^{-1} p, -\frac{1}{2}) \end{pmatrix}$$ (68)

where $\Lambda p$ is the spatial component of $\Lambda p$, $D(\phi_W(\pm p, \Lambda))$ are the unitary representations of the three dimensional Wigner rotation corresponding to the Lorentz transformation $\Lambda$. $u(\pm \Lambda^{-1} p, \sigma)$ can be calculated by using eqs. (64) & (67). Our task is, therefore, now confined to obtaining the Wigner rotation corresponding to $\Lambda$.

Hence, our next step is to construct the Wigner rotation and identify the Wigner angle corresponding to the Lorentz boost given by eq. (65). To keep the calculations as simple as possible so as not to obscure the physical content, we consider the standard vector as $l \equiv (M, 0, 0, 0)$ and subject it to a Lorentz boost $\kappa_k$ directed along the $x^3$ axis and identified by the rapidity parameter $\varpi$. Its matrix representation is

$$\kappa_k = \begin{pmatrix} \cosh \varpi & 0 & 0 & \sinh \varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \varpi & 0 & 0 & \cosh \varpi \end{pmatrix}$$ (69)

Under the effect of this transformation, the standard vector transforms as $k \equiv \kappa_k l = (M \cosh \varpi, 0, 0, M \sinh \varpi)$. We, now, have

$$\kappa_k^{-1} = \begin{pmatrix} \cosh \varpi & 0 & 0 & -\sinh \varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \varpi & 0 & 0 & \cosh \varpi \end{pmatrix}$$ (70)
\[ \kappa^{-1}_k \Lambda = \begin{pmatrix} \cosh \varpi \cosh \tau & \cosh \varpi \sinh \tau & 0 & -\sinh \varpi \\ \sinh \tau & \cosh \tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \varpi \cosh \tau & -\sinh \varpi \sinh \tau & 0 & \cosh \varpi \end{pmatrix} \] 

(71)

and

\[ \Lambda^{-1}k = (M \cosh \varpi \cosh \tau, -M \cosh \varpi \sinh \tau, 0, M \sinh \varpi) \]

so that

\[ \kappa^{-1}_k \Lambda = \begin{pmatrix} \cosh \varpi \cosh \tau & -\cosh \varpi \sinh \tau & 0 & \sinh \varpi \\ -\cosh \varpi \sinh \tau & 1 + \alpha (M \cosh \varpi \sinh \tau)^2 & 0 & -\alpha M^2 \cosh \varpi \sinh \varpi \sinh \tau \\ 0 & 0 & 1 & 0 \\ \sinh \varpi & -\alpha M^2 \cosh \varpi \sinh \varpi \sinh \tau & 0 & 1 + \alpha M^2 \sinh^2 \varpi \end{pmatrix} \]

(72)

where

\[ \alpha = \frac{(\cosh \varpi \cosh \tau - 1)}{M^2 (\cosh^2 \varpi \cosh^2 \tau - 1)} = \frac{1}{M^2 (\cosh \varpi \cosh \tau + 1)} \]

(73)

Instead of making explicit calculations of the Wigner rotation \( \eta_k = \kappa^{-1}_k \Lambda \kappa^{-1}_{\Lambda^{-1}k} \) which are quite cumbersome, we examine the impact of \( \eta_k \) on the spatial basis vectors \((i, j, k) \equiv ((0, 1, 0, 0)^T, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T)\). The results are

\[ \eta_i = \eta_k (0, 1, 0, 0) = \kappa^{-1}_k \Lambda \left( -\cosh \varpi \sinh \tau, 1 + \alpha M^2 \cosh^2 \varpi \sinh^2 \tau, 0, -\alpha M^2 \cosh \varpi \sinh \varpi \sinh \tau \right) \]

\[ = \begin{pmatrix} 0 \\ \frac{\cosh \varpi + \cosh \tau}{\cosh \varpi \cosh \tau + 1} \\ 0 \end{pmatrix}, \frac{\sinh \varpi \sinh \tau}{\cosh \varpi \cosh \tau + 1} \]

(74)

\[ \eta_j = \eta_k (0, 0, 1, 0) = (0, 0, 1, 0) \]

(75)

\[ \eta_k = \eta_k (0, 0, 0, 1) = \kappa^{-1}_k \Lambda \left( \sinh \varpi, -\alpha M^2 \cosh \varpi \sinh \varpi \sinh \tau, 0, 1 + \alpha M^2 \sinh^2 \varpi \right) \]

\[ = \begin{pmatrix} 0 \\ \frac{\sinh \varpi \sinh \tau}{\cosh \varpi \cosh \tau + 1} \\ 0 \end{pmatrix}, \frac{\cosh \varpi + \cosh \tau}{\cosh \varpi \cosh \tau + 1} \]

(76)
The above expressions for the three transformed spatial basis vectors vindicate that the Wigner rotation is a spatial rotation that takes the form of a rotation about the \( x^2 \) axis and further, identify the Wigner angle as

\[
\phi_W = \tan^{-1} \frac{\sinh \varpi \sinh \tau}{\cosh \varpi + \cosh \tau}
\]  

(77)

Using eq. (77), we can write the unitary representation of the three dimensional Wigner rotation as

\[
D^{1/2} (\phi_W (p, \Lambda)) = \begin{pmatrix}
\cos \phi_W & -\sin \phi_W \\
\sin \phi_W & \cos \phi_W
\end{pmatrix}
\]  

(78)

From eqs. (68) and (78), we arrive at expression for the transformation of the Dirac spinors and hence, of the Bell state as

\[
u' \left( p, \frac{1}{2} \right) = \cos \phi_W u \left( \Lambda^{-1} p, \frac{1}{2} \right) - \sin \phi_W u \left( \Lambda^{-1} p, -\frac{1}{2} \right)
\]

(79)

\[
u' \left( p, -\frac{1}{2} \right) = \sin \phi_W u \left( \Lambda^{-1} p, \frac{1}{2} \right) + \cos \phi_W u \left( \Lambda^{-1} p, -\frac{1}{2} \right)
\]

(80)

Conclusions

We have, thus, shown in this paper that the constituent particles of the maximally entangled Bell states, when subject to Lorentz boosts undergo a momentum dependent Wigner rotation (which is a rotation in spatial coordinates and hence, is unitary). The entanglement fidelity is, therefore, preserved under such transformations Explicit calculation for the Wigner rotation corresponding to a boost along the \( x^1 \) axis is performed and the it is shown to be a rotation about the \( x^2 \) axis through an angle that is momentum dependent and is given by eq. (77). As is shown, the rotation matrix, being unitary, restores unitarity in the transformation. It is also shown that the spins in the Bell state undergo a reorientation in the direction of the boost when observed from a frame moving with a constant velocity with respect to the rest frame. The point to be noted here is that since a multi particle state transforms as the direct product of single particle states, the transformed multiparticle state consists of a linear combination of spin states with the transformed momenta. It follows that if we obtain the density matrix of the “reduced state” by tracing over one of the states, we will still get a maximally entangled (mixed) density matrix implying that the entanglement fidelity is not disturbed under such Lorentz transformations as are considered here.

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Partial Swapping, Unitarity and No-signalling

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Abstract: It is a well known fact that a quantum state $|\psi(\theta, \phi)\rangle$ is represented by a point on the Bloch sphere, characterized by two parameters $\theta$ and $\phi$. In a recent work we already proved that it is impossible to partially swap these quantum parameters. Here in this work we will show that this impossibility theorem is consistent with principles like unitarity of quantum mechanics and no signalling principle.

Keywords: Quantum Information Theory; Partial Swapping; impossibility theorem; Unitarity of Quantum Mechanics
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1. Introduction

In quantum information theory understanding the limits of fidelity of different operations has become an important area of research. Noticing these kind of operations which are feasible in classical world but have a much restricted domain in quantum information theory started with the famous 'no-cloning' theorem [1]. The theorem states that one cannot make a perfect replica of a single quantum state. Later it was also shown by Pati and Braunstein that we cannot delete either of the two quantum states when we are provided with two identical quantum states at our input port [2]. In spite of these two famous 'no-cloning' [1] and 'no-deletion' [2] theorem there are many other 'no-go' theorems like 'no-self replication' [3], 'no-partial erasure' [4], 'no-splitting' [5] and many more which have come up.

Recently in ref [6], we introduce a new no-go theorem, which we refer as 'no partial

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swapping’ of quantum information. Since we know that the information content in a qubit is dependent on the angles azimuthal and phase angles $\theta$ and $\phi$, then the partial swapping of quantum parameters $\theta$ and $\phi$ is given by,

$$\begin{align*}
|A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| & \rightarrow |A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| \\
|A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| & \rightarrow |A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)|
\end{align*}$$

(1)

(2)

However in ref [6] we showed that this operation is impossible in the quantum domain from the linear structure of quantum theory.

In this work we once again claim this impossibility from two different principles namely [i] unitarity of quantum mechanics [ii] no signalling principle. The organization of the work is as follows: In the first section we will prove this impossibility from the unitarity of quantum mechanics. In the second section we will do the same from the principle of no signalling. Then the conclusion follows.

2. Partial Swapping: Unitarity of Quantum Mechanics

Let us consider a set $S$ consisting of two non orthogonal states $S = \{|A(\theta_1, \phi_1)|, |B(\theta_2, \phi_2)|\}$

Let us assume that hypothetically it is possible to partially swap the parameters of these two states $|A(\theta_1, \phi_1)|, |B(\theta_2, \phi_2)|$.

First of all we will assume that swapping of phase angles of two quantum states are possible, keeping the azimuthal angles fixed. Therefore the transformation describing such an action is given by,

$$\begin{align*}
|A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| & \rightarrow |A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| \\
|A(\theta_2, \phi_2)||\bar{A}(\theta_2, \phi_2)| & \rightarrow |A(\theta_2, \phi_2)||\bar{A}(\theta_2, \phi_2)|
\end{align*}$$

(3)

To preserve the unitarity of the above transformation, will preserve the inner product.

$$\langle A(\theta_1, \phi_1)|A(\theta_2, \phi_2)\rangle|\bar{A}(\theta_1, \phi_1)||\bar{A}(\theta_2, \phi_2)\rangle = \langle A(\theta_1, \phi_1)|A(\theta_2, \phi_2)\rangle|\bar{A}(\theta_1, \phi_1)||\bar{A}(\theta_2, \phi_2)\rangle$$

(4)

The above equality will not hold for all values of $(\theta, \phi)$. The equality will hold if

$$i \tan \frac{\theta_1}{2} \tan \frac{\phi_2}{2} = \tan \frac{\theta_1}{2} \tan \frac{\phi_2}{2} \text{ or } ii (\phi_2 - \phi_1) = (\phi_2 - \phi_1) \pm 2k\pi, \text{ where } k \text{ is an integer.}$$

These two conditions characterizes the set of states on the Bloch sphere for which the partial swapping of the phase angles are possible. However in general this is not possible for all possible values of $\theta, \phi$.

Let us now assume that partial swapping of azimuthal angles are possible, without altering the phase angles of the quantum states.

$$\begin{align*}
|A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| & \rightarrow |A(\theta_1, \phi_1)||\bar{A}(\theta_1, \phi_1)| \\
|A(\theta_2, \phi_2)||\bar{A}(\theta_2, \phi_2)| & \rightarrow |A(\theta_2, \phi_2)||\bar{A}(\theta_2, \phi_2)|
\end{align*}$$

(5)

Now once again, in order to preserve the unitarity of such a transformation we arrive at the same conditions, (i) and (ii). This clearly indicates the fact that there are certain class
of states on the Bloch sphere for which partial swapping of phase angles and azimuthal angles are possible. However in this context we cannot say that this is true for all such values of \( \theta \) and \( \phi \) on the Bloch sphere. Therefore it is evident that the unitarity of quantum mechanics, doesn’t allow partial swapping of quantum parameters for all such pairs of non-orthogonal states on the Bloch sphere.

3. Partial Swapping: Principle of No signalling

Suppose we have two identical singlet states \( |\chi\rangle \) shared by two distant parties Alice and Bob. Since the singlet states are invariant under local unitary operations, it remains same in all basis. The states are given by

\[
|\chi\rangle|\chi\rangle = \frac{1}{2}(|\psi_1(\theta_1, \phi_1)\rangle|\bar{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\rangle - |\bar{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\rangle|\psi_1(\theta_1, \phi_1)\rangle)
\]

where \( \{ |\psi_1\rangle, |\bar{\psi}_1\rangle \} \) and \( \{ |\psi_2\rangle, |\bar{\psi}_2\rangle \} \) are two sets of mutually orthogonal spin states (qubit basis). Alice possesses the first particle while Bob possesses the second particle. Alice can choose to measure the spin in any one of the qubit basis namely \( \{ |\psi_1\rangle, |\bar{\psi}_1\rangle \} \), \( \{ |\psi_2\rangle, |\bar{\psi}_2\rangle \} \).

The theorem of no signalling tells us that the measurement outcome of one of the two parties is invariant under local unitary transformation done by other party on his or her qubit. The density matrix \( \rho_B = tr \rho_{AB} = tr[(U_A \otimes I_B)\rho_{AB}(U_A \otimes I_B)^\dagger] \) is invariant under local unitary operation by the other party. Hence the first party cannot distinguish two mixtures due to the unitary operation done at remote place.

At this point one may ask if Alice (Bob) partially swap the quantum parameters of her (his) particle and if Bob (Alice) measure his (her) particle in either of the two basis then is there any possibility that Alice (Bob) know the basis in which Bob (Alice) measures his (her) qubit or in other words, is there any way by which Alice (Bob) using a perfect partial swapping machine can distinguish the statistical mixture in her (his) subsystem resulting from the measurement done by Bob (Alice). If Alice (Bob) can do this then signalling will take place, which is impossible. Note that whatever measurement Bob (Alice) does, Alice (Bob) does not learn the results and her (his) description will remain as that of a completely random mixture, i.e., \( \rho_{A(B)} = \frac{I}{2} \otimes \frac{I}{2} \). In other words we can say that the local operations performed on his (her) subspace has no effect on Alice’s (Bob’s) description of her (his) states.

Let us consider a situation where Alice is in possession of a hypothetical machine which can partially swap quantum parameters \( \theta \) and \( \phi \).

Let us first of all consider the case where with the help of the machine we can partially swap the phase angles keeping the azimuthal angles of the states fixed. The action of
such a machine is given by,

$$|\psi_i(\theta_i, \phi_i)||\tilde{\psi}_i(\bar{\theta}_i, \bar{\phi}_i)| \longrightarrow |\psi_i(\theta_i, \phi_i)||\tilde{\psi}_i(\bar{\theta}_i, \bar{\phi}_i)|$$

(7)

where \((i = 1, 2)\). Now if after the action of such a transformation on Alice’s qubit, the entangled state initially shared between these two parties takes the form,

$$|\chi\rangle_{PS}|\chi\rangle_{PS} = \frac{1}{2}([|\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)]_A(|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)]_B +$$

$$([|\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)]_A(|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)]_B -$$

$$([|\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)]_A(|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)]_B -$$

$$([|\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)]_A(|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)]_B)$$

$$= \frac{1}{2}([|\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)]_A(|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)]_B +$$

$$([|\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)]_A(|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)]_B -$$

$$([|\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)]_A(|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)]_B -$$

$$([|\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)]_A(|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)]_B)$$

(8)

where A, B denotes the particles in Alice’s and Bob’s possession respectively. Now, if Bob does his measurement on \{\(|\psi_1\rangle, |\tilde{\psi}_1\rangle\}\ qubit basis, then the reduced density matrix describing Alice’s subsystem is given by,

$$\rho_A = \frac{1}{4}(|\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)|\langle\psi_1(\theta_1, \phi_1)\psi_1(\theta_1, \phi_1)| +$$

$$|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)|\langle\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)| +$$

$$|\psi_1(\theta_1, \phi_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)|\langle\psi_1(\theta_1, \phi_1)\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)| +$$

$$|\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\psi_1(\theta_1, \phi_1)|\langle\tilde{\psi}_1(\bar{\theta}_1, \bar{\phi}_1)\psi_1(\theta_1, \phi_1)|$$

(9)

Interestingly if Bob does his measurement in \{\(|\psi_2\rangle, |\tilde{\psi}_2\rangle\}\ qubit basis, then the density matrix representing Alice’s subsystem is given by,

$$\rho_A = \frac{1}{4}(|\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)|\langle\psi_2(\theta_2, \phi_2)\psi_2(\theta_2, \phi_2)| +$$

$$|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)|\langle\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)| +$$

$$|\psi_2(\theta_2, \phi_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)|\langle\psi_2(\theta_2, \phi_2)\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)| +$$

$$|\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\psi_2(\theta_2, \phi_2)|\langle\tilde{\psi}_2(\bar{\theta}_2, \bar{\phi}_2)\psi_2(\theta_2, \phi_2)|$$

(10)

It is clearly evident that the equations (9) and (10) are not identical in any respect and henceforth we will conclude that Alice can distinguish the basis in which Bob has performed the measurement, This is impossible in principle as this will violate the causality. Hence we arrive at a contradiction with the assumption that the partial swapping of phase angle is possible.

Next we see that whether the partial swapping of azimuthal angles is consistent with the principle of no signalling or not. If we assume that the partial swapping of phase angles is possible keeping the azimuthal angles fixed, then its action is given by,

$$|\psi_i(\theta_i, \phi_i)||\tilde{\psi}_i(\bar{\theta}_i, \bar{\phi}_i)| \longrightarrow |\psi_i(\theta_i, \phi_i)||\tilde{\psi}_i(\bar{\theta}_i, \bar{\phi}_i)|$$

(11)
Let us assume that this hypothetical machine is in possession of Alice, and she applies the transformation (11) on her particles as a result of which the entangled state (6) takes the form,

$$|\chi\rangle_{PS} = \frac{1}{2}(|\tilde{\psi}(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|_A(|\tilde{\psi}(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|_B + (|\tilde{\psi}(\theta_1, \phi_1)\bar{\psi}(\theta_1, \phi_1)|_A(|\bar{\psi}(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|_B - (|\psi(\theta_1, \phi_1)\bar{\psi}(\theta_1, \phi_1)|_A(|\bar{\psi}(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|_B - (|\bar{\psi}(\theta_1, \phi_1)\bar{\psi}(\theta_1, \phi_1)|_A(|\bar{\psi}(\theta_1, \phi_1)\bar{\psi}(\theta_1, \phi_1)|_B + (|\bar{\psi}(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_A(|\bar{\psi}(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_B - (|\psi(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_A(|\bar{\psi}(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_B - (|\bar{\psi}(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_A(|\bar{\psi}(\theta_2, \phi_2)\bar{\psi}(\theta_2, \phi_2)|_B)$$

If Bob does his measurement in any one of the two basis \{\ket{\psi}, \ket{\tilde{\psi}}\} and \{\ket{\bar{\psi}}, \ket{\bar{\tilde{\psi}}}\}, then the respective density matrix representing Alice’s subsystem is given as,

$$\rho_A = \frac{1}{4}(|\psi(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|\psi(\theta_1, \phi_1)|\psi(\theta_1, \phi_1)| + |\tilde{\psi}(\theta_1, \phi_1)\tilde{\psi}(\theta_1, \phi_1)|\tilde{\psi}(\theta_1, \phi_1)|\tilde{\psi}(\theta_1, \phi_1)| + |\psi(\theta_1, \phi_1)\tilde{\psi}(\theta_1, \phi_1)|\psi(\theta_1, \phi_1)|\tilde{\psi}(\theta_1, \phi_1)| + |\tilde{\psi}(\theta_1, \phi_1)\psi(\theta_1, \phi_1)|\tilde{\psi}(\theta_1, \phi_1)|\psi(\theta_1, \phi_1)| = \frac{1}{4}(|\psi(\theta_2, \phi_2)\psi(\theta_2, \phi_2)|\psi(\theta_2, \phi_2)|\psi(\theta_2, \phi_2)| + |\tilde{\psi}(\theta_2, \phi_2)\tilde{\psi}(\theta_2, \phi_2)|\tilde{\psi}(\theta_2, \phi_2)|\tilde{\psi}(\theta_2, \phi_2)| + |\psi(\theta_2, \phi_2)\tilde{\psi}(\theta_2, \phi_2)|\psi(\theta_2, \phi_2)|\tilde{\psi}(\theta_2, \phi_2)| + |\tilde{\psi}(\theta_2, \phi_2)\psi(\theta_2, \phi_2)|\tilde{\psi}(\theta_2, \phi_2)|\psi(\theta_2, \phi_2)|$$

Alice can easily distinguish two statistical mixtures and as a consequence of which she can easily understand in which basis Bob has performed his measurement. This is not possible in principle as this will violate causality. Hence we conclude that the partial swapping of azimuthal angles is not possible.

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References

Time scale synchronization between two different time-delayed systems

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Abstract: In this paper we consider time scale synchronization between two different time-delay systems. Due to existence of intrinsic multiple characteristic time scales in the chaotic time series, the usual definition for the calculation of phase failed. To define the phase, we have used empirical mode decomposition and the results are compared with those from continuous wavelet transform. We investigate the generalized synchronization between these two different chaotic time delay systems and find the existence condition for the generalized synchronization. It has been observed that the generalized synchronization is a weaker than the phase synchronization. Due to the presence of scaling factor in the wavelet transform it has more flexibility for application.

Keywords: Time scale; Synchronization; Krasovskii-Lyapunov Theory; Time Delay System

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1. Introduction

Synchronization of chaotic oscillators is one of the fundamental phenomena in nonlinear dynamics. Various types of synchronization in chaotic systems have been classified[1], such as complete synchronization(CS), generalized synchronization(GS), lag synchronization(LS) and phase synchronization(PS). Among them, PS refers to the condition where the phases between two chaotic oscillators are locked, or the weaker condition where the mean frequencies between two chaotic oscillators are locked[2]. But the case of phase synchronization between two different time-delayed systems have not yet been identified and addressed. A main problem here is to define even the notion of phase in chaotic time delay system due to the intrinsic multiple characteristic time scales in these systems[3].

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In that case, its trajectories on the complex plane may show multiple centers of rotation and broadband power spectrum of the signal indicates multiple Fourier modes, which creates a difficulty in the estimation of phase. There is no unique method to determine the phase in complex chaotic oscillators and different definition of phase are found in [4]. All the synchronization types mentioned above are associated with each other, but the relationship between them is not completely clarified yet for time delay system.

In this paper, we propose a universal method to detect all types of synchronization and find a relationship between them. The main idea of this approach is to analyze the behavior of different time scale of the coupled time delay system. To study the time scale synchronization between two different time delay systems, we use empirical mode decomposition (EMD) [5] and continuous wavelet transform (CWT) [6]. Using CWT, we introduce the continuous set of time scales $s$ and the instantaneous phases $\phi_s(t)$ associated with them. In other words, $\phi_s(t)$ is a continuous function of time $t$ and time scale $s$. For the case of synchronization, the time series generated by these systems involve time scales $s$ that are necessarily correlated and satisfy the phase locking condition

$$|\phi_{s_1}(t) - \phi_{s_2}(t)| < \text{constant}$$  

The structure of this paper is as follows. In section 2, we discuss the method of EMD. In section 3, we consider the phase synchronization of two different chaotic time-delayed systems with unidirectional coupling by EMD. The CWT method are discussed in section 4. In section 5, the synchronization between Logistic and Mackey-Glass time-delayed systems are discussed. The GS versus time scale synchronization are discussed in section 6. We summarize our results in section 7.

2. Empirical Mode Decomposition

The study of synchronization basically requires the analysis of the signal or data which is available in the form of a time series. Many applications that involve signal or data processing require the use of transforms such as the Fast Fourier Transform (FFT) or Discrete Fourier Transform (DFT). These transforms allow a signal or data set that satisfies certain conditions to be converted to the frequency domain. A new signal processing technology called the Hilbert-Huang Transform (HHT) [5] has recently found that accurately analyzes physical signals. It calculate instantaneous frequencies based on the EMD method when intrinsic mode functions (IMFs) are generated for complex data. Then a Hilbert transform converts the local energy and instantaneous frequency derived from the IMFs to a full energy-frequency-time distribution of the data (i.e., a Hilbert spectrum). Finally, the physical signal is filtered by reconstruction from selected IMFs and a curve can be fitted to the filtered signal. Such a curve fitting might not have been possible with the original, unfiltered signal.

To use EMD method to decompose a function $f(t)$ as a linear combination of IMF’s $\psi_n$, the first step is to choose a time scale which is inherent in the function and has physical meaning. We also require to know the time between successive zero-crossings,
successive extrema and successive curvature extrema. Here \( n = 0 \), \( f_0 = f \), \( h_0 = f_n \) and \( k = 0 \) are taken. The upper envelope is constructed for \( h_k \). For this all the local extrema are identified and all the local maxima are fitted by a cubic spline interpolation for use as the upper envelope \( U(t) \). Similarly by taking all the local minima the lower envelope \( L(t) \) is constructed. The functions \( U(t) \) and \( L(t) \) should envelop the data between them, i.e., \( L(t) \leq h_k(t) \leq U(t) \), for all \( t \). Their mean is denoted by \( m_k(t) \) and the \( k \)-th component is defined as

\[
h_{k+1} = h_k - m_k
\]

If \( h_{k+1} \) is not an IMF, \( h_k \) is replaced by \( h_{k+1} \) and the envelopes are again constructed. Otherwise, \( h_{k+1} \) is defined as IMF \( \psi_n \) and the residual is \( f_{n+1} = f_n - \psi_n \). If the convergence criterion is not satisfied then \( n \) is increased and the whole procedure is repeated.

The convergence criterion is tested by considering the residual either smaller than a predetermined value or a monotonic function. Adding all the IMF’s together with the residual slow trend reconstructs the original signal without information loss or distortion.

Explicit procedure of EMD for a given signal \( x(t) \) can be summarized\[7\] as follows.

(i) All extrema of \( x(t) \) should be identified.

(ii) Interpolation should be done between the minima (respectively maxima) ending up with some envelope \( e_{\text{min}}(t) \) (respectively \( e_{\text{max}}(t) \)).

(iii) The mean \( m(t) = (e_{\text{min}}(t) + e_{\text{max}}(t))/2 \) should be computed.

(iv) The detail \( d(t) = x(t) - m(t) \) should be extracted.

(v) The residual \( m(t) \) should be iterated.

In practice, a sifting process refines the above procedure which amounts to first iterating steps (i) to (iv) upon the detail signal \( d(t) \), until the latter can be considered as zero-mean according to some stopping criterion. Once this is achieved, the detail can be referred to as IMF. The corresponding residual is computed and step (v) applies. The number of extrema decreases while going from one residual to the next and the whole decomposition is completed within a finite number of modes.

The phase variable \( \phi(t) \) can be easily estimated from a scalar time series \( x(t) \). But the problem arises when the signal possesses a multi component or a time varying spectrum. In that case, its trajectory on the complex plane may show multiple centers of rotation and an instantaneous phase cannot be defined easily \[4\]. The power spectrum of the signal indicates multiple Fourier modes, which creates a difficulty in the estimation of phase. To overcome this difficulty we have used the algorithm of EMD. EMD ensures that the complex plane of \( C_j(t) \) rotate around a unique rotation center, the resulting signal can be considered as a proper rotation mode and the phase can be defined.

We have decomposed the original chaotic signals \( x(t) \) and \( y(t) \) as

\[
x(t) = \sum_{j=1}^{M} C_j(t) + R(t) \quad \text{and} \quad y(t) = \sum_{j=1}^{N} C'_j(t) + R'(t)
\]

where \( R(t) \) and \( R'(t) \) are residuals of the signals \( x(t) \) and \( y(t) \) respectively. The functions \( C_j(t) \) (and \( C'_j(t) \)) are nearly orthogonal to each other. Thus each mode generates a proper
rotation on the complex plane with the analytic signal

\[ C_j(t) = A_j(t)e^{i\phi_j(t)} \quad \text{and} \quad C'_j(t) = A'_j(t)e^{i\phi'_j(t)} \]

and the two phases \( \phi_j \) and \( \phi'_j \) of the signals \( C_j \) and \( C'_j \) respectively are obtained.

Thus the phases \( \phi_j \) and \( \phi'_j \) are obtained by using Hilbert transform of each \( C_j \) and \( C'_j \) respectively and the frequencies \( \omega_j \) and \( \omega'_j \) are obtained by averaging the instantaneous phases \( \frac{d\phi_j}{dt} \) and \( \frac{d\phi'_j}{dt} \) respectively, separately for each mode. The instantaneous frequencies of each mode can vary with time and the fast oscillations present in the signal are in general extracted into the lower and the slow oscillations into the higher modes so that \( \omega_0 > \omega_1 > \omega_2 > ... > \omega_M \) and \( \omega'_0 > \omega'_1 > \omega'_2 > ... > \omega'_N \). Moreover, the mode amplitudes usually decay fast with \( j \) so that the signal can be decomposed into a small number of empirical modes. In this context one should note that the phase synchronization between the drive and response systems can always be characterized either as a phase locking or a weaker condition of frequency locking. One should note that phase locking and mean frequency locking are two independent conditions to characterize phase synchronization [1]. In the synchronized state the phase difference between the oscillators is bounded and the frequency difference is zero or at least close to zero. By using the EMD method, the transition to phase synchronization is basically analyzed as a process of merging of the frequencies of the corresponding modes and the phase difference is bounded as the coupling strength is increased. Further, when synchronization is set, different types of phase interactions may simultaneously arise at specific time scales. Then the two signals \( x(t) \) and \( y(t) \) can be written in polar form as

\[
x(t) = \sum_{j=1}^{M} A_j(t)\exp[i \int_0^t \Omega_j(t)dt] \quad \text{and} \quad y(t) = \sum_{j=1}^{N} A'_j(t)\exp[i \int_0^t \Omega'_j(t)dt]
\]

3. Synchronization between Logistic and Mackey-Glass time-delayed systems via EMD method: From phase to lag synchronization

Let us consider the more complicated example when it is impossible to correctly introduce the instantaneous phase \( \phi(t) \) of chaotic signal \( x(t) \). It is clear, that for such cases the traditional methods of the phase synchronization detection fail and it is necessary to use the other techniques, e.g. indirect measurement. As the first attempt to study PS between two different coupled time-delayed chaotic systems, we couple two such chaotic time-delayed systems in our model. The drive system is a delay Logistic system[8] and response system as time-delayed Mackey-Glass system[9].
To illustrate it we consider the following unidirectionally coupled drive \( x(t) \) and response \( y(t) \) systems as

\[
\dot{x}(t) = -ax(t) + rx(t - \tau_1)[1 - x(t - \tau_1)] \quad (2a) \\
\dot{y}(t) = -\alpha y(t) + \frac{\beta y(t - \tau_2)}{1 + y^{10}(t - \tau_2)} + k(x(t) - y(t)) \quad (2b)
\]

where \( k \) is a coupling strength. We choose the values of parameter as \( a = 15, r = 54, \tau_1 = 5, \alpha = 1, \beta = 2, \tau_2 = 10 \). In the absence of coupling strength the drive system \( x(t) \) and response system \( y(t) \) exhibit chaotic attractors. It is necessary to note that under these control parameter values none of the direct measurement methods for phase permits us to define the phase of chaotic signal correctly in the whole range of coupling parameter \( k \) variation.

The behavior of phase difference \([\Delta(\phi)]\) at different intrinsic time scales for different coupling strength \( k = 2.0 \) and \( k = 6.0 \) are shown in fig. (1a) and fig. (1b) respectively. We observed that in fig. (1a) for low coupling strength \( k = 2.0 \) the phase difference between different intrinsic mode are not bounded. The phase difference \( \phi_j(t) - \phi'_j(t) \) are not bounded for almost all intrinsic time scales. At this case the intrinsic time scales of both the systems are not correlated. For further increase of coupling strength (i.e. \( k = 6.0 \)), some of intrinsic time scales of the first chaotic oscillator becomes correlated with the other intrinsic modes of the second oscillator and the phase synchronization[PS] occur [in fig.(1b)]. With further increase of coupling (i.e.\( k = 13.53 \)), all intrinsic time scales of two systems are correlated. Phase difference between any two time scales are correlated with each other[Fig. (1c)]. With increase the value of coupling, PS converted to lag synchronization[LS]. The LS between oscillators means that all intrinsic modes are correlated. From the condition of LS we have \( x(t - \tau) \simeq y(t) \) and therefore \( \phi_j(t - \tau) \simeq \phi'_j(t) \) where \( \tau \) is the time lag. This time lag depends on the coupling parameter. At high coupling parameter, the time lag decreases and LS transfered to CS.

4. Continuous wavelet transform(CWT)

An alternative approach for the analysis of phase of a complicated time series is the wavelet transform. Several people have already worked on this approach [6]. To elaborate the basis of wavelet technique, consider a time series \( x(t) \). The behavior of such systems can be characterized by a continuous phase set define wavelet transform of the chaotic signal \( x(t) \); 

\[
W(s_0, t_0) = \int_{-\infty}^{\infty} x(t)\psi_{s_0,t_0}^* dt
\]

where the asterisk means complex conjugation and

\[
\psi_{s_0,t_0}(t) = \frac{1}{\sqrt{s_0}} \psi_0 \left( \frac{t - t_0}{s_0} \right)
\]

is the wavelet function obtained from the mother wavelet \( \psi_{s_0,t_0}(t) \). The time scale \( s_0 \) determines the width of \( \psi(t) \), where \( t_0 \) stands for time shift of the wavelet function.
Here it should be noted that the time scale $s_0$ is replaced by the frequency of Fourier transform and can be considered as the quantity inverse to it. In this paper we have used Morlet wavelet which is given by:

$$\psi_0(\eta) = \frac{1}{\sqrt{\pi}} \exp(i\omega_0 \eta) \exp\left(-\frac{\eta^2}{2}\right)$$

The wavelet parameter $\omega_0 = 2\pi$ ensures the relation $s = \frac{1}{f}$ between the time scale $s$ and frequency $f$ of Fourier transform. One then considers

$$W(s_0, t_0) = |W(s_0, t_0)| e^{i\phi_{s_0}(t_0)}$$

which determines the wavelet surface characterizing the behavior of the system for every time scale $s_0$ at any time $t_0$. The magnitude of $W(s_0, t_0)$ represents the relative presence and magnitude of the corresponding time scale $s_0$ at $t_0$. Usually it is very standard to consider the integral energy distribution over all time as

$$\langle E(s_0) \rangle = \int |W(s_0, t_0)|^2 dt_0$$

The phase $\phi_{s_0}(t) = \arg W(s_0, t_0)$ also proves to be naturally defined for time scale $s_0$. So that the behavior of each time scale $s_0$ can be identified using the phase $\phi_{s_0}(t)$. We now apply this idea to the two time series obtained from the two chaotic systems. It is observed that the time scales accounting for the greatest fraction of the wavelet spectrum energy $\langle E(s_0) \rangle$ are obviously synchronized first. For other time scales there is no synchronization. Actually we should have $|\phi_{s_1}(t) - \phi_{s_2}(t)| < \text{constant}$, for some $s$, leading to phase locking in the situation of phase synchronization.

5. Synchronization between the Logistic and Mackey-Glass time-delayed systems via CWT

We now apply continuous wavelet transform to the two time series obtained from the two chaotic time-delayed systems. It is observed that the time scales accounting for the greatest fraction of the wavelet spectrum energy $\langle E(s_0) \rangle$ are obviously synchronized first. For the other time scales there is no synchronization. Actually we should have $|\phi_{s_1}(t) - \phi_{s_2}(t)| < \text{constant}$, for some time-scale $s$, leading to phase locking in the situation of phase synchronization.

We consider the dynamics of different time scale $s$ of two different coupled time-delayed systems (2) when the coupling parameter value increases. If there is no phase synchronization between the oscillators, their dynamics remain uncorrelated for all time scales $s$. The dynamics of the coupled system (2) when the coupling parameter $k$ is sufficiently small $k = 2.0$ are shown in figure (2a) and (2b). The power spectrums $\langle E(s) \rangle$ of the Logistic and Mackey-Glass time delay systems are different from each other [fig. 2(a)], but the maximum value of the power spectrum for both the systems are occurred to the same time scale $s_0 = 10.2$. It is clear that the phase difference
\( \phi_{s_1}(t) - \phi_{s_2}(t) \) are not bounded for all the time scales \( s_0 = 3.0, 10.2, 12.0 \) [Fig. 2(b)]. This means that the systems under consideration do not involve synchronized time scales. Therefore the systems are unsynchronized.

As the coupling increases, the systems are brought to PS regime as illustrates in figure (2c, 2d). The power spectrum of the response system is near similar to the power spectrum of driving system [in fig. 2(c)]. It is seen that phase locking occurs at the time scale \( s_0 = 10.2 \) corresponding to the maximum energy in the wavelet spectrum, \( < E(s) > \) [in fig 2(d)]. But the other time scales \( s_0 = 3.0, 12.0 \), the phase difference are same as before [Fig. 2(b)] i.e. not bounded. As soon as any of the time scales of the driving system becomes correlated with another time scale of the response system (e.g. when the coupling parameter increases), phase synchronization occurs. The time scale \( s_0 \) is characterized by the largest value of energy in the wavelet spectrum \( < E(s) > \) is more likely to be correlated first. The other time scales remain uncorrelated as before. The phase synchronization between chaotic systems leads to phase locking (1) at the correlated time scales \( s_0 \). With a further increase in the coupling parameter, the unsynchronized time scales become synchronized. The number of time scales for which the phase locking occurs increases and one can say that the degree of synchronization grows. For coupling parameter \( k = 13.53 \), we observe that the normalized energy spectrum totally overlap [Fig. 2(e)]. The time scales \( s_0 = 3.0, 12.0 \) which are not synchronized in the previous [Fig. 2(b), 2(d)] are synchronized. The phase difference remain bounded for all time scales \( s_0 \) where we have shown this variation for low \( s_0 \) and also for high \( s_0 \) value [in Fig. 2(f)].

The occurrence of lag synchronization between time delay system means that all the time scales are correlated. The lag synchronization condition \( x(t - \tau) \approx y(t) \) implies that \( \omega_x(s, t - \tau) \approx \omega_y(s, t) \) and therefor \( \phi_{s_1}(t - \tau) \approx \phi_{s_2}(t) \). In that case phase locking condition(1) is satisfied for all time scales. For instance, where the coupling parameter \( k \) is sufficient large then lag synchronization of coupled system (2) occurs. In this case the power spectrum of two system are coincident with each other and the phase locking condition satisfied for all time scales. It is to be noted that the phase difference \( \phi_{s_1}(t) - \phi_{s_2}(t) \) will not be zero in the case of lag synchronization. This difference depends on time lag \( \tau \). We can therefore say that the time-scale synchronization is the most general synchronization and phase, lag synchronization are particular cases of time-scale synchronization.

6. Generalized synchronization versus time scale synchronization

Let us consider another type of synchronization behavior, the so-called generalized synchronization(GS). It has been shown in the above section that the PS, LS and CS are the particular type of time scale synchronization. All the above type synchronization are depend on the number of synchronized time scales. But the relation between the PS and GS is not clear. Several work are studied the problem, how the PS and GS are corre-
lated with each other. In this section we will study the GS of the coupled time-delayed system. In a paper[10] the CS and GS of one way, linearly coupled Mackey-Glass system is studied. In recent paper[11] the GS between two unidirectionally linearly and nonlinearly coupled chaotic nonidentical Ikeda models are discussed. For this purpose, we use the auxiliary system method to detect GS[10]; that is, given another identical driven auxiliary system \( z(t) \), GS between \( x(t) \) and \( y(t) \) is established with the achievement of CS between \( y(t) \) and \( z(t) \). In fact, the auxiliary method allows us to find the local stability condition of the GS[10].

To illustrate the above procedure we consider the coupled Mackey-Glass system as

\[
\dot{y}(t) = -\alpha y(t) + \frac{\beta y(t - \tau_2)}{1 + y^{10}(t - \tau_2)} + k(x(t) - y(t)) \tag{3a}
\]

\[
\dot{z}(t) = -\alpha z(t) + \frac{\beta z(t - \tau_2)}{1 + z^{10}(t - \tau_2)} + k(x(t) - z(t)) \tag{3b}
\]

We derive the existence conditions of the CS in coupled time-delayed systems (3) with the help of Krasovskii-Lyapunov functional approach. We denote the error signal \( \Delta = y - z \). Then the error dynamics can be written as

\[
\dot{\Delta} = -r(t)\Delta + s(t)\Delta \tag{4}
\]

where \( r(t) = \alpha + k \), \( s(t) = f'(y_{\tau_2}), f(y_{\tau_2}) = \frac{\beta y(t - \tau_2)}{1 + y^{10}(t - \tau_2)} \). The driving and response subsystem described by (3) are synchronized if the fixed point \( \Delta = 0 \) of system (4) is stable. By the Krasovskii-Lyapunov theory, a positively defined functional was introduced as,

\[
V(t) = \frac{\Delta^2}{2} + \mu \int_{-\tau}^{0} \Delta^2(t + \theta)d\theta \tag{5}
\]

where \( \mu > 0 \) is an arbitrary positive parameter. According to [12-14], the sufficient stability condition for the trivial solution \( \Delta = 0 \) of the time delayed equation (4) is

\[
r(t) > |s(t)| \quad \text{with} \quad \mu = \frac{|s(t)|}{2} \tag{6}
\]

For a particular problem, two cases may arises in equation(6), for the first case \( s \) is constant and \( r(t) \) is variable and the second case \( r \) is constant and \( s(t) \) is variable. The second case arise in our case where \( s(t) \) is a variable. For the general cases the stability condition \( \mu = \frac{|s(t)|}{2} \) is not always true because \( \mu \) is a parameter and \( s(t) \) is a variable. The above problem can be removed if we define \( \mu \) as a function of time and the derivative of \( \mu \) can be considered in the expression of \( \dot{V} \).

Suppose \( \mu = \zeta(t) > 0 \), then the Krasovskii-Lyapunov functional can be taken as[12-14]

\[
V(t) = \frac{1}{2}\Delta^2(t) + \zeta(t) \int_{-\tau}^{0} \Delta^2(t + \xi)d\xi
\]

Then

\[
\dot{V}(t) = -r(t)\Delta^2 + s(t)\Delta \dot{\Delta} + \zeta(t)(\Delta^2 - \Delta^2) + \]

\[
\]
\[ \dot{\zeta}(t) \int_{-\tau}^{t} \Delta^2(t + \xi) d\xi \]

If \( \dot{\zeta}(t) \leq 0 \) then we have

\[ \dot{V}(t) \leq -[r(t) - s^2(t)/4\zeta(t) - \zeta(t)]\Delta^2 \]

We obtain the stability condition as

\[ r(t) - s^2(t)/4\zeta(t) - \zeta(t) > 0 \]

\( i.e. \ r(t) > h(s, \zeta) \) where \( h(s, \zeta) = s^2(t)/4\zeta(t) + \zeta(t) \)

For any function of \( s(t) \), \( h(s, \zeta) \) is a function of \( \zeta(t) \) and has an absolute minimum for \( \zeta(t) = \frac{s(t)}{2} \) and \( h_{\text{min}}(s, \zeta) = |s(t)| \). Thus \( h(s, \zeta) \geq |s(t)| \) for any \( s \) and \( \zeta > 0 \). The stability condition for synchronization is \( r(t) > |s(t)| \). Then the stability condition for the trivial solution \( \Delta = 0 \) of linear time delay system is \( r(t) > |s(t)| \) for \( \zeta(t) = |s(t)| \).

The stability condition for CS is \( \alpha + k > |f'(y_{\tau})| \) that is \( k > \max|f'(y_{\tau})| - 1 \), where the maximum is defined on the trajectory of the driving system. Therefore, GS between (2a) and (2b) exists if \( k > 3.05 \). Figure(3a) shows the CS between (3a) and (3b). GS between (2a) and (2b) shown in figure(3b).

Thus, the GS of the unidirectional coupled time-delayed systems (2) appears as the time scale synchronized dynamics, as another types of synchronization does not occurs before. At this coupling parameter \( k = 4.0 \) the PS does not occurs [Fig. 3c, 3d]. The above results are similar to the results in Ref [15], in which the GS between two unidirectional coupled Rossler systems are occurred while the PS has not been observed. It is also clear why the PS has not been observed in our case. The instantaneous phases \( \phi_{s1}(t) - \phi_{s2}(t) \) of the chaotic signal \( x(t) \) and \( y(t) \) determined by CWT for the time scales \( s_0 = 3.0, 10.2, 12.0 \) but only the time scale \( s_0 = 10.2 \) are synchronized[Fig. 3d]. The other time scale are not synchronized. The instantaneous phases does not allow to detect PS in that case although the synchronization of time scales occurs.

**Conclusion**

In conclusion, we have considered the time scale synchronization between two different time-delayed systems by means of EMD and CWT and compared their results. Since the two systems are in chaotic regime, the time series contain multiple Fourier modes. The several definition for detection of phase are failed. We observe that while the EMD method separates the complex signal in various IMF’s corresponding to a definite frequency, the CWT approach does the same but with respect to different scales. Actual comparison between EMD and CWT is possible only through statistical methods. One can observe that different synchronization (CS, LS, PS and GS) come from universal position, i.e. time scale synchronization is a common type of synchronization where CS, LS, PS and GS are the particular cases of time scale synchronization. We have investigated the relation between GS and PS. We observe that GS is a weaker than PS. PS could be...
stranger than GS, and they can also occur for chaotic time delay systems. The sufficient condition for generalized synchronization are studied analytically and also we have shown numerically the effectiveness of the synchronized system. According to our paper one can see that it will be a unified framework for different types of chaotic synchronization for any dynamical systems. These two methods are also applicable for experimental data because it does not require any information about the dynamical systems.

References

Fig. 1 Phase difference between the IMF's of systems (2a) and (2b) at (a) $k = 2.0$, (b) $k = 6.0$ and (c) $k = 13.53$, where 1 represents the phase difference between the corresponding $C_1$ and $C'_1$ IMF's, 2 between $C_2$ and $C'_3$ IMF's, 3 between $C_3$ and $C'_4$ IMF's, 4 between $C_4$ and $C'_4$ IMF's, 5 between $C_6$ and $C'_4$ IMF's.
Fig. 2 a) The normalized energy distribution in the wavelet spectrum $<E(s_0)>$ for the drive (solid line) and response (dashed line) of coupled system (2), b) the dependence of the phase difference $\phi_{s_1}(t) - \phi_{s_2}(t)$ on time $t$ for different time scales $s$. The coupling parameter between the oscillators is $k = 3.0$. There is no phase synchronization between the systems. c) The normalized energy distribution in the wavelet spectrum $<E(s_0)>$ for the drive (solid line) and response system (dashed line), d) phase difference $\phi_{s_1}(t) - \phi_{s_2}(t)$ for coupling system (2). The coupling parameter between the oscillators is $k = 6.0$. The time scale $s_0 = 10.2$ are correlated with each other and synchronization is observed. e) The normalized energy distribution in the wavelet spectrum $<E(s_0)>$ for the drive (solid line) and response system (dashed line), f) phase difference $\phi_{s_1}(t) - \phi_{s_2}(t)$ for coupling system (2). The coupling parameter between the oscillators is $k = 13.53$. All the time scales are correlated and synchronization is observed.
**Fig. 3** a) Complete synchronization between $y(t)$ and $z(t)$ in system 3, b) generalized synchronization between $x(t)$ and $y(t)$ in system 2. c) normalized energy distribution and d) phase difference $\phi_{s_1}(t) - \phi_{s_2}(t)$ at $k = 4.0$. 
Thermodynamic Fluctuation Theory and Gravitational Clustering of Galaxies

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Abstract: We study the phase transitions occurring in the gravitational clustering of galaxies on the basis of thermodynamic fluctuation theory. This is because the fluctuations in number and energy of the particles are constantly probing the possibility of a phase transition. A calculation of various moments of the fluctuating thermodynamic extensive parameters like the number and energy fluctuations, has been performed. The correlated fluctuations \(\langle \Delta N \Delta U \rangle\), have shown some interesting results. For weak correlations, their ensemble average is positive, indicating that a region of density enhancement typically coincides with a region of positive total energy. Its perturbed kinetic energy exceeds its perturbed potential energy. Similarly an underdense region has negative total energy since it has preferentially lost the kinetic energy of the particles that have fled. For larger correlations the overdense regions typically have negative total energy, underdense regions have positive total energy. The critical value at which this switch occurs is the critical temperature \(T = T_C\), whose value has been calculated analytically. At this critical value \(T_C\), a positive \(\langle \Delta N \rangle\) is just as likely to be associated with a positive or a negative \(\Delta U\).

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1. Introduction

Galaxies cluster on very large scales under the influence of their mutual gravitation and the characterization of this clustering is a problem of current interest. Observations indicate that while the large-scale distribution of galaxies appears to be essentially uniform, however, small-scale distribution is appreciably influenced by the well known tenancy

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towards clustering. The universe is homogeneous and isotropic on scales \( \geq 100\text{-}200 \text{ Mpc} \) where as on smaller scales its fundamental units- galaxies cluster together to form groups, clusters and even super clusters.

Our universe emerged from a singularity at a very high temperature about \( 2 \times 10^{10} \) years ago. At such temperatures all matter behaves like photons, hence the initial state was a chaotic gaseous inferno of high energy elementary particles and photons. As the universe expanded, the temperature dropped and the heavier particles annihilated and decayed in to the less massive stable particles (protons, neutrons and neutrinos). At a time of about \( 10^{-4} \) seconds, the temperature had dropped to below that needed to make the heavy particles such as protons. At a time of one second this era ended when temperatures fell below \( 10^{10} \) K.

After particle formation, most of the energy in the universe was in the form of light which we call radiation era. Near the beginning of this era, cosmic nucleosynthesis took place. Nucleosynthesis began with the production of deuterium. The presence of deuterium then led to other, even faster reactions in which deuterons combined to produce helium. At the end of about five minutes, the nucleosynthesis process was complete. The end product of this phase of cosmic evolution was hydrogen, deuterium and helium. The universe expanded through the nucleosynthesis phase too fast to go for any more fusion than this. After 2000 years when matter (hydrogen and helium) began to dominate the universe and radiation became a secondary constituent, the matter era began. The evolution of this phase is governed by the standard flat model and the relationship \( R \approx t^{2/3} \). At about 3000K, the matter and radiation decoupled and both evolved independently. Thereafter the radiation cooled to about 2.7K cosmic background radiation observed today and the gaseous matter underwent a critically important transformation- it could form the first galaxies. Prior to decoupling the radiation pressure kept the matter distribution smooth. Afterwards small inhomogeneities could grow and condense into the first gravitationally bound systems.

The clustering of galaxies is amply borne out in three-dimensional red-shift surveys, such as the CFA survey and the southern sky redshift survey (SSRY). Different techniques such as Correlation functions [1, 2, 3], Percolation [4] and distribution functions [5, 6] have been introduced to understand the large scale structure of universe. The theories of the cosmological many body galaxy distribution function have been developed mainly from a thermodynamic point of view. Comparison of gravitational thermodynamics to the cosmological many body problem has been discussed on the basis of N-body computer simulation results [7, 8], along with other theoretical arguments [9] and these support it further. In the present paper we investigate the problem of gravitational clustering of galaxies from the point of view of thermodynamics using the fluctuation theory.

2. Thermodynamic Fluctuation theory and Phase Transitions

The phase transition refers to a change, usually abrupt, from one type of dominant symmetry to another. The fluctuations in a system are constantly probing the possibility of a
phase transition. The fluctuations are an intrinsic part of the thermodynamic equilibrium. Within any sub-region of the system and from region to region, the macroscopic qualities will fluctuate around their average values. Although fluctuations are necessarily small, they can become enormous in gravitational systems. Fluctuations occur between regions, sub-regions or cells of the system—they are not generally isolated. They exchange energy, volume or number of particles with one another or with a much larger reservoir. Away from a phase transition, the fluctuations are usually small and there is no difference between the canonical and microcanonical ensembles. When fluctuations are large we have the case for non-linear galaxy clustering. Since both energy and particles (galaxies) can move across the cell boundaries, the grand canonical ensemble is the relevant one to use. These fluctuations can completely explore new set of states at ease. Often the system changes its character completely and permanently by falling into a state of much lower total energy and altered symmetry i.e phase transition occurs.

A macroscopic system undergoes incessant and rapid transitions among its microstates. Some cells find a microstate of higher entropy or lower free energy and fall into such a microstate. A trapped cell in the new microstate modifies the thermodynamic functions of its surrounding cells. Many such cells may form nuclei and their microstate may propagate through out the entire system i.e; a phase transition occurs. Its consequences are dramatic. Ice turns to water, water boils to steam, crystals change structure and magnetic domains collapse.

In this work we investigate the problem of phase transitions occurring in the gravitational clustering of galaxies from the point of view of thermodynamic fluctuation theory and statistical mechanics. The work done by [10], ensures the application of statistical mechanics to gravitational clustering of galaxies. We consider a large system, which consists of an ensemble of cells of the same volume \( V \) (much smaller than the total volume) and average density \( \bar{n} \). Both the number of galaxies and their total energy will vary among these cells, that are presented by a grand canonical ensemble, in which galaxies interact pairwise and their distribution is statistically homogeneous over large regions.

### 3. Gravitational Partition Function

For the cosmological many body system where linear and non-linear clustering of galaxies and their haloes are present over many scales simultaneously, grand canonical ensembles are needed.

The partition function of a system of \( N \) particles each of mass \( m \) interacting gravitationally with potential energy \( \phi \), having momenta \( p_i \) and average temperature \( T \) is

\[
Z_N(T,V) = \frac{1}{\Lambda^{3N} N!} \int \exp \left[ - \sum_{i=1}^{N} \frac{p_i^2}{2m} + \phi_{ij}(r_1, r_2, \cdots, r_N) \right] d\rho^{3N} dr^{3N}
\]

Here \( N! \) takes the distinguishability of classical particles into account and \( \Lambda \) normalizes the phase space volume cell. The integral over phase space in equation (1) has been
evaluated analytically by using virial expansion method and is given by:

\[ Z_N(T, V) = \frac{1}{N!} \left( \frac{2\pi m T}{\Lambda^2} \right)^{\frac{3N}{2}} V^N \left[ 1 + \beta \bar{n} T^{-3} \alpha(\epsilon/R_1) \right]^{N-1} \]  

with \( \beta = \frac{3}{2} (Gm^2)^3 \) and \( \alpha(\epsilon/R_1) \) as:

\[ \alpha(\epsilon/R_1) = \sqrt{1 + (\epsilon/R_1)^2 + (\epsilon/R_1)^2 \ln \frac{(\epsilon/R_1)}{1 + \sqrt{1 + (\epsilon/R_1)^2}}} \]  

Here \( R_1 \) is the radius of the cell and \( \epsilon/R_1 = 0 \) corresponds to point mass.

The free energy is given by:

\[ F = -T \ln Z_N(T, V) \]  

Using equation (2) in equation (4), we have:

\[ F = NT \ln \left( \frac{N}{V} T^{-3/2} \right) - NT \ln \left( 1 + \beta \bar{n} T^{-3} \alpha(\epsilon/R_1) \right) - \frac{3}{2} NT \ln \left( \frac{2\pi m \Lambda^2}{N^2} \right) \]  

Again from [6, 10], we have:

\[ b = \frac{\beta \bar{n} T^{-3}}{1 + \beta \bar{n} T^{-3}} \]  

\[ P = \frac{b}{1 + \beta T^{-4}} \]  

The value of \( b \) measures the influence of gravitational correlation potential energy and ranges between 0 and 1.

Using equation (6) in (7) we have:

\[ P = \frac{NT}{V} (1 - b) \]  

Where \( \bar{n} = N/V \) is the average number density of \( N \) point mass galaxies in a volume \( V \).

In the thermodynamic fluctuation theory, the mean square deviation is widely used and convenient measure of the magnitude of the fluctuation. The mean square deviation is also called the second moment of the distribution and is expressed as [11]

\[ Z_N(T, V) = \frac{1}{\Lambda^{3N} N!} \int \exp \left[ -\sum_{i=1}^{N} \frac{p_i^2}{2m} + \phi_{ij}(r_1, r_2, \cdots, r_N) \right] T^{-1} \]  

\[ dp^N dr^{3N} \]  

Where \( p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_s, X_{s-1}, \ldots, X_t \)

Where \( F_0 = \frac{1}{T}, F_1 = \frac{p}{T} \) and \( F_2 = \frac{p^2}{T} \) etc.

and \( X_0 = U, X_1 = V, X_2 = N \)

A calculation of various moments of thermodynamic extensive parameters like number and energy fluctuations has been performed for single component point mass galaxies clustering gravitationally in an expanding universe. We derive the moments in terms of
the temperature $T$, and a critical temperature $T_C$ has been calculated at which a transition occurs from positive to negative total energy. The critical temperature $T_C$ has been equated with the peculiar velocities of galaxies, thereby giving a prediction that our result can be tested in the laboratory by either N-body computer simulations or by observing the velocity catalogues of galaxies or their velocity distribution functions.

4. Thermodynamic Moments

The second order moment is obtained by putting $j = 2$, $k = 2$ in equation (9).

$$\langle \Delta X_2 \Delta X_2 \rangle = - \left[ \frac{\partial X_2}{\partial F_2} \right]_{T,V}$$

$$\langle \Delta N^2 \rangle = T \left[ \frac{\partial N}{\partial \mu} \right]_{T,V}$$

$$\left[ \frac{\partial N}{\partial \mu} \right]_{T,V} = \left[ \frac{\partial N}{\partial P} \right]_{T,V} \left[ \frac{\partial P}{\partial \mu} \right]_{T,V}$$

$$\left[ \frac{\partial N}{\partial \mu} \right]_{T,V} = \frac{N}{V} \left[ \frac{\partial N}{\partial P} \right]_{T,V}$$

Where $\frac{\partial P}{\partial \mu} = \frac{N}{V}$

From equation (8), we have

$$\left[ \frac{\partial P}{\partial N} \right]_{T,V} = \frac{T}{V} \left[ 1 - b - N \frac{\partial b}{\partial N} \right]$$

From equation (6),

$$\frac{\partial b}{\partial N} = \frac{b(1 - b)}{N}$$

Using equation (15) in equation (14), we have

$$\left[ \frac{\partial P}{\partial N} \right]_{T,V} = \frac{T}{V} (1 - b)^2$$

Substituting in equation (13), we have:

$$\left[ \frac{\partial N}{\partial \mu} \right]_{T,V} = \frac{N}{T(1 - b)^2}$$

Also substituting equation (17) in equation (11), the result leads to,

$$\langle \Delta N^2 \rangle = \frac{N}{(1 - b)^2}$$

Again using equation (6), we get

$$\langle \Delta N^2 \rangle = N \left( 1 + \beta \bar{n} T^{-3} \right)^2$$
Similarly third order moment is given by,

$$\langle \Delta X_i \Delta X_j \Delta X_k \rangle = \left[ \frac{\partial^2 X_i}{\partial F_j \partial F_k} \right]$$ (20)

The third order moment is obtained by putting \(i=2, j=2\) and \(k=2\) in equation (20)

$$\langle \Delta N^3 \rangle = N (1 + \beta \bar{n} T^{-3})^4 \left[ 1 + \frac{2 \beta \bar{n} T^{-3}}{1 + \beta \bar{n} T^{-3}} \right]$$ (21)

Finally the fourth order moment is given by,

$$\langle \Delta X_i \Delta X_j \Delta X_k \Delta X_l \rangle = -\frac{\partial^3 X_i}{\partial F_j \partial F_k \partial F_l} + \frac{\partial X_i}{\partial F_j} \frac{\partial X_j}{\partial F_k} \frac{\partial X_k}{\partial F_l} + \frac{\partial X_k}{\partial F_l} \frac{\partial X_l}{\partial F_j} + \frac{\partial X_l}{\partial F_j} \frac{\partial X_j}{\partial F_k} \frac{\partial X_k}{\partial F_l}$$ (22)

The fourth order moment is obtained by putting \(i=j=k=2\) in equation(22):

$$\langle \Delta N^4 \rangle = N (1 + \beta \bar{n} T^{-3})^6 \left[ 1 + \frac{8 \beta \bar{n} T^{-3}}{1 + \beta \bar{n} T^{-3}} + 6 \left( \frac{\beta \bar{n} T^{-3}}{1 + \beta \bar{n} T^{-3}} \right)^2 \right] + 3 \langle \Delta N^2 \rangle^2$$ (23)

Since in the grand canonical ensemble of sub-regions or cells, both number of particles and energy can fluctuate between different cells, so it is important to consider the combination of moments, in which both the quantities can fluctuate. In this paper an attempt in this direction is performed.

The combination of moments is obtained by putting \(j=2, k=0\) in equation (9)

$$\langle \Delta N \Delta U \rangle = \frac{1}{T} \left[ \frac{\partial U}{\partial \mu} \right]_{1/T, P/T}$$ (24)

$$\left[ \frac{\partial U}{\partial \mu} \right]_{T,V} = \left[ \frac{\partial U}{\partial b} \right] \left[ \frac{\partial b}{\partial \mu} \right]$$ (25)

The internal energy is given by

$$U = \frac{3}{2} NT(1 - 2b)$$ (26)

$$\left[ \frac{\partial U}{\partial b} \right]_{T,V} = \frac{3}{2} T \frac{\partial}{\partial b} [N(1 - 2b)]$$ (27)

Using equation (15), we get.

$$\left[ \frac{\partial U}{\partial b} \right]_{T,V} = \frac{3}{2} \frac{NT}{b(1-b)} [1 - 4b + 2b^2]$$ (28)

Using equation(6), we have

$$\left[ \frac{\partial b}{\partial \mu} \right] = \frac{b}{T(1-b)}$$ (29)
Substituting equations (28) and (29) in equation (25), we get

\[
\frac{\partial U}{\partial \mu}_{T,V} = \frac{3N(1 - 4b + 2b^2)}{2(1 - b)^2}
\]  

(30)

Substituting equation (30) in equation (24), and using equation (6), we get

\[
\langle \Delta N \Delta U \rangle = \frac{3}{2} NT(1 + \beta \bar{n}T^{-3})^2 \left[ 1 - 4 \frac{\beta \bar{n}T^{-3}}{1 + \beta \bar{n}T^{-3}} + 2 \left\{ \frac{\beta \bar{n}T^{-3}}{1 + \beta \bar{n}T^{-3}} \right\}^2 \right]
\]

(31)

The correlated fluctuations \(\langle \Delta N \Delta U \rangle\) are especially interesting. The average energy \(\Delta U\) can be either positive or negative. For weak correlations, their ensemble average is positive, indicating that a region of density enhancement typically coincides with a region of positive total energy. Its perturbed kinetic energy exceeds its perturbed potential energy. Similarly, an underdense region has negative total energy, since it has preferentially lost the kinetic energy of the particles that have fled. For larger correlations the overdense regions typically have negative total energy; underdense regions usually have positive total energy. The critical value at which this switch occurs is the critical temperature \(T = T_c\), whose value has been calculated analytically. At this critical value of \(T_C\), a positive \(\Delta N\) is just as likely to be associated with a positive or a negative \(\Delta U\).

The critical value is obtained as

\[\langle \Delta N \Delta U \rangle = 0\]

Using equation (31), we get the result as

\[T_C = (2\beta T^{-3})^{\frac{1}{2}}\]

(32)

where \(\beta = \frac{3}{2}(Gm^2)^3\) is a positive constant. Equation (32) is especially important because it has direct observational consequences by relating it to observed catalogues and to computer simulations.

5. Results and Discussion

We have applied the thermodynamic fluctuation theory to study the phase transitions occurring in the gravitational clustering of galaxies in an expanding universe. The calculations of various moments of number and energy fluctuations have been performed in this paper. We have also calculated the correlated fluctuations \(\langle \Delta N \Delta U \rangle\) in which both total number of particles (galaxies) as well as their energy can fluctuate between different cells or subsystems of the grand canonical ensemble. The results show that there is a critical temperature \(T_C\) at which the transition from positive to negative total energy of the system occurs. We define this as a phase transition. Recently the Lee-Yang theory has been applied to the gravitational clustering of galaxies and a critical temperature \(T_C\) [12] has been obtained at which a phase transition occurs in the clustering process. The results of Lee-Yang theory match with the results of the fluctuation theory because the expression for critical temperature is same in both the cases i.e. the phase transition occurs at a unique value of critical temperature \(T_C\), given by equation (32).
The critical temperature $T_C$ at which the phase transition occurs in the cosmological many body problem has been calculated analytically here. We can equate it with the kinetic energy of peculiar motion of galaxies.

$$[T_C]_j = \frac{2}{3N} \frac{1}{2} m_j \sum_{k=1}^{N_j} v_k^2, j = 1, 2, \ldots \quad (33)$$

Where $T_j$, $N_j$ and $m_j$ are the temperature, the number and the mass respectively of the $j$th component of galaxies.

Peculiar velocities may arise from interactions between individual galaxies as well from collective interaction between a galaxy and a cluster. In the early stages, the more massive galaxies cluster more rapidly and their velocity dispersion increases faster than that of less massive galaxies. Next the less massive galaxies cluster around the massive ones. This implies that in the early stages the massive galaxies speed up the clustering but in the later stages the distinction between different masses diminishes. In other words, although the mass of an individual galaxy is important at early stages, collective effects become more important at late stages and the effects of mass spectrum are substantially reduced.

Equation (33) gives a prediction of observing the phase transition occurring in the gravitational clustering of galaxies in an expanding universe, by either N-body computer simulations or by observing the velocity catalogues or the distribution functions of galaxies. It is therefore very important to see that whether it is a phase transition from the uniform expanding phase to the centrally condensed inhomogeneous state or from a poisson distribution to a correlated distribution, slowly developing on large and large scales as the universe expands. Again it may well be a transition from individual interactions to collective effects between galaxies. From the observations one should be able to calculate $T_C$ in terms of the velocities of galaxies that will lead us to many new aspects of the cosmological N-body gravitational clustering problem.

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References


Neutrino Oscillation Probability from Tri-Bimaximality due to Planck Scale Effects

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Abstract: Current neutrino experimental data on neutrino mixing are well describes by Tri-bi-maximal mixing, which is predicts $\sin^2 \theta_{12} = 1/3$, zero $U_{e3}$ and $\theta_{23} = 45^o$. We consider the planck scale operator on neutrino mixing. We assume that the neutrino masses and mixing arise through physics at a scale intermediate between planck scale and the electroweak braking scale. We also assume, that just above the electroweak breaking scale neutrino mass are nearly degenerate and the mixing is tri-bi-maximal. Quantum gravity (Planck scale) effects lead to an effective $SU(2)_L \times U(1)$ invariant dimension-5 Lagrangian symmetry involving Standard Model. On electroweak symmetry breaking, this operator gives rise to correction to the neutrino masses and mixings these additional terms can be considered as perturbation to the tri-bimaximal neutrino mass matrix. We compute the deviation of the three mixing angles and oscillation probability. We find that the only large change in solar mixing angle and % change in maximum $P_{\mu e}$ is about 10%.

Keywords: Quantum Gravity; Neutrino Oscillation Probability; Planck Scale Effects; Tri-bi-maximal Mixing

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1. Introduction

Recent advance in neutrino physics observation mainly of astrophysical observation suggested the existence of tiny neutrino mass. The experiments and observation has shown evidences for neutrino oscillation. The solar neutrino deficit has been observed [1,2,3,4], the atmospheric neutrino anomaly has been found [5,6,7], and currently almost confirmed by KamLAND [8], and hence indicate that neutrino massive and there is mixing in lepton sector, this indicate to imagine that there occurs CP violation in lepton sector. Several
physicist has considered whether we can see CP violation effect in lepton sector through long baseline neutrino oscillation experiments. The neutrino oscillation probabilities in general depend on six parameters two independent mass squared difference $\Delta_{21}$ and $\Delta_{31}$, there mixing angle, $\theta_{12}$, $\theta_{13}$, $\theta_{23}$, and one CP violating phase $\delta$. There are two large mixing angle $\theta_{12}$, $\theta_{23}$ and one small ($\theta_{13}$), and two mass square difference $\Delta_{ij} = m_j^2 - m_i^2$, with $m_{ij}$ the neutrino masses.

Where

$$\Delta_{21} = \Delta_{\text{solar}},$$ (1)

$$\Delta_{31} = \Delta_{\text{atm}}.$$ (2)

The angle $\theta_{12}$ and $\theta_{23}$ represent the neutrino mixing angles corresponding to solar and atmospheric neutrino oscillation. Much progress has been made towards determining the values of the three mixing angle. In this paper we discuss the effect of Planck’s scale on neutrino mixing and neutrino oscillation probability.

2. Neutrino Mixing Angle and Mass Squared Differences due to Planck Scale Effects

To calculate the effects of perturbation on neutrino observables. The calculation developed in an earlier paper [12]. A natural assumption is that unperturbed ($0^{th}$ order mass matrix) $M$ is given by

$$M = U^* \text{diag}(M_i) U^\dagger,$$ (3)

where, $U_{ei}$ is the usual mixing matrix and $M_i$, the neutrino masses is generated by Grand unified theory. Most of the parameter related to neutrino oscillation are known, the major expectation is given by the mixing elements $U_{e3}$. We adopt the usual parametrization.

$$\frac{|U_{e2}|}{|U_{e1}|} = \tan \theta_{12},$$ (4)

$$\frac{|U_{\mu3}|}{|U_{\tau3}|} = \tan \theta_{23},$$ (5)

$$|U_{e3}| = \sin \theta_{13}.$$ (6)

In term of the above mixing angles, the mixing matrix is

$$U = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) R(\theta_{23}) \Delta R(\theta_{13}) \Delta^* R(\theta_{12}) \text{diag}(e^{ia_1}, e^{ia_2}, 1).$$ (7)

The matrix $\Delta = \text{diag}(e^{i\delta_1}, 1, e^{-i\delta_2})$ contains the Dirac phase. This leads to CP violation in neutrino oscillation $a1$ and $a2$ are the so called Majoring phase, which effects the
neutrino less double beta decay. $f_1$, $f_2$ and $f_3$ are usually absorbed as a part of the definition of the charge lepton field. Planck scale effects will add other contribution to the mass matrix that gives the new mixing matrix can be written as [12]

$$U' = U (1 + i \delta \theta),$$

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} + i \begin{pmatrix} U_{e2} \delta \theta_{12}^* + U_{e3} \delta \theta_{23}^*, & U_{e1} \delta \theta_{12} + U_{e3} \delta \theta_{23}^*, & U_{e1} \delta \theta_{13} + U_{e3} \delta \theta_{23}^* \\ U_{\mu 2} \delta \theta_{12}^* + U_{\mu 3} \delta \theta_{23}^*, & U_{\mu 1} \delta \theta_{12} + U_{\mu 3} \delta \theta_{23}^*, & U_{\mu 1} \delta \theta_{13} + U_{\mu 3} \delta \theta_{23}^* \\ U_{\tau 2} \delta \theta_{12}^* + U_{\tau 3} \delta \theta_{23}^*, & U_{\tau 1} \delta \theta_{12} + U_{\tau 3} \delta \theta_{23}^*, & U_{\tau 1} \delta \theta_{13} + U_{\tau 3} \delta \theta_{23}^* \end{pmatrix}.$$  

(8)

Where $\delta \theta$ is a hermitian matrix that is first order in $\mu$ [12,13]. The first order mass square difference $\Delta M_{ij}^2 = M_i^2 - M_j^2$ get modified [12,13] as

$$\Delta M_{ij}^{\prime 2} = \Delta M_{ij}^2 + 2 (M_i Re(m_{ii}) - M_j Re(m_{jj})),$$

(9)

where

$$m = \mu U' \lambda U,$$

$$\mu = \frac{v^2}{M_{pl}} = 2.5 \times 10^{-6} eV.$$

The change in the elements of the mixing matrix, which we parametrized by $\delta \theta$[12], is given by

$$\delta \theta_{ij} = \frac{i Re(m_{jj})(M_i + M_j) - Im(m_{jj})(M_i - M_j)}{\Delta M_{ij}^{\prime 2}}.$$

(10)

The above equation determine only the off diagonal elements of matrix $\delta \theta_{ij}$. The diagonal element of $\delta \theta_{ij}$ can be set to zero by phase invariance. Using Eq(8), we can calculate neutrino mixing angle due to Planck scale effects,

$$\frac{|U'_{e2}|}{|U'_{e1}|} = \tan \theta_{12}',$$

(11)

$$\frac{|U'_{\mu 3}|}{|U'_{\tau 3}|} = \tan \theta_{23}',$$

(12)

$$|U'_{e3}| = \sin \theta_{13}'$$

(13)

For degenerate neutrinos, $M_3 - M_1 \approx M_3 - M_2 \gg M_2 - M_1$, because $\Delta_{31} \approx \Delta_{32} \gg \Delta_{21}$. Thus, from the above set of equations, we see that $U'_{e1}$ and $U'_{e2}$ are much larger than $U'_{e3}$, $U'_{\mu 3}$ and $U'_{\tau 3}$. Hence we can expect much larger change in $\theta_{12}$ compared to $\theta_{13}$ and $\theta_{23}$. As one can see from the above expression of mixing angle due to Planck scale effects,
depends on new contribution of mixing $U' = U(1 + i\delta \theta)$. We assume that, just above the electroweak breaking scale, the neutrino masses are nearly degenerate and the mixing are Tri-bimaximal, with the value of the mixing angle as $\theta_{12} = 35^\circ$, $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. Taking the common degenerate neutrino mass to be 2 eV, which is the upper limit coming from tritium beta decay [9]. We compute the modified mixing angles using Eqs (11)-(13). We have taken $\Delta_{31} = 0.002eV^2[10]$ and $\Delta_{21} = 0.00008eV^2[11]$. For simplicity we have set the charge lepton phases $f_1 = f_2 = f_3 = 0$. Since we have set the $\theta_{13} = 0$, the Dirac phase $\delta$ drops out of the zeroth order mixing angle. Next section, we discuss the neutrino oscillation probability under Planck scale effects.

3. Neutrino Oscillation Probability Under Planck Scale Effects

The flux of solar neutrino observed by the Homestake detector was on third of that predicted by Standard Solar Model (SSM). The phenomenon of neutrino oscillation can be used to explain neutrino deficit. suppose an electron neutrino is produced at $t = 0$. A set of neutrino mass eigen state at $t = 0$ as

$$\left| \nu(t = 0) > |\nu_e > = cos \theta_{12} |\nu_1(0) > + sin \theta_{12} |\nu_2(0) > \right. \). \quad (14)$$

After time $t$ it becomes

$$\left| \nu(t = t) > |\nu_\mu > = cos \theta_{12} e^{-iE_1 t} |\nu_1(0) > + sin \theta_{12} e^{-iE_2 t} |\nu_2(0) > \right. \). \quad (15)$$

Then the oscillation probability becomes

$$P(\nu_e \rightarrow \nu_\mu) = sin^2 2\theta_{12} sin^2 \left( \frac{1.27 \Delta_{21} L}{E} \right), \quad (16)$$

and the survival probability

$$P(\nu_e \rightarrow \nu_e) = 1 - sin^2 2\theta_{12} sin^2 \left( \frac{1.27 \Delta_{21} L}{E} \right). \quad (17)$$

In the above two equation units of $\Delta_{21} = m_2^2 - m_1^2$ is $ev^2L$ (baseline length) is in meter and $E$ is neutrino energy in MeV. For a maximum oscillation case the phase term in eq(16), $\left( \frac{1.27 \Delta_{21} L}{E} \right)$ equal to $\frac{\pi}{2}$, then oscillation probability only depend on $\theta_{12}$

$$P(\nu_e \rightarrow \nu_\mu) = sin^2 2\theta_{12}. \quad (18)$$

The oscillation probability due to Planck scale effects is

$$P(\nu_e \rightarrow \nu_\mu) = sin^2 2\theta'_{12}, \quad (19)$$

In the above Eq(19), $\theta'_{12}$ is the mixing angle due to Planck scale effects.
4. Numerical Results

We assume that, just above the electroweak breaking scale, the neutrino masses are nearly degenerate and the mixing are Teri-bi maximal, with the value of the mixing angle as \( \theta_{12} = 35^\circ, \theta_{23} = \pi/4 \) and \( \theta_{13} = 0 \). Taking the common degenerate neutrino mass to be 2 eV, which is the upper limit coming from tritium beta decay \[9\]. We compute the modified mixing angles using Eqs (11)-(13). We have taken \( \Delta_{31} = 0.002eV^2 \)[10] and \( \Delta_{21} = 0.00008eV^2 \)[11]. For simplicity we have set the charge lepton phases \( f_1 = f_2 = f_3 = 0 \). Since we have set the \( \theta_{13} = 0 \), the Dirac phase \( \delta \) drops out of the zeroth order mixing angle. We compute the modified mixing angles as function of \( a_1 \) and \( a_2 \) using Eq(11). In table 1, we list the modified neutrino mixing angle \( \theta'_{12} \) and maximum \( P(\nu_e \to \nu_\mu) \) oscillation probability for some sample of \( a_1 \) and \( a_2 \). From Table 1, we see that planck scale effects change the \( \theta_{12} \) from the Tri-bimaximal value of \( \theta_{12} = 35^\circ \) to a value close the the best fit value of the data \[15,16\]. The Planck scale effects give rise the correction to neutrino mass matrix on electroweak symmetry breaking. It is imperative to cheack that these correction do not spoil the good agreement between the experiments fits and the predcetion of the tri-bimaximal mixing scenario. It is expected that dynamics at a higher scale generates the neutrino mass matrix, which will eventually provides the presently observed neutrino mass and mixing. In an attractive scenario, the neutrino mixing pattern generated by high scale dynamics is predicted to be tri-bimaximal. However the solar neutrino data show that the mixing angle \( \theta_{12} \) is substantially less than 35\(^\circ\). It is argued in the literature that renormalization group evolution effects from the higher scale to electroweak scale, can bring down the value of \( \theta_{12} \) from 35\(^\circ\) to a value which is within the experimentally acceptable range. However, for a large range of neutrino parameters, the renormalization group evolution leads to negligible change in the neutrino mass matrix. Then it become imperative to search for such alternate mechanism for which the necessary reduction in \( \theta_{12} \) can be achieved.

5. Conclusions

In this paper, we studied, how Planck scale effects the mixing and oscillation probability. The effective dimension-5 operator from Planck scale [12], leads to correction in neutrino mass matrix at the electroweak symmetry breaking scale. We compute the change in the mixing angle due to additional mass terms for the case of Tri-bimaximal. The change in \( \theta_{12} \) is more than 3\(^\circ\)from the Tri-bimaximal value. Therefore corresponding maximum change in oscillation probability is about 10\%. The change of \( \theta_{12} \) occurs of course, for degenerate neutrino mass with a common mass of about 2 eV. Cosmology constraints, from WMAP measurement [14] impose an upper limit of 0.7eV on neutrino mass. Then the change in the value of \( \theta_{12} \) is smaller. One summarizing statement of this work might be the following, due to Planck scale effects only \( \theta_{12} \) deviated by 3.5\(^\circ\) and other mixing angle have very small deviation and maximum change of \( P(\nu_e \to \nu_\mu) \) oscillation probability is about 10\%, this can be achieved by our calculation of “Tri-Bimaximal” neutrino mixing.
Table 1 Modified mixing angles and maximum $P(\nu_e \rightarrow \nu_\mu)$ oscillation probabilities for some sample of $a_1$ and $a_2$. Input value are $\Delta_{31} = 0.002eV^2$, $\Delta_{21} = 0.00008eV^2$, $\theta_{12} = 35^\circ$, $\theta_{23} = 45^\circ$, $\theta_{13} = 0^\circ$.

References


Representation of su(1,1) Algebra and Hall Effect

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Abstract: In this paper we consider the Schwinger and Heisenberg representation of su(1,1) algebra under Hall effect. In presence of magnetic field, we obtain the generators of su(1,1) algebra in terms of ladder operators, and magnetic field for the one and two bosons system. Also the Casimir operator for both systems are obtained by ladder operators.

Keywords: su(1,1) algebra; Hall Effect; Schwinger Representation; Heisenberg Representation; Casimir Operator; Non-commutative Space

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1. Introduction

When a magnetic field placed transverse to the current of particles we have the Hall effect in system. On the other hand the Hall effect is correspond to the non-commutative geometry. The non-commutative geometries have become the subject literature on the q-deformed groups $E_q(2)$ and $SU_q(2)$ which are the automorphism groups of the quantum plane $zz^* = qz^*z$ and the quantum sphere respectively[1,2,3], these lead us to show effect of non-commutativity in $su(1,1)$ algebra. In another word we would like to consider bosons moving in two dimensional non-commutative space when uniform external magnetic field is present. In the usual case this system leads to Hall effect. Indeed, we know that in representation algebra one can obtains Hall effect in terms of an effective magnetic field [4]. Two bosons system is the same as non-relativistic linear singular harmonic oscillator. It realizes the unitary irreducible representation $D^+(l + 1)$ of $SU(1,1)$ group and the $su(1,1)$ algebra in non-commutative space, which is corresponding to the Hall effect.

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First, we are going to introduce the \( su(1, 1) \approx sp(2, R) \approx so(2, 1) \) algebra which is defined by the following commutation relations,

\[
\begin{align*}
[K_1, K_2] &= -iK_0, \\
[K_0, K_1] &= iK_2, \\
[K_2, K_0] &= iK_1.
\end{align*}
\] (1)

So we can choose generators of \( su(1, 1) \) algebra as \( K_+ \), \( K_- \) and \( K_0 \), and we have,

\[
K_{\pm} = (K_1 \pm iK_2),
\] (2)

which satisfy with following relations,

\[
\begin{align*}
[K_+, K_-] &= -2K_0, \\
[K_0, K_{\pm}] &= \pm K_{\pm},
\end{align*}
\] (3)

and the Casimir operator is given by,

\[
C = K_0^2 - \frac{1}{2}(K_+ K_- + K_- K_+) = \frac{1}{4}(N + 1)(N - 1).
\] (4)

If the eigenvalues of the operator \( C \) is denoted by,

\[
C = l(l + 1),
\] (5)

the irreducible of \( su(1, 1) \) algebra are characterized by a total boson number,

\[
N = -(2l + 1),
\] (6)

this operator commutes with all of the \( K \)'s. Since the \( SU(1, 1) \) group is non-compact, all its unitary irreducible representations are infinite dimensional.

In Section 2, we review the Schwinger representation of \( su(1, 1) \) algebra for one and two bosons system \([5,6]\). In order to construct a spectrum generating algebra for system with the finite number of bound states we introduce a set of boson creation and annihilation operators \([7]\), and then we consider Hall effect and obtain generators of algebra and Casimir operator for both systems.

In section 3, we repeat our formalism in non-commutative geometries as Heisenberg representation of \( su(1, 1) \) algebra \([8]\).

### 2. Schwinger Representation of \( su(1, 1) \) Algebra

Here, we review single and two bosons system and see the generators of \( su(1, 1) \) algebra in terms of ladder operators. Finally, we consider Hall effect and find generators of \( su(1, 1) \) algebra in terms of ladder operators and \( B \) field which \( su(1, 1) \) algebra lives invariant. In
Schwinger representation of $su(1, 1)$ algebra for one boson system, we have generators of $su(1, 1)$ algebra in terms of creation and annihilation operators,

\[
K_1 = \frac{1}{4}(a^\dagger a^\dagger + aa),
\]

\[
K_2 = \frac{1}{4\epsilon}(a^\dagger a^\dagger - aa),
\]

and

\[
K_+ = \frac{1}{2}a^\dagger a^\dagger,
\]

\[
K_- = \frac{1}{2}aa,
\]

\[
K_0 = \frac{1}{4}(2a^\dagger a + 1) = \frac{1}{4}(aa^\dagger + a^\dagger a),
\]

with

\[
[a, a^\dagger] = 1,
\]

equations (7) and (8) satisfy commutative relations (1) and (3). In this case the Casimir operator reduces identically to,

\[
C = -\frac{3}{16}.
\]

Now we introduce two bosons system with operators $a_1$ and $a_2$ which obey the usual commutation relations,

\[
[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1,
\]

\[
[a_1, a_2^\dagger] = [a_2, a_1^\dagger] = 0,
\]

\[
[a_1^\dagger, a_2^\dagger] = [a_1^\dagger, a_2] = 0,
\]

\[
[a_1, a_1] = [a_1, a_2] = 0.
\]

The bilinear combinations $a_1^\dagger a_1$, $a_1 a_2$, $a_1^\dagger a_2^\dagger$ and $a_2^\dagger a_2$ generate the $SU(1, 1)$ group. So, we have,

\[
K_1 = \frac{1}{2}(a_1^\dagger a_2^\dagger + a_2 a_1),
\]

\[
K_2 = \frac{1}{2\epsilon}(a_1^\dagger a_2^\dagger - a_2 a_1),
\]

and,

\[
K_+ = a_1^\dagger a_1^\dagger,
\]

\[
K_- = a_2 a_1,
\]

\[
K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1).
\]

The total boson number operator of this algebra is given by,

\[
N = a_1^\dagger a_1 - a_2^\dagger a_2.
\]
in this case the Casimir operator is,

\[ C = \frac{1}{4} (a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}. \]  

(15)

In the next step, we introduce Hall effect and see that how changes the above operations. The ladder operators and generators of algebra in presence of magnetic field will be changed but the commutation relation (1) and (3) are not changed. In that case for single boson one can find,

\[ a = -2 \frac{\partial}{\partial Z} + \frac{B}{2} \bar{Z}, \]
\[ a^\dagger = 2 \frac{\partial}{\partial \bar{Z}} + \frac{B}{2} Z, \]  

(16)

where \( Z \) and \( \bar{Z} \) are complex coordinates in poincaré space and \( B \) is magnetic field. Now equation (9) become,

\[ [a, a^\dagger] = -2B. \]  

(17)

Therefore we find the generators algebra as follow,

\[ K_1 = \frac{1}{8B} (a^\dagger a^\dagger + aa), \]
\[ K_2 = -\frac{i}{8B} (a^\dagger a^\dagger - aa), \]  

(18)

and

\[ K_+ = \frac{1}{4B} a^\dagger a^\dagger, \]
\[ K_- = \frac{1}{4B} aa, \]
\[ K_0 = \frac{1}{4B} (B - a^\dagger a). \]  

(19)

One can obtain the equation (19) in terms of complex coordinate in poincaré space by the following expressions,

\[ K_+ = \frac{1}{B} \partial_{\bar{Z}}^2 + \frac{1}{2} Z \partial_Z + \frac{B}{16} Z^2, \]
\[ K_- = \frac{1}{B} \partial_Z^2 - \frac{1}{2} \bar{Z} \partial_{\bar{Z}} + \frac{B}{16} \bar{Z}^2, \]
\[ K_0 = \frac{1}{B} \partial_{\bar{Z}} \partial_Z + \frac{1}{4} (Z \partial_Z - \bar{Z} \partial_{\bar{Z}}) - \frac{B}{16} Z \bar{Z}. \]  

(20)

We see that the Casimir operator is same as equation (10).

Now we consider two bosons system, so creation and annihilation operators are,

\[ a_i = -2 \partial_{Z_i} + \frac{B}{2} \bar{Z}_i, \]
\[ a_i^\dagger = 2 \partial_{\bar{Z}_i} + \frac{B}{2} Z_i, \]  

(21)
therefore the commutation relations will be as,

\[
\begin{align*}
\left[a_i, a_i^\dagger\right] &= -2B, \\
\left[a_i, a_j\right] &= 0, \\
\left[a_i, a_i\right] &= \left[a_i, a_j\right] = 0,
\end{align*}
\]

(22)

where i=1,2.

In order to preserve the su(1, 1) algebra, the generators can be written by,

\[
\begin{align*}
K_1 &= \frac{1}{4B} (a_1^\dagger a_2^\dagger + a_2 a_1), \\
K_2 &= -\frac{i}{4B} (a_1^\dagger a_2^\dagger - a_2 a_1),
\end{align*}
\]

(23)

and

\[
\begin{align*}
K_+ &= \frac{1}{2B} a_1^\dagger a_2, \\
K_- &= \frac{1}{2B} a_2 a_1, \\
K_0 &= \frac{1}{4B} (2B - a_1^\dagger a_1 - a_2^\dagger a_2).
\end{align*}
\]

(24)

and the number operators is,

\[
N = \frac{1}{2B} (a_1^\dagger a_1 - a_2^\dagger a_2).
\]

(25)

The same as one boson case, we can write generators in terms of complex coordinates,

\[
\begin{align*}
K_+ &= \frac{2}{B} \partial_{Z_1} \partial_{Z_2} + \frac{B}{8} Z_1 Z_2 + \frac{1}{2} (Z_1 \partial_{Z_2} + Z_2 \partial_{Z_1}), \\
K_- &= \frac{2}{B} \partial_{Z_2} \partial_{Z_1} + \frac{B}{8} Z_1 Z_2 - \frac{1}{2} (\bar{Z}_1 \partial_{Z_2} + \bar{Z}_2 \partial_{Z_1}), \\
K_0 &= \frac{1}{B} (\partial_{Z_1} \partial_{Z_2} + \partial_{Z_2} \partial_{Z_1}) - \frac{B}{16} (Z_1 \bar{Z}_1 + Z_2 \bar{Z}_2) \\
&\quad + \frac{1}{4} (Z_1 \partial_{Z_1} + Z_2 \partial_{Z_2} - \bar{Z}_1 \partial_{\bar{Z}_1} - \bar{Z}_2 \partial_{\bar{Z}_2}).
\end{align*}
\]

(26)

But, we see that the Casimir operator is depend to B, so we have,

\[
C = \frac{1}{16B^2} (a_1^\dagger a_1 - a_2^\dagger a_2)^2 - \frac{1}{4}.
\]

(27)

this is maybe interesting result that Casimir operator in the case of one boson system in presence magnetic field and without magnetic field have same form as C = \[-\frac{3}{16}\]. But in the case of two bosons system Casimir operator in presence of magnetic field (Hall effect) have a different form with respect to without magnetic field B.

In the next section, we work in Heisenberg representation of su(1, 1) algebra.
3. Heisenberg Representation of $su(1, 1)$ Algebra

In this section first we express Heisenberg representation of $su(1, 1)$ algebra, and then introduce Hall effect. The $SU(1, 1)$ group is considered as the automorphism group of the Heisenberg algebra $H$. The commutative relations between generators in this case are like equation (1) and (3), but its generators and creation and annihilation operators are different. So, in this representation we have $Z^*$ and $Z$, so creation and annihilation operators are respectively,

$$Z^* = \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x}),$$
$$Z = \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x}),$$

and their commutation relation is,

$$[Z, Z^*] = 1,$$  \hspace{1cm} (29)

where $Z$, $Z^*$ and 1 are basic elements of one dimensional Heisenberg algebra in the 3-dimensional vector space. Now we like to turn on magnetic field and find creation and annihilation operators,

$$Z^* = \frac{1}{\sqrt{2}}(2Bx - \frac{\partial}{\partial x}),$$
$$Z = \frac{1}{\sqrt{2}}(2Bx + \frac{\partial}{\partial x}),$$

and their commutation relation is,

$$[Z, Z^*] = 2B.$$  \hspace{1cm} (31)

The generators of $su(1, 1)$ algebra satisfy equations (1) and (3) can be written by,

$$K_1 = \frac{1}{8B}(ZZ + Z^*Z^*),$$
$$K_2 = \frac{i}{8B}(Z^*Z^* - ZZ),$$

and

$$K_+ = \frac{ZZ}{4B},$$
$$K_- = \frac{Z^*Z^*}{4B},$$
$$K_0 = \frac{1}{4B}(B - ZZ).$$

(32)

(33)

We obtain generators of $su(1, 1)$ algebra in Heisenberg representation in terms of magnetic field $B$. 

Conclusion

In this study we discuss about Hall effect in $su(1, 1)$ algebra for Schwinger and Heisenberg representation. In presence of magnetic field we know that algebra is invariant, therefore we obtain generators of algebra in terms of ladder operators and magnetic field $B$. Earth-shaking result found in this paper is that magnetic field is not affect on Casimir operator in the case of single boson system for Schwinger representation of $su(1, 1)$ algebra, it means that Hall effect for Casimir operator vanish in that system. But for two bosons system it depends to $B$ field and not invariant under Hall effect.

References

Some LRS Bianchi Type VIₐ Cosmological Models with Special Free Gravitational Fields

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Abstract: The properties of the free gravitational fields and their invariant characterizations are discussed and also obtained LRS Bianchi type VI₀ cosmological models imposing different conditions over the free gravitational fields. Models thus formed are then discussed in detail with respect to their physical and kinematical parameters in the last section of the paper. © Electronic Journal of Theoretical Physics. All rights reserved.

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1. Introduction

Homogeneous cosmological models filled with perfect fluid have already been widely studied in the framework of general relativity. The perfect fluid, however, is supposed to be only a portion of the total constituents of the universe. Cosmological models representing matter and radiation are of particular interest as they depict early stages of the universe when neutrino and photon decoupling take place and parts of these behave like unidirectional streams moving with the velocity of light. Bianchi type V universes are studied by Roy and Singh [1,2] for such material configurations. These universes are a generalization of the open FRW models which eventually isotropize. The standard cosmological models of Friedmann satisfactorily describe the universe at least since the epoch of the last scattering. However a number of questions regarding the early stages of the evolution and the observed inhomogeneities and anisotropies on a smaller scale can not be explained.

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within this framework. It is, therefore, one looks for the possibility of obtaining models of a less restrictive nature displaying anisotropy and inhomogeneity. Considerable work has been done in obtaining various Bianchi type models and their inhomogeneous generalizations. Barrow [3] has pointed out that Bianchi VI\textsubscript{0} universes give a better explanation of some of the cosmological problems like primordial helium abundance and they also isotropize in a special sense. Looking to the importance of Bianchi type VI\textsubscript{0} universes, Tikekar and Patel [4], Bali et al. [5,6], Pradhan and Bali [7] have investigated Bianchi type VI\textsubscript{0} string cosmological models in presence and absence of magnetic field. Bianchi type VI universes representing different types of material distributions have been studied by a number of authors namely Wainwright et al. [8], Dunn and Tupper [9,10], Collins and Hawking [11], Hughston and Jacob [12], Ellis and MacCallum [13], Collins [14], Ruban [15], Tupper [16], MacCallum [17], Uggla and Rosquist [18], Coley and Dunn [19], Wainwright and Anderson [20], Lorentz-Petzold [21,22,23] and De Rop [24].

The role of the free gravitational field in the determination of the flow of matter in the cosmological models has been emphasized by Ellis [25]. It is well known that near the singularities, the curvature of spacetime becomes dominating so that the evolution of the universe in its early stages is largely affected by the character of the free gravitational field. The study of models with specific types of the free gravitational field has also become necessary due to the speculative existence of copious amount of gravitational radiation following the big-bang. Models for which the free gravitational field is of the ‘electric type’ are considered to have their Newtonian analogues while a ‘magnetic type’ free gravitational field is purely general relativistic. Since there is an intimate relationship between the flow patterns of the fluid and the free gravitational field, it is tempting to impose adhoc conditions on the latter. Bianchi type VI\textsubscript{0} magnetic type solutions in presence of a viscous fluid and an incident magnetic field were obtained by Roy and Singh [26]. Bianchi VI electric type solutions are given for different material distributions by Roy and Banerjee [27,28].

In the present paper we have redefined the two specific types of free gravitational fields which we term as the ‘electric type’, the ‘magnetic type’ and also recall the idea of the ‘gravitational wrench’ of unit ‘pitch’. The concept of the ‘gravitational wrench’ of unit ‘pitch’ was first explored by Roy and Banerjee [29]. We receive this idea here again and give a brief discussion about the properties of these free gravitational fields and their invariant characterizations.

In classical electromagnetic theory, the electromagnetic field has two independent invariants $F_{ij}F^{ij}$ and $^*F_{ij}F^{ij}$. The classification of the field is characterized by the property of the scalar $k^2 = (F_{ij}F^{ij})^2 + (^*F_{ij}F^{ij})^2$. When $k = 0$ the field is said to be null and for any observer $|E| = |H|$ and $E, H = 0$ where $E$ and $H$ are the electric and magnetic vectors respectively. When $k \neq 0$, the field is non-null and there exists an observer for which $mE = nH$, $m$ and $n$ being scalars. It is easy to see that if $^*F_{ij}F^{ij} = 0, F_{ij}F^{ij} \neq 0$ then either $E = 0$ or $H = 0$. These we call the magnetic and electric fields, respectively. If $F_{ij}F^{ij} = 0, ^*F_{ij}F^{ij} \neq 0$ then $E = \pm H$. This we
call the ‘electromagnetic wrench’ with unit ‘pitch’. In this case Maxwell’s equations lead to an empty electromagnetic field with constant electric and magnetic intensities. In the case of gravitational field, the number of independent scalar invariants of the second order is fourteen. The independent scalar invariants formed from the conformal curvature tensor are four in number. In the case of Petrov type D spacetimes, the number of independent scalar invariants are only two, viz. \(C_{hijk}C^{hijk}\) and \(*C_{hijk}C^{hijk}\). Analogous to the electromagnetic case, the electric and magnetic parts of free gravitational field for an observer with velocity \(v^i\) are defined as \(E_{\alpha\beta} = C_{\alpha j\beta i}v^j v^i\) and \(H_{\alpha\beta} = *C_{\alpha j\beta i}v^j v^i\) (Ellis [25]). It is clear from the canonical form of the conformal curvature for a general type D spacetime that there exists an observer for which \(E_{\alpha\beta} = (n/m)H_{\alpha\beta}\), where \(m,n\) being integers and \(m \neq 0\). The field is said to be purely magnetic type for \(n = 0, m \neq 0\). In this case we have \(E_{\alpha\beta} = 0\) and \(H_{\alpha\beta} \neq 0\). The physical significance for the gravitational field of being magnetic type is that the matter particles do not experience the tidal force. When \(m \neq 0\) and also \(n \neq 0\), we call that there is a ‘gravitational wrench’ of unit ‘pitch’ \(|n/m|\) in the free gravitational field [29]. If ‘pitch’ is unity then we have \(E_{\alpha\beta} = \pm H_{\alpha\beta}\).

The spacetime having a symmetry property is invariant under a continuous group of transformations. The transformation equations for such a group of order \(r\) is given by

\[
X^i = f^i(x^1, ..., x^r, a^1, ..., a^r) \tag{1}
\]

which satisfy the differential equations

\[
\frac{\partial X^i}{\partial x^\alpha} = \xi^i_{(\beta)}(X)A^\beta_a(a), (\alpha, \beta = 1, ..., r) \tag{2}
\]

where \(a^1, ..., a^r\) are \(r\) essential parameters. The vectors \(\xi^i_{(\alpha)}\) are the Killing vectors for the group \(G_r\) of isometry satisfying the Killing’s equation

\[
\xi_{(\alpha)ij} + \xi_{(\alpha)ji} = 0. \tag{3}
\]

A subspace of spacetime is said to be the surface of transitivity of the group if any point of this space can be transformed to another point of it by the action of this group. A spacetime is said to be spatially homogeneous if it admits a group \(G_r\) of isometry which is transitive on three dimensional space-like hypersurfaces. The group \(G_3\) of isometry was first considered by Bianchi [30] who obtained nine different types of isometry group known as the Bianchi types. The space-time which admits \(G_4\) group of isometry is known as locally rotationally symmetric (LRS) which always has a \(G_3\) as its subgroup belonging to one of the Bianchi types provided this \(G_3\) is simply transitive on the three dimensional hypersurface \(t = \text{constant}\).

Here we have considered an LRS Bianchi type VI_0 spacetime and obtained models with free gravitational field of purely ‘magnetic type’ and also in the presence of ‘gravitational wrench’ of unit ‘pitch’ in the free gravitational field. It is found that the ‘magnetic’ part of the free gravitational field induces shear in the fluid flow, which is zero in the case of a ‘electric’ type free gravitational field representing an unrealistic distribution in this case. In section 2, solutions representing LRS Bianchi type VI_0 cosmological models with
perfect fluid imposing different conditions on the free gravitational field are obtained. In the last section, we discuss the models so formed with respect to their physical and kinematical parameters.

2. Formation of the Line Element

We consider an LRS Bianchi type VI\(_0\) universe for which

\[ ds^2 = \eta_{ab} \theta^a \theta^b \] (4)

where \( \theta^1 = Adx, \theta^2 = Bexp^x dy, \theta^3 = Bexp^{-x} dz, \theta^4 = dt; A \) and B being functions of t alone. The non-vanishing physical components of \( C_{abcd} \) for the line element (1) are given by

\[ C_{2323} = \frac{1}{6} \left[ \frac{2\dddot{A}}{A} - \frac{2\dddot{B}}{B} - \frac{2\dot{A}\ddot{B}}{AB} + \frac{2\dot{B}^2}{B^2} + \frac{4}{A^2} \right] \]

\[ = -\frac{1}{2} C_{3131} = C_{1212} = -C_{1414} = \frac{1}{2} C_{2424} = -C_{3434} \] (5)

\[ C_{2314} = -\frac{1}{A} \left[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right] = -\frac{1}{2} C_{3124} = C_{1234}. \] (6)

The Einstein field equations

\[ R^a_b - \frac{1}{2} R \delta^b_a + \Lambda \delta^b_a = -8\pi T^b_a \] (7)

for the line-element (4), it leads to

\[ \frac{2\dddot{B}}{B} + \frac{\dddot{B}^2}{B^2} + \frac{1}{A^2} + \Lambda = -8\pi T^1_1, \] (8)

\[ \frac{\dddot{A}}{A} + \frac{\dddot{B}}{B} + \frac{\dot{A}\ddot{B}}{AB} - \frac{1}{A^2} + \Lambda = -8\pi T^2_2 = -8\pi T^3_3, \] (9)

\[ \frac{2\dddot{A}}{AB} + \frac{\dddot{B}^2}{B^2} - \frac{1}{A^2} + \Lambda = -8\pi T^4_4 \] (10)

and

\[ 0 = -8\pi T^b_a \quad (a \neq b). \] (11)

The matter in general will represent an anisotropy fluid with time-lines as the flow-lines of the fluid. The kinematical parameters of the fluid flow are given by

\[ \theta = \frac{\dot{A}}{A} + \frac{2\dot{B}}{B}, \] (12)

\[ \sigma = \frac{1}{\sqrt{3}} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right). \] (13)

We now solve the equations (8) – (10) under the following two alternative possibilities:
**Case – I:** Free gravitational field is Magnetic type ($m \neq 0$, $n = 0$ i.e. $H_{\alpha\beta} \neq 0$, and $E_{\alpha\beta}=0$). From (5), we have

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{2}{A^2} = 0. \quad (14)$$

We assume the matter to be a perfect fluid with comoving flow vector $v^a$;

$$T^b_a = (\in + p)v^a v^b + p\delta^b_a. \quad (15)$$

with $v^a = \delta^a_0$, $\in$ and $p$ being respectively, the density and the thermodynamic pressure of the fluid. We have

$$T_1^1 = T_2^2 = T_3^3 = p, T_4^4 = -\in. \quad (16)$$

The field equation (8) together with equation (9) gives

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{B}}{B} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) - \frac{2}{A^2} = 0. \quad (17)$$

From equations (14) and (17), we further have two independent equations

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = 0, \quad (18)$$

$$\frac{\dot{B}}{B} \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) - \frac{2}{A^2} = 0. \quad (19)$$

If we write $A/B$ as $U$, respectively, we get from equations (18) and (19)

$$UU_{\xi\xi} + \dot{A}UU_{\xi} - 2U^2_{\xi} = 0 \quad (20)$$

and

$$U^2_{\xi} - \dot{A}UU_{\xi} + 2U^2 = 0. \quad (21)$$

where $\xi$ is defined by

$$\frac{d}{dt} = \frac{1}{A} \frac{d}{d\xi}. \quad (22)$$

Equations (20) and (21) lead to a second order differential equation

$$UU_{\xi\xi} - U^2_{\xi} + 2U^2 = 0. \quad (23)$$

On substituting $U = e^\lambda$, the above equation reduces to the form

$$\lambda_{\xi\xi} + 2 = 0, \quad (24)$$

which yields,

$$\lambda = c_1\xi - \xi^2 + c_2, \quad (25)$$
where $c_1$ and $c_2$ are arbitrary constants. Thus
\[ U = c_3 e^{-\tau(\tau - c_1)}, \] (26)

where $\tau$ stands for $\xi$.

Equations (21), (22) and (26) together give
\[ A = \frac{c_4}{(c_1 - 2\tau)} e^{-\tau(\tau - c_1)}, \] (27)
\[ B = \frac{c_5}{c_1 - 2\tau}, \] (28)

where $\tau$ is given by
\[ \frac{d\tau}{dt} = \frac{1}{c_4} (c_1 - 2\tau) e^{\tau(\tau - c_1)}, \] (29)

c_3, c_4 and c_5 being arbitrary constants. The metric (4) then reduces to the form
\[ ds^2 = \frac{c_4^2}{(c_1 - 2\tau)^2} e^{-2\tau(\tau - c_1)} \left[ -d\tau^2 + dx^2 + e^{2\tau(\tau - c_1)} \{ e^{2x} dy^2 + e^{-2x} dz^2 \} \right]. \] (30)

Expressions for $\epsilon$ and $p$ are respectively given by
\[ 8\pi p = -3 c_4^2 [4 - (c_1 - 2\tau)^2] e^{2\tau(\tau - c_1)} - \Lambda \] (31)
and
\[ 8\pi \epsilon = 3 c_4^2 [4 + (c_1 - 2\tau)^2] e^{2\tau(\tau - c_1)} + \Lambda. \] (32)

The kinematical parameters given by equation (12) and equation (13) take the form
\[ \theta = \frac{1}{c_4} \{ 6 + (c_1 - 2\tau)^2 \} e^{\tau(\tau - c_1)}, \] (33)
\[ \sigma = \frac{1}{c_4 \sqrt{3}} (c_1 - 2\tau)^2 e^{\tau(\tau - c_1)}. \] (34)

**Case II:** There is a ‘gravitational wrench’ of unit ‘pitch’ in the free gravitational field i.e. $E_{\alpha\beta} = \pm H_{\alpha\beta}$.

In this case, we have
\[ E_{\alpha\beta} = e H_{\alpha\beta}, \quad e^2 = 1. \] (35)

Equation (35) together with equations (5) and (6) gives
\[ \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{B}}{B} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \frac{2}{A^2} = -3e \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right). \] (36)

For a perfect fluid medium, with the help of the equations (17) and (36), we have
\[ \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} = -\frac{3e}{2A} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right), \] (37)
\[
\frac{\dot{B}}{B} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) - \frac{2}{A^2} = \frac{3e}{2A} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right). \tag{38}
\]

On assuming \(A/B = U\), the equations (37) and (38) respectively reduce to the forms
\[
U_{\xi\xi} + \dot{A} U_{\xi} = -\frac{3e}{2} U_{\xi} \tag{39}
\]
and
\[
\dot{A} U U_{\xi} - U_{\xi}^2 - 2U^2 = \frac{3e}{2} U U_{\xi} \tag{40}
\]
where \(\xi\) is defined by (22). Equations (39) and (40) lead to a second order differential equation
\[
U U_{\xi\xi} + 3e U U_{\xi} + U_{\xi}^2 + 2U^2 = 0. \tag{41}
\]
On substituting \(U = \exp(\lambda)\), equation (41) take the form
\[
\lambda_{\xi\xi} + 2\lambda_{\xi}^2 + 3e\lambda_{\xi} + 2 = 0 \tag{42}
\]
which yields,
\[
U^4 = (2\tau^2 + 3e\tau + 2)^{-1} \exp \left( \frac{6e}{\sqrt{7}} \tan^{-1} \frac{4\tau + 3e}{\sqrt{7}} \right) \tag{43}
\]
where, \(\tau = \frac{d\lambda}{d\xi}\). From (40), we have
\[
\dot{A} = \frac{(2\tau^2 + 3e\tau + 4)}{2\tau}. \tag{44}
\]
Equations (22) and (42) together lead to
\[
A^2 = \frac{(2\tau^2 + 3e\tau + 2)^{1/2}}{\tau^2} \exp \left( \frac{3e}{\sqrt{7}} \tan^{-1} \frac{4\tau + 3e}{\sqrt{7}} \right), \tag{45}
\]
\[
B^2 = \frac{(2\tau^2 + 3e\tau + 2)}{\tau^2} \tag{46}
\]
where, \(\tau\) is given by
\[
\frac{d\tau}{dt} = -\frac{(2\tau^2 + 3e\tau + 2)}{A}. \tag{47}
\]
Metric (4) then takes the form
\[
\begin{align*}
\text{d}s^2 &= \frac{(2\tau^2 + 3e\tau + 2)^{1/2}}{\tau^2} \exp \left( \frac{3e}{\sqrt{7}} \tan^{-1} \frac{4\tau + 3e}{\sqrt{7}} \right) \left[ -\frac{d\tau^2}{(2\tau^2 + 3e\tau + 2)^2} + dx^2 \\
&+ (2\tau^2 + 3e\tau + 2)^{1/2} \exp \left( -\frac{3e}{\sqrt{7}} \tan^{-1} \frac{4\tau + 3e}{\sqrt{7}} \right) \left\{ e^{2\tau} dy^2 + e^{-2\tau} dz^2 \right\} \right]
\end{align*} \tag{48}
\]
Expressions for \(\epsilon, p, \theta\) and \(\sigma\) for the above metric are
\[
8\pi \epsilon = \frac{(12e\tau^3 + 39\tau^2 + 72e\tau + 48)}{4(2\tau^2 + 3e\tau + 2)^{1/2}} \exp \left( -\frac{3e}{\sqrt{7}} \tan^{-1} \frac{4\tau + 3e}{\sqrt{7}} \right) + \Lambda, \tag{49}
\]
\[8\pi p = \frac{(12e\tau^3 + 29\tau^2 + 72e\tau - 48)}{4(2\tau^2 + 3e\tau + 2)^{1/2}} \exp \left(\frac{-3e}{\sqrt{7}} \tan^{-1}\left(\frac{4\tau + 3e}{\sqrt{7}}\right)\right) - \Lambda, \quad (50)\]

\[\theta = \frac{(2\tau^2 + 9e\tau + 12)}{2(2\tau^2 + 3e\tau + 2)^{1/4}} \exp \left(\frac{-3e}{2\sqrt{7}} \tan^{-1}\left(\frac{4\tau + 3e}{\sqrt{7}}\right)\right), \quad (51)\]

\[\sigma = \frac{\tau^2}{\sqrt{3}} \left(2\tau^2 + 3e\tau + 2\right)^{1/4} \exp \left(\frac{-3e}{2\sqrt{7}} \tan^{-1}\left(\frac{4\tau + 3e}{\sqrt{7}}\right)\right). \quad ... (52)\]

3. Discussion

The model (30) starts expansion with a big-bang singularity from \(\tau = -\infty\) and it goes on expanding till \(\tau = \frac{c_1}{2}\) where \(\tau = -\infty\) and \(\tau = \frac{c_1}{2}\) respectively correspond to the cosmic time \(t = 0\) and \(t = \infty\). The model is found to be realistic everywhere in this time interval for \(\Lambda > -\frac{12}{c_1^2} \exp \left(-\frac{c_1^2}{2}\right)\). The model behaves like a steady-state de-Sitter type universe at late times where the physical and kinematical parameters \(\epsilon, p\) and \(\theta\) tend to a finite value however shear vanishes there. The model has a non-zero anisotropy at the initial stage of its expansion which goes on decreasing with time and asymptotically it dies out completely. The model has a point singularity at \(t = 0\). The weak and strong energy conditions are also satisfied identically within this time span.

The model (48) is realistic for \(e = +1\). It starts expansion with a single shot explosion from a point singularity state \(\tau = \infty\) and goes on expanding till \(\tau = 0\) where \(\tau = \infty\) and \(\tau = 0\) respectively correspond to the cosmic time \(t = 0\) and \(t = \infty\). At \(t = 0\) the model has a point singularity. The energy conditions \(\epsilon > 0\) and \(\epsilon \geq p\) are satisfied throughout the time span for \(6\sqrt{2} \exp \left(-\frac{3\sqrt{7}}{\sqrt{7}} \tan^{-1}\frac{3}{\sqrt{7}}\right) + \Lambda > 0\). At \(t = \infty\), the model enters a steady-state de-Sitter type phase where the physical and kinematical parameters \(\theta, \epsilon\) and \(p\) approach a finite value while the shear vanishes. Initially at \(t = 0\), the model has non-zero anisotropy which goes on decreasing with time and finally the model gets isotropized at large times where \(\frac{\sigma}{\theta}\) vanishes. The necessary covariant condition for the existence of gravitational waves in cosmology explained by Maartens et al. [31], is \(D^b E_{ab} = 0 \neq \text{curl} E_{ab}, D^b H_{ab} = 0 \neq \text{curl} H_{ab}\). In the present model (48), the above two conditions are not satisfied which indicates as a result no presence of the gravitational wave.

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References

The Motion of A Test Particle in the Gravitational Field of A Collapsing Shell

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Abstract: We use the Israel formalism to describe the motion of a test particle in the gravitational field of a collapsing shell. The formalism is developed in both of Schwarzschild and Kruskal coordinates.

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1. Introduction

The dynamics of thin shells of matter in general relativity has been discussed by many authors. Our approach is similar to that used by Israel [1] and Kuchar [2] for studying the collapse of spherical shells. Israel [1] found invariant boundary conditions connecting the extrinsic curvature of a shell in space-times on both sides of it with the matter of this shell. In this paper we study the motion of a test particle in the gravitational field of a collapsing shell. In Section 2 we give the general formalism. In Section 3 this formalism is applied to the thin shell in Schwarzschild space-time and test particle. The equations of motion for shell and particle in Kruskal coordinates are given in Section 4, and the equations of motion of a shell and particle in Minkowski space are given in Section 5. Finally Section 6 is devoted to the numerical results and discussion. Also adopt the units such that \( c = G = 1 \).

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2. Formalism

The junction conditions for arbitrary surfaces and the equation for a shell have been well understood since the works of Israel [1], Barrabes [3], de la Cruz and Israel [4], Kuchar [2], Chase [5], Balbinot and Poisson [6], and Lake [7].

The time-like hypersurface \( \Sigma \), which divides the Riemannian space-time \( M \) into two regions, \( M^- \) and \( M^+ \), represents the history of a thin spherical shell of matter in general. The regions \( M^- \) and \( M^+ \) are covered by the mutually independent coordinate systems \( X^\alpha_- \) and \( X^\alpha_+ \). The hypersurface \( \Sigma \) represents the boundary of \( M^- \) and \( M^+ \) respectively; consequently the intrinsic geometry of \( \Sigma \) induced by the metrics of \( M^- \) and \( M^+ \) must be the same. Let \( \Sigma \) be parameterized by intrinsic coordinates \( \zeta^a \),

\[
X^\alpha_\pm = X^\alpha_\pm(\zeta^a) \tag{1}
\]

(Greek letters refer to 4-dimensional indices, Latin letters refer to 3-dimensional indices on \( \Sigma \). The signature of the metric is +2, and the Newtonian gravitational constant and light velocity are equal to unity). The basic vectors \( e^\alpha_a = \frac{\partial}{\partial \zeta^a} \) tangent to \( \Sigma \) have the components

\[
e^\alpha_{a\pm} = \frac{\partial X^\alpha_{\pm}}{\partial \zeta^a} \tag{2}
\]

with respect to the two four-dimensional coordinate systems in \( M^- \) and \( M^+ \). Their scalar products define the metric induced on the hypersurface \( \Sigma \),

\[
g^\pm_{ab} = \delta^\mu_\nu e^\mu_{a\pm} e^\nu_{b\pm} \tag{3}
\]

The metric induced by the metrics of both regions \( M^- \) and \( M^+ \) must be identical, \( g^\pm_{ab}(\zeta) = g_{ab}(\zeta) \). The condition is stated independently of the coordinate systems in \( M^- \) and \( M^+ \). The unit normal vector \( n \) to \( \Sigma \):

\[
n \cdot n|_{\pm} = 1 \tag{4}
\]

is directed from \( M^- \) and \( M^+ \). The manner in which \( \Sigma \) is bent in space \( M^- \) and \( M^+ \) is characterized by the three-dimensional extrinsic curvature tensor

\[
K^\pm_{ab} = -n^a_\alpha D^\alpha_{b\pm} = e^\alpha_{a\pm} \frac{Dn^\alpha_{b\pm}}{\partial \zeta^b} \tag{5}
\]

where \( \frac{D}{\partial \zeta^b} \) represents the absolute derivative with respect to \( \zeta^b \). The surface energy-momentum tensor \( t_{ab} \) is determined by the jump \( [K_{ab}] = K^+_{ab} - K^-_{ab} \). \( \Sigma \) represents the history of a surface layer (a singular hypersurface of order one) if \( K^+_{ab} \neq K^-_{ab} \). The Einstein equation determines the relations between the extrinsic curvature \( K^\pm_{ab} \) and the three-dimensional intrinsic energy-momentum tensor \( t_{ab} = t_{a\beta} e^\alpha_a e^\beta_b \) is given by the Lanczos equation
\[ [K_{ab}] = -8\pi(t_{ab} - \frac{1}{2}tg_{ab}) \]  

(6)

where \( t = t^a_a \). We can write this relation in the form

\[ t_{ab} = -\frac{1}{8\pi}([K_{ab}] - g_{ab}[K]) \]  

(7)

where \([K] = g^{ab}[K_{ab}]\). These are the field equations for the shell.

3. The Motion of Shell and Test Particle

3.1 The Motion of One Shell in The Schwarzschild Space

The shell is spherically symmetric. Therefore, the space-time outside the shell can be described by the line element,

\[ ds^2 = -f dt^2_+ + f^{-1}dr^2_+ + r^2 d\Omega^2 \]  

(8)

where

\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \]

is the line element on the unit sphere and

\[ f = 1 - \frac{2m}{r} \]

where \( m \) is the gravitational mass in the exterior space. Inside the shell space-time is flat, i.e., \( f = 1 \). As exterior and interior coordinates, we use \( X^a_+ = (t_+, r_+, \theta, \phi) \) and \( X^a_- = (t_-, r_-, \theta, \phi) \) respectively. The intrinsic coordinates on \( \Sigma \) are the proper time \( \tau \) measured by the co-moving observer on the shell and the spherical angles \( \theta, \phi : \xi^a = (\tau, \theta, \phi) \). Let us suppose that \( \Sigma \) is determined by

\[ r_\pm = R(\tau), \; t_\pm = t_\pm(\tau), \; \theta_\pm = \theta, \; \phi_\pm = \phi \]

(condition (3) implies the continuity of \( r \) on the shell). We get from (2) and (4)

\[ e^a_{\tau \pm} = (\dot{t}_\pm, \dot{R}, 0, 0), \]
\[ e^a_{\theta \pm} = (0, 0, 1, 0), \]
\[ e^a_{\phi \pm} = (0, 0, 0, 1) \]  

(9.a)

and

\[ n_{a\pm} = (-\dot{R}, \dot{t}_\pm, 0, 0) \]  

(9.b)

where the dot represents the derivative with respect to proper time, and \( R \) is the radius of the shell. Thus the three-dimensional metric tensor on \( \Sigma \) is
From (5) we get the extrinsic curvature $K^\pm_{ab}$ in $M^-$ and $M^+$,

\[
K^-_{\tau\tau} = \dot{R}t^- - \ddot{R}t^-,
\]

\[
K^-_{\theta\theta} = \dot{R}t^-,
\]

\[
K^-_{\phi\phi} = \dot{R}t^- \sin^2 \theta,
\]

and

\[
K^+_{\tau\tau} = -t_+ \ddot{R} + \ddot{R}t_+ - \frac{1}{2} t_+ [t_+ f' f - 3 f^{-1} \dot{f} \ddot{R}^2],
\]

\[
K^+_{\theta\theta} = f R t_+,
\]

\[
K^+_{\phi\phi} = f R t_+ \sin^2 \theta
\]

We suppose that the form of the three-dimensional energy-momentum tensor is

\[
t_{ab} = P g_{ab} + (P + \sigma) U_a U_b
\]

where $\sigma$ is the surface density and $P$ is the surface pressure. Therefore the components of $t_{ab}$ are

\[
t_{\tau\tau} = - \frac{1}{4\pi R} (t_+ f - t_-)
\]

\[
t_{\theta\theta} = - \frac{R^2}{8\pi} [H + f \dot{t}_+ \left(3 \frac{R^2}{2f} - \frac{1}{2} f \dot{t}_+^2 \right) + \frac{1}{R} (t_- - f t_+)]
\]

\[
t_{\phi\phi} = t_{\theta\theta} \sin^2 \theta
\]

where

\[
H = \ddot{R} (t^- - t_+) + \dot{R} (\ddot{t}_+ - \ddot{t}^-)
\]

Because $\tau$ is the proper time on the shell the conditions

\[
\ddot{t}^- = 1 + \dot{R}^2
\]

\[
(f \dot{t}_+)^2 = f + \dot{R}^2
\]

must be fulfilled. Supposing $t_{\tau\tau} = \sigma$ and $t_{\theta\theta} = t_{\phi\phi} = P$, we get from (12) and (11),

\[
4\pi R \sigma = \sqrt{1 + \dot{R}^2} \mp \sqrt{f + \dot{R}^2}
\]

\[
8\pi R^2 P = \frac{1}{\sqrt{1 + \dot{R}^2}} \left[ m \sqrt{1 + \dot{R}^2} + L \right]
\]
where
\[ L = R(R\ddot{R} - \sqrt{1 + R^2\sqrt{f + \dot{R}^2}})(\sqrt{1 + R^2} \mp \sqrt{f + \dot{R}^2}) \]

All relations (8 – 13) are valid both above and under the horizon. Above the horizon \( \dot{t}_+ > 0 \) and \( f > 0 \), so that \( f \dot{t}_+ = \sqrt{f + \dot{R}^2} \) and the upper sign is valid in (13a). Under the horizon \( f < 0 \) and \( \dot{t}_+ \) can be either positive or negative. Consequently, if \( \dot{t}_+ > 0 \) the sign (+) must be taken in (13). The sign changes for \( R = \frac{M^2}{2m} \) where \( M = 4\pi R^2 \sigma \). We put (13a) into (13b) to get the relation between the surface pressure and the surface density in the form
\[ P = \frac{m}{8\pi R^2 \sqrt{f + \dot{R}^2}} + \frac{R\ddot{R} - \sqrt{1 + R^2\sqrt{f + \dot{R}^2}}}{2\sqrt{1 + R^2\sqrt{f + \dot{R}^2}}} \sigma \] (14)

In the case of dust \( P = 0 \) and \( U^a = (1, 0, 0) \); then equation (13a) where \( 4\pi R^2 \sigma = M = \text{const.} \),
\[ \sqrt{1 + \dot{R}^2} \mp \sqrt{f + \dot{R}^2} = \frac{M}{R} \] (15)
is an integral of motion of (14), and it fully determines the motion of the shell.

3.2 The Motion of a Test Particle

The motion of a test particle is given by the geodesic equation. The velocity of the particle is
\[ \dot{r} = \pm \sqrt{c_1 - f_p} \] (16)
where \( f_p = 1 - \frac{2m}{r} \) and \( c_1 \) is determined according to our choice of the initial value. Therefore, the equations of motion of the test particle in Kruskal coordinates are
\[ \ddot{r} = -\frac{m}{r^2} \] (17.a)
\[ \ddot{v}_p = A_p \dot{v}_p [r \ddot{r} + (1 + \frac{r}{2m}) \dot{r}^2 + \frac{r}{4m}] + v_p \left[ \frac{\dot{r}}{8m} (1 + \frac{r}{2m}) \right] \] (17.b)
\[ \ddot{v}_p = A_p \dot{v}_p [r \ddot{r} + (1 + \frac{r}{2m}) \dot{r}^2 + \frac{r}{4m}] + u_p \left[ \frac{\dot{r}}{8m} (1 + \frac{r}{2m}) \right] \] (17.c)

where
\[ A_p = \frac{1}{8m^2 (v_p \dot{u}_p - u_p \dot{v}_p)} e^{\frac{r}{2m}} \]

Equation (17.a) represents the acceleration of the particle if it is outside the shell and if \( r \) is far from the centre of the shell. If the particle is inside the shell, in flat-space, its acceleration is zero and its velocity is the same as we obtained during passage through the surface of the shell.
3.3 The Intersection of The Shell and Particle

Let us take a particle moving in the gravitational field of a collapsing thin shell. We suppose that the particle interacts with the shell only gravitationally. In this case it must be fulfilled that the projection of the four-velocity of the particle before the intersection $U^\alpha_+ n^\alpha_+$ on $n^\alpha_+$ is equal to the projection of the four-velocity of the particle after the intersection $U^\alpha_- n^\alpha_-$ on $n^\alpha_-$.

\[ U^\alpha_+ n^\alpha_+ |_{(before)} = U^\alpha_- n^\alpha_- |_{(after)} . \]

Also, the projection of the four-velocities of the particle before and after crossing on the tangent $e^\alpha_\alpha$ are equal,

\[ e^\alpha_+ U^\alpha_+ |_{(before)} = e^\alpha_- U^\alpha_- |_{(after)} . \]

From these relations we get the velocity of the particle after crossing the shell from outside

\[ \dot{\hat{r}}_+ \equiv \dot{r}_+ = \psi^2 \ddot{R}_- [\dot{\hat{u}}_p - \dot{\hat{u}}_p] + \dot{t}_- [\dot{\hat{v}}_p - \dot{\hat{v}}_p] \] (18)

Similarly, if the particle crosses the shell from inside,

\[ \dot{\hat{u}}_p = \dot{r}_+ - \dot{R}_- \dot{\hat{u}}_p + t_+ [\dot{\hat{v}}_p - \dot{\hat{v}}_p] \] (19)

Because $\tau_p$ is a proper time of the test particle, the conditions

\[ t_{p-} = \sqrt{1 + \dot{r}_p^2}, \]
\[ \dot{t}_{p+} = f_p^{-1} \sqrt{f_p + \dot{r}_p^2}, \]
\[ \dot{\hat{v}}_p = \sqrt{\dot{u}_p^2 + \psi_p^2}, \]
\[ \dot{t}_- = \sqrt{1 + \dot{R}_-^2} \] (20)

must be fulfilled. At the point of intersection $R = r = R_0$, where $R_0$ is the radius of the shell at the point of intersection. The suffix ‘+’ means a value of the quantity in the region to which the normal vector is directed at the points of the shell and ‘-’ means the quantity on the other side of the shell.

3.4 The Motion of a Shell and Particle in a Flat Space

The radius of the shell changes smoothly on the surface of the shell, $R_+(\tau) = R_-(\tau)$, therefore the equation of the motion of the shell in Minkowski space is

\[ \ddot{R}_- = -\sqrt{\left(\frac{M}{2R} + \frac{m}{M}\right)^2 - 1} \]

and
\[ \dot{t}_- = \sqrt{1 + \dot{R}^2}. \]

The acceleration of the particle inside the shell will be \( \ddot{r}_- = 0 \) and \( \ddot{t}_- = 0 \). Therefore, the velocity is \( \dot{r}_- = \text{const.} \equiv \dot{r}_p \) and \( \dot{t}_- = \text{const.} \equiv \dot{t}_p \), where \( \dot{r}_p \) and \( \dot{t}_p \) are determined from (18) at the point of intersection. This represents

\[ r_- = \dot{r}_p \tau_p + R_0, \]

and

\[ t_- = \dot{t}_p \tau_p + t_{01} \]

where \( t_{01} \) is the Minkowskian time and \( R_0 \) is the radius of the shell at the point of intersection.

To find the space-time point at which the particle’s four-velocity intersects the hypersurface representing the shell we use the Kruskal coordinate \( v \) as the independent variable.

4. The Motion of A Shell and Particle in Kruskal Coordinates

The motion of the shell can be described by

\[ R' = G\left[-Rve^\frac{n}{R} \pm \frac{u}{q} \sqrt{q^2 R^2 e^\frac{n}{R} - 2mR \phi e^\frac{n}{2m}}\right] \quad (21.a) \]

\[ u' = \frac{1}{u}(v + \frac{RR'}{8m^2 e^\frac{n}{2m}}) \quad (21.b) \]

where

\[ G = \frac{8m^2 q^2 \psi^2}{(64m^4 u^2 + R^2 q^2 \psi^2 e^\frac{n}{2m})}, \]

\[ q = -\sqrt{\left(\frac{M}{2R} + \frac{m}{M}\right)^2 - 1}, \]

\[ \phi \equiv \psi^2 - u^2 = (1 - \frac{R}{2m}) e^\frac{n}{2m}. \]

The motion of the test particle is described by

\[ r' = G_p\left[-rv_p e^\frac{n}{R} \pm \frac{u_p}{q_p} \sqrt{q_p^2 r^2 e^\frac{n}{R} - 2m \phi_p e^\frac{n}{2m}}\right] \quad (22.a) \]

\[ u'_p = \frac{1}{u_p}(v_p + \frac{rr'}{8m^2 e^\frac{n}{2m}}) \quad (22.b) \]

where

\[ G_p = \frac{8m^2 q_p^2 \psi_p^2}{(64m^4 u_p^2 + r^2 q_p^2 \psi_p^2 e^\frac{n}{2m})}, \]
\[ u_p = \sqrt{v_p^2 + \left(\frac{r}{2m} - 1\right)e^{\frac{r}{2m}}}, \]
\[ \phi_p = v_p^2 - u_p^2 = (1 - \frac{r}{2m})e^{\frac{r}{2m}}, \]
\[ q_p = -\sqrt{\frac{2m}{r} - 1 + c_1}. \]

The constant \( c_1 \) is determined by the initial value. We shall study the shell with \( m = M \) (the shell which starts to collapse from infinity with vanishing velocity \( R' \)). We integrate its equation of motion, starting from some initial radius \( R_0 \). For the particle we assume an initial radius \( r_0 \) and the value of the constant \( c_1 \) is determined by the condition \( r' = R' \) in the initial moment. Therefore,

\[ c_1 = 1 - \frac{2m}{r_0} + \frac{m}{R_0} + \left(\frac{m}{2R_0}\right)^2 \]  

as the consequence of (15) and (16). The relation between Minkowskian time, Schwarzschild time and independent variable \( v \) are

\[ t_\tilde{=} = \left(\frac{M}{2R} + \frac{m}{M}\right)\sqrt{\tilde{\psi}^2(1 - u'^2)} \]  

(24.a)

and

\[ t' = \frac{4m(u - vu')}{u^2 - v^2} \]  

(24.b)

where \( R' = \frac{dR}{dv} \) and \( \dot{v} = \frac{dv}{d\tau} \).

5. The Motion of A Shell and Particle in Minkowski Space

Since \( R_\tilde{=} = R_+ \) on the surface of the shell and \( v = v_a \) at the point of intersection is the independent variable, therefore the motion of the shell is

\[ \tilde{R}_\tilde{=} = q\sqrt{\tilde{\psi}^2(1 - u'^2)} \]  

(25.a)

where

\[ q = -\sqrt{(\frac{M}{2R} + \frac{m}{M})^2 - 1} \]

with

\[ \tilde{t}_\tilde{=} = \left(\frac{M}{2R} + \frac{m}{M}\right)\sqrt{\tilde{\psi}^2(1 - u'^2)} \]  

(25.b)

where \( \tilde{R} = \frac{dR}{dv} \) and \( \tilde{t} = \frac{dt}{dv} \). The motion of the particle is

\[ \tilde{r}_\tilde{=} = r'_p \tilde{t}_\tilde{=} \]  

(25.c)

where \( r'_p \) is the velocity of the particle after crossing the shell from outside determined from (18) at the point of intersection and given by
\[ r'_p = \frac{Q}{\sqrt{1 + Q^2}} \]  \hspace{1cm} (25.d)

where

\[ Q = H \left[ \frac{M}{2R} + \frac{m}{M} (u'_p - u') + \frac{\bar{R}_-(1 - u'u'_p)}{\psi^2(1 - u^2)} \right] \]

and

\[ H = \frac{1}{\sqrt{(1 - u^2)(1 - u_p'^2)}}. \]

The particle can intersect the shell again and come out to the exterior Schwarzschild space. From (19) we can determine the velocity \( u'_{p+} \) of the particle after crossing,

\[ u'_{p+} = \frac{Z\sqrt{\psi^2}}{\sqrt{1 + Z^2\psi^2}} \]  \hspace{1cm} (26)

where

\[ Z = \frac{(r'_{p-} - \bar{R}_-) + u'(1 - r'_{p-}\bar{R}_-)}{\sqrt{(1 - \bar{R}_-^2)(1 - r_{p-}'^2)(\psi^2(1 - u^2))}}. \]

In this case \( q_p \) in equation (22.b) will change to \( q_{p1} \) given by

\[ q_{p1} = -\sqrt{\frac{2m}{r} - 1 + c_2} \]  \hspace{1cm} (27.a)

Therefore,

\[ r' = q_{p1}\sqrt{\psi_p^2(1 - u_p'^2)} \]  \hspace{1cm} (27.b)

Determine \( c_2 \) from

\[ c_2 = f_b + \frac{2me^{-\frac{\alpha}{m}}(\omega u_p - v_p)\omega}{r_b(1 - \omega^2)} \]

where

\[ \omega = \frac{1}{u_p}(v_p + \frac{r_b u'_p}{8m^2} \frac{r_b}{e^{\frac{\alpha}{m}}}) \]

where \( R_- = r_- = r_b \) and \( f_b = 1 - \frac{2m}{r_b} \). Inserting \( c_2 \) into (27) and solving differential equations (21) and (22) we get the motion of the shell and of the particle.
6. Numerical Solution and Discussion

Let us solve differential equations (21), (22) and (24) with these initial values

\[ r_g = 2m = 200, \quad R_0 = 5r_g, \quad r_0 = 1.001R_0 \]
\[ t_0 = t_0^-, = 0, \quad v = 0 \]

From (23) we get the value of \( c_1 \) and \( u_0, u_{p0} \) from

\[ u_0 = \sqrt{v^2 + \left( \frac{R_0}{2m} - 1 \right) \frac{n_0}{\pi m}} \]
\[ u_{p0} = \sqrt{v^2 + \left( \frac{r_0}{2m} - 1 \right) \frac{n_0}{\pi m}}. \]

Since the velocity of the particle is greater than the velocity of the shell, the particle crosses the shell. In the flat space inside the shell the particle moves at a constant velocity, while the shell is accelerated. Consequently, they will intersect again, and so on; i.e., the particle oscillates around the shell. We found many intersections before they collapse to singularity. This is shown in Figure 1.

During the collapse, the lapses of the proper time for the shell and the particle \( (\Delta \tau_{s(i)} = \tau_{s(i)} - \tau_{s(i-1)}, \quad \Delta \tau_{p(i)} = \tau_{p(i)} - \tau_{p(i-1)}) \) between the points of intersection are diminishing. The rate of change of the proper time \( (\Delta \tau_{s(i)} / \Delta \tau_{p(i)}) \) between the points of intersection is diminishing, too. The shell goes to singularity faster than the test particle from the point of view of the co-moving observers.

References

Fig. 1 The x-axis is the time $v$ between the shell and particle, and the y-axis is the distance between the shell and particle. Conditions of collapse: the mass of the shell is 100 solar masses; the starting radius of the shell is 10 times the mass; the starting radius of the particle is 1.001 times the starting radius of the shell; the starting velocity of the particle is greater than the shell.