EJTP **6, No. 20** (2009) 367–384 *Electronic Journal of Theoretical Physics*

Riemann Zeta Function Zeros Spectrum

Igor Hrnčić*

Ludbreška 1b, HR42000 Varaždin, Croatia

Received 22 January 2009, Accepted 5 February 2009, Published 20 February 2009

Abstract: This paper shows that quantum chaotic oscillator Hamiltonian $H = px$ generates Riemann zeta function zeros as energy eigenvalues assuming validity of the Riemann hypothesis. We further put this on a firmer ground proving rigorously the Riemann hypothesis. We next introduce reformulation of special theory of relativity by which chaotic oscillator motion described via Hamiltonian $H = px$ is generated by gravitational potential, thus linking chaotic motion and Riemann zeta function to gravity.

Keywords: Quantum Chaos, Chaotic Oscillator, Riemann Zeta Function, Riemann Hypothesis, Special Theory of Relativity, Gravity
PACS (2008): 05.45.Mt; 03.65.-w; 03.65.Ge; 02.30.Gp; 03.30.+p

1. Introduction to Chaotic Quantum Oscillator

Chaos is all around us. Consider a flame from the cigarette lighter, for instance. Or fractal patterns in a leaf on a tree. Nature is chaotic.

One of the simplest chaotic systems is chaotic oscillator [1]. One is simply to complexify the harmonic oscillator Hamiltonian $H = p^2 + x^2$ by substituting $x \rightarrow ix$, turning it thus into chaotic Hamiltonian

$$H = p^2 - x^2 \tag{1}$$

After performing canonic rotation

$$\begin{aligned} p &\rightarrow p - x \\ x &\rightarrow p + x \end{aligned} \tag{2}$$

upon (1), we reach chaotic oscillator Hamiltonian in simpler form

$$H = px \tag{3}$$

* ihrncic1@yahoo.com

Classically, Hamilton equations for this Hamiltonian are

$$\begin{aligned} \dot{x} &= \tfrac{\partial H}{\partial p} = x \\ \dot{p} &= -\tfrac{\partial H}{\partial x} = -p \end{aligned} \tag{4}$$

These equations integrate to

$$\begin{aligned} x\left(t\right) &= x_0 e^{t} \\ p(t) &= p_0 e^{-t} \end{aligned} \tag{5}$$

These equations represent unstable trajectories reflecting the fact that system described by Hamiltonian $H = px$ is chaotic.

To rewrite Hamiltonian (3) for quantum system, we are simply to symmetrize it as follows,

$$H = \frac{1}{2}\left(px + xp\right) \tag{6}$$

Hamiltonian (6) is obviously hermitean. Its differential form is

$$H = -i\left(x\frac{d}{dx} + \frac{1}{2}\right) \tag{7}$$

Coordinate representation eigenfunctions $\psi(x)$ satisfying Schroedinger eigenequation

$$H\psi(x) = E\psi(x) \tag{8}$$

with energy E being constant of motion for given orbit are

$$\psi(x) = \frac{A}{x^{1/2-iE}} \tag{9}$$

Impulse representation eigenfunction $\phi(p)$ is simply Fourier transform of coordinate eigenfunction $\psi(x)$,

$$\phi(p) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\psi(x)e^{-ipx}dx = \frac{A}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\frac{e^{-ipx}}{x^{1/2-iE}}dx \tag{10}$$

To evaluate this integral, we have to handle singularity of integrand at $x = 0$. To do this, it is appropriate to require $x \to |x|$. This enables us to calculate integral (10) and we find

$$\phi(p) = \frac{A2^{iE}}{|p|^{1/2+iE}}\frac{\Gamma\left(\frac{1}{4}+\frac{iE}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{iE}{2}\right)} = \frac{A}{\sqrt{2\pi}\left|\frac{p}{2\pi}\right|^{1/2+iE}}e^{2i\theta(E)} \tag{11}$$

Here function $\theta(E) = \arg\Gamma\left(\frac{1}{4}+\frac{iE}{2}\right) - \frac{t}{2}\log\pi$ is actually the argument of the Riemann zeta function on the critical line $\Re s = 1/2$.

We have encountered the Riemann zeta function while investigating the chaotic oscillator. Riemann zeta function zeros density function has mean value [1]

$$\langle\mathcal{N}(E)\rangle = \frac{\theta(E)}{\pi} + 1 = \frac{E}{2\pi}\log\left(\frac{E}{2\pi}\right) + \frac{7}{8} + O\left(\frac{1}{E}\right) \tag{12}$$

Next we count quantum states of chaotic oscillator with Hamiltonian (6). Mean number of states of energy lesser than E is simply area under the contour $H = E$ in phase space (x, p). For Hamiltonian (6) this area is unbound because classical motion (5) generated by H is not bound. To overcome this obstacle, we have to regularize position x and impulse p by trucating them, $x > x_{\mathrm{min}}$, $p > p_{\mathrm{min}}$. We can always truncate any physical system [2]. When thus calculated, mean density of states of H is exactly (12). Hence, there are asymptotically as many states of H as Riemann zeta function nontrivial zeros in the mean. Because of this fact we deliberately chose $x \to |x|$ to be the appropriate continuation over integrand singularity when evaluating integral (11).

Further connection of chaotic quantum oscillator with Riemann zeta function comes from considering an appropriate boundary condition [1] for wave-functions (9) and (11). Intriguing relation coming from this boundary condition is [1]

$$x^{1/2}\zeta\left(1/2 - iE\right)\psi(x) - p^{1/2}\zeta\left(1/2 + iE\right)\phi(p) = 0 \tag{13}$$

Here x and p are ordinary real numbers. This condition does not seem to generate Riemann zeta function nontrivial zeros because of the difference in signs of energy E in zeta functions in (13), nor can be easily explained as a boundary condition geometrically, since it mixes both x and p.

This is as far as one gets when considering chaotic quantum oscillator given by Hamiltonian $H = px$ so far.

2. Riemann Zeta Function Zeros Spectrum

We next consider equation (13). Let us rewrite it as

$$x^{1/2}\zeta\left(1/2 - iE\right)\psi(x) = p^{1/2}\zeta\left(1/2 + iE\right)\phi(p) \tag{14}$$

Since energy E being constant for given eigen-state $\psi(x)$, we notice that left hand side of Eq. (14) is function of x only. Similarly, right hand side of Eq. (14) is function of p only. These two sides being equal, we conclude that they are both equal to some constant $C(E)$, ie.

$$\begin{aligned} x^{1/2}\zeta\left(1/2 - iE\right)\psi(x) &= C(E) \\ p^{1/2}\zeta\left(1/2 + iE\right)\phi(p) &= C(E) \end{aligned} \tag{15}$$

These equations are now easily turned into a boundary condition.

With wave-function $\psi(x)$ defined in (9), first of equations (15) becomes

$$Ax^{iE}\zeta\left(1/2 - iE\right) = C(E) \tag{16}$$

Since the system we are considering is chaotic, positions x are not constants of motion, but change with time. So there is only one way to fullfill requirement (16), namely to have $C(E) = 0$. Then the value of x does not metter. Condition

$$C(E) = 0 \tag{17}$$

is fullfilled as soon as

$$\zeta\left(1/2 - iE\right) = 0 \tag{18}$$

This condition, being part of the boundary condition for wave-functions of chaotic quantum oscillator, obviously generates Riemann zeta function nontrivial zeros.

Hence, spectrum of chaotic quantum oscillator given by Hamiltonian (6) consists solely of imaginary parts of not necessarily all of the Riemann zeta function nontrivial zeros on the critical line $\Re s = 1/2$.

There is still one interesting point about this result. Hilbert and Polya showed that if one manages to find Hermitean Hamiltonian such that eigenenergies of that Hamiltonian be imaginary parts of all of the Riemann zeta function nontrivial zeros, then this way one actually proves the Riemann hypothesis, saying that all the Riemann zeta function nontrivial zeros have real part $\Re s = 1/2$. Similar to condition (18).

We actually did not prove the Riemann hypothesis, because condition (18) involves only the Riemann zeta function nontrivial zeros on the critical line $\Re s = 1/2$ and does not concern rest of them from entire critical strip $0 < \Re s < 1$. However, if one is able to prove the Riemann hypothesis independently of results exposed so far, ie. without concerning Hamiltonian $H = px$, say by purely analytic methods, then Hamiltonian $H = px$ would prove to be the candidate for Hamiltonian Hilbert and Polya wrote about.

Actually, we are able to prove the Riemann hypothesis rigorously by purely analytic and almost elementary methods, and this is what we do next.

3. Introduction to Riemann Zeta Function

Euler was the first to consider the Riemann zeta function. He was also the first to consider the prime number counting function $\pi(r)$ counting primes $p \leq r$.

Riemann gave the complete analytic treatment of zeta function [3,4,5,6] defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{19}$$

in the complex half plane $\Re s > 1$. He showed that zeta function continues analytically over entire complex plane as a meromorphic function with a pole at $s = 1$. Riemann showed that zeta function nontrivial zeros $\rho = \sigma + it$ are situated symmetrically with respect to point $s = 1/2$ in the strip $0 \leq \Re s \leq 1$.

I find neccessary to point to a fact that there are no zeta function zeros[7] on the line $\Re s = 1$. Therefore, there are no zeta function zeros on line $\Re s = 0$ being centrally symmetric to the line $\Re s = 1$ with respect to point $s = 1/2$.

Hence, we may consider region $0 < \Re s < 1$ to be the critical strip where all the nontrivial zeta function zeros are located. Let us therefore henceforth refer to the strip $0 < \Re s < 1$ as to the critical strip.

Primes counting function behaves asymptotically as

$$\pi(r) = Li(r) + \mathcal{O}(r^{\theta} \log r) \tag{20}$$

with $1/2 \leq \theta < \sigma$. Hence, if $\sigma = 1/2$ then the uncertainty in θ is minimal and $\theta = 1/2$. Hence the importance of the Riemann hypothesis,

The Riemann hypothesis: *Zeta function nontrivial zeros have real part equal to* 1/2.

One of integral representations of zeta function valid for any s from the critical strip $0 < \Re s < 1$ is[8]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^p - 1} - \frac{1}{p} \right) p^{s-1} dp \tag{21}$$

4. Definition and Lemma

4.1 Definition

Define family $\mathcal{F}$ as family of functions $F(s)$ analytic and single-valued in complex variable s in critical strip $0 < \Re s < 1$, satisfying following three conditions:

Condition A Zeros $\rho = \sigma + it$ of $F(s)$ are located symmetrically with respect to point $s = 1/2$ in the critical strip $0 < \Re s < 1$.

Condition B There exists real function $f(p)$, continuous in real variable $p \in R_+$, such that Mellin transform

$$F(s) = \int_0^\infty f(p) p^{s-1} dp \tag{22}$$

exists for s in the critical strip.

Condition C $\lim\limits_{p \to 0^+} p^\sigma \left(1 + \log^2 p\right) \frac{df(p)}{dp} = 0$

Hence, by the theory of Mellin transformation, we conclude from Eq. (22) that $f(p) = \mathcal{O}(1)$ as $p \to 0^+$ for $0 < \Re s < 1$.

We notice that condition ***C*** follows from $f(p) = \mathcal{O}(1)$ as $p \to 0^+$ for $0 < \Re s < 1$. We keep condition ***C*** as a distinct condition simply for having it at our disposal explicitely.

We also notice by inspecting Mellin transform (22) that function $F(s)$ is real on real axis. Hence by the Schwarz principle of reflection we conclude that $F(\bar{s}) = \bar{F}(s)$. Therefore nontrivial zeros ρ are distributed symmetrically with respect to real axis.

4.2 Lemma

In order to prove the Riemann hypothesis we should start by proving the following lemma.

Lemma (Riemann hypothesis for Mellin transforms analytic in critical strip with zeros symmetric with respect to $s = 1/2$): *Let $F(s) \in \mathcal{F}$. Then all zeros $\rho = \sigma + it$ of $F(s)$ in critical strip have real part σ equal to* 1/2.

To prove this lemma, let us introduce continuous increasing parametrization $p(q)$, $0 \leq q \leq 1$, such that $p(0) = 0$, $\lim\limits_{q \to 1^-} p(q) = +\infty$.

With such parametrization we reparametrize integral (22) yielding

$$F(s) = \int_0^1 f(p(q))p(q)^{s-1}p'(q)dq \tag{23}$$

Let us decompose this integral into its real and imaginary parts,

$$\begin{aligned} \Re F(s) &= \int_0^1 f(p(q))p(q)^{x-1}\cos(y\log p(q))p'(q)dq \\ \Im F(s) &= \int_0^1 f(p(q))p(q)^{x-1}\sin(y\log p(q))p'(q)dq \end{aligned} \tag{24}$$

with $x = \Re s$, $y = \Im s$, $s = x + iy$ and using the real logarithm.

Hence, by the mean value theorem, we conclude that for any s from the critical strip there exist at least one a and b, $0 \leq a, b \leq 1$, such that Eq. (24) becomes

$$\begin{aligned} \Re F(s) &= f(p(a))p(a)^{x-1}\cos(y\log p(a))p'(a) \\ \Im F(s) &= f(p(b))p(b)^{x-1}\sin(y\log p(b))p'(b) \end{aligned} \tag{25}$$

Equation (25) is numerical, meaning we cannot differentiate (25) and expect to yield any result since equation (25) simply states that for any s there exist some a and b such that number on the right in (25) equals number on the left in (25). Left and righ hand side of (25) represent two numbers, not functions.

When we change $s = x+iy$ we consequently change a nad b. We are therefore naturally inclined to introduce new functions $a(x, y)$ and $b(x, y)$ such that Eq. (25) becomes

$$\begin{aligned} \Re F(s) &= f(p(a(x,y)))p(a(x,y))^{x-1}\cos(y\log p(a(x,y)))p'(a(x,y)) \\ \Im F(s) &= f(p(b(x,y)))p(b(x,y))^{x-1}\sin(y\log p(b(x,y)))p'(b(x,y)) \end{aligned} \tag{26}$$

Equation (26) defines functions $a(x, y)$ and $b(x, y)$ implicitely.

Let us next introduce parametrization $p(q) = e^{\tan(\pi(q-1/2))}$. It satisfies all requirements parametrization $p(q)$ is required to satisfy. Namely, it is continuous, increasing, $p(0) = 0$, $\lim\limits_{q\to 1^-} p(q) = +\infty$ and it maps interval $0 \leq q \leq 1$ onto R_+. Parametrization $p(q) = e^{\tan(\pi(q-1/2))}$ is bijective as soon as we restrict values of q to $0 \leq q \leq 1$.

Let us further inspect properties of parametrization $p(q) = e^{\tan(\pi(q-1/2))}$. We notice that $p'(q) = \pi\frac{e^{\tan(\pi(q-1/2))}}{\cos^2(\pi(q-1/2))} = \pi p(q)\left(1 + \log^2(p(q))\right)$. We further notice that $\lim\limits_{q\to 0^+} p(q)^{x-1}p'(q) = \pi \lim\limits_{q\to 0^+} \frac{e^{x\tan(\pi(q-1/2))}}{\cos^2(\pi(q-1/2))} = 0$ for any $0 < x < 1$.

Hence whenever $F(s)$ is in $\mathcal{F}$, by inspecting (26) having $\lim\limits_{q\to 0^+} p(q)^{x-1}p'(q) = 0$, $p'(q) = \pi p(q)\left(1 + \log^2(p(q))\right)$ and $f(0) = \mathcal{O}(1)$, we conclude that

$$\begin{aligned} \Re F(\rho) &= \pi f(p(0))p(0)^{\sigma}\cos(t\log p(0))\left(1 + \log^2(p(0))\right) = 0 \\ \Im F(\rho) &= \pi f(p(0))p(0)^{\sigma}\sin(t\log p(0))\left(1 + \log^2(p(0))\right) = 0 \end{aligned} \tag{27}$$

in zeros $\rho = \sigma + it$ of function $F(s)$, whenever we set $a(\sigma, t), b(\sigma, t) = 0$. Whenever we set $a(x, y), b(x, y) = 0$ in (26) we reproduce result $F(s) = 0$ as demonstrated in (27).

Therefore we can require $a(\sigma, t), b(\sigma, t) = 0$ for all zeros $\rho = \sigma + it$ from the critical strip, as we will.

I find useful to pay some attention to our choice $a(\sigma, t), b(\sigma, t) = 0$. I'd like to stress the fact that by the mean value theorem points a and b are completely arbitrary as long as Eq. (26) is satisfied for given s. Hence any $a, b \in [0, 1]$ that reproduce result $F(s) = 0$ as in Eq. (27) are admissible at any zero ρ. We use this arbitrarity to set $a(\sigma, t), b(\sigma, t) = 0$ for all zeros $\rho = \sigma + it$. Once we made our choice for $a(\sigma, t), b(\sigma, t)$, we explicitely do not allow any other value for $a(\sigma, t), b(\sigma, t)$ for simple fact that one value for $a(\sigma, t), b(\sigma, t)$ is enough to completely describe function $F(s)$ at any point $s = \sigma + it$.

Next we demonstrate that derivative $\frac{\partial p(a(\sigma,t))}{\partial y}$ exists at zeros $\rho = \sigma + it$. To show this, we suppose that derivative $\frac{\partial p(a(\sigma,t))}{\partial y}$ indeed exists at zeros $\rho = \sigma + it$. We differentiate (27) formally with respect to y as a product of functions and yield

$$\begin{gathered} \frac{\partial \Re F(\rho)}{\partial y} = \\ = \pi[f'(p)p^{\sigma} \cos(t \log p)(1 + \log^2 p) + \\ + \sigma f(p)p^{\sigma-1} \cos(t \log p)(1 + \log^2 p) - \\ - t f(p)p^{\sigma-1} \sin(t \log p)(1 + \log^2 p) + \\ + 2 f(p)p^{\sigma-1} \cos(t \log p) \log p] \frac{\partial p}{\partial y} - \\ - \pi f(p)p^{\sigma} \sin(t \log p)(1 + \log^2 p) \log p \end{gathered} \tag{28}$$

as $p \to 0^+$. Variable p in (28) stands for function $p(a(x, y))$ at $a(\sigma, t) = 0$ as a shorthand abbreviation.

We notice that $\lim\limits_{p \to 0^+} p^x \left(1 + \log^2 p\right) \log p = 0$ for any $0 < x < 1$. We further notice that by condition $\boldsymbol{C}$ the first summand in square brackets in (28) vanishes.

Thus we are left with

$$\begin{gathered} \frac{\partial \Re F(\rho)}{\partial y} = \\ = \pi f(p)p^{\sigma-1} \cos(t \log p) \left[(\sigma - t \tan(t \log p)) \left(1 + \log^2 p\right) + 2 \log p\right] \frac{\partial p}{\partial y} \end{gathered} \tag{29}$$

Let us next suppose that

$$\sigma - t \tan(t \log p) \neq 0 \tag{30}$$

as $p \to 0^+$.

We notice that $\lim\limits_{p \to 0^+} \frac{1 + \log^2 p}{\log p} = 2 \lim\limits_{p \to 0^+} \log p$ by virtue of the L'Hospital theorem. Hence, since $\sigma - t \tan(t \log p) \neq 0$ we find terms 1 and $2 \log p$ neglectable compared to $\log^2 p$ as $p \to 0^+$. Hence Eq. (29) becomes

$$\frac{\partial \Re F(\rho)}{\partial y} = \\ = 2\pi f(p) p^{\sigma-1} \cos(t \log p)(\sigma - t \tan(t \log p)) \log^2 p \frac{\partial p}{\partial y} \tag{31}$$

At this point we notice that $\frac{\partial \Re F(\rho)}{\partial y}$ exists. We further notice that product $p^{\sigma-1} \log^2 p$ is unbounded as $p \to 0^+$. Further, $f(p) = \mathcal{O}(1)$ as $p \to 0^+$. We also supposed that $\sigma - t \tan(t \log p) \neq 0$. Thus Eq. (31) suggests

$$\cos(t \log p) \frac{\partial p}{\partial y} = 0 \tag{32}$$

This is true if either $\frac{\partial p}{\partial y} = 0$ or if $\cos(t \log p) = 0$.

If $\frac{\partial p}{\partial y} = 0$ we have proven that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial y}$ exists as $p \to 0^+$.

If $\cos(t \log p) = 0$, then $\sin(t \log p) = \pm 1$ and Eq. (31) becomes

$$\frac{\partial \Re F(\rho)}{\partial y} = \mp 2t\pi f(p) p^{\sigma-1} \log^2 p \frac{\partial p}{\partial y} \tag{33}$$

Since $\frac{\partial \Re F(\rho)}{\partial y}$ exists, since product $p^{\sigma-1} \log^2 p$ is unbounded as $p \to 0^+$ and since $f(p) = \mathcal{O}(1)$ as $p \to 0^+$, Eq. (33) suggests again that $\frac{\partial p}{\partial y} = 0$.

Therefore if $\sigma - t \tan(t \log p) \neq 0$ we conclude that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial y}$ exists as $p \to 0^+$.

We next consider assumption

$$\sigma - t \tan(t \log p) = 0 \tag{34}$$

From (34) we conclude that $\tan(t \log p) = \sigma/t$. Having finite σ and nonvanishing t we notice that $\cos(t \log p) \neq 0$.

Consider (34). We know that there is some $r > 0$, $r \in R$, such that $\sigma - t \tan(t \log p)$ tends to zero as p^r as $p \to 0^+$. Hence, factor $(\sigma - t \tan(t \log p)) \log^2 p$ in (29) behaves as $p^r \log^2 p$ as $p \to 0^+$ under assumption $\sigma - t \tan(t \log p) = 0$.

Therefore we next consider limit $\lim_{p \to 0^+} p^r \log^2 p$ and find by the use of the L'Hospital rule that $\lim_{p \to 0^+} p^r \log^2 p = \frac{2}{r^2} \lim_{p \to 0^+} p^r = 0$ for any $r > 0$.

We should also consider the case with $\sigma - t \tan(t \log p)$ tending to zero identically. Then we find $(\sigma - t \tan(t \log p)) \log^2 p = 0$ again, identically.

Hence, by putting $(\sigma - t \tan(t \log p)) \log^2 p = 0$ back to Eq. (29) we conclude

$$\frac{\partial \Re F(\rho)}{\partial y} = 2\pi f(p) p^{\sigma-1} \cos(t \log p) \log p \frac{\partial p}{\partial y} \tag{35}$$

At this point we again notice that $\frac{\partial \Re F(\rho)}{\partial y}$ exists. We further notice that product $p^{\sigma-1} \log p$ is unbounded as $p \to 0^+$. Further, $f(p) = \mathcal{O}(1)$ as $p \to 0^+$ and $\cos(t \log p) \neq 0$. Thus Eq. (35) suggests $\frac{\partial p}{\partial y} = 0$ as $p \to 0^+$.

Therefore we conclude that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial y}$ exists for every zero $\rho = \sigma + it$.

Next we show that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ exists at zeros $\rho = \sigma + it$. To show this, we suppose that derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ indeed exists at zeros $\rho = \sigma + it$. We differentiate (27) formally with respect to x as a product of functions and yield

$$\begin{gathered} \frac{\partial \Re F(\rho)}{\partial x} = \\ = \pi[f'(p)p^{\sigma}\cos(t\log p)(1+\log^2 p)+ \\ +\sigma f(p)p^{\sigma-1}\cos(t\log p)(1+\log^2 p)- \\ -tf(p)p^{\sigma-1}\sin(t\log p)(1+\log^2 p)+ \\ +2f(p)p^{\sigma-1}\cos(t\log p)\log p]\frac{\partial p}{\partial x}- \\ -\pi f(p)p^{\sigma}\cos(t\log p)(1+\log^2 p)\log p \end{gathered} \tag{36}$$

as $p \to 0^+$. Variable p in (36) again stands for function $p(a(x,y))$ at $a(\sigma,t) = 0$ as a shorthand abbreviation.

We notice that $\lim\limits_{p\to 0^+} p^x \left(1+\log^2 p\right)\log p = 0$ for any $0 < x < 1$. We further notice that by condition $\boldsymbol{C}$ the first summand in square brackets in (36) vanishes.

Thus we are left with

$$\begin{gathered} \frac{\partial \Re F(\rho)}{\partial x} = \\ = \pi f(p)p^{\sigma-1}\cos(t\log p)\left[(\sigma - t\tan(t\log p))\left(1+\log^2 p\right)+2\log p\right]\frac{\partial p}{\partial x} \end{gathered} \tag{37}$$

We notice that Eqs. (29) and (37) are identical as soon as we substitute $\frac{\partial}{\partial y} \to \frac{\partial}{\partial x}$ in (29). Hence, by following steps (29) through (35) with $\frac{\partial}{\partial y} \to \frac{\partial}{\partial x}$, we find that partial derivative $\frac{\partial p(a(\sigma,t))}{\partial x}$ exists for every zero $\rho = \sigma + it$.

Hence we conclude that partial derivatives $\frac{\partial p(a(\sigma,t))}{\partial x}$ and $\frac{\partial p(a(\sigma,t))}{\partial y}$ exist at any zero $\rho = \sigma + it$.

We would like now to examine the behavior of ratio $F(s)/F(1-s)$ at zeros $s = \rho$ and $1-s = 1-\rho$. This ratio involves ratio $p(a(\sigma,y))/p(a(1-\sigma,y))$ at zeros $s = \rho$ and $1-s = 1-\rho$. So let us continue proving lemma by inspecting the following limit,

$$\lim_{y\to t}\frac{p(a(\sigma,y))}{p(a(1-\sigma,y))} \tag{38}$$

We first notice that point $1-\sigma+it$ is a zero as soon as $\sigma+it$ is a zero. This is so because zeros are located symmetrically with respect to point $s = 1/2$ as well as with respect to real axis – and therefore symmetrically with respect to critical line $\Re s = 1/2$ as well – for any function $F(s) \in \mathcal{F}$. Therefore both $p(a(\sigma,t))$ and $p(a(1-\sigma,t))$ equal zero following discussion of Eq. (27).

Limit (38) may or may not exist. If it does not exist, it does not exist because it is unbounded, since $p(a(\sigma,t))$ being continuous at any zero ρ and since zeros ρ and $1-\rho$ being of the same order, and hence we conclude that if limit (38) does not exist, then reciprocal limit, namely $\lim\limits_{y\to t}\frac{p(a(1-\sigma,y))}{p(a(\sigma,y))}$, vanishes, therefore exists.

Let us first assume that $\lim_{y\to t}\frac{p(a(\sigma,y))}{p(a(1-\sigma,y))}$ exists. Since $p(a(\sigma,t))=0$ for any zero $\rho=\sigma+it$ and since $p(a(\sigma,t))$ being continuous at every zero $\rho=\sigma+it$, we conclude that we can employ L'Hospital rule upon limit (38) and yield

$$\begin{aligned}&\lim_{y\to t}\frac{p(a(\sigma,y))}{p(a(1-\sigma,y))}=\\&=\lim_{y\to t}\frac{\frac{\partial}{\partial y}p(a(\sigma,y))}{\frac{\partial}{\partial y}p(a(1-\sigma,y))}\end{aligned}\tag{39}$$

choosing complex number s to approach zero ρ keeping real part $x=\sigma$ constant.

Limit involving partial derivatives in (39) may be ill defined as well – for instance, partial derivatives in (39) vanish. However, we are not interested in the exact value of this limit. All that we require is the formal identity (39) as it is.

We notice that

$$\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}\tag{40}$$

exists as soon as limit $\lim_{y\to t}\frac{p(a(\sigma,y))}{p(a(1-\sigma,y))}$ exists, for
$\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}=\lim_{y\to t}\left[\frac{p(a(\sigma,y))}{p(a(1-\sigma,y))}\right]^{\sigma}\frac{1}{p(a(1-\sigma,y))^{1-2\sigma}}$ exists as soon as $\sigma>1/2$.

Since condition $\boldsymbol{A}$ required zeros to be located symmetrically with respect to point 1/2, and since zeros are distributed symmetrically with respect to real axis, therefore zeros appear symmetrically with respect to the critical line $\Re s=x=1/2$, therefore we conclude that we may restrict our analysis to zeros with $\sigma>1/2$ without loss of generality.

There is of course one more possibility left, namely $\sigma=1/2$, but then the lemma is proved automatically.

Hence, let us assume $\sigma>1/2$ without loss of generality. We notice that $\lim_{q\to 0^+}p(q)^{\sigma}=0$ for any $0<\sigma<1$. Hence we may employ the L'Hospital rule upon limit $\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}$ still choosing complex number s to approach zero ρ keeping real part $x=\sigma$ constant. This way,

$$\begin{aligned}&\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}=\\&=\frac{\sigma}{1-\sigma}\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}\frac{p(a(1-\sigma,y))}{p(a(\sigma,y))}\frac{\frac{\partial}{\partial y}p(a(\sigma,y))}{\frac{\partial}{\partial y}p(a(1-\sigma,y))}\end{aligned}\tag{41}$$

Result (39) when employed upon (41) implies

$$\begin{aligned}&\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}=\\&=\frac{\sigma}{1-\sigma}\lim_{y\to t}\frac{p(a(\sigma,y))^{\sigma}}{p(a(1-\sigma,y))^{1-\sigma}}\end{aligned}\tag{42}$$

Since limit on the left hand side of (42) is identical to the one on the right hand side of (42), equation (42) demands $\sigma=1/2$, since (42) leads to

$$1 = \frac{\sigma}{1-\sigma} \tag{43}$$

We should also consider case with $\lim_{y \to t} \frac{p(a(1-\sigma,y))}{p(a(\sigma,y))} = 0$. We simply repeat steps (39) through (43) with ρ and $1-\rho$ interchanged and for $\sigma < 1/2$. The result is the same, namely $\sigma = 1/2$.

Since we supposed $\sigma \neq 1/2$ and have arrived at a contradiction, we conclude that $\sigma = 1/2$.

This proves the lemma.

5. Proof of the Riemann Hypothesis

Let us consider integral representation (21). Let us define function $\chi(s)$ according to

$$\chi(s) = \zeta(s)\Gamma(s) = \int_0^\infty \left(\frac{1}{e^p - 1} - \frac{1}{p} \right) p^{s-1} dp \tag{44}$$

We notice that gamma function $\Gamma(s)$ is finite and nonvanishing in the critical strip. We hence conclude that functions $\chi(s)$ and $\zeta(s)$ have common zeros.

Hence chi function zeros $\rho = \sigma + it$ are situated symmetrically with respect to point $s = 1/2$. Thus it satisfies condition $\boldsymbol{A}$.

Both $\Gamma(s)$ and $\zeta(s)$ are analytic in critical strip. Hence we conclude that $\chi(s)$ is analytic in critical strip.

We also notice that (44) represents a Mellin transform for s in critical strip. Hence $\chi(s)$ satisfies condition $\boldsymbol{B}$.

Further, $p^\sigma \left(1 + \log^2 p\right) \frac{d}{dp} \left(\frac{1}{e^p-1} - \frac{1}{p} \right) = p^\sigma \left(1 + \log^2 p\right) \left(\frac{1}{p^2} + \frac{1}{2} \frac{1}{1-\cosh(p)} \right) = 0$ as $p \to 0^+$ for any $0 < \sigma < 1$, hence for any zero $\rho = \sigma + it$ of function $\chi(s)$. This is so because by L'Hospital rule $\frac{p^2}{2(1-\cosh(p))} = -1$ as $p \to 0^+$. This way we've checked the truth of condition $\boldsymbol{C}$ for $\chi(s)$ as defined by (44).

Therefore we conclude that $\chi(s)$ belongs to family $\mathcal{F}$.

Function $\chi(s)$ being in $\mathcal{F}$, we conclude that nontrivial zeros $\rho = \sigma + it$ of $\chi(s)$, and therefore the very same nontrivial zeros ρ of $\zeta(s)$, satisfy lemma, i.e. $\sigma = 1/2$. This proves the Riemann hypothesis.

6. Introduction to 3-relativity

The invention of 3-relativity was motivated by the fact that Klein-Gordon equation is a second order differential equation. Since Klein-Gordon equation being co-ordinate representation of on-shell relation $E^2 = p^2 + m^2$, we are linearizing the on-shell relation.

One way to linearize on-shell relation is to do it the Dirac way. Result is Dirac equation. However, Dirac equation describes fermions only. Therefore, this paper finds another way to linearize on-shell condition. This linearization turns to be suitable for bosons as well as for fermions.

Let us begin by first considering virtual photons. To do so, let us pay attention to Bremsstrahlung where single particle emits a photon. Let the particle of rest-mass m move with 3-impulse p and let it emit a photon. Let the particle continue to move with some 3-impulse q after emission of a photon.

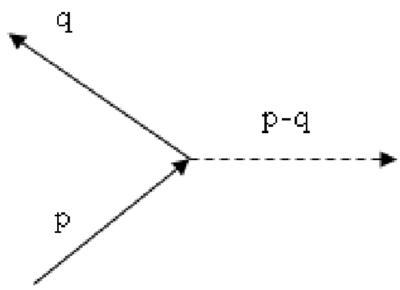

Fig. 1 *Bremsstrahlung diagram*

The conservation of impulse dictates photon to carry 3-impulse $p - q$. Such photon should have energy $p-q$ as measured in natural units. On the other hand, conservation of energy demands photon to have energy $\sqrt{p^2 + m^2} - \sqrt{q^2 + m^2}$. Therefore, if energy being conserved, emitted photon must lack some energy given its impulse. Such photon is called a virtual photon. The explanation is that virtual particles in general happen to exist only on account of borrowing the lacking energy from Dirac sea i.e. from quantum vacuum. Namely, if virtual particle borrows energy ΔE from vacuum, it may exist approximately in the mean for only $\Delta t \approx \frac{\hbar}{\Delta E}$ seconds by the virtue of Heisenberg uncertainty relation. This is quite a realistic scenario if virtual particle is absorbed by another particle while still being in existence during period of Δt. Hence, virtual particles exit Feynman diagrams and re-enter another Feynman diagram within Δt seconds. Virtual particles have to enter another diagram within Δt seconds – otherwise they would sink back into Dirac sea and thus violate conservation of 4-impulse.

Let us return back to Bremsstrahlung. Consider a particle emitting a photon. This photon exits this diagram depicting Bremsstrahlung and is therefore virtual. It is bound to enter another diagram within Δt seconds. The crucial argument is – in reality, in experiment, emitted photons do not have to interact for infinitely long time.

Consider this fact – all photons have been emitted by some particle. Every single photon in Universe once left some diagram. Therefore, all the photons are virtual. Having all the photons virtual, we are facing the enormously important question – how can electromagnetic forces be long-ranged? All the experiments done during last 200 years conform with the fact that photons have infinite range. The magnificent Maxwell theory, being a reference for every other theory in physics, predicts that photons are long-ranged. Maxwell theory implied conclusions of special theory of relativity. And yet, the very same special theory of relativity predicts that photon should be short ranged.

Let us consider photo-effect now. Photo-effect is really Bremsstrahlung with CT

inversion. Therefore, it is easy to notice that conservation of energy requires incoming photon to be virtual, whatever the nature of charged particle absorbing it.

The solution suggested by contemporary QFT to this problem is to require photo-effect to be followed by Bremsstrahlung, Fig. 2.

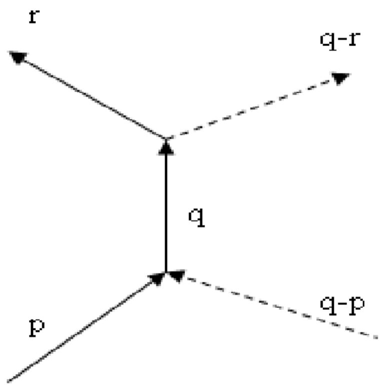

Fig. 2 *Photo-effect followed by Bremsstrahlung*

This way, one can blame all the virtuality on electron, and not on photons.

The way out of this puzzle was to claim the existence of real photons. These are the ones that do not enter any diagram and are free. However, when we measure a real photon, the very act of measuring the real photon is in fact interaction of a real photon with apparatus. Hence, real photon enters a diagram at the moment of measuring. Entering the diagram, it has to be virtual in order not to violate conservation of energy. So, it seems that there are no real photons. If there were real photons, then the electrons in apparatus should be virtual for a while. Virtual particles are undetectable per definitionem. The question arising is – what are we measuring, then? And what is it that enables measuring? Certainly not the undetectable virtual apparatus.

The situation is quite different if the mediator is massive. All the relativistic calculations still demand massive particle to be virtual in order to have energy conserved, but only in Bremsstrahlung as depicted in Fig. 1. The difference with respect to photons is that massive mediators are short-ranged. Massive mediators cannot originate from Bremsstrahlung depicted in Fig. 2 because such mediators are long-ranged since being real and not virtual. Therefore, massive mediators are short-ranged only with Bremsstrahlung as in Fig. 1.

These arguments are compelling enough to motivate us to find the answer to this puzzle – the puzzle of long-ranged virtual photons. This paper shows that there is another way to define special theory of relativity. We define special theory of relativity in 3-dimensional euclidean space, in contrast to 4-dimensional Minkowski space-time. This 3-relativity predicts the same results as 4-relativity as long as there are no interactions. This may seem trivial. However, the difference between 3-relativity and 4-relativity shows when considering interactions – and therefore when considering virtual particles. The results of 3-relativity show that photons do not violate energy conservation and indeed are long-ranged. This result does not stand for massive mediators, so the theory of strong and weak interactions stands, as viewed in 3-relativity. So let us introduce 3-relativity.

7. 3-relativity

Consider following manipulation

$$E^2 = p^2 + m^2 = (|p| + im)(|p| - im) \tag{45}$$

This factorization suggests that energy E should be rewritten as

$$E = |p| + im \tag{46}$$

If we choose to have $p, m \in R$, then energy E in Eq. (46) is complex and Eq. (1) becomes

$$|E|^2 = E\bar{E} = p^2 + m^2 = (|p| + im)(|p| - im) \tag{47}$$

Suppose that we could do this for each component of 3-impulse p. Let indices $i, j, k, l = 1, 2, 3$ label spatial components of a vector. Let us define Eq. (44) to be

$$\mathcal{E}_k = |p_k| + im_k \tag{48}$$

with $\mathcal{E}_k$ denoting the k-component of complex energy 3-vector $\mathcal{E}$. This equation is linearization of Klein-Gordon equation.

Let vectors p, $\mathcal{E}$ and m be spanned by quantities q_k that satisfy $\{q_k, q_l\} = 2\delta_{kl}$. For instance, quantities q_k may be cartesian unit vectors e_i. So the energy 3-vector as defined in Eq. (48) is

$$q_k(\mathcal{E}_k - |p_k| - im_k) = 0 \tag{49}$$

assuming summation over dummy index $k = 1, 2, 3$. This equation turns into Dirac equation for $q_k = \sigma_k$ with σ^k being Pauli matrices. It also turns into an equation similar to Dirac equation, but describing not fermions but bosons, as soon as we choose mutually commuting vectors q_k, such that $[q_k, q_l] = 0$. We will discuss this in a short while. Let us for the moment mimic Dirac equation and suppose that operator for conjugate wave function similar to one given in Eq. (49) is

$$q_l(\bar{\mathcal{E}}_l - |p_l| + im_l) = 0 \tag{50}$$

with $\{q_k, q_l\} = 2\delta_{kl}$.

Multiply Eq. (50) by Eq. (49) from the left, multiply Eq. (49) by Eq. (50) from the right, and sum these two products together. The result is

$$\frac{1}{2}\{q_k, q_l\}(\mathcal{E}_k - |p_k| - im_k)(\bar{\mathcal{E}}_l - |p_l| + im_l) = 0 \tag{51}$$

where we used the fact that $\mathcal{E}_k$, $\bar{\mathcal{E}}_k$, p_k and m_k all mutually commute. Since $\{q_k, q_l\} = 2\delta_{kl}$ we conclude from Eq. (51)

$$(\mathcal{E}_k - |p_k| - im_k)(\bar{\mathcal{E}}_k - |p_k| + im_k) = 0 \tag{52}$$

with summation over dummy index k. When multiplied through, Eq. (52) results in

$$\bar{\mathcal{E}}_k \mathcal{E}_k - p_k p_k - m^2 = 0 \tag{53}$$

with summation over dummy k assumed and with definition

$$m_k m_k = m^2 \tag{54}$$

with summation over dummy k.

We notice that Eq. (53) represents on-shell relation

$$E^2 - p_k p_k - m^2 = 0 \tag{55}$$

with

$$\bar{\mathcal{E}}_k \mathcal{E}_k = |\mathcal{E}|^2 = E^2 \tag{56}$$

with E denoting energy as defined in Einstein's 4-relativity.

So we conclude that we found another way to define special theory of relativity. The advantage of this approach is that we are able to express energy-impulse relation in linear form now, Eq. (48). In Einstein's formulation there was only one energy E and three impulses p_k and any statement about energy E had impact on all of impulses p_k. Now, we have defined a portion of energy $\mathcal{E}_k$ for each impulse component p_k. This allows more detailed analysis of energy-impulse relationship than the use of 4-impulse. On the other hand, we have concealed conserved stationary quantities – 4-energy E and rest mass m, and thus have lost explicit form of these constants of motion.

We can linearize not only the energy-impulse on-shell relation, but also the time-space on-shell relation – the metric. For as soon as we define complex time 3-vector $\mathcal{T}_k$ and substitute $\mathcal{E}_k \to \mathcal{T}_k$ and $p_k \to x_k$ and $m_k \to \tau_k$ with $\tau_k \tau_k = \tau^2$ in Eqs. (48) through (53) with τ being proper time, we find that

$$\mathcal{T}_k = |x_k| + i\tau_k \tag{57}$$

leads to

$$\mathcal{T}_k \bar{\mathcal{T}}_k = x_k x_k + \tau^2 \tag{58}$$

with summation over dummy index k. Eq. (58) is the metric 4-element as soon as we define

$$\mathcal{T}_k \bar{\mathcal{T}}_k = |\mathcal{T}|^2 = t^2 \tag{59}$$

with t denoting time as defined in Einstein's 4-relativity.

We notice that metric (58) is invariant to any transformation of 3-time $\mathcal{T}_k$ and 3-space x_k as long as Eq. (57) stands invariant. Any such transformaion is Lorentz transformation for 3-relativity.

It is interesting to notice appearance of absolute values $|p|$ and $|x|$ in Eqs. (48) and (57) describing relativistic particle in 3-relativity, as well as in Eq. (11) describing quantum chaotic oscillator.

8. Virtual Particles in 3-relativity

I would like to show that there is a difference between 3-relativity and 4-relativity when describing a virtual photon.

Let us for simplicity use only one spatial dimension, say x. Consider a particle of impulse p_i and rest mass m scattering with outgoing impulse p_f whilst emitting a photon of impulse f.

In 4-relativity impulses balance out according to

$$p_i = p_f + f \tag{60}$$

Energy of incoming particle is $\sqrt{p_i^2 + m^2}$, energy of outgoing particle is $\sqrt{p_f^2 + m^2}$ and energy of a photon is f. Energy balance therefore reads

$$\sqrt{p_i^2 + m^2} = \sqrt{p_f^2 + m^2} + f \tag{61}$$

Eqs. (60) and (61) are not solvable simultaneously, so the photon must tunnel through the Dirac sea borrowing extra energy ϵ from the vacuum according to uncertainty principle

$$\epsilon \Delta t \approx \hbar \tag{62}$$

The tunneling allows photon to be in existence only for Δt seconds. Such virtual photon can traverse only a distance $\Delta x = c\Delta t$.

This scenario is unphysical. This mechanism of production of a virtual photon is not realistic because electromagnetic forces are of infinite range and are not limited to ranges of Δx.

Let us describe this situation in 3-relativity. Impulses balance as in Eq. (60). Energy of an incoming particle is $|p_i| + im$. Energy of an outgoing particle is $|p_f| + im$. Energy of a photon is $|f|$. Hence,

$$|p_i| + im = |p_f| + im + |f| \tag{63}$$

Eqs. (60) and (63) are actually identities as soon as $sign(p_i) = sign(p_f) = sign(f)$. We notice that photon is no longer virtual. Real photon now has infinite range as it should.

The situation changes if emitted particle being massive. Let the emitted particle have impulse f and mass n. Then energy should balance according to

$$|p_i| + im = |p_f| + im + |f| + in \tag{64}$$

Eqs. (60) and (64) are not solvable simultaneously because of extra mass in, so the emitted massive particle has to be virtual thus being short-ranged. Since in Standard model massive mediators should be short-ranged, we conclude that 3-relativity predicts physically acceptable results.

9. Chaotic Oscillator and Gravity

Consider a particle of rest mass m in potential $V(x)$. Its energy H in 3-relativity can be written as

$$H = p + im + V(x) \tag{65}$$

Chaotic oscillator is also a one-particle system in some potential $V(x)$ and is completely described given Hamiltonian $H = px$. Product px already contains all information on potential $V(x)$, although potential $V(x)$ does not appear in it explicitely.

Hamiltonian $H = px$ is not 3-relativistic because it does not take rest mass energy into account. It's 3-relativistic form is

$$H = px + im \tag{66}$$

This represents particle's energy and is constant of motion. Hamiltonian $H = px + im$ obviously has complex eigenvalues and is no longer hermitean. Since rest mass m being constant, quantity

$$K \equiv H - im = px \tag{67}$$

is also a constant of motion for a given orbit, and is represented by hermitean operator.

Equations (65) and (66) say that

$$px = p + V(x) \tag{68}$$

Potential $V(x)$ is therefore

$$V(x) = px - p = K - \frac{K}{x} \tag{69}$$

This is one-dimensional gravitational potential plus some constant potential K being dependent on rest mass m and is the kinetic part of energy assigned to a given orbit.

Since any Hamiltonian can be split into a sum of Hamiltonians, we are free to interpret Eq. (66) as radial part of energy of a particle in potential $V(r)$, identifying x with radial distance r in spherical co-ordinates. Thus, potential in which a particle is moving chaotically as expressed by Hamiltonian's kinetic radial part $H_r = K = p_r r$ is

$$V(r) = -\frac{K}{r} \tag{70}$$

up to constant potential K. This is gravitational potential. Therefore, gravity makes particle move chaotically following unstable orbits as dictated by Hamiltonian $H_r = p_r r$.

This is quite interesting conclusion, since it links gravity with Riemann zeta function zeros and chaos. Constant K is not a global constant, but is constant only along given orbit. This shows that this result cannot be the complete information about the given system. The point we wish to stress here is not the complete theory, but rather single interesting detail about gravity and chaotic oscillator. Complete treatment demands more thorough analysis, of course.

10. Conclusions

This paper showed that chaotic system described by Hamiltonian $H = px$ has imaginary parts of Riemann zeta function nontrivial zeros as eigenenergies, although there is still left to prove that spectrum of H consists of all of the Riemann zeta function nontrivial zeros. By proving rigorously the Riemann hypothesis by analytic methods, this paper shows that Hamiltonian $H = px$ is very likely the Hamiltonian Hilbert and Polya wrote about, namely the Hamiltonian that might serve to prove the Riemann hypothesis without purely analytic and number-theoretical methods. We further introduced reformulation of special theory of relativity by describing it over three-dimensional space rather than in Minkowski four-dimensional space-time. In this relativistic representation, the potential driving a particle to move chaoticaly with energy $H = px$ proves to behave as gravitational potential, thus linking theory of chaos to gravity and to Riemann zeta function nontrivial zeros.

References

[1] M.V. Berry and J.P. Keating, *The Riemann Zeros and Eigenvalue Asymptotics*, http://www.hpl.hp.com/techreports/98/HPL-BRIMS-98-26.pdf

[2] I. Hrncic, *Existence of Yang-Mills Theory with Vacuum Vector and Mass Gap*, EJTP, http://www.ejtp.com/articles/ejtpv5i18p33.pdf

[3] E. Bombieri, *Problems of the Millenium: the Riemann Hypothesis*, http://www.claymath.org/millenium/Riemann_Hypothesis/riemann.pdf (2008).

[4] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, 1859, in Gesammelte Werke, Teubner, Leipzig (1872).

[5] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, Inc., San Diego, New York (1974).

[6] E. C. Titchmarsh, *The Theory of the Riemann Theta Function*, Oxford Univ. Press, London, New York (1951).

[7] J. Hadamard, *Sur la distribution des zéros de la function $\zeta(s)$ et ses conséquences arithmétiques*, Bull. Soc. Math. France 14 (1896) pp. 199-220.

[8] B.E. Petersen, *Riemann Zeta Function*, http://www.orst.edu/~peterseb/zeta.pdf