Self Force on A Point-Like Source Coupled with Massive Scalar Field

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Abstract: The problem of determining the radiation reaction force experienced by a scalar charge moving in flat spacetime is investigated. A consistent renormalization procedure is used, which exploits the Poincaré invariance of the theory. Radiative parts of Noether quantities carried by massive scalar field are extracted. Energy-momentum and angular momentum balance equations yield Harish-Chandra equation of motion of radiating charge under the influence of an external force. This equation includes effect of particle’s own field. The self force produces a time-changing inertial mass.

Keywords: Renormalization; Symmetry and Conservation Laws

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1. Introduction

The problem of calculating the motion of an isolated point-like charge coupled to massive scalar field in flat spacetime is an old one which is currently receiving renewed interest. Classical equations of motion of a point particle interacting with a neutral massive vector field were first found by Bhabha [1] following a method originally developed by Dirac [2] for the case of electromagnetic field. In this method the finite force and self-force terms in the equations of motion are obtained from the conservation laws for the energy-momentum tensor of the field. It was extended by Bhabha and Harish-Chandra [3] to particles interacting with any generalized wave field and was applied to the motion of a simple pole of massive scalar field by Harish-Chandra [4].

The principal new feature of the field which carries rest mass in addition to energy and momentum is that it is nonlocal. (The field depends not only on the current state of motion of the source but on its past history.) Physically it is due to the fact that the...
massive field propagates at all speeds smaller than the velocity of light.

In [5, 6, 7, 8] a consistent theory of action at a distance was formulated in a case of point sources coupled to massive scalar or vector fields. In electrodynamics of Wheeler and Feynman [9], the interaction is assumed to be symmetric in time. The symmetric case is the only one for which the equations of motion follow from a variational principle. However, they do not contain any terms describing radiation damping. Such terms do appear if the assumption of complete absorption is applied to these equations. By this the authors [9] mean that the total advanced field of all the particles in the Universe equals their total retarded field.

The equations of motion obtained from the field-theoretical and action-at-a-distance point of view are different: the integrals over entire world line of the particle substitute for integrals over the past motion which appear in case of purely retarded fields [1, 4]. In [10, 11, 12] the total cross sections for the scattering of the various kinds of mesons by a heavy particle (nucleon) were calculated\(^2\) and compared with those obtained within the Bhabha and Harish-Chandra approach. The predictions following from the two approaches should be exploited to furnish an experimental decision between the two theories. It is worth noting that in case of the retarded interactions, Havas [5] and Crownfield and Havas [13] obtain the Bhabha [1] and Harish-Chandra [4] equations.

In the present paper we calculate energy-momentum and angular momentum carried by outgoing massive scalar waves. Effective equations of motion of radiating scalar source will be obtained via the consideration of energy-momentum and angular momentum balance equations. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction. The verification is not a trivial matter, since the Klein-Gordon field generated by the scalar charge holds energy near the particle. This circumstance makes the procedure of decomposition of the Noether quantities into bound and radiative parts unclear.

In [14, 15, 16] Cawley and Marx study the massive scalar radiation from a point source with a prescribed world line. The authors assume that the particle accelerates only over a portion of the world line which corresponds to a finite proper time interval. They evaluate the energy-momentum which flows across a fixed three-dimensional sphere of large radius \(R\). The radiation part of energy-momentum carried by massive scalar field was extracted which depends on \(R\) explicitly. It casts serious doubt on the validity of the result. In the present paper we apply a consistent splitting procedure which obeys the spirit of Dirac scheme of decomposition of electromagnetic potential into singular (symmetric) and regular (radiative) components [2].

Recently [17], Quinn has obtained an expression for the self-force on a point-like particle coupled to a massless scalar field arbitrarily moving in a curved spacetime. It is worth noting that in curved background massless waves propagate not just at speed of light, but at all speeds smaller than or equal to the speed of light. (It can be understood as the result of interaction between the radiation and the spacetime curvature.) Therefore, the particle may "fill" its own field, which will act on it just like an external field. In [18]

\(^2\) In the present paper we shall not identify neither fields nor sources with any currently known particles.
Quinn establishes that the total work done by the scalar self-force matches the amount of energy radiated away by the particle.

Using Quinn’s general expression, Pfenning and Poisson [19] calculate the self-force experienced by a point scalar charge moving in a weakly curved spacetime. It is characterized by a generic Newtonian potential \( \Phi \) which determines the small deviation of the metric \( g_{\alpha\beta} \) with respect to the Minkowski values \( \eta_{\alpha\beta} = \text{diag}(-1,1,1,1) \). Potential \( \Phi \) behaves as \(-M/r\) at large distances \( r \) from the bounded mass distribution of total mass \( M \). In contrast to the electromagnetic case, the equations of motion of scalar charge does not provide conservation of the rest mass (see also [17, 18]). In Refs.[20, 21] this phenomenon is studied for various kinds of cosmological spacetimes.

In this paper we consider the radiation reaction problem for a point particle that acts as a source for a massive scalar field in Minkowski spacetime. It is organized as follows. In Section 2. we recall the Green’s functions associated with the Klein-Gordon wave equation. Convolving them with the point-like source, we derive the retarded scalar potential and field strengths as well as their advanced counterparts. In Section 3. we decompose the momentum 4-vector carried by massive scalar field into singular and regular parts. All diverging terms have disappeared into the procedure of mass renormalization while radiative terms survive. In analogous way we analyze the angular momentum of the Klein-Gordon scalar field. The radiative parts of Noether quantities carried by field and already renormalized particle’s individual momentum and angular momentum constitute the total energy-momentum and total angular momentum of our particle plus field system. In Section 4. we derive the effective equations of motion of radiating scalar charge via analysis of balance equations. We show that it coincides with the Harish-Chandra equation [4]. In Section 4. we discuss the result and its implications.

2. Scalar Potential and Field Strengths of A Point-Like Scalar Charge

The dynamics of a point-like charge coupled to massive scalar field is governed by the action [22, 23]

\[
I_{\text{total}} = I_{\text{part}} + I_{\text{int}} + I_{\text{field}}. \tag{1}
\]

Here

\[
I_{\text{field}} = -\frac{1}{8\pi} \int d^4y \left( \eta^{\alpha\beta} \varphi_\alpha \varphi_\beta + k_0^2 \varphi^2 \right) \tag{2}
\]

is an action functional for a massive scalar field \( \varphi \) in flat spacetime. We shall use the metric tensor \( \eta^{\alpha\beta} = \text{diag}(-1,1,1,1) \) and its inverse \( \eta_{\alpha\beta} = \text{diag}(-1,1,1,1) \) to raise and lower indices, respectively. The mass parameter \( k_0 \) is a constant with the dimension of reciprocal length. The integration is performed over all the spacetime. The particle action is

\[
I_{\text{part}} = -m_0 \int d\tau \sqrt{-\dot{z}^2} \tag{3}
\]

where \( m_0 \) is the bare mass of the particle which moves on a world line \( \zeta : \mathbb{R} \to M_4 \) described by relations \( z^\alpha(\tau) \) which give the particle’s coordinates as functions of proper
time; \( \dot{z}^{\alpha}(\tau) = dz^{\alpha}(\tau)/d\tau \). Finally, the interaction term is given by

\[
I_{\text{int}} = g \int d\tau \sqrt{-\ddot{z}^2} \varphi(z)
\]  

(4)

where \( g \) is scalar charge carried by a four-dimensional Dirac distribution supported on \( \zeta \); charge’s density is zero everywhere, except at the particle’s position where it is infinite.

The action (1) is invariant under infinitesimal transformations (translations and rotations) which constitute the Poincaré group. According to Noether’s theorem, these symmetry properties yield conservation laws, i.e. those quantities that do not change with time.

Variation on field variable \( \varphi \) of action (1) yields the Klein-Gordon wave equation

\[
(\Box - k_0^2) \varphi(y) = -4\pi \rho(y),
\]

(5)

where \( \Box = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \) is the D’Alembert operator. We consider a scalar field satisfying eq.(5) in Minkowski spacetime with a point particle source

\[
\rho(y) = g \int_{-\infty}^{+\infty} d\tau \delta^{(4)}(y - z(\tau)).
\]

(6)

A solution to eq.(5) can be expressed as

\[
\varphi(y) = \int d^4 x G(y, x) \rho(x).
\]

(7)

The relevant wave equation for the Green’s function \( G(y, x) \) is

\[
(\Box - k_0^2) G(y, x) = -4\pi \delta^{(4)}(y - x),
\]

(8)

where \( \delta^{(4)}(y - x) \) is a four-dimensional Dirac functional in \( \mathbb{M}_4 \). The retarded Green’s function \([1, 22, 23]\)

\[
G^{\text{ret}}(y, x) = \theta(y^0 - x^0) \left[ \delta(\sigma) - \frac{k_0}{\sqrt{-2\sigma}} J_1(k_0 \sqrt{-2\sigma}) \theta(-\sigma) \right]
\]

(9)

consists of singular part (this proportional to \( \delta(\sigma) \)) and smooth part (that proportional to \( \theta(-\sigma) \)). The former possesses support only on the past light cone of the field point \( y \) while the latter represents a function supported within the past light cone of \( y \). By \( \sigma \) we denote Synge’s world function in flat space-time \([23]\)

\[
\sigma(y, x) = \frac{1}{2} \eta^{\alpha\beta}(y^\alpha - x^\alpha)(y^\beta - x^\beta)
\]

(10)

which is equal to half of the squared length of the geodesic connecting two points in \( \mathbb{M}_4 \), namely “base point” \( x \) and “field point” \( y \). \( \theta(y^0 - x^0) \) is step function defined to be one if \( y^0 > x^0 \), and defined to be zero otherwise, so that \( G^{\text{ret}}(y, x) \) vanishes in the past of \( x \). \( \theta(-\sigma) \) is the step function of \( -\sigma(y, x) \) and \( J_1 \) is the first order Bessel’s function of \( k_0 \sqrt{-2\sigma} \).
We substitute eq.(6) for the scalar density $\rho(x)$ in the right-hand side of eq.(7). Massive scalar waves propagate at all speeds smaller than or equal to the speed of light. Hence the retarded potential at each point $y$ of Minkowski space $\mathbb{M}_4$ consists of a local term as well as non-local one. The local term is evaluated at the retarded instant $\tau^{\text{ret}}(y)$ which is determined by the intersection of the world line with the past cone of the field point $y$. The non-local term defines contribution from cone’s interior. It reflects the circumstance that the retarded field at $y$ is generated also by the point source during its history prior $\tau^{\text{ret}}(y)$.

Convolving the retarded Green’s function (9) with the charge density (6) we construct the massive scalar field [4, 5, 23]:

$$\varphi^{\text{ret}}(y) = \frac{g}{r} - g\int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau \frac{k_0 J_1(k_0 \sqrt{-(K \cdot K)})}{\sqrt{-(K \cdot K)}}$$

(11)

where $J_1$ is the first order Bessel’s function of $k_0 \sqrt{-2\sigma}$ which is rewritten as $k_0 \sqrt{-(K \cdot K)}$. By $K^\mu = y^\mu - z^\mu(\tau)$ we denote the unique timelike (or null) vector pointing from the emission point $z(\tau) \in \zeta$ to a field point $y \in \mathbb{M}_4$. The upper limit of the integral is the root of algebraic equation $\sigma(y, z(\tau)) = 0$ which satisfies causality condition $y^0 - z^0(\tau^{\text{ret}}) > 0$. By $r$ we mean the retarded distance

$$r(y) = -\eta_{\alpha\beta}(y^\alpha - z^\alpha(\tau^{\text{ret}})) u^\beta(\tau^{\text{ret}}).$$

(12)

Because the speed of light is set to unity, it is also the spatial distance between $z(\tau^{\text{ret}}) \text{ and } y$ as measured in this momentarily comoving Lorentz frame where 4-velocity $u^\beta(\tau^{\text{ret}}) = (1, 0, 0, 0)$.

Scalar field strengths are given by the gradient of the potential (11). Let us differentiate the local term. Because $y$ and $z(\tau^{\text{ret}})$ are linked by the light-cone mapping, a change of field point $y$ generally comes with a change $\tau^{\text{ret}}$. Suppose that $y$ is displaced to the new field point $y + \delta y$. The new emission point $z(\tau^{\text{ret}} + \delta \tau^{\text{ret}})$ satisfies the algebraic equation $\sigma(y + \delta y, z(\tau^{\text{ret}} + \delta \tau^{\text{ret}})) = 0$. Expanding this to the first order of infinitesimal displacements $\delta y$ and $\delta \tau^{\text{ret}}$, we obtain $K_\alpha \delta y^\alpha + r\delta \tau^{\text{ret}} = 0$, or

$$\frac{\partial \tau^{\text{ret}}}{\partial y^\alpha} = -\frac{K_\alpha}{r(y)}.$$

(13)

This relation allows us to differentiate the retarded distance (12) in the local Coulomb-like term involved in eq.(11).

Now we differentiate the non-local term in the potential (11). Apart from the integral

$$f^{(\theta)}_\mu = g\int_{-\infty}^{\tau^{\text{ret}}(y)} d\tau k_0^2 \frac{d}{d\Xi} \left(\frac{J_1(\Xi)}{\Xi}\right) k_0 \frac{K_\mu}{\sqrt{-(K \cdot K)}}$$

(14)

the gradient $f_{\text{tail}, \mu} = f^{(\theta)}_\mu + f^{(\delta)}_\mu$ contains also local term

$$f^{(\delta)}_\mu = g k_0^2 \left. \frac{J_1(\Xi) K_\mu}{\Xi r} \right|_{\tau = \tau^{\text{ret}}}$$

(15)
which is due to time-dependent upper limit of integral in eq.(11). Because of asymptotic behaviour of the first order Bessel’s function with argument $\Xi := k_0 \sqrt{-(K \cdot K)}$ the local term $f^{(\delta)}_{\mu}$ is finite on the light cone where $\Xi = 0$. It diverges on the particle’s trajectory only.

To simplify the non-local contribution as much as possible we use the identity

$$\frac{k_0}{\sqrt{-(K \cdot K)}} = \frac{1}{(K \cdot u)} \frac{d\Xi}{d\tau}$$

in the integral (14) and perform integration by parts. On rearrangement, we add it to the expression (15). The term which depends on the end points only annuls $f^{(\delta)}_{\mu}$. Finally, the gradient of potential (11) becomes

$$\frac{\partial \phi_{\text{ret}}(y)}{\partial y^\mu} = -g \left[ 1 + \frac{(K \cdot a)}{r^3} K_\mu + g \frac{u^\mu}{r^2} + g \int_{-\infty}^{\tau_{\text{ret}}(y)} \frac{d\tau}{\Xi} k_0^2 J_1(\Xi) \left[ \frac{1 + (K \cdot a)}{r^2} K_\mu - \frac{u^\mu}{r} \right] \right]$$

where $\Xi := k_0 \sqrt{-(K \cdot K)}$. As it is in the potential itself, particle’s position, velocity, and acceleration in the local part are referred to the retarded instant $\tau_{\text{ret}}(y)$ while ones under the integral sign are evaluated at instant $\tau \leq \tau_{\text{ret}}(y)$. The non-local part arises from source contributions interior to the light cone. This part of field is called the “tail term”. The invariant quantity

$$r = -(K \cdot u)$$

is an affine parameter on the time-like (null) geodesic that links $y$ to $z(\tau)$; it can be loosely interpreted as the time delay between $y$ and $z(\tau)$ as measured by an observer moving with the particle.

The advanced Green’s function is non-zero in the past of emission point $x$:

$$G^{\text{adv}}(y, x) = \theta(-y^0 + x^0) \left[ \delta(\sigma) - \frac{k_0}{\sqrt{-2\sigma}} J_1(k_0 \sqrt{-2\sigma}) \theta(-\sigma) \right].$$

The advanced force

$$\frac{\partial \phi^{\text{adv}}(y)}{\partial y^\mu} = -g \left[ -\frac{1 + (K \cdot a)}{r^3} K_\mu + g \frac{u^\mu}{r^2} + g \int_{\tau^{\text{adv}}(y)}^{+\infty} \frac{d\tau}{\Xi} k_0^2 J_1(\Xi) \left[ \frac{1 + (K \cdot a)}{r^2} K_\mu - \frac{u^\mu}{r} \right] \right]$$

is generated by the point charge during its entire future history following the advanced time associated with $y$. Particle’s characteristics in the local part are referred to the instant $\tau^{\text{adv}}(y)$.

3. **Bound and Radiative Parts of Noether Quantities**

In this Section we decompose the energy-momentum and angular momentum carried by massive scalar field into the bound and radiative parts. The bound terms will be absorbed
by particle’s individual characteristics while the radiative terms exert the radiation reaction. We do not calculate the flows of the massive scalar field across a thin tube around a world line of the source. To extract the appropriate finite parts of energy-momentum and angular momentum we apply the scheme developed in Refs. [24, 25]. In these papers the radiation reaction problem for an electric charge moving in flat spacetime of three dimensions is considered. A specific feature of 2 + 1 electrodynamics is that both the electromagnetic potential and electromagnetic field are non-local: they depend not only on the current state of motion of the particle, but also on its past (or future) history. The scalar potential (11) as well as the scalar field strengths (17) and (20) behave analogously.

Decomposition of Noether quantities into bound and radiative components satisfies the following conditions [24, 25]:

- proper non-accelerating limit of singular and regular parts;
- proper short-distance behaviour of regular part;
- Poincaré invariance and reparametrization invariance.

The first point means that in specific case of rectilinear uniform motion regular parts should vanish because of non-accelerating charge does not radiate. By “proper short-distance behaviour” we mean the finiteness of integrand near the coincidence limit where point of emission placed on the world line tends to the field point which also lies on ζ. (The bound parts of non-local conserved quantities in 2 + 1 electrodynamics contain one integration over the world line while radiative ones are integrated over ζ twice.)

The scalar potential (11) and the scalar field strengths (17) and (20) contain local terms as well as non-local ones. Local part of energy-momentum carried by massive scalar field is obtained in [4, 5, 26]. It is equal to one-half of the well-known Larmor rate of radiation integrated over the world line:

$$p_{\text{loc}, R}^\mu = \frac{g^2}{3} \int_{-\infty}^{\tau} ds a^2(s) u^\mu(s).$$  \hspace{1cm} (21)

Similarly, the local part of radiated angular momentum is equal to the one-half of corresponding quantity in classical electrodynamics [27]:

$$M_{\text{loc}, R}^{\mu\nu} = \frac{g^2}{3} \int_{-\infty}^{\tau} ds a^2_s [z^\mu_s u^\nu_s - z^\nu_s u^\mu_s] + \frac{g^2}{3} \int_{-\infty}^{\tau} ds [u^\mu_s a^\nu_s - u^\nu_s a^\mu_s].$$  \hspace{1cm} (22)

There are singular terms associated with the Coulomb-like potential taken on particle’s world line (see A, eq.(A.8)). Inevitable infinity is absorbed by “bare” mass within the renormalization procedure.

To find the “tail” parts of radiated Noether quantities sourced by the interior of the light cone we deal with the field defined on the world line only. Following the scheme presented in [24, 25] we build our construction upon the tail part of the field strengths (17) evaluated at point \(z(\tau_1) \in \zeta\):

$$f_{\text{tail}, \mu}^{\text{ret}} = \frac{\partial \varphi_{\text{tail}}(y)}{\partial y^\mu} \bigg|_{y = z(\tau_1)} = g \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 J_1(\xi) \left[ \frac{1 + (q \cdot a_2)}{r_2^2} q_\mu - \frac{u_2,\mu}{r_2^2} \right].$$  \hspace{1cm} (23)
Here $q^\mu = z_1^\mu - z_2^\mu$ defines the unique timelike 4-vector pointing from an emission point $z(\tau_2) \in \zeta$ to a field point $z(\tau_1) \in \zeta$. Index 1 indicates that particle’s position, velocity, or acceleration is referred to the instant $\tau_1 \in [-\infty, \tau]$ while index 2 says that the particle’s characteristics are evaluated at instant $\tau_2 \leq \tau_1$. We use the notations $\xi = k_0 \sqrt{-(q \cdot q)}$ and $r_2 = -(q \cdot u_2)$.

Next we consider the “advanced” counterpart of the expression (23):

$$f_{\text{tail},\mu}^{\text{adv}} = \frac{g}{2} \int_{-\infty}^\tau \mathrm{d}\tau_1 \int_{-\infty}^\tau \mathrm{d}\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[ \frac{1}{r_2^2} (q \cdot a_2) q^\mu - \frac{u_2^\mu}{r_2} - \frac{1}{r_1^2} (q \cdot a_1) q^\mu + \frac{u_1^\mu}{r_1} \right].$$

(24)

It is intimately connected with the gradient of $\varphi_{\text{tail}}^{\text{adv}}(y)$ evaluated at point $y = z(\tau_1)$. Note that the advanced force (20) is generated by the point charge during its entire future history. In (24) the domain of integration is the portion of the world line which corresponds to the interval $\tau_2 \in [\tau_1, \tau]$ where $\tau$ is the so-called “instant of observation”. This instant arise naturally in [24, 25] where an interference of outgoing waves at the plane of constant value of $y^0$ is investigated. Its role is elucidated in [24, Figs.2-4] and [25, Figs.1,2].

We postulate that non-local part of energy-momentum carried by outgoing radiation is one-half of work done by the retarded tail force minus one-half of work performed by the advanced one, taken with opposite sign:

$$p_{\text{tail},\mu}^R = -\frac{g}{2} \left( \int_{-\infty}^\tau \mathrm{d}\tau_1 f_{\text{tail},\mu}^{\text{ret}} - \int_{-\infty}^\tau \mathrm{d}\tau_1 f_{\text{tail},\mu}^{\text{adv}} \right).$$

(25)

It is obvious that the “advanced” domain of integration, $\int_{-\infty}^\tau \mathrm{d}\tau_1 \int_{-\infty}^\tau \mathrm{d}\tau_2$, is equivalent to $\int_{-\infty}^\tau \mathrm{d}\tau_2 \int_{-\infty}^\tau \mathrm{d}\tau_1$. It can be replaced by the “retarded” one, $\int_{-\infty}^\tau \mathrm{d}\tau_1 \int_{-\infty}^\tau \mathrm{d}\tau_2$, via interchanging of indices “first” and “second” in the integrand. The “tail” part of energy-momentum carried by outgoing radiation becomes

$$p_{\text{tail},R}^\mu = \frac{g^2}{2} \int_{-\infty}^\tau \mathrm{d}\tau_1 \int_{-\infty}^\tau \mathrm{d}\tau_2 k_0^2 \frac{J_1(\xi)}{\xi} \left[ -\frac{1}{r_2^2} (q \cdot a_2) q^\mu + \frac{u_2^\mu}{r_2} - \frac{1}{r_1^2} (q \cdot a_1) q^\mu + \frac{u_1^\mu}{r_1} \right]$$

(26)

where $r_a = -(q \cdot u_a)$. It is noteworthy that all the moments are before the observation instant $\tau$, and the retarded causality is not violated.

In the specific case of a uniformly moving source $q^\mu = u^\mu(\tau_1 - \tau_2)$ and $r_a = \tau_1 - \tau_2$ for both $a = 1$ and $a = 2$. Hence the bracketed integrands in eq. (26) is identically equal to zero. The local parts of radiation (21) and (22) vanish if $u^\mu = \text{const}$. As could be expected, nonaccelerating scalar charge does not radiate.

Now we evaluate the short-distance behaviour of the expression under the double integrals in eq.(26). Let $\tau_1$ be fixed and $\tau_1 - \tau_2 := \Delta$ be a small parameter. With a degree of accuracy sufficient for our purposes

$$\sqrt{-(q \cdot q)} = \Delta$$

$$q^\mu = \Delta \left[ u_1^\mu - \frac{a_1^\mu \Delta}{2} + \frac{\dot{a}_1^\mu \Delta^2}{6} \right]$$

$$u_2^\mu = u_1^\mu - a_1^\mu \Delta + \frac{\dot{a}_1^\mu \Delta^2}{2}.$$
Substituting these into integrands of the double integrals of eq.(26) and passing to the limit $\Delta \to 0$ yields vanishing expression. Hence the subscript “R” stands for “regular” as well as for “radiative”.

In analogous way we construct the non-local part of radiated angular momentum. First of all we introduce the torque of the retarded tail force (23) and its advanced counterpart:

$$m_{\text{ret}}^{\tau,\mu} = z_{1,\mu} f_{\tau,\mu}^{\text{ret}} - z_{1,\nu} f_{\tau,\nu}^{\text{ret}}$$
$$m_{\text{adv}}^{\tau,\mu} = z_{1,\mu} f_{\tau,\mu}^{\text{adv}} - z_{1,\nu} f_{\tau,\nu}^{\text{adv}}$$

(28)

The desired expression is equal to the one-half of integral of $m_{\text{ret}}^{\tau,\mu}$ over the world line up to observation instant $\tau$ minus one-half of integral of $m_{\text{adv}}^{\tau,\mu}$, taken with opposite sign:

$$M_{\tau,\mu}^{\mu} = \frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau} d\tau_2 k_0^2 J_1(\xi) \left[ \frac{1 + (q \cdot a_2)}{r_2^2} \left( z_{1,\mu} z_{1,\nu} - z_{1,\nu} z_{1,\mu} \right) + \frac{z_{1,\mu}^2 u_{1,\nu} - z_{1,\nu}^2 u_{1,\nu}}{r_2} \right]$$
$$+ \frac{1 - (q \cdot a_1)}{r_1^2} \left( z_{1,\mu} z_{2,\nu} - z_{1,\nu} z_{2,\mu} \right) + \frac{z_{2,\mu}^2 u_{1,\nu} - z_{2,\nu}^2 u_{1,\nu}}{r_1}. \quad (29)$$

In the specific case of constant velocity this expression vanishes. Substituting eqs.(27) in the integrand passing to the limit $\Delta \to 0$ leads to zero.

We postulate that bound part of energy-momentum carried by non-local part of massive scalar field is one-half of sum of work done by the retarded and the advanced tail forces:

$$p_{\tau,\mu}^S = \frac{-g}{2} \left( \int_{-\infty}^{\tau} d\tau_1 f_{\tau,\mu}^{\text{ret}} + \int_{-\infty}^{\tau} d\tau_1 f_{\tau,\mu}^{\text{adv}} \right)$$
$$= \frac{-g^2}{2} \int_{-\infty}^{\tau} dsk_0^2 J_1(\xi) \left. q_{\mu}(\tau, s) \right| \frac{1}{r_\tau}. \quad (30)$$

The bound part of angular momentum also contains only one integration over the fragment of particle’s world line:

$$M_{\tau,\mu}^S = \frac{-g}{2} \left( \int_{-\infty}^{\tau} d\tau_1 m_{\tau,\mu}^{\text{ret}} + \int_{-\infty}^{\tau} d\tau_1 m_{\tau,\mu}^{\text{adv}} \right)$$
$$= \frac{g^2}{2} \int_{-\infty}^{\tau} dsk_0^2 J_1(\xi) \frac{z_{\tau,\mu} z_{s,\nu} - z_{\tau,\nu} z_{s,\mu}}{r_\tau}. \quad (31)$$

Here index $\tau$ indicates that particle’s position, velocity, or acceleration is referred to the observation instant $\tau$ while index $s$ says that the particle’s characteristics are evaluated at instant $s \leq \tau$. We denote $r_\tau = -(q \cdot u_\tau)$.

If $u^\mu = \text{const}$ that $\xi = k_0(\tau - s)$ and $q^\mu / r_\tau = u^\mu$. Since

$$\int_{-\infty}^{\tau} ds \frac{J_1[k_0(\tau - s)]}{\tau - s} = 1, \quad (32)$$

the field generated by a uniformly moving charge contributes an amount $p_{\tau,\mu}^S = -1/2g^2k_0u^\mu$ to its energy-momentum. This finding is in line with that of Appendix A where is established that if the particle is permanently at rest, the scalar meson field adds $-1/2g^2k_0$.
Putting the field point \( z \) between Bessel's function of order zero and of order one in eq.(11) yields
\[
\varphi^{\text{ret}}(y) = g \int_{-\infty}^{\tau_{\text{ret}}(y)} d\tau J_0(\Xi) \frac{1 + (K \cdot a)}{r^2} \tag{33}
\]
after integration by parts. The authors state that the Klein-Gordon source does not emanate massless radiation. Following their approach, we rewrite the scalar field strengths (17) as follows:
\[
\frac{\partial \varphi^{\text{ret}}(y)}{\partial y^\mu} = g \int_{-\infty}^{\tau_{\text{ret}}(y)} d\tau J_0(\Xi) \left\{ -3 \frac{[1 + (K \cdot a)]^2}{r^4} K_\mu - \frac{(K \cdot a)}{r^3} K_\mu + 3 \frac{1 + (K \cdot a)}{r^3} u_\mu + \frac{a_\mu}{r^2} \right\}. \tag{34}
\]
Putting the field point \( z(\tau_1) \in \zeta \) and the emission point \( z(\tau_2) \in \zeta \), we obtain the scalar self-field:
\[
F^{\text{ret}}_{\mu} = g \int_{-\infty}^{\tau_1} d\tau_2 J_0(\xi) \left\{ -3 \frac{[1 + (q \cdot a_2)]^2}{r^4} q_\mu - \frac{(q \cdot a_2)}{r^3} q_\mu + 3 \frac{1 + (q \cdot a_2)}{r^3} u_{2,\mu} + \frac{a_{2,\mu}}{r^2} \right\}. \tag{35}
\]
Similarly one can construct its “advanced” counterpart which is generated by the point source during its history after \( \tau_1 \) up to the observation instant \( \tau \).

Our next task is to extract the radiation part of energy-momentum carried by Cawley’s scalar field (33). Since the radiation does not propagate with the speed of light, the Larmor-like term (21) does not appear. The tail contribution to the radiation
\[
p^{\text{R}}_{\mu} = -\frac{g}{2} \left( \int_{-\infty}^{\tau} d\tau_1 F^{\text{ret}}_{\mu} - \int_{-\infty}^{\tau} d\tau_1 F^{\text{adv}}_{\mu} \right) \tag{36}
\]
\[
= -\frac{g^2}{2} \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 k_0^2 J_0(\xi) \left\{ -3 \frac{[1 + (q \cdot a_2)]^2}{r^4} q_\mu - \frac{(q \cdot a_2)}{r^3} q_\mu + \frac{a_{2,\mu}}{r^2} \right\} \left\{ 3 \frac{[1 - (q \cdot a_1)]^2}{r^4} q_\mu + \frac{(q \cdot a_1)}{r^3} q_\mu \right\} - 3 \frac{1 - (q \cdot a_1)}{r^4} u_{1,\mu} + \frac{a_{1,\mu}}{r^2} \right\}.
\]

Is the choice of the force (23) the only one? If not there exists an alternative expression for radiated energy-momentum. It is interesting to apply our decomposition procedure to the massive scalar field as it is described in Refs.[14, 15, 16]. Cawley and Marx [14] remove the local Coulomb-like infinity is the only divergence stemming from the pointness of the source.

If there is one more question to be answered: is the bound angular momentum \( M_{\text{tail},S} = z_0^\mu p_{\text{tail},S} - z_0^\nu p_{\text{tail},S} \) in case of uniform motion. We suppose, that the bound parts (30) and (31) of energy-momentum and angular momentum, respectively, are permanently “attached” to the charge and are carried along with it. It is worth noting that they possess the proper short-distance behaviour and, therefore, do not diverge. The “local” Coulomb-like infinity is the only divergence stemming from the pointness of the source.
is meaningful only. Let us study the short-distance behaviour. Having inserted the relations (27), we see that the double integral is ill defined because the integrand diverges at the edge \( \tau_2 = \tau_1 \) of the integration domain \( D_\tau = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1 \in ]-\infty, \tau[ , \tau_2 \leq \tau_1 \} \). It is because the Coulomb-like divergency moves under the integral sign (cf. eqs.\,(11) and (33)).

In the following Section we check the formula (26) and (29) via analysis of energy-momentum and angular momentum balance equations. Analogous equations yield correct equation of motion of radiating charge in conventional 3+1 electrodynamics \([28, 29]\) as well as in six dimensions \([30]\). It is reasonable to expect that conservation laws result correct equation of motion of point-like source coupled with massive scalar field where radiation back reaction is taken into account.

4. Equation of Motion of Radiating Charge

The equation of motion of radiating pole of massive scalar field was derived by Harish-Chandra \([4]\) in 1946. (An alternative derivation was produced by Havas and Crownfield in \([13]\).) Following the method of Dirac \([2]\), Harish-Chandra enclosed the world line of the particle by a narrow tube, the radius of which will in the end be made to tend to zero. The author calculates the flow of energy and momentum out of the portion of the tube in presence of an external field. The condition was imposed that the flow depends only on the states at the two ends of the tube (the so-called “inflow theorem”, see \([31, 3]\)). After integration over the tube along the world line and a limiting procedure, the equation of motion was derived. In our notation it looks as follows:

\[
\begin{align*}
  m_0 a^\mu_\tau - \frac{g^2}{3} \left( \dot{a}^\mu_\tau - a^2 u^\mu_\tau \right) - \frac{g^2}{2} k_0^2 u^\mu_\tau + g^2 \int_{-\infty}^\tau \, ds k_0^4 J_2(\xi) \frac{q^\mu}{\xi^2} + g^2 \frac{d}{d\tau} \left( u^\mu_\tau \int_{-\infty}^\tau \, ds k_0^2 J_1(\xi) \right) \\
  = g \eta^{\mu\alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} + g \frac{d}{d\tau} (u^\mu_\tau \varphi_{\text{ext}})
\end{align*}
\]

(37)

where \( m_0 \) is an arbitrary constant identified with the mass of the particle and \( \varphi_{\text{ext}} \) is the scalar potential of the external field evaluated at the current position of the particle. \( J_2(\xi) \) is the second order Bessel’s function. In this Section the Harish-Chandra equation will be obtained via analysis of energy-momentum and angular momentum balance equations.

In previous Section we introduce the radiative part \( p_R = p_{\text{loc}, R} + p_{\text{tail}, R} \) of energy-momentum carried by the field. We proclaim that it alone exerts a force on the particle. We assume that the bound part, \( p_S \), is absorbed by particle’s 4-momentum so that “dressed” charged particle would not undergo any additional radiation reaction. Already renormalized particle’s individual three-momentum, say \( p_{\text{part}} \), together with \( p_R \) constitute the total energy-momentum of our composite particle plus field system: \( P = p_{\text{part}} + p_R \). We suppose that the gradient of the external potential matches the change of \( P \) with
time:

\[ \dot{p}^\mu_{\text{part}}(\tau) = -\dot{p}^\mu_R + g \eta^{\mu \alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} \]

\[ = -\frac{g^2}{3} a^2(\tau) u^\mu + \frac{g^2}{2} \int_{-\infty}^\tau ds k_0^2 \frac{J_1(\xi)}{\xi} \left[ \frac{1}{r_s^2} q^\mu - \frac{u_s^\mu}{r_s} + 1 - \frac{q \cdot a_s}{r_s^2} q^\mu - \frac{u_s^\mu}{r_s} \right] \]

\[ + g \eta^{\mu \alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha}. \]  

(38)

The overdot means the derivation with respect to proper time \( \tau \).

Our next task is to derive expression which explain how three-momentum of “dressed” charged particle depends on its individual characteristics (velocity, position, mass etc.). We do not make any assumptions about the particle structure, its charge distribution and its size. We only assume that the particle 4-momentum \( p_{\text{part}} \) is finite. To find out the desired expression we analyze conserved quantities corresponding to the invariance of the theory under proper homogeneous Lorentz transformations. The total angular momentum, say \( M \), consists of particle’s angular momentum \( z \wedge p_{\text{part}} \) and radiative part of angular momentum carried by massive scalar field:

\[ M^{\mu \nu} = z^{\mu} p^{\nu}_{\text{part}}(\tau) - z^{\nu} p^{\mu}_{\text{part}}(\tau) + M^{\mu \nu}_R(\tau). \]  

(39)

We assume that the torque \( z^{\mu} \partial^\nu \varphi_{\text{ext}} - z^{\nu} \partial^\mu \varphi_{\text{ext}} \) of the potential external force matches the change of \( M \) with time. Having differentiated (39) where the radiated angular momentum \( M^{\mu \nu}_R = M^{\mu \nu}_{R,\text{loc}} + M^{\mu \nu}_{R,\text{tail}} \) is determined by eqs.(22) and (29), and inserting eq.(38) we arrive at the equality

\[ u^{\tau} \wedge \left( p_{\text{part}} + \frac{g^2}{3} a^{\tau} + \frac{g^2}{2} \int_{-\infty}^\tau ds k_0^2 \frac{J_1(\xi)}{\xi} q^\tau \right) = 0. \]  

(40)

Apart from usual velocity term, the 4-momentum of “dressed” particle contains also a contribution from field:

\[ p^{\mu}_{\text{part}} = m u^\mu - \frac{g^2}{3} a^\mu - \frac{g^2}{2} \int_{-\infty}^\tau ds k_0^2 \frac{J_1(\xi)}{\xi} q^\mu. \]  

(41)

The local part is the scalar analog of Teitelboim’s expression [32] for individual 4-momentum of a “dressed” electric charge in conventional electrodynamics. The integral term is then nothing but the bound part (30) of energy-momentum carried by the massive scalar field.

The expression for the scalar function \( m(\tau) \) is find in B via analysis of differential consequences of conservation laws. We derive that already renormalized dynamical mass \( m \) depends on particle’s evolution before the observation instant \( \tau \):

\[ m = m_0 + g^2 \int_{-\infty}^\tau ds k_0^2 \frac{J_1(\xi(\tau,s))}{\xi(\tau,s)} - g \varphi_{\text{ext}}. \]  

(42)

The constant \( m_0 \) can be identified with the renormalization constant in action (3) which absorbs Coulomb-like divergence stemming from local part of potential (11). It is of great
importance that the dynamical mass, \( m \), will vary with time: the particle will necessarily gain or lost its mass as a result of interactions with its own field as well as with the external one. The field of a uniformly moving charge contributes an amount \( g^2 k_0 \) to its inertial mass.

To derive the effective equation of motion of radiating charge we replace \( \dot{p}_\text{part} \) in left-hand side of eq.(38) by differential consequence of eq.(41). We apply the formula

\[
\frac{\partial}{\partial \tau} \int_{-\infty}^{\tau} ds f(\tau, s) = \int_{-\infty}^{\tau} ds \left( \frac{\partial f}{\partial \tau} + \frac{\partial f}{\partial s} \right) .
\]

(43)

At the end of a straightforward calculations, we obtain

\[
ma_\mu + \dot{m}u_\mu = \frac{g^2}{3} \left( \dot{a}_\mu - a_\tau^2 u_\mu \right) + g^2 \int_{-\infty}^{\tau} ds k_0^2 J_1(\xi) \left[ \frac{1 + (q \cdot a_s)}{r_s^2} - q^\mu - \frac{u_s^\mu}{r_s} \right] + 2g \left( \eta^{\mu\alpha} \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha} \right)
\]

(44)

where dynamical mass \( m(\tau) \) is defined by eq.(42). The local part of the self-force is one-half of well-known Abraham radiation reaction vector while the non-local one is then nothing but the tail part of particle’s scalar field strengths (17) acting upon itself (see eq.(23)). Indeed, since the massive field does not propagate with the velocity of light, the charge may “fill” its own field, which will act on it just like an external field.

Now we compare this effective equation of motion with the Harish-Chandra equation (37). The latter can be simplified substantially. Having used the recurrent relation

\[
J_2(\xi) = \frac{J_1(\xi)}{\xi} - \frac{dJ_1(\xi)}{d\xi}
\]

(45)

between Bessel functions of order two and of order one, after integration by parts we obtain

\[
g^2 \int_{-\infty}^{\tau} ds k_0^2 J_2(\xi) q^\mu = \frac{g^2}{2} k_0^2 u_\mu - g^2 \int_{-\infty}^{\tau} ds k_0^2 J_1(\xi) \left[ \frac{1 + (q \cdot a_s)}{r_s^2} - q^\mu - \frac{u_s^\mu}{r_s} \right].
\]

(46)

We also collect all the total time derivatives involved in Harish-Chandra equation (37). The term \( m(\tau)u_\mu \) arises under the time derivative operator, where time-dependent function \( m(\tau) \) is then nothing but the dynamical mass (42) of the particle. On rearrangement, the Harish-Chandra equation of motion (37) coincides with the equation (44) which is obtained via analysis of balance equations. It is in favour of the renormalization scheme for non-local theories developed in [24, 25].

To clear physical sense of the effective equation of motion (44) we move the velocity term \( \dot{m}u_\mu \) to the right-hand side of this equation:

\[
m(\tau)a_\mu = \frac{g^2}{3} \left( \dot{a}_\mu - a_\tau^2 u_\mu \right) + f^\mu_{\text{self}} + f^\mu_{\text{ext}}.
\]

(47)

According to [22], the scalar potential produces the Minkowski force

\[
f^\mu_{\text{ext}} = g \left( \eta^{\mu\alpha} + u_\mu u_\alpha^\tau \right) \frac{\partial \varphi_{\text{ext}}}{\partial z^\alpha}
\]

(48)
which is orthogonal to the particle’s 4-velocity. The self-force

\[
f^{\mu}_{self} = g^2 \int_{-\infty}^{\tau} \text{d}sk_0 \frac{J_1(\xi)}{\xi} \left[ \frac{1 + (q \cdot a_s)}{r_s^2} (q^\mu - r_s u_\sigma^\mu) - \frac{u_s^\mu + (u_s \cdot u_\tau) u_\mu}{r_s} \right] \tag{49}
\]
is constructed analogously from the tail part of gradient (17) of particle’s own field (11) supported on the world line \( \zeta \). The own field contributes also to particle’s inertial mass \( m(\tau) \) defined by eq.(42).

**Conclusions**

In the present paper, we find the radiative parts of energy-momentum and angular momentum carried by massive scalar field coupled to a point-like source. Scrupulous analysis of energy-momentum and angular momentum balance equations yields the Harish-Chandra equation of motion of radiating scalar pole. This equation includes the effect of particle’s own field as well as the influence of an external force.

To remove divergences stemming from the pointness of the particle we apply the regularization scheme originally developed for the case of electrodynamics in flat spacetime of three dimensions [24, 25]. It summarizes a scrupulous analysis of energy-momentum and angular momentum carried by non-local electromagnetic field of a point electric charge. The simple rule allows us to identify that portion of the radiation which arises from source contributions interior to the light cone.

Energy-momentum and angular momentum balance equations for radiating scalar pole constitute system of ten linear algebraic equations in variables \( p^\mu_{part}(\tau) \) and their first time derivatives \( \dot{p}^\mu_{part}(\tau) \) as the functions of particle’s individual characteristics (velocity, acceleration, charge etc.). The system is degenerate, so that solution for particle’s 4-momentum includes arbitrary scalar function, \( m(\tau) \), which can be identified with the dynamical mass of the particle. Besides renormalization constant, the mass includes contributions from particle’s own field as well as from an external field.

This is a special feature of the self force problem for a scalar charge. Indeed, the time-varying mass arises also in the radiation reaction for a pointlike particle coupled to a massless scalar field on a curved background [17]. The phenomenon of mass loss by scalar charge is studied in [20, 21]. Similar phenomenon occurs in the theory which describe a point-like charge coupled with massless scalar field in flat spacetime of three dimensions [33]. The charge loses its mass through the emission of monopole radiation.

**A Energy-momentum of the scalar massive field of uniformly moving particle**

The simplest scalar field is generated by an unmoved source placed at the coordinate origin. Setting \( z = (t, 0, 0, 0) \) and \( u = (1, 0, 0, 0) \) in eq.(33), one can derive the static potential [1, 14]:

\[
\varphi(y) = g \frac{\exp(-k_0r)}{r} \tag{A.1}
\]
where \( r = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \) is the distance to the charge. It is the well-known Yukawa field.

In this Appendix we calculate the energy-momentum

\[
p^\nu_{sc}(\tau) = \int_\Sigma d\sigma \mu T^{\mu\nu}
\]

carried by the scalar massive field due to a uniformly moving pointlike source \( g \). The stress-energy tensor \( \hat{T} \) is given by \([14, 15, 16]\)

\[
4\pi T_{\mu\nu} = \frac{\partial \varphi}{\partial y^\mu} \frac{\partial \varphi}{\partial y^\nu} - \frac{\eta_{\mu\nu}}{2} \left( \eta^{\alpha\beta} \frac{\partial \varphi}{\partial y^\alpha} \frac{\partial \varphi}{\partial y^\beta} + k_0^2 \varphi^2 \right)
\]

and \( \Sigma \) is an arbitrary space-like three-surface.

It is convenient to choose the simplest plane \( \Sigma_t = \{y \in M_4 : y^0 = t\} \) associated with unmoving observer. We start with the spherical coordinates

\[
y^0 = s + r, \quad y^i = r n^i
\]

where \( n^i = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \) and \( s \) is the parameter of evolution. To adopt them to the integration surface \( \Sigma_t \) we replace the radius \( r \) by the expression \( t - s \).

On rearrangement, the final coordinate transformation \((y^0, y^1, y^2, y^3) \rightarrow (t, s, \phi, \theta)\) looks as follows:

\[
y^0 = t, \quad y^i = (t - s)n^i.
\]

The surface element is given by

\[
d\sigma_0 = (t - s)^2 ds d\Omega
\]

where \( d\Omega = \sin \theta d\theta d\phi \) is an element of solid angle.

After trivial calculation one can derive the only non-trivial component of energy-momentum (A.2) is

\[
p^0_{sc} = \frac{1}{4\pi} \int_{-\infty}^{t} ds (t - s)^2 \int d\Omega \frac{1}{2} \left[ \sum_i \left( \frac{\partial \varphi}{\partial y^i} \right)^2 + k_0^2 \varphi^2 \right]
\]

\[
= \frac{g^2}{2} \left[ k_0 \exp[-2k_0(t - s)] + k_0 \exp[-2k_0(t - s)] \right]_{s \rightarrow t}^{s \rightarrow -\infty}
\]

\[
= \lim_{\varepsilon \rightarrow 0} \frac{g^2}{2\varepsilon} \left( \frac{g^2}{2} \right) k_0
\]

where \( \varepsilon \) is positively valued small parameter.

Having performed Poincaré transformation, the combination of translation and Lorentz transformation, we find the energy-momentum carried by massive scalar field of uniformly moving charge:

\[
p^\mu_{sc} = \lim_{\varepsilon \rightarrow 0} \frac{g^2}{2\varepsilon} u^\mu - \frac{g^2}{2} k_0 u^\mu.
\]

The divergent Coulomb-like term is absorbed by the “bare” mass \( m_0 \) involved in action integral (1) while the finite term contributes to the particle’s individual 4-momentum (41).
B Derivation of the dynamical mass

The scalar product of particle 4-velocity on the first-order time-derivative of particle 4-momentum (38) is as follows:

\[
(p_{\text{part}} \cdot u_\tau) = \frac{g^2}{3} a_\tau^2 + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k^2_0 J_1(\xi) \frac{1 + (q \cdot a_s)}{r_s^2} \left( \frac{u_s \cdot u_\tau}{r_s} + \frac{g \cdot a_\tau}{r_\tau} \right) + g \frac{d\varphi_{\text{ext}}}{d\tau}.
\]  
(B.1)

Since \((u \cdot a) = 0\), the scalar product of particle acceleration on the particle 4-momentum (41) does not contain the scalar function \(m_\infty\) which arises in eq.(A.8).

\[
(p_{\text{part}} \cdot a_\tau) = -\frac{g^2}{3} a_\tau^2 - \frac{g^2}{2} \int_{-\infty}^{\tau} ds k^2_0 \frac{J_1(\xi)}{\xi} \frac{(q \cdot u_\tau)}{r_\tau}.
\]  
(B.2)

Summing up (B.1) and (B.2) we obtain the non-local expression:

\[
\frac{d}{d\tau}(p_{\text{part}} \cdot u_\tau) = \frac{g^2}{2} \int_{-\infty}^{\tau} ds k^2_0 \frac{J_1(\xi)}{\xi} \left( \frac{(q \cdot u_\tau)}{r_\tau} \right) + g \frac{d\varphi_{\text{ext}}}{d\tau}.
\]  
(B.3)

We rewrite the expression under the integral sign as the following combination of partial derivatives in time variables:

\[
k^2_0 \frac{J_1(\xi)}{\xi} \frac{\partial}{\partial s} \left( \frac{(q \cdot u_\tau)}{r_\tau} \right) = \frac{\partial}{\partial s} \left( k^2_0 \frac{J_1(\xi)}{\xi} \frac{(q \cdot u_\tau)}{r_\tau} \right) - \frac{\partial}{\partial \tau} \left( k^2_0 \frac{J_1(\xi)}{\xi} \right).
\]  
(B.4)

This circumstance allows us to integrate the expression (B.3) over \(\tau\):

\[
(p_{\text{part}} \cdot u_\tau) = -m_0 + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k^2_0 \int_{-\infty}^{\tau_1} d\tau_2 \left[ \frac{\partial}{\partial \tau_2} \left( k^2_0 \frac{J_1(\xi)}{\xi} \frac{(q \cdot u_\tau)}{r_\tau} \right) - \frac{\partial}{\partial \tau_1} \left( k^2_0 \frac{J_1(\xi)}{\xi} \right) \right] + g \varphi_{\text{ext}}.
\]  
(B.5)

To integrate the second term in between the square brackets we substitute \(\int_{-\infty}^{\tau_1} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1\) for \(\int_{-\infty}^{\tau_1} d\tau_1 \int_{-\infty}^{\tau_2} d\tau_2\). The external potential is referred to the observation instant \(\tau\).

Alternatively, the scalar product of 4-momentum (41) and 4-velocity is as follows:

\[
(p_{\text{part}} \cdot u_\tau) = -m + \frac{g^2}{2} \int_{-\infty}^{\tau} ds k^2_0 \frac{J_1[\xi(\tau, s)]}{\xi(\tau, s)}.
\]  
(B.6)

Having compared these expressions we obtain:

\[
m = m_0 + g^2 \int_{-\infty}^{\tau} ds k^2_0 \frac{J_1[\xi(\tau, s)]}{\xi(\tau, s)} - g \varphi_{\text{ext}}.
\]  
(B.7)

We suppose that the renormalization constant \(m_0\) already absorbs the Coulomb-like infinity which arises in eq.(A.8).
References

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