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Classification of Electromagnetic Fields in non-Relativist Mechanics

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Abstract: We study the classification of electromagnetic fields using the equivalence relation on the set of all 4-potentials of the Schrödinger equation. In the general case we find the relations among the equivalent fields, currents, and charge densities. Particularly, we study the fields equivalent to the null field. We show that the non-stationary state function for a particle in arbitrary uniform time-dependent magnetic field is equivalent to a plane wave. We present that the known coherent states of a free particle are equivalent to the stationary states of an isotropic oscillator. We reveal that the only constant magnetic field is not equivalent to the null field (contrary to a constant electrical field) and we find other fields that are equivalent to the constant magnetic field. We establish that one particular transformation of the free Schrödinger equation puts a plane wave and Green's function in a equivalence relation.

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1. Exposition of the Problem

It is known that the Schrödinger equation for a particle in the electromagnetic field in Cartesian coordinates

$$\imath\hbar\frac{\partial\Psi}{\partial t}=\hat{H}\Psi \quad \text{with} \quad \hat{H}=\frac{1}{2m}\left[-\imath\hbar\nabla-e\vec{A}\left(t,\vec{r}\right)\right]^2+e\varphi\left(t,\vec{r}\right). \tag{1}$$

must be completed with the following known characteristics: a physical interpretation of the wave function and the fact that the 4-potential must be real so that the Hamilton

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operator is hermitian (if not the normalization integrals could not be constant). These characteristics are intrinsic of the physics problem.

The Schrödinger equation completed by their intrinsic characteristics we call a physical problem in non-Relativist Mechanics. In this paper we realize the classification on the set of all physical problems using the Shapovalov's approach [1]. It is evident that for different intrinsic characteristics we obtain different equivalence relations and, consequently, different classifications.

Traditional studies add to the list of known intrinsic characteristics the structure of the electromagnetic field: $\vec{E}$ and $\vec{B}$ must be fixed. This imposition reduces the group of equivalence to the Cartesian product of the gauge invariance group and the Galilean transformations group (see, for example, [2]).

On the contrary, Shapovalov's approach does not include the structure of the electromagnetic field in the intrinsic characteristics. This allows us to find a broader equivalence group, and gives the new exact solutions of different physical problems.

The following theorem defines the equivalence group of the Schrödinger equation (we call it *The Shapovalov group*):

Shapovalov's Theorem 1. The Schrödinger equation in the Euclidian space completed by mentioned intrinsic characteristics admits following equivalence group: $G_{sh} = R \otimes T \otimes V \otimes \Gamma \otimes Y$. Where Γ is the gauge invariance group and $R\left(\theta_1\ (t), \theta_2\,(t)\,, \theta_3\,(t)\right)$; $T\left(c_1\ (t), c_2\ (t), c_3\ (t)\right)$ and $V\left(s\ (t)\right)$ are the rotations, translations, and scale change groups. The Shapovalov group contains 7 arbitrary time-dependent functions: $s\,(t)\,, \theta_i\,(t)\,, c_i\,(t)$ and $\frac{dt'}{dt} := s^2 > 0$, and Y is the discrete invariance group of the Schrödinger equation from Section 4.5.

The Shapovalov group is very effective for finding new solutions. We note that the traditional equivalence group is a subgroup of the $G_{sh} : \theta_i, c_i, s = const.$

For a specification of the Shapovalov group, we compare two Schrödinger equations such as (1) with different 4-potentials: $\{\vec{A}(t,\vec{r}\,), \varphi(t,\vec{r}\,)\}$ and $\{\vec{A}'\,(t,\vec{r}\,)\,, \varphi'\,(t,\vec{r}\,)\}$. Additionally, we designate $\{t', \vec{r}'\}$ to be the variables of the second equation, this allows us to distinguish equivalent fields. The equivalence relations must conserve the structure (1) of the Schrödinger equation. When using only this imposition we found [1] that the Shapovalov group of the Schrödinger equation is the following :

$$t' = t'\,(t) \qquad \left[\frac{dt'}{dt} := s^2 > 0\right] \tag{2}$$

$$\vec{r}' = s\,(t)\,a\,(t)\,\vec{r} + c\,(t) \tag{3}$$

$$\Psi'\,(t', \vec{r}') = \Psi\,(t, \vec{r})\,s^{-3/2}. \tag{4}$$

The seven arbitrary time-dependent functions of the Shapovalov group mean the following: $s(t)$ represents a freedom of time-scale and the coordinate-scale choice. Three functions $c_i(t)$ define three independent displacements with arbitrary time-dependent linear velocities. The orthogonal matrix $a(t)$ describes the rotations with three arbitrary time-dependent angular velocities. In Section 4 we study the subclasses of the null field equivalence class generated by each one of these three types of equivalence relations.

If a reference system were inertial, the Shapovalov group puts it in equivalence with the non-inertial systems. This fact is used frequently, for example, to describe the circular movement of a classic particle with a constant linear velocity. After being transferred to a rotating reference system, this physical problem is reduced to a free particle (see, for instance, [3], [4]). We emphasize that in the traditional approach such obvious equivalence does not exist. At least, this fact shows a need for extension of the traditional equivalence group.

2. Equivalent Potentials and Fields

Using the fact that the form (1) of the Schrödinger equation must be invariant and the equivalence relations of the Shapovalov group (2), (3) and (4), we obtain the equivalent potentials:

$$A'_k = s^{-1} \sum_j a_{kj} A_j - \frac{m}{e} s^{-2} x_k, \tag{5}$$

$$\varphi' = \frac{1}{s^2}\varphi + \sum_{k,j} \frac{1}{s^3} \dot{x}'_k a_{kj} A_j - \sum_k \frac{m}{2e}\frac{1}{s^4} \dot{x}'_k \dot{x}'_k, \tag{6}$$

where $\dot{x}'_k := \frac{\partial x'_k}{\partial t}$. Rigorously, the Shapovalov group is specified through formulas (2), (3), (4), (5), (6), and (39). Now, we can easily calculate the relation among equivalent electromagnetic fields:

$$s^2 B'_i(t', \vec{r}') = \sum_k a_{ik} B_k(t, \vec{r}) + \frac{m}{e} \sum_{l,k,j} \varepsilon_{ilk} \dot{a}_{lj}(t) a_{kj}(t), \tag{7}$$

$$s^2 \vec{E}'(t', \vec{r}') = \frac{1}{s} a \vec{E}(t, \vec{r}) - \dot{\vec{r}}' \times \vec{B}' + \frac{m}{e} \left(\frac{\dot{\vec{r}}'}{s^2} \right)^{\cdot}. \tag{8}$$

For a compact presentation, we introduce Larmor's notation: $\omega := \frac{eB}{2m}$ and the magnetic field's anti-symmetrical 3×3 matrix: $F_{ij} := \sum_k \varepsilon_{ijk}\omega_k$. The relation (7) between equivalent magnetic fields in matrix form appears as:

$$s^2 F'(t', \vec{r}') = aF(t, \vec{r})a^T + \dot{a}a^T. \tag{9}$$

If, for example, we have only time-dependent magnetic fields: $B' = B'(t')$ and $B = B(t)$ they both are equivalent (a is an orthogonal matrix):

$$\dot{a}(t) = s^2 F'(t')a - aF(t). \tag{10}$$

Particularly, if both magnetic fields are constant, the following orthogonal matrix provides the equivalent relations:

$$a(t) = \exp\{F't'(t) - Ft\}; \qquad t'(0) = 0. \tag{11}$$

Then we observe that naturally the equivalent fields (7) and (8) simultaneously verify the first pair of the Maxwell equations. It is easy to find that:

$$s^3\nabla' \cdot \vec{B}' = \nabla \cdot \vec{B}, \quad s^5\left[\nabla' \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'}\right] = sa\left[\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right] - \left(div\vec{B}\right)\dot{\vec{r}}'.$$

If one referential system does not have a "magnetic charge", than no other equivalent system will have it. The second pair of the Maxwell equations gives the relations between the equivalent densities of charge and current:

$$s^4\left(\frac{\rho'(t',\vec{r}')}{\epsilon_0} - \frac{2m}{e}\vec{\omega}'^2\right) = \frac{\rho(t,\vec{r})}{\epsilon_0} - \frac{2m}{e}\vec{\omega}^2 + s^2\dot{\vec{r}}'\, rot'\vec{B}' - \frac{3m}{e}f(t), \tag{12}$$

$$\vec{J}'(t',\vec{r}') + \epsilon_0\frac{\partial \vec{E}'}{\partial t'} = \frac{1}{s^3}a\left(\vec{J}(t,\vec{r}) + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right). \tag{13}$$

Where $\omega(t)$ is the Larmor frequency, $\vec{E}$ and $\vec{E}'$ verify (8) and

$$f(t) := s\left(\frac{1}{s}\right)^{\cdot\cdot}. \tag{14}$$

3. Fields Equivalent to the Null Field

Here we study the null field equivalence class. If $\vec{E}' = \vec{B}' = 0$, then relations (7), (8), and (10) allow us to describe all fields of this equivalence class:

$$\vec{B} = \vec{B}(t)$$

defined by (7)

$$\dot{a} = -aF(t); \tag{15}$$

$$\vec{E}(t,\vec{r}) = \frac{m}{e}\left[b(t)\vec{r} + \vec{d}(t)\right], \tag{16}$$

where we denote the matrix:

$$b(t) := -sa^T\left[\frac{(sa)^{\cdot}}{s^2}\right]^{\cdot} = f(t)I + \dot{F} - F^2, \tag{17}$$

with $f(t)$ from (14) , I unity matrix, and the vector

$$\vec{d}(t) := -sa^T\left[\frac{\dot{\vec{c}}}{s^2}\right]^{\cdot}. \tag{18}$$

We conclude from (15) and (16) that the widest field equivalent to the null field can contain an arbitrary time-dependent magnetic field, whereas the electric field is linear related to $\vec{r}$ (with the matrix of time-dependent coefficients) :

$$\vec{B} = \vec{B}(t) \tag{19}$$

$$\vec{E}(t,\vec{r}) = \left(\frac{m}{e}\right)\left[\vec{r}\times\dot{\vec{\omega}} + (\vec{\omega}\times\vec{r})\times\vec{\omega} + f(t)\vec{r} + d(t)\right], \tag{20}$$

where the second term of $\vec{E}$ has the form of the centrifugal force. We observe that the class has non-orthogonal fields $\vec{E}\cdot\vec{B} \neq 0$, but all are equivalent to one orthogonal. From (12) and (13) we find the null equivalent densities of charge and current:

$$e\rho(t) = 2m\epsilon_0\omega^2(t) + 3m\epsilon_0 f(t), \tag{21}$$

$$\vec{J}(t,\vec{r}) = -\epsilon_0\frac{\partial\vec{E}}{\partial t} = -\frac{\epsilon_0 m}{e}\left[\dot{b}\vec{r} + \dot{\vec{d}}\right]. \tag{22}$$

4. Particular Cases

4.1 Exclusively Rotation Equivalence ($s = 1,\ \vec{c} = 0$)

Now we treat the subclass of the fields which are equivalent to the null field using only the rotating equivalent reference systems. From (19) and (20) we have:

$$\vec{B} = \vec{B}(t), \tag{23}$$

$$\vec{E}(t,\vec{r}) = \left(\frac{m}{e}\right)\vec{r}\times\dot{\vec{\omega}} + \left(\frac{m}{e}\right)(\vec{\omega}\times\vec{r})\times\vec{\omega}. \tag{24}$$

The following 4-potential creates such a field:

$$\vec{A} = \left(\frac{m}{e}\right)(\vec{\omega}\times\vec{r}), \qquad \varphi = -\left(\frac{m}{2e}\right)(\vec{\omega}\times\vec{r})^2,$$

where the scalar potential represents the centrifugal energy. We observe that the electric and magnetic fields are not orthogonal in the general case: $\vec{E}\cdot\vec{B} = \left(\frac{1}{2}\right)\left(\dot{\vec{B}}\times\vec{B}\right)\cdot\vec{r}$, but they are if, for example, $\dot{\vec{B}} \parallel \vec{B}$. Hamilton's operator of this system has the form:

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta - \vec{\omega}(t)\cdot\hat{\vec{L}}, \tag{25}$$

where Δ is the Laplacian and $\hat{\vec{L}} := \vec{r}\times\hat{\vec{p}}$ is the operator of the angular momentum. The formula (25) corresponds exactly to the classic mechanics theorem of transformation of energy after passing from an inertial reference system to a system in rotation: $E_{rotation} = E_{inertial} - \vec{\omega}\cdot\vec{L}$ (see for instance [4]). If the field (23) and (24) is equivalent to the null field, then the system with Hamilton's operator (25) has a symmetry that corresponds to one of a free particle. The first integrals of motion of (25) are:

$$\hat{\vec{X}}_1 = a(t)\hat{\vec{p}}, \qquad \hat{\vec{X}}_2 = a(t)\left[-\frac{t}{m}\hat{\vec{p}} + \vec{r}\right], \qquad \hat{\vec{X}}_1 = a(t)\hat{\vec{L}}. \tag{26}$$

These first order operators of symmetry are unique for the system and constitute the base of all superior-order symmetry operators. The corresponding integrals of motion of a free particle are: $\hat{\vec{X}}'_i = a^T \hat{\vec{X}}_i$ $(i = 1, 2, 3)$. It is known that the eigenvectors of the linear momentum and the initial coordinate operators

$$\hat{\vec{X}}'_1 = \hat{\vec{p}}', \qquad \hat{\vec{X}}'_2 = -\left(\frac{t'}{m}\right)\hat{\vec{p}}' + \hat{\vec{r}}', \tag{27}$$

are respectively a plane wave and Green's function (we remove the primes on top of wave functions and variables):

$$\Psi_{plane\ wave} = C \exp\left[-\imath \frac{E}{\hbar} + \imath \frac{\vec{p}}{\hbar} \cdot \vec{r}\right]; \qquad \Psi_{Green} = \frac{C}{t^{3/2}} \exp\left[\imath \frac{m(\vec{r} - \vec{\lambda})^2}{2\hbar t}\right]. \tag{28}$$

In Section 4.5 we show that an equivalence exists among these functions and among the operators (27). Now, using the known solutions (28), we can construct new solutions to the Schrödinger equation for the field (23) and (24) for an arbitrary $\vec{B}(t)$. The first operator of (26) $\hat{\vec{X}}_1$ has the following eigenfunction:

$$\Psi(t, \vec{r}) = C \exp\left[\imath \frac{\vec{\lambda}^T a(t) \vec{r}}{\hbar} - \imath \frac{\lambda^2}{2m\hbar} t\right]. \tag{29}$$

This solution describes a non-stationary state, it is equivalent to a plane wave for a free particle (stationary state function). The constant $\frac{\lambda^2}{2m}$ is quasi-energy, the expression $\vec{p}_0 := a^T(t)\lambda$ is understood to be a quasi-momentum. We did not need to solve the Schrödinger equation for a particle in an arbitrary time-dependent field (23), (24). But we directly wrote its solution (29) without solving this equation, only using an equivalence relation of the Shapovalov group. We observe that contrary to the null field, for a particle in the field (23) and (24) the vector momentum is not a first integral of motion.

The Hamilton's operator (25) and the vector operator $\hat{\vec{X}}_2$ of (26) provide the solution of the Schrödinger equation that corresponds to the Green's function:

$$\Psi(t, \vec{r}) = \frac{C}{t^{3/2}} \exp\left[\imath \frac{m(\vec{r} - \vec{r}_0(t))^2}{2\hbar t}\right], \tag{30}$$

where $\vec{r}_0(t) := a^T(t)\vec{\lambda}$, $\vec{\lambda}$ is the initial coordinate in the inertial reference system; $\vec{r}_0(t)$ is the same initial coordinate in a rotating reference system. The vector λ is an arbitrary constant vector (eigenvalue of $\hat{\vec{X}}_2$). An orthogonal matrix $a(t)$ verifies (15) with the arbitrary vector function $\vec{\omega}(t)$. It is known that the solution (30) can be orthonormalized to the Dirac delta function, but its interpretation is difficult. It is a case of equivalence of a non-stationary solution to another non-stationary one.

The third vector operator of symmetry (26) $\hat{\vec{X}}_3$ does not allow a direct construction of a solution because as its components do not commute, it is necessary to use the set equivalent to: $\hat{H}'$, $\hat{\vec{L}}'^2$, $\hat{L}'_z$.

4.2 Exclusively Euclidean Translational Equivalence (the Unity Matrix $a = I$ and $s = 1$)

The subclass of the Euclidean translational displacements $(\vec{c} \neq 0)$ relates to the equivalence between reference systems moving with arbitrary linear velocities. For example, the only time-dependent uniform electrical field (16) defined by vector $\vec{d}(t)$ from (18), with the uniform corresponding current (22) is equivalent to the null field without a current. Naturally, Galileo's equivalence belongs to the Shapovalov group (see Section 4.5).

4.3 Isotropic Oscillator (Exclusively Time Scale Equivalence: the Unity Matrix $a = I$ and $\vec{c} = 0$)

Let be $\vec{B} = 0 \implies b(t)$ is a constant matrix. Using formulas (19 and 20), for $\vec{c} = 0$ and $\vec{\omega} = 0$, we find the field of the isotropic oscillator: $E = \left(\frac{m}{e}\right) f(t)r$. In case of $f(t) = const$, it corresponds to the potential energy $U = \frac{m\omega_0^2 r^2}{2}$, with $\omega_0 = const$. We note that constant ω_0 does not have any relation to the Larmor frequency ω (which is null here). From this form of potential energy and definition (14) we obtain the well known relation:

$$\ddot{\left(\frac{1}{s}\right)} + \omega_0^2 \left(\frac{1}{s}\right) = 0.$$

Solving it, we find the time-scale equivalence (scale change group of Shapovalov's theorem):

$$\frac{dt'}{dt} := s^2 = sec^2(\omega_0 t) \quad \implies \quad \omega_0 t' = \tan(\omega_0 t), \qquad \vec{r}' = s(t)\vec{r}. \tag{31}$$

This equivalence relation provides an equivalence between the free Schrödinger equation

$$\imath\hbar \frac{\partial \Psi'}{\partial t'} = \hat{H}'\Psi' = -\frac{\hbar^2}{2m}\Delta'\Psi', \tag{32}$$

and the equation for the isotropic oscillator:

$$\imath\hbar \frac{\partial \psi_{nlm}(t,\vec{r})}{\partial t} = \hat{H}\psi_{nlm}(t,\vec{r}) := \left[-\frac{\hbar^2}{2m}\Delta + \frac{m\omega_0^2 r^2}{2}\right]\psi_{nlm}(t,\vec{r}).$$

The solutions of the last equation are known (for instance, [5], [7]). We observe that these states are defined by the following set of integrals of movement: $\hat{H}$, $\hat{\vec{L}}^2$, $\hat{L}_z$. In addition, in the equivalence relation (31), it is necessary to use gauge invariance:

$$\Psi'_{nlm}(t',\vec{r}') = \frac{\psi_{nlm}(t,\vec{r})}{s^{3/2}} \exp\left[\imath \frac{m}{2\hbar}\frac{\dot{s}}{s} r^2\right]; \tag{33}$$

with the wave function:

$$\psi_{nlm}(t,\vec{r}) = \varphi_{nlm}(\vec{r}) \exp\left[-\imath \frac{E_{nl}}{\hbar} t\right]; \tag{34}$$

$$E_{nl} = \hbar\omega_0 \left[2n + l + \frac{3}{2}\right]; \qquad n,\ l = 0,\ 2,\ ...; \qquad m = 0,\ \pm 1,\ ...,\ \pm l;$$

$$\varphi_{nlm}(r, \theta,\ \phi) = \frac{1}{\xi} R_{nl}(\xi) Y_{lm}(\theta,\ \phi); \quad \xi := r\sqrt{\frac{m\omega_0}{\hbar}}.$$

Here the functions $Y_{lm}(\theta,\ \phi)$ are the spherical harmonics and $R_{nl}(\xi)$ are the radial functions expressed by confluent hypergeometric functions. The non-stationary solutions Ψ' (33) of the free Schrödinger equation (32) are known as coherent states and are equivalent to the stationary states of the isotropic oscillator. For example, the function of the fundamental state that verifies the equation (32) can be written (we abandon the primes on top of the variables and the wave function):

$$\Psi_{000}(t, \vec{r}) = \frac{C}{\sqrt{4\pi}\,(1 + \omega_0^2 t^2)^{3/4}} \ \exp\left[-\frac{m\omega_0}{2\hbar}\frac{1}{1 + \imath\omega_0 t}\vec{r}^2 - \imath\frac{3}{2} arctan(\omega_0 t)\right]. \tag{35}$$

We observe that to establish the equivalence of the free particle's coherent states to the stationary states of an isotropic oscillator, the time scale equivalence relation (31) is indispensable. For another equivalent solution to (35) of the free Schrödinger equation see (40). The equivalence between a free particle and an isotropic oscillator of one dimension is discussed in [8].

If $f(t)$ from (14) is not a constant, we have an arbitrary time-dependent radial dilation. In this case, new solutions of other physical problems can be constructed.

4.4 Constant Magnetic Field

In the null field equivalent class (15) and (16), we find a field with a constant magnetic component:

$$\vec{B} = (0,\ 0,\ B) = const; \qquad \vec{E}(t, \vec{r}) = \frac{m}{e}\left[f(t)\vec{r} + \omega^2 (x_1,\ x_2,\ 0)\right] + \vec{\varepsilon}(t). \tag{36}$$

Here $f(t)$ is defined by (14), $\vec{\varepsilon}(t)$ is an arbitrary time-dependent vector function. The first term of the electrical component corresponds to an arbitrary time-dependent dilation; particularly to an isotropic oscillator from Section 4.3 with an arbitrary time-dependent frequency. The second term corresponds to the constant centrifugal force field and the third term results from Euclidean translational freedom ($c(t) \neq 0$). Here are two particular cases of interest:

$$if \ \ f = 0 \quad (s = 1) \quad \Longrightarrow \quad \vec{B} = (0,\ 0,\ B)\,; \quad \vec{E}(t, \vec{r}) = \frac{m\omega^2}{e}\,(x_1,\ x_2,\ 0)\,; \tag{37}$$

$$if \ f := s\left(\frac{1}{s}\right)^{\cdot\cdot} = -\omega_0^2 \quad \Longrightarrow \quad \vec{B} = (0,\ 0,\ B)\,; \quad \vec{E}(t, \vec{r}) = -\frac{m\omega^2}{e}\,(0,\ 0,\ x_3)\,.$$

We can see that only constant magnetic field does not belong to the null field equivalent class, but always some electrical component of the field is present. By using formulas (7) and (8), we find that only constant magnetic field is equivalent to the constant centripetal force field:

$$\vec{B}' = (0,\ 0,\ B') = const; \quad \vec{E}' = 0 \qquad \Leftrightarrow \qquad \vec{B} = 0; \quad \vec{E}(t,\vec{r}) = -\frac{m\omega'^2}{e}\,(x_1,\ x_2,\ 0)\,. \tag{38}$$

This result explains the formula (37). If $f = const$, $\varepsilon = const$, the field (36) defines the widest stationary field equivalent to the null field.

4.5 Free Schrödinger Equation's Nucleus

It is interesting to find the set of equivalence relations that put the null field in equivalence to itself (see also [6]). By choosing $\vec{E}' = \vec{B}' = \vec{E} = \vec{B} = 0$, we easily find that such a nucleus is the Galilean group:

$$t' = \alpha^2 t + \beta; \quad \vec{r}' = \alpha a\vec{r} + \vec{v}_0 t + \vec{r}_0; \quad \Psi' = \alpha^{-3/2}\Psi$$

with arbitrary constants $\alpha,\ \beta,\ \vec{v}_0,\ \vec{r}_0,$ and a constant orthogonal matrix a. In addition, one isolated equivalence operation exists and leaves the free Schrödinger equation invariant:

$$t' = -\frac{1}{t}; \quad \vec{r}' = \frac{1}{t}\vec{r}; \quad \Psi'(t',\vec{r}') = \Psi(t,\vec{r})t^{3/2}\ \exp\left[-\imath\frac{m}{2\hbar}\frac{r^2}{t}\right], \tag{39}$$

$$\frac{t}{\Psi}\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right]\Psi = -\frac{t'}{\Psi'}\left[i\hbar\frac{\partial}{\partial t'} + \frac{\hbar^2}{2m}\Delta'\right]\Psi'.$$

Let us denote this transformation e. It is easy to show that $\{e, e^2, e^3, e^4\}$ constitutes the discrete invariance group of the Schrödinger equation Y from the Shapovalov's theorem. This transformation has a very interesting property: it transforms Green's function (28) to the plane wave propagating in the direction $-\vec{\lambda}$:

$$\Psi'(t',\vec{r}') = C'\ \exp\left[-\imath\omega' t' - \imath\vec{k}'\vec{r}'\right], \quad \hbar\omega' := \frac{m\lambda^2}{2}; \quad \hbar\vec{k}' := m\vec{\lambda}.$$

And inversely, the equivalence relation (39) transforms a plane wave into Green's function. Of course, the relation (39) also puts in equivalence the operator of linear momentum $\hat{\vec{p}}'$ and the operator of initial coordinate: $-(\frac{t}{m})\,\hat{\vec{p}} + \vec{r}$ (see formula (27)). This operation is similar to passing to momentum space. This transformation does not make sense if $\vec{\lambda} = 0$.

We easily find that relation (39) does not change the angular momentum: $\vec{L} = \hat{\vec{L}}'$. Consequently, the well-known states of a free particle with a constant angular momentum are invariant in respect to relation (39).

Another example, relation (39) puts the coherent state (35) in equivalence with the following function:

$$\Psi'_{000}(t',\vec{r}') = \frac{C}{\sqrt{4\pi}\,(\omega_0^2 + t'^2)^{3/4}}\ \exp\left[-\frac{m}{2\hbar}\frac{1}{\omega_0 + \imath t'}\vec{r}'^2 - \imath\frac{3}{2}arctan(\frac{\omega_0}{t'})\right]. \tag{40}$$

It is a new solution of the free Schrödinger equation (32).

Also, we can note that formula (33) with the relations $\frac{dt'}{dt} := s^2(t)$ and $\vec{r}' = s(t)\vec{r}$ contains the operation of equivalence (39) if we just impose the transformation of the Green's function to a plane wave or inverse.

4.6 Classic Particle

The Shapovalov group was found by Shapovalov and Sukhomlin (1974) [1] who also enumerated all cases of separation of variables in a parabolic equation and proved that all of these cases recur in quantum mechanic and in classical Hamilton Jacobi approach.

In 1980, S. Benenti and M. Francaviglia [9] applied the Shapovalov group to the Hamilton-Jacobi equation and G. Reid [10] extended it in 1986 to the space of n-dimensions.

A study of the Hamilton-Jacobi equation gives the same equivalence group that the Schrödinger equation (2), (3), (4) does. In fact, the equivalent potentials and the fields are the same as (5), (6) and (7), (8).

In particular, we refer to the results to the equivalence study in some cases. Using (31) we establish the relation between equivalent actions as in Section 4.3:

$$W'(t',\vec{r}') = W(t,\vec{r}) + \frac{m}{2}\frac{\dot{s}}{s}\vec{r}^2, \tag{41}$$

where $W(t,\vec{r})$ corresponds to the isotropic oscillator and $W'(t',\vec{r}')$ to coherent states of a free particle.

Finally:

$$\begin{aligned} W'(t',r') = &-\frac{\alpha}{\omega_0}\cos^{-1}\left(\sqrt{\frac{1}{2\alpha}\frac{m\omega_0^2}{1+\omega_0^2 t'^2}}r'\right) + \frac{m\omega_0}{2}r'\sqrt{\frac{1}{1+\omega_0^2 t'^2}\left(\frac{2\alpha}{m\omega_0^2} - \frac{r'^2}{1+\omega_0^2 t'^2}\right)} \\ &+ \frac{m\omega_0}{2}\frac{t'}{1+\omega_0^2 t'^2}r'^2 - \frac{\alpha}{\omega_0}\arctan(\omega_0 t'). \end{aligned} \tag{42}$$

Here α is energy. As in Section 4.5 we can verify that the nucleus of the Hamilton-Jacobi equation is specified by the same relation (39) as gauge invariance:

$$W'(t',\vec{r}') = W(t,\vec{r}) - \frac{m}{2}\frac{r^2}{t}, \tag{43}$$

where $W(t,\vec{r})$ is (42) without the primes. The solution of Hamilton-Jacobi that corresponds to (40), presents:

$$\begin{aligned} W'(t',r') = &-\frac{\alpha}{\omega_0}\cos^{-1}\left(\sqrt{\frac{1}{2\alpha}\frac{m\omega_0^2}{\omega_0^2+t'^2}}r'\right) + \frac{m\omega_0}{2}r'\sqrt{\frac{1}{\omega_0^2+t'^2}\left(\frac{2\alpha}{m\omega_0^2} - \frac{r'^2}{\omega_0^2+t'^2}\right)} \\ &+ \frac{m\omega_0}{2t'}\frac{r'^2}{\omega_0^2+t'^2} - \frac{\alpha}{\omega_0}\arctan(\frac{\omega_0}{t'}). \end{aligned} \tag{44}$$

Conclusions

(1) The null field is equivalent to the uniform magnetic field with arbitrary time dependence. The corresponding electric component (linear related to $\vec{r}$) is defined by (16) or (20). The equivalence relation between such a magnetic field and the null field is similar to the passing from an initial reference system to one which is in rotation defined by an orthogonal matrix from (15).

(2) The null field equivalence class has several non-orthogonal fields.

(3) The arbitrary-uniform charge distribution (time-dependent or not) is equivalent to the null charge density according to (21). Its time dependence comes from two types of equivalence: one from rotation of a reference system and the other from time scale equivalence. It is possible to have a null charge distribution simultaneously for both fields: one of type (15), (16) and the null field. Its current densities can be null also. The widest current density equivalent to the null current is linear related to $\vec{r}$ (22).

(4) The time scale equivalence ($\dot{s} \neq 0$) has an important role in the equivalence relations on all physical problem sets (see the isotropic oscillator, Section 4.3). Particularly it is known that the non-stationary coherent states of a free particle (33) are equivalent to the stationary states of a harmonic oscillator (34). Another example is when concrete relations establish the equivalence between a non-dissipative wave packet and one with dissipation [5].

(5) Only Euclidean translational equivalence ($\vec{c}(t) \neq 0$) establishes a correspondence between arbitrary time-dependent uniform electrical fields and the null field (see Section 4.2).

(6) Since the free Schrödinger equation has both stationary and non-stationary solutions, in the null field equivalence class we find four types of equivalence relations: the stationary wave functions equivalent to stationary or non-stationary states. Also non-stationary states could be equivalent to one other stationary or non-stationary wave functions. For example, the non-stationary wave function (29) is equivalent to a stationary one (plane wave (28)); the non-stationary function (30) is equivalent to another non-stationary one (Green's function of the free Schrödinger equation (28)).

(7) The widest stationary field equivalent to the null field is defined by formula (36) with conditions $f = const, \quad \vec{\varepsilon} = const$.

(8) The formulas (36) and (38) show that only a constant magnetic field does not belong to the null field equivalent class (contrary to a constant electrical field).

(9) The known isolated equivalence relation (39) puts the free Schrödinger equation in equivalence to itself. The plane wave is equivalent by means of this relation to Green's function. It is similar to the rotation in a phase space.

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