Noncommutative Geometry and Modified Gravity

N. Mebarki and F. Khelili *

Laboratoire de Physique Mathematique et Subatomique, Mentouri University, Constantine, Algeria

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Abstract: Using noncommutative deformed canonical commutation relations, a model of gravity is constructed and a schwarchild like static solutions are obtained. As a consequence, the Newtonian potential is modified and it is shown to have a form similar to the one postulated by Fishbach et al. to explain the proposed fifth force. More interesting is the form of the gravitational acceleration expression proposed in the modified Newtonian dynamics theories (MOND) which is obtained explicitly in our model without any ad hoc assumptions.

Keywords: General Relativity; Gravity Models; Modified theories of gravity; Noncommutative Geometry

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1. Introduction

Some physicists think that the Einstein gravitation theory has to be modified at small as well as large scales. In fact, there are certain cosmological and astrophysical arguments suggesting the modification of general relativity. Studying the galaxies cluster mass, Zwicky has noticed that the resulted mass deduced from the measurement of the galaxies velocity is 10 to 100 times greater than the observed matter mass. Moreover, other cosmological problems such as galaxies curvature anomalies, gravitational mirage, galaxy formation, homogeneous and uniform structure of our universe, flatness, horizon problem, etc...[1] – [6] suggest that either the Newtonian gravitation theory is not applicable at the cosmological scale or 90% of the universe mass is not observable (about 25% dark matter and 71% dark energy which do not have electromagnetic interactions) and does manifest only by gravitational effects. Determining the nature of these dark matter and energy [17] – [19] is one of the problems and tasks of the modern cosmol-
ogy and particle physics. Instead of admitting the existence of dark matter and energy some theoreticians believe that these cosmological anomalies are due essentially to the fact that the Newtonian gravitation is incomplete. To account for these observations at large scales, the gravitational force has to be much bigger than the one given by the Newton approximation. For this, some models such as the Modified Newtonian Dynamics (MOND)\[20\] – [22], Tensor-Vector-Scalar (TeVeS) [23] etc..., were proposed allowing the reproduction of spiral galaxies rotational curves without recourse to the dark matter scenario. It is important to mention that within Einstein Gravitational theory it was possible to develop relative astrophysics and cosmology in Riemann spacetime successfully describing the basic structures of observable Universe. However the difficulties of classical theoretical cosmology and up-to date state of art in observation cosmology result in new problems of fundamental physics. One way to solve these problems suggested by many authors is to generalize Gravitation theory. Moreover, Einstein’s theory of relativity has not been tested on cosmological scales, and so one might contemplate if the observed acceleration could be the first direct indication of our lack of understanding of gravity. The goal of this paper is to study and understand some qualitative aspects of the space noncommutativity through a gravity model based on noncommutative deformed canonical commutation relations. In section2, we present the formalism, construct the noncommutative action and derive the noncommutative Schwarchild like static solutions. In section3, we discuss the modified Newtonian potential and finally in section4, we give some of qualitative results and draw our conclusions.

2. Formalism

In what follows, we take $\hbar = c = 1$ and consider a noncommutative geometry, characterized by the space-time coordinates $\hat{x}^{\mu}$ and momenta $\hat{p}^{\mu}$ which are non commuting operators satisfying the following matrices valued commutation relations:

$$[\hat{x}_{\mu} , \hat{x}_{\nu}] = 0$$

$$[\hat{x}_{\mu} , \hat{p}_{\nu}] = i(\delta_{\mu\nu}I + \theta_{\mu\nu})$$

and

$$[\hat{p}_{\mu}, \hat{p}_{\nu}] = 0$$

($I$ is 4X4 identity matrix ) where $\theta_{\mu\nu}$ are matrices valued tensor under ordinary general coordinates transformations and taken to be proportional to the Dirac matrices $\gamma_{\mu\nu}$ in a curved space-time such that:

$$\theta_{\mu\nu} = \frac{1}{4}\xi(x)\gamma_{\mu\nu} = \frac{1}{4}\xi(x)[\gamma_{\mu}, \gamma_{\nu}]$$
(here $\xi(x)$ is a scalar function of the space-time variable $x_\mu$). Notice that although the above commutation relations do not fit into the case where the noncommutativity parameters are c-numbers, there is nothing fundamentally wrong with this choice.

2.1 Non commutative Gravity Model

The operators $\hat{x}_\nu$ and $\hat{p}_\nu$ have as representations:

$$\hat{x}_\nu = x_\nu, \quad \hat{p}_\nu = -i\hat{\partial}_\nu$$

(5)

where the noncommutative matrix derivative $\hat{\partial}_\nu$ has as expression

$$\hat{\partial}_\nu = \partial_\nu + i\theta_{\nu\alpha} \partial^\alpha$$

(6)

and

$$\partial^\alpha = \hat{g}^{\mu\alpha} \partial_\mu$$

(7)

($x_\nu$ and $\partial_\nu$ are the ordinary coordinates and derivative respectively). $\hat{g}^{\mu\alpha}$ is the inverse of the noncommutative symmetric metric $\hat{g}_{\mu\alpha}$ (which is not a matrix) such that

$$\hat{g}_{\nu\mu} \hat{g}^{\mu\alpha} = \delta^\alpha_\nu$$

(8)

The modified affine connection (which is not Riemannian) denoted by $\hat{\Gamma}^\nu_{\mu\lambda}$ takes the form:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2} \hat{g}^{\mu\nu} \left( \partial_{\beta} \hat{g}_{\nu\alpha} + \partial_{\alpha} \hat{g}_{\nu\beta} - \partial_{\nu} \hat{g}_{\alpha\beta} \right)$$

(9)

which can be rewritten as:

$$\hat{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \tilde{\Gamma}^\mu_{\alpha\beta}$$

(10)

where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \hat{g}^{\mu\nu} \left( \partial_{\beta} \hat{g}_{\nu\alpha} + \partial_{\alpha} \hat{g}_{\nu\beta} - \partial_{\nu} \hat{g}_{\alpha\beta} \right)$$

(11)

and

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \frac{i}{2} \hat{g}^{\mu\nu} \left( \theta_{\beta\sigma} \partial^\sigma \hat{g}_{\nu\alpha} + \theta_{\alpha\sigma} \partial^\sigma \hat{g}_{\nu\beta} - \theta_{\nu\sigma} \partial^\sigma \hat{g}_{\alpha\beta} \right)$$

(12)

Here $\tilde{\Gamma}^\mu_{\alpha\beta}$ represents a non metricity like a tensor. We remind the reader that in differential geometry, the affine connection on a differential manifold with a metric can be always decomposed into the sum of a Levi-Civita (metric) connection, a non metricity tensor and a torsion. This is the case of theories with more complicated geometrical structure like the Riemann-Cartan space with general metric-affine spaces (curvature, torsion and non-metricity) and the Weyl-Cartan space which is a connected differentiable manifold with a Lorenz metric obeying the Weyl non-metricity condition. In the Riemannian space
of general relativity the metric and the connection (which are considered respectively as a potential and strength of the gravitational field) are linked through the requirement of metric homogeneity (metricity condition). The latter assures that the length of a vector transported parallel in any direction remains invariant. Regarding the noncommutative matrices curvature and Ricci tensors $\hat{R}_{\sigma}^{\lambda\mu\nu}$ and $\hat{R}_{\mu\nu}$, they are given by:

$$\hat{R}_{\alpha\beta\lambda} = \partial_{\beta}\hat{\Gamma}_{\alpha\lambda} - \partial_{\lambda}\hat{\Gamma}_{\alpha\beta} + \hat{\Gamma}_{\sigma\beta}^{\mu}\hat{\Gamma}_{\alpha\lambda}^{\sigma} - \hat{\Gamma}_{\sigma\lambda}^{\mu}\hat{\Gamma}_{\alpha\beta}^{\sigma}$$  \hspace{1cm} (13)

and

$$\hat{R}_{\mu\nu} = \hat{R}_{\mu\lambda\nu}^{\lambda} = \partial_{\nu}\hat{\Gamma}_{\lambda\mu} - \partial_{\lambda}\hat{\Gamma}_{\mu\nu} + \hat{\Gamma}_{\mu\sigma}^{\lambda}\hat{\Gamma}_{\lambda\nu}^{\sigma} - \hat{\Gamma}_{\nu\sigma}^{\lambda}\hat{\Gamma}_{\lambda\mu}^{\sigma}$$  \hspace{1cm} (14)

We can also define a noncommutative matrix Einstein tensor $\hat{G}_{\mu\nu}$ as:

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}$$ \hspace{1cm} (15)

and where the non commutative matrix scalar curvature $\hat{R}$ is defined as:

$$\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu}$$ \hspace{1cm} (16)

Now, we define the noncommutative Hilbert-Einstein $I_{\text{NCG}}$ by

$$I_{\text{NCG}} = \frac{1}{64\pi\kappa}\int d^{4}x\sqrt{\hat{g}}\hat{g}^{\mu\nu}Tr\hat{R}_{\mu\nu}$$ \hspace{1cm} (17)

Where $\kappa$ is the gravitational constant, $\hat{g}$ stands for $|\det(\hat{g}_{\mu\nu})|$ and $Tr$ is the trace over the gamma Dirac matrices. Since $Tr\theta^{\beta\sigma} = 0$, thus the terms linear in $\theta^{\beta\sigma}$ do not contribute in the expression of $Tr\hat{R}_{\mu\nu}^{\alpha\beta\lambda}$ and therefore:

$$\hat{R}_{\mu\nu} \equiv Tr\hat{R}_{\alpha\lambda} = \hat{R}_{\alpha\lambda} + Tr\tilde{R}_{\alpha\lambda}$$ \hspace{1cm} (18)

where $\tilde{R}_{\mu\nu}$ and $\hat{R}_{\mu\nu}$ are given by:

$$\tilde{R}_{\mu\nu} = \left( i\theta_{\beta\sigma}\partial^{\sigma}\tilde{\Gamma}_{\mu\nu}^{\beta} - i\theta_{\mu\sigma}\partial^{\sigma}\tilde{\Gamma}_{\beta\nu}^{\mu} + \tilde{\Gamma}_{\sigma\beta}^{\lambda}\tilde{\Gamma}_{\mu\nu}^{\sigma} - \tilde{\Gamma}_{\sigma\nu}^{\lambda}\tilde{\Gamma}_{\mu\beta}^{\sigma} \right)$$ \hspace{1cm} (19)

and

$$\hat{R}_{\mu\nu} = 4\left( \partial_{\nu}\hat{\Gamma}_{\lambda\mu}^{\lambda} - \partial_{\lambda}\hat{\Gamma}_{\mu\nu}^{\lambda} + \hat{\Gamma}_{\mu\sigma}^{\lambda}\hat{\Gamma}_{\lambda\nu}^{\sigma} - \hat{\Gamma}_{\nu\sigma}^{\lambda}\hat{\Gamma}_{\lambda\mu}^{\sigma} \right)$$ \hspace{1cm} (20)

It is worth to mention that the principle of a least action leads to the following noncommutative Einstein field equation in the vacuum:

$$\hat{G}_{\mu\nu} \equiv \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}_{\alpha\lambda} = 0$$ \hspace{1cm} (21)

which is equivalent to:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$ \hspace{1cm} (22)
where $G_{\mu\nu}$ has the form:

$$G_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \bar{R}^\alpha_\alpha$$  \hspace{1cm} (23)

and $T_{\mu\nu}$ is an effective matter energy-momentum tensor induced by the non commutativity of the space and has as expression:

$$T_{\mu\nu} = -\frac{1}{\kappa} (Tr \bar{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} Tr \bar{R}^\alpha_\alpha)$$  \hspace{1cm} (24)

This means that the non commutativity deform the space and generate a more complex structure with a non metricity contributing to the field equations and induce an effective macroscopic matter energy-momentum tensor as an additional source of gravity. This is not surprising about the role of the deformed canonical commutation relations in quantum mechanics where it was shown in ref.[27], that there exist an intimate connection to the curved space. Moreover, a suitable choice of the position-momentum commutator can elegantly describe many features of gravity, including the IR/UV correspondence and dimensional reduction (holography)[28]. In what follows, we set:

$$\xi^2 (x) = 4\eta^2 \sigma (x)$$  \hspace{1cm} (25)

($\eta \ll 1$ is a constant parameter which characterizes the noncommutativity). Since $\hat{g}_{\nu\alpha}$ is a solution of the field equations, we assume to have the form:

$$\hat{g}_{\nu\alpha} = g_{\nu\alpha} + \eta^2 g_{\nu\alpha}^{(1)}$$  \hspace{1cm} (26)

where $g_{\nu\alpha}$ is the classical Einstein Riemannian metric and $g_{\nu\alpha}^{(1)}$ is a non commutative correction to be determined later. Furthermore, to simplify our calculation, we assume that the only non vanishing matrix valued parameters are $\theta_{01}$ (of course our qualitative results remain valid in the general case). Then, it is easy to show that at the $O(\eta^2)$, one has:

$$\frac{1}{4} Tr \theta_{01} \theta_{01} \approx \eta^2 \sigma (x) (g_{01} g_{01} - g_{00} g_{11})$$  \hspace{1cm} (27)

It is important to mention that the noncommutative Hilbert-Einstein action given in eq.(17) is invariant under general coordinate transformations.

### 2.2 Nocommutative Schwarzchild metric

In the case of Schwarzchild metric, one has in spherical coordinates system (where $0 \rightarrow t$, $1 \rightarrow r$, etc..): $g_{01} = 0$ and $g_{00} g_{11} = -1$. Therefore, eq. (27) takes the simple form:

$$\frac{1}{4} Tr \theta_{01} \theta_{01} \approx \eta^2 \sigma (x)$$  \hspace{1cm} (28)

For static solutions, tedious but straightforward calculations lead to the following non vanishing components of $Tr \bar{R}_{\alpha\lambda}$:

$$Tr \bar{R}_{00} = -\frac{1}{2} \eta^2 e^{-2\lambda} \sigma \left\{ -\lambda'' + \frac{1}{2} \lambda^2 + \frac{4}{r} \lambda' + \frac{1}{2} \lambda' \nu' + \frac{2}{r} \nu' \right\} + \frac{1}{4} \eta^2 e^{-2\lambda} \sigma' \left( \lambda' + \frac{4}{r} \right)$$  \hspace{1cm} (29)
\[ Tr\tilde{R}_{11} = -\frac{1}{2} \eta^2 e^{-(\lambda+\nu)} \sigma \left\{ \lambda'' - \frac{1}{2} \lambda'^2 + \frac{2}{r} \lambda' - \frac{1}{2} \lambda'' \nu' \right\} - \frac{1}{4} \eta^2 e^{-(\lambda+\nu)} \sigma' \lambda' \]  
\[ (30) \]

\[ Tr\tilde{R}_{22} = -\frac{1}{2} \eta^2 e^{-2\lambda} e^{-\nu} \sigma \left\{ 2 - r \lambda' - r \nu' \right\} - \frac{1}{2} \eta^2 e^{-2\lambda} e^{-\nu} \sigma' r \]  
\[ (31) \]

\[ Tr\tilde{R}_{33} = Tr\tilde{R}_{22} \sin^2 \theta \]  
\[ (32) \]

Using a perturbative expansion around the classical solutions \( \lambda_0 \) and \( \nu_0 \) at the \( O(\eta^2) \):

\[ \lambda \approx \lambda_0 + \eta^2 \lambda_1 \quad \nu \approx \nu_0 + \eta^2 \nu_1 \]  
\[ (33) \]

Direct calculations give the following noncommutative Einstein equations in the vacuum:

\[ -\frac{1}{2} \nu''_1 - \frac{1}{r} \nu'_1 - \frac{1}{4} (3 \nu'_1 - \lambda_1) \nu'_0 - \sigma \left( \frac{1}{2} \nu''_0 - \frac{1}{r} \nu'_0 \right) + \frac{1}{4} \left( \frac{4}{r} - \nu'_0 \right) \sigma' = 0 \]  
\[ (34) \]

\[ \frac{1}{2} \nu''_1 - \frac{1}{r} \lambda'_1 + \frac{1}{4} (3 \nu'_1 - \lambda'_1) \nu'_0 + \sigma \left( \frac{1}{2} \nu''_0 + \frac{1}{r} \nu'_0 \right) + \frac{1}{4} \nu'_0 \sigma' = 0 \]  
\[ (35) \]

and

\[ \lambda'_1 - \nu'_1 + \frac{2}{r - b} \lambda_1 + \frac{2}{r} \sigma + \sigma' = 0 \]  
\[ (36) \]

where \( \nu_0 \) and \( \lambda_0 \) are given by the classical Schwarchild like static solutions that is:

\[ e^{\nu_0} = e^{-\lambda_0} = 1 - \frac{b}{r} \]  
\[ (37) \]

from eqs.(34) and (35) we get:

\[ \lambda'_1 + \nu'_1 - 2 \nu'_0 \sigma - \sigma' = 0 \]  
\[ (38) \]

Now, eqs.(36) and (37) lead to:

\[ \lambda'_1 + \frac{1}{r - b} \lambda_1 + \left( \frac{2}{r} - \frac{1}{r - b} \right) \sigma = 0 \]  
\[ (39) \]

and its solution has the form:

\[ \lambda_1 = \frac{C}{r - b} - \frac{1}{r - b} \int \left( 1 - \frac{2b}{r} \right) \sigma dr \]  
\[ (40) \]

from eq.(38) we deduce that:

\[ \nu_1 = -\lambda_1 + 2 \int \nu'_0 \sigma dr + \sigma + D \]  
\[ (41) \]

Using eq.(40) we obtain the following second order differential equation:

\[ [(r - b) \nu_1]'' = (r - b) \sigma' + 3 \sigma'' + 2 \nu'_0 \sigma \]  
\[ (42) \]
(the notation ′′ means space coordinates second derivative). Eq.(34) together with the differential equation of the classical solution \( \nu_0 \):

\[
\frac{1}{2} \nu''_0 + \frac{1}{2} \nu'^2_0 + \frac{1}{r} \nu'_0 = 0
\]  

(43)

imply that:

\[
[(r-b) \nu_1]'' = \frac{2}{r} (r-b) \sigma' - 2 (r-b) \nu''_0 \sigma
\]  

(44)

Moreover, as the classical solutions give:

\[
2 (r-b) \nu''_0 = \frac{2}{r^2} (r-b) - \frac{2}{(r-b)}
\]  

(45)

eqs.(40), (44) and (45) lead to the following second order differential equation:

\[
(r-b) \sigma'' + \left(1 + \frac{2b}{r} \right) \sigma' - \frac{2b}{r^2} \sigma = 0
\]  

(46)

where the general solution has the form:

\[
\sigma (r) = A \frac{r^2}{(r-b)^2} + B \frac{r}{(r-b)^2} (r \ln r + b)
\]  

(47)

If we set the integration constant \( B = 0 \) (for the sake of simplicity), eqs.(40) and (41) read:

\[
\lambda_1 = -A + \frac{C - Ab}{r-b} - \frac{Ab^2}{(r-b)^2}
\]  

(48)

and

\[
\nu_1 = 2A + D - \frac{C - Ab}{r-b} + \frac{Ab^2}{(r-b)^2}
\]  

(49)

where \( A, C, D \) are integration constants and:

\[
b = 2\kappa M.
\]  

(50)

(\( \kappa \) and \( M \) are the Newton constant and the body mass producing the gravitational field). Thus, in the noncommutative space, the Schwarzschild metric takes the final form:

\[
ds^2 = - \left(1 - \frac{b}{r} \right) e^\nu_1 dt^2 + \left(1 - \frac{b}{r} \right)^{-1} e^{\nu_1 + \lambda_1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\]  

(51)

Notice that in the classical limit where \( \eta \to 0 \), eq.(51) is reduced to the classical Schwarzschild metric.
3. Newtonian Potential in the Curved Noncommutative Space

Recently, some experiments were undertaken to measure the gravitational constant (Australian mine experiment by S.Franks, Greenland ice bring experiment by Fischbach et al....). It was found that the measured value is smaller by about 2% than the predicted one. Then, it was concluded that there is a possibility of the existence of a fifth force. The latter is repulsive with a small action range $\lambda \sim 1cm - 100m$ and depends on the substances nature of the bodies. Fiscbach et al. have proposed to modify the Newtonian potential $V(r)$ to have the form:

$$V(r) = -\frac{\kappa M}{r} \left\{ 1 + \alpha \exp \left( -\frac{r}{\lambda} \right) \right\}$$  \hspace{1cm} (52)

where the second term gives rise to the fifth force with strength $\alpha \sim 0.01$. Moreover, to reproduce the spiral galaxies rotational curves, the MOND (Modified Newtonian Dynamics) theories, postulate that at the limit of very weak accelerations, the gravitational acceleration $g_M$ takes the form:

$$g_M \approx \sqrt{\alpha_0 g_N}$$  \hspace{1cm} (53)

where the Newtonian gravitational acceleration $g_N$ has the form:

$$g_N = \frac{\kappa M}{r^2}$$  \hspace{1cm} (54)

and $\alpha_0 \approx 1.2 \times 10^{-10} m/s$. In 1983 Milgrom has proposed other modifications to this law namely [20] – [22]:

$$g_M \approx g_N \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\alpha_0}{g_N}} \right)$$  \hspace{1cm} (55)

or

$$g_M \approx g_N \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4\alpha_0}{g_N}} \right)^{\frac{1}{2}}$$  \hspace{1cm} (56)

In our case, from the noncommutative Schwarzschild metric of eq(51) we deduce that :

$$g_{00} = - (1 + 2V(r)) = - \left( 1 - \frac{r_g}{r} \right) e^{\eta_1 \nu_1} \approx - \left( 1 - \frac{r_g}{r} \right) \left( 1 + \eta^2 \nu_1 \right)$$  \hspace{1cm} (57)

with

$$\nu_1 = 2A + D - \frac{C - Ar_g}{r - r_g} + \frac{Ar_g^2}{(r - r_g)^2}$$  \hspace{1cm} (58)

Thus, the modified Newtonian potential produced by a body of mass $M$ and radius $R$ at a distance $r$ ($r > R$) takes the form:

$$V(r) = -\frac{\kappa M}{r} + \eta^2 \frac{1}{r} \left( E + \frac{A}{r - r_g} + E \right)$$  \hspace{1cm} (59)
where the first and the second terms (proportional to $\eta^2$) are the classical Newtonian potential and the contribution of the space noncommutativity respectively. Here $E, \bar{E}$ and $\bar{A}$, are integration constants and $\rg$ is the Schwarzchild radius ($\rg \approx 3$ km for the sun and $\rg \approx 0.88$ cm for the earth etc.). Notice that there is a singularity at $r = \rg$ for very massive bodies (black holes) where $\rg > R$, whereas for small masses where $\rg < R$ the singularity is absent. The modified Newtonian potential of eq.(59), can be rewritten in the form:

$$V(r) = -\frac{\kappa M}{r} \left\{ 1 + \eta^2 \alpha \left(1 + \frac{r}{\lambda} + \frac{\gamma}{r - \rg}\right) \right\}$$

(60)

with $\alpha, \lambda$ and $\gamma$ are new constants. Now, for small values of $\alpha$ and $\gamma$ and the noncommutative parameter $\eta$, the modified Newtonian potential takes the following expression:

$$V(r) = -\frac{\kappa M}{r} \left\{ \frac{1}{2} + \frac{1}{2} \tilde{\alpha} \exp \left(-\eta^2 \frac{r}{\lambda}\right) \right\}$$

(61)

with

$$\tilde{\alpha} = \exp 2\eta^2 \alpha \left(1 + \frac{\gamma}{r - \rg}\right)$$

(62)

and

$$\tilde{\lambda} = -\frac{\lambda}{2\alpha}$$

(63)

(for $\gamma \approx 0$ we have $\tilde{\alpha} \approx 1 + \eta^2 \alpha$). Notice that the expression in eq.(61) of the potential is similar to the one postulated by Fishbach et al.[23] – [25] to explain the origin of the fifth force. In our case the value of $\tilde{\alpha}$ is a function of $r$, except for $\gamma \approx 0$. Moreover, the sign of the second term which gives rise to the fifth force depends on the parameters $\alpha, \lambda, \gamma$, and the variable $r$. The resulted force can be repulsive as well as attractive. Moreover, eq.(61) can be rewritten in the form:

$$V(r) = -\frac{\kappa(r) M}{r}$$

(64)

where the factor $\kappa(r)$ behaves like a running Newton constant.

$$\kappa(r) = \left[ 1 + \eta^2 \alpha \left(1 + \frac{r}{\lambda} + \frac{\gamma}{r - \rg}\right) \right] \kappa$$

(65)

Thus, eq.(61) can be interpreted as a Newton potential with a running coupling due to the noncommutativity of the space [26].

4. Results and Conclusions

In what follows, to get qualitative results, study the behavior of the modified Newtonian force according to the possible cases and make our analysis clear and simplified, we take for illustration (in arbitrary units) $\rg = 1, \bar{E} = -1, E = 0, \pm 1$ and $\bar{A} = 0, \pm 1$: 
4.1 $\eta^2 = 0$, or $E = 0, \overline{A} = 0$:

In this case, the noncommutative Newtonian potential is reduced to the classical one. Figure(1) displays the behavior of the Newtonian force $F(r)$ acting on a body of mass unity ($m = 1$) as a function of the distance $r$ ($F(r) = -\frac{1}{r^2}$).

\[\text{Fig. 1}\]

4.2 $E > 0, \overline{A} > 0$:

In this case and in arbitrary units, the behavior of the noncommutative Newtonian potential $V(r)$ is presented in fig.2 and the corresponding Newtonian force $F(r)$ acting on a body of mass $m = 1$ is displayed in fig.3. We distinguish eight main regions:

\(a\) For $0 \leq r \leq \tilde{r}_1 (\tilde{r}_1 \approx 0.9)$, the non commutative Newtonian potential (NCNP) is negative. It is an increasing function of $r$. It varies from $V(0) \approx -\infty$ to $V(\tilde{r}_1) \approx -1.223$. The related non commutative force (NCF) is attractive and its intensity decreases from $|F(0)| = +\infty$ to $|F(\tilde{r}_1)| \approx 0$.

\(b\) For $\tilde{r}_1 \leq r \leq r_1 (r_1 \approx 0.92)$, the NCNP is negative. It is an increasing function of $r$ until a maximum value $V(r_1) \approx -1.222$. The NCF is repulsive and its intensity increases from $|F(\tilde{r}_1)| = 0$ to $|F(r_1)| \approx 0.16$.

\(c\) For $r_1 \leq r \leq r_g$, the NCNP remains negative but a decreasing function of $r$ and gives a repulsive NCF where the intensity increases form $|F(r_1)| \approx 3.3$ to $|F(r_g)| = +\infty$.

\(d\) For $r_g \leq r \leq r_2 (r_2 \approx 1.01)$, the NCNP is positive. It is a decreasing function of $r$ and it vanishes at a distance $r_2 \approx 1.01$. The NCF is a repulsive force where the intensity increases from $|F(r_g)| = 0$ to $|F(r_2)| \approx 97.03$.

\(e\) For $r_2 \leq r \leq r_3 (r_3 \approx 1.09)$, the NCNP is negative. It is a decreasing function of $r$. It varies from $V(r_2) = 0$ to a minimal value $V(r_3) \approx -1.02$. The corresponding NCF is repulsive where the intensity decreases from $|F(r_2)| \approx 97.03$ to $|F(r_3)| \approx 0.19$.

\(f\) For $r_3 \leq r \leq r_4 (r_4 \approx 1.1)$, the NCNP is negative. It is an increasing function of $r$. It varies from $V(r_3) \approx -1.02$ to value $V(r_4) \approx -0.99$. The NCF is a repulsive force with an intensity decreasing from $|F(r_3)| \approx 0.19$ to $|F(r_4)| \approx 0$.

\(g\) For $r_4 \leq r \leq r_5 (r_5 \approx 1.25)$, the NCNP is negative. It is an increasing function of $r$. It varies from $V(r_4) \approx -0.99$ to $V(r_5) \approx -0.83$. The NCF is an attractive force and
its intensity increases from $|F(r_4)| \approx 0$ to $|F(r_5)| \approx 0.46$.

$h)$ For $r_5 \leq r \leq +\infty$, the NCNP has a behavior which looks like the classical Newtonian one. Thus, it is negative and increasing function of $r$. It gives an attractive NCF with an intensity decreasing from $|F(r_5)| \approx 0.46$ to $|F(+\infty)| = 0$.

4.3 $E > 0$, $\overline{A} < 0$:

In this case and in arbitrary units, the behavior of the NCNP is presented in fig.4 and the corresponding Newtonian force $F(r)$ acting on a body of $m = 1$ is displayed in fig.5. We distinguish four main regions:

$a)$ for $0 \leq r \leq r_6$ ($r_6 \approx 0.82$ ), the NCNP is negative and an increasing function of $r$. It varies from $V(0) = -\infty$ to $V(r_6) \approx -1.15$. This gives an attractive NCF where the intensity decreases from $|F(0)| = +\infty$ to $|F(r_6)| \approx 1.79$.

$b)$ for $r_6 \leq r \leq r_7$ ($r_7 \approx 0.99$ ), the NCNP is negative and an increasing function of $r$. It varies from $V(r_6) \approx -1.15$ to $V(r_7) \approx 0$. This will give an attractive NCF where the intensity increases from $|F(r_6)| \approx 1.79$ to $|F(r_7)| \approx 101.02$.

$c)$ for $r_7 \leq r \leq r_g$, the NCNP is positive and an increasing function of $r$. It varies from
$V(r_\gamma) \approx 0$ to $V(r_g) \approx +\infty$. The related NCF is an attractive force where the intensity increases from $|F(r_\gamma)| \approx 101.02$ to $F(r_g) = +\infty$.

d) for $r_g \leq r \leq +\infty$, the NCNP has a behavior which looks like the classical one. Thus, it is negative and increasing function of $r$. This gives an attractive NCF with an intensity decreasing from $|F(r_g)| = +\infty$ to $F(+\infty) = 0$.

Notice that, if we add the logarithmic term in the expression of $\sigma(r)$ see eq.(47) (which diverges for $r \to \infty$), we find the following expression for the noncommutative potential:
\[ 2V(r) = -\frac{b}{r} - \eta^2 \frac{1}{r} \left( C - (A - B)r - Bb \ln r - \frac{1}{r - b} \left( Ab^2 + Bb^2 \right) \right) \]
\[ + \eta^2 \left( 1 - \frac{b}{r} \right) \left\{ \ln \frac{r}{(r - b)^2} \left( Bb^2 - Bbr + Br^2 \right) + B \ln (r - b) - B \ln r \right\} \]
\[ + \eta^2 \left( 1 - \frac{b}{r} \right) \left( 2 \ln r \frac{Br}{2b^2 - 4br + 2r^2} + \frac{B}{2b^2 - 4br + 2r^2} \right) \]
\[ + \eta^2 \left( 1 - \frac{b}{r} \right) \left( A \frac{r^2}{(r - b)^2} + B \frac{r}{(r - b)^2} (r \ln r + b) + D \right) \] (66)

For large values of \( r \), it reduces to:

\[ 2V(r) \approx -\frac{r_g}{r} + B\eta^2 \left[ \ln (r - r_g) + \frac{r^2}{(r - r_g)^2} \ln r \right] \approx -\frac{r_g}{r} + 2B\eta^2 \ln r \] (67)

Thus, the modified Newtonian acceleration behaves like:

\[ g_M \approx \frac{GM}{r^2} \left( 1 + B_1\eta^2 r \right) \approx g_N \left( 1 + \eta^2 \frac{B}{g_N} \right) \] (68)

or equivalently:

\[ g_M \approx g_N \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4a_0}{g_N}} \right) \] (69)

Therefore, we obtain the same form for the gravitational acceleration as in the modified
Newtonian dynamics theories (MOND) at low energies (large scales).

As a conclusion, using noncommutative deformed canonical commutation relations,
we have constructed a non Riemannian model of gravity with non metricity like tensor
and complex geometric structure. As a consequence, the non commutative Schwarchild
like static solutions yield to a modification of the Newtonian potential. The latter is
shown to have a form similar to the one postulated by Fishbach et al. to explain the
fifth force. One can also interpret the resulted potential as the classical Newtonian
potential with a running Newton coupling constant. More interesting, is the form of
the gravitational acceleration (obtained in our noncommutative space approach without
any ad hoc assumption) which looks like the one proposed in the modified Newtonian
dynamics theories (MOND).

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