Quantized Fields Around Field Defects

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Abstract: A heuristic exercise exploring analogies between different field theories. Similarities between the crystal defects and other various fields help to create a model to quantize these fields. The charge of the electromagnetic field, and the electromagnetic waves are used as examples.

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1. Introduction

1.1 Physical Introduction

We can observe similarities between the phenomena appearing in solid states and in electrodynamics or particle physics. In the crystal lattice of the solid states various kinds of crystal defects can exist, for instance the line-defects such as the edge dislocations and the screw dislocations. [1][2] Around the crystal defects, the bulk material is intact all over, and inside these materials we can use such simple formulas which become unusable and invalid at the place of the figurable lattice defects. Drawing a parallel between these, we will consider a field in which in the "solid" domain we can use some rule, except of some "figurable" field defects. The "solid" domain works in a way, that the initial conditions and the boundary conditions determines the inside of the domain. On the other hand, the "figurable" domain can easily fit every kind of initial and boundary conditions, and the continuation in time and space is hardly predictable. Of course there are some rules existing in this domain too, but applying these rules, we get near chaos results. If the boundary of this kind of "solid" field is over determined, then it will inevitably contain "figurable" field defects. Since the "figurable" field defects have very

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unpredictable behavior, we will try to characterize them using the "solid" field encircling them.

1.2 Mathematical introduction

1.2.1 Quantized integrals

Let there be a field described by two physical properties: $C$, $S$. and let there be a soft constraint condition between them:

$$C^2 + S^2 = 1$$

(I use the expression: "soft constraint", if the constraint criterion is not too stiff, and valid only mostly and approximately.) $C^2 + S^2$ is mainly equal to 1 but not always. In some regions, the equation is not valid. We can close out the most invalid points of our region, so we will use multiply-connected regions, where the constraint-criterion is approximately valid. The $C$ and $S$ properties depend on each others. In a simply-connected region, we can describe our field with a phase of a wave. Let be $C = \cos(\phi)$ and $S = \sin(\phi)$. We know the two originally used properties, and we want to calculate the phase. We will create a vector from the following determinants:

$$v_i = \text{Det} \begin{bmatrix} C; \partial_i C \\ S; \partial_i S \end{bmatrix}$$

The determinants are created from the properties of the field and its partial derivatives with respect to space variables.

$$v_i = \text{Det} \begin{bmatrix} \cos(\phi); -\sin(\phi) \partial_i \phi \\ \sin(\phi); \cos(\phi) \partial_i \phi \end{bmatrix} = (\cos^2(\phi) + \sin^2(\phi)) \partial_i \phi = \partial_i \phi$$

We can calculate the phase difference between two points:

$$\Delta \phi = \int v_i dx^i = \int \partial_i \phi dx^i$$

In a simply-connected region, if we calculate the integral on a closed curve, we will get zero, but in the case of multiply-connected regions we will get quantized results. The result can be $2\pi * n$, where $n$ is a positive or negative or zero whole number.

1.2.2 Additions of fields with defects

We have defects, and we would like to have the field of the sum of the defects. If we are in the case of line surrounded defects, and our trajectory in the mapped field consists of
only a single loop, let’s create two times two rotational operators for the mapped space. These operators are constitute closed group [3].

\[
U = \begin{bmatrix}
C; -S \\
S; C
\end{bmatrix}
\] (5)

The field with two defects can be generated from the single fields multiplying by the operators which represent the single fields. The result represents a field with multiple defects. We can repeat this procedure several times, so we can create fields with multiple defects. Of course we can multiply the result with the operators of the defectless field. In this two dimensional case the operators are commutative. In general: If we can project the subspace of the soft constraint condition to a grid, where the cell of the grid corresponds to the single encircling, then we can use vectors in the space of the grid. The defects are similar to the well known up and down counting operators [4].

2.

2.1 Line surrounded defects.

If we are on a surface, we can walk around a point-defect always staying inside the ”solid” field. If we are in a three-dimensional space, we can walk around a line-defect on a curved line, while if we are in a time and space, we can walk around a surface or a moving curve. We will try to create properties similar to the Burgers-vectors [5] in this field. We would like any optional closed curve integrals which walk around inside the ”solid” field to be quantized, and in a defectless case the result must be zero. Let there be a field described by a vector composed of two physical properties. It creates a two dimensional mapped space.

\[
\Psi = (A, B)
\] (6)

and let there be a soft constraint condition between them:

\[
C(\Psi) = 0
\] (7)

\[
v_i = Det \begin{bmatrix}
F_A(A, B); \partial_i A \\
F_B(A, B); \partial_i B
\end{bmatrix}
\] (8)

Where \(F_A(A, B)\) and \(F_B(A, B)\) are functions. The determinants are created from the properties of the field and its partial derivatives with respect to space variables. The curl
[6][7] of this vector is the following:

\[
\text{curl}\mathbf{v} = \nabla \times \mathbf{v} = (g^n \times g^m) \text{Det} \left[ \begin{array}{c} \partial_n F_A(A, B); \partial_m A \\ \partial_n F_B(A, B); \partial_m A \\ \partial_n F_A(A, B); \partial_m B \\ \partial_n F_B(A, B); \partial_m B \end{array} \right]
\]

(9)

\[
= \delta^{nm} (g^p \otimes g^q) \text{Det} \left[ \begin{array}{c} \partial_n F_A(A, B); \partial_m A \\ \partial_n F_B(A, B); \partial_m A \\ \partial_n F_A(A, B); \partial_m B \\ \partial_n F_B(A, B); \partial_m B \end{array} \right]
\]

(9)

Where the \( g^i \) is a contravariant base-vector, and \( \delta \) is the Kronecker’s symbol. We will see, that the result of a curve line integral around a defect is quantized:

\[
\oint v_i dx_i = \oint \text{Det} \left[ \begin{array}{c} F_A(A, B); \partial_A \\ F_B(A, B); \partial_B \\ \end{array} \right] dx_i
\]

(10)

Every point of our space is connected to a point of the mapped space, so while we walk around on a curve in our real space, it is a closed curve line at the same time in the mapped space too, which can walk around multiple times. The result of the closed line integral is equal with the integral made in the mapped space. For example the curve determined by the soft constraint condition in the mapped space can contain a loop. If it is without a loop, our integral has to be zero, but otherwise, it can be quantized. The size of the quantum step or steps depend on the loop curve determined by the soft constraint condition in the mapped space. The closed curve integral is the multiple of the single loop integral of the projected mapped space, in consequence of the fact, that the projected closed curve can be not only a single times encircled line, but a multiplied times encircled closed curve, so the result of the integral can be the result of the single times encircled case multiplied by a positive or negative or zero whole number.

2.2 Defect surrounded by surfaces.

2.2.1 Example: The Electric displacement field

In a three dimensional space a closed surface can encircle a point defect, or in a four dimensional spacetime it can encircle a trajectory. We will try to create quantized integrals. Let there be

\[
\beta = (\beta_1, \beta_2, \beta_3)
\]

(11)
a three dimensional property mapped field. Let there be a soft constraint condition again:

\[
C(\beta) = 0
\]

(12)
Let’s create the following electric displacement vectors in the spacetime:

\[ \mathbf{D} = \delta^{jk}_{mn} \frac{1}{2} (g^n \otimes g^m) \text{Det} \begin{bmatrix} F_1(\beta); \partial_j \beta_1; \partial_k \beta_1 \\ F_2(\beta); \partial_j \beta_2; \partial_k \beta_2 \\ F_3(\beta); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} \] (13)

Where \( F_1(\beta), F_2(\beta) \) and \( F_3(\beta) \) are functions.

\[ D_{ij} = \text{Det}(F(\beta), \partial_i \beta, \partial_j \beta) \] (14)

In our model the electric and magnetic field transforms according to the general relativity. A relativistic four-tensor describes the electromagnetic field.

\[ \mathcal{D} = \begin{bmatrix} 0; H_x; H_y; H_z \\ -H_x; 0; D_z; -D_y \\ -H_y; -D_z; 0; D_x \\ -H_z; D_y; -D_x; 0 \end{bmatrix} \] (15)

So the closed surface integral:

\[ \oint D_{nm} da^{nm} = \oint \text{Det} \begin{bmatrix} F_1(\beta); \partial_j \beta_1; \partial_k \beta_1 \\ F_2(\beta); \partial_j \beta_2; \partial_k \beta_2 \\ F_3(\beta); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} da^{jk} \] (16)

in this model is quantized. Every point of our real space is connected to a point of the mapped space, so while we integral around on a closed surface in our real space, it is a closed surface integral in the mapped space too. The result of the closed surface integral is equal with the integral made in the mapped space. If the surface of the mapped space is closed, the integral can differ from zero. In the case of certain kind of surface, it is possible to wrap around the defect in the mapped space multiple too. The size of the quantum step depends on the surface of property space determined by the soft constraint condition. In this case in a similar way as in the two dimensional case:

\[ \mathbf{J} = \nabla \times \mathbf{D} = \delta^{ijk}_{pnm} \frac{1}{2} (g^n \otimes g^m \otimes g^p) \text{Det} \begin{bmatrix} \partial_i F_1(\beta); \partial_j \beta_1; \partial_k \beta_1 \\ \partial_i F_2(\beta); \partial_j \beta_2; \partial_k \beta_2 \\ \partial_i F_3(\beta); \partial_j \beta_3; \partial_k \beta_3 \end{bmatrix} \] (17)

We can create a pseudo-vector for the four-vector electric current density, which is the following:

\[ J^s = \frac{1}{6} \frac{1}{\sqrt{g}} \varepsilon^{sijk} j_{ijk} = \frac{1}{2} \frac{1}{\sqrt{g}} \varepsilon^{sijk} \nabla_i D_{jk} \] (18)
Where \( \nabla_j \) represents the covariant differential, and \( g \) is the determinant of the metric-tensor divided by \(-c^2\).

In details:

\[
\mathbf{J} = (\rho; J_x; J_y; J_z)
\]  

(19)

Of course this satisfies the electric charge continuity [8], a trivial way, as we can see it using the equation (17):

\[
\nabla \cdot \mathbf{J} = \nabla_s J^s = \partial_s J^s + J^s \Gamma^m_{ns} = 0
\]  

(20)

2.3 Defect encircled by volume

2.3.1 The pure theory

In a four dimensional spacetime one closed volume can encircle one event defect. We will try to make quantized the event defect. Let there be

\[
\gamma = \text{Det}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)
\]  

(21)

a four-vector in the mapped space, where we project our field. Let there be a soft constraint condition again, which decreases the four dimensional freedom in the mapped space to three: \( C(\gamma) = 0 \) Now, we create the following three-indexed tensor in the real spacetime:

\[
\mathbf{v} = \delta_{nm}^{ij}(g^n \otimes g^m \otimes g^l)\text{Det}\begin{bmatrix}
F_0; \partial_j \gamma_0; \partial_k \gamma_0; \partial_l \gamma_0 \\
F_1; \partial_j \gamma_1; \partial_k \gamma_1; \partial_l \gamma_1 \\
F_2; \partial_j \gamma_2; \partial_k \gamma_2; \partial_l \gamma_2 \\
F_3; \partial_j \gamma_3; \partial_k \gamma_3; \partial_l \gamma_3
\end{bmatrix}
\]  

(22)

The quantized event count can be calculated based on the volume-boundary integral encircling the event The event count will be the particle count difference:

\[
\Delta N = \oint v_{ijk} dV^{ijk}
\]  

(23)

2.3.2 The Magnetic induction field

Till now, in our model of the electromagnetic field, it was not possible to create a vectorpotential. We will create the electric intensity vector \( \mathbf{E} \) and the magnetic induction vector \( \mathbf{B} \) from the following vectorpotential and scalarpotential.

\[
\mathbf{A} = \frac{1}{2} g^i \text{Det}(\mathbf{G}(\alpha); \partial_i \alpha)
\]  

(24)

Where \( \mathbf{G}(\alpha) \) is function of \( \alpha \). The vectorpotencial in four-vector form is [9]:

\[
\mathbf{A} = (\Phi, A_x, A_y, A_z)
\]  

(25)
In the usual way, if let there be: The magnetic induction field as a tensor derived from the vectorpotential:

\[ \mathbf{B} = \nabla \times \mathbf{A} \]  
(26)

\[ B_{mn} = \frac{1}{2} \text{Det} \left( \partial_m G(\alpha); \partial_n \alpha \right) - \frac{1}{2} \text{Det} \left( \partial_n G(\alpha); \partial_m \alpha \right) \]  
(27)

(There is a soft constraint condition too: \( \mathbf{E} = \mathbf{D} \) and \( \mathbf{B} = \mathbf{H} \).)

\[ \mathbf{B} = \begin{bmatrix} 0; & -E_x; & -E_y; & -E_z \\ E_x; & 0; & B_z; & -B_y \\ E_y; & -B_z; & 0; & B_x \\ E_z; & B_y; & -B_x; & 0 \end{bmatrix} \]  
(28)

In this case, as a trivial result, the magnetic induction field is charge free.

\[ \nabla \times \mathbf{B} = 0 \]  
(29)

The magnetic flux:

\[ \Phi = \int B_{ij} \, da_{ij} = \oint A_i \, ds^i \]  
(30)

In parallel with the method used at the electric field due to the switched on soft constraint condition, we can move only on a loop in the mapped space, and if we suppose, that the magnetic field likes the magnetic induction vector and the electric intensity vector to be zero, then the magnetic flux is quantized. Then it is the same case as studied before, at the chapter ”Line surrounded defects.” We know, that the magnetic flux can really be quantized, for example in supraconductors, where neither electric intensity vector nor magnetic induction vector is present. It is the Josephson-effect [10]. The flux quantum seems to be independent of the type of the material.

2.3.3 Boundary integral of the Event-current-density

To have quantized integrals, our boundary circle integrals of the real spacetime are needed to be boundary circle integrals in the mapped space too. For example the following expression complies this requirement.

\[ L = \mathbf{ED} - \mathbf{BH} + \mathbf{JA} - \Phi \rho = \text{div}(\mathbf{HA} - \mathbf{D} \Phi) - \partial_t (\mathbf{DA}) \]  
(31)

(This is a soft constraint condition.) In this case:

\[ \int (\partial_n L^n) dx^4 = \oint L^n dV_n \]  
(32)

\[ L \propto \epsilon^{ijk} \partial_n \text{Det} \begin{vmatrix} E(\beta); & 0; & \partial_i \beta; & \partial_j \beta; & \partial_k \beta \\ 0; & G(\alpha); & \partial_i \alpha; & \partial_j \alpha; & \partial_k \alpha \end{vmatrix} \]  
(33)
This integral may be quantized.

\[ \Delta N = \int L^s dV_n \]  

(34)

The event-current-density pseudo-vector:

\[ L^s = -\frac{1}{2} \frac{1}{\sqrt{g}} \epsilon^{ijkl} A_i D_{jk} \]  

(35)

2.3.4 The Energy-impulse tensor

If we use (18) and (35) then the supposed Energy-Impulse tensor can be written in the following indexed form:

\[ T^p_q = \nabla_q L^p - J^p A_q + \delta^p_q (J^s A_s - \nabla_s L^s) \]  

(36)

where the upper index is pseudo. The energy-impulse tensor needn’t to be symmetrical, because it is only a subsystem, and shouldn’t be symmetrical, due to the photon count change. The force density of the electromagnetic field can be derived from the Energy-Impulse tensor:

\[ f_i = \nabla_j T^j_i = B_{is} J^s + A_s \nabla_i J^s \]  

(37)

The expression of the change of the impulse:

\[ \Delta P_k = \int T^j_k dV_j \]  

(38)

So the impulse is on an infinite volume:

\[ P_k = \int T^j_k dV_j = \int \nabla_k L^s - J^s A_k + \delta^s_k (J^j A_j - \nabla_j L^j) dV_s \]  

(39)

Without the members which are proportional to the currents or the events, we will get the photon impulse:

\[ P_{k[\text{photon}]} = \int \nabla_k L^s dV_s \]  

(40)

2.3.5 The moment of momentum

The torque causes the change of the moment of momentum. The torque density derived from (37) with an additional torque density is:

\[ \boldsymbol{M} = \boldsymbol{x} \times \boldsymbol{f} + \boldsymbol{A} \times \boldsymbol{J} = g^i \partial_j (\cdot g_p T^p_i (\boldsymbol{x} \times \boldsymbol{g}^i)) + T^j_i (\boldsymbol{g}^i \times \boldsymbol{g}^j) + \boldsymbol{A} \times \boldsymbol{J} = g^i \partial_j (\cdot g_p T^p_i (\boldsymbol{x} \times \boldsymbol{g}^i)) + (\nabla \times \boldsymbol{L}) \]  

(41)

The first member is the change of the orbital moment of momentum. The second member is the change of the spin of the photon.

A corresponding operator was found for the spin. It was derived from a quantized property with the help of a rotational operator.

As we can see, in our model both the Energy-impulse and the Spin of the photon, can be derived from the quantized integral of the Event-current-density.
3. Conclusion

The soft constraint criterions in the mapped space can cause quantized integrals, where the quantum step depends on the subspaces which depend on the soft constraint criterions. The members of the Energy-Impulse tensors derived from the event-current-density are similar to the Bosons contrasted with the members derived from the current-densities. It was possible to use the method in the simple example of the electromagnetic field. As a result of our model, the electric charge, magnetic flux and the photon-properties are quantized, and magnetic monopoles are not exists.

References


