

Group Properties of the Black Scholes Equation & its Solutions

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Abstract: Several techniques of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. are being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc.. The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton and which constitutes the cornerstone of contemporary valuation theory. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions. In this paper, we apply the well-entrenched group theoretic methods to obtain various solutions of the Black Scholes equation for the pricing of contingent claims. We also examine the infinitesimal symmetries of the said equation and explore group transformation properties. The structure of the Lie algebra of the Black Scholes equation is also studied.

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1. Introduction

The origin of the association between physics and finance, though, can be traced way back to the seminal works of Pareto [1] and Batchlier [2], the former being instrumental in establishing empirically that the distribution of wealth in several nations follows a power law with an exponent of 1.5, while the latter pioneered the modeling of speculative prices by the random walk and Brownian motion. The cardinal contribution of physicists to the

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world of finance came from Fischer Black & Myron Scholes through the option pricing formula [3] which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton [4]. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions.

The theory of stochastic processes constitutes the “golden thread” that unites the disciplines of physics and finance. Modeling of non relativistic quantum mechanics as energy conserving diffusion processes is, by now, well known [5]. Unification of the general theory of relativity and quantum mechanics to enable a consistent theory of quantum gravity has also been attempted on “stochastic spaces” [6]. Time evolution of stock prices has been, by suitable algebraic manipulations, shown to be equivalent to a diffusion process [7].

2. The Black Scholes Model

The Black Scholes valuation theory constitutes the cornerstone of modern finance. The model, as initially propounded, envisaged the formulation of a partial differential equation for the pricing of an European call option by creating a portfolio that exactly replicated the payoff of the option and the value of whose constituents was known. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max[S(T) - E, 0] = [S(T) - E]^+$ where $S(T)$ is the stock price on the exercise date and E is the exercise price.

The theory behind this valuation methodology is well disseminated and can be found in any text on financial derivatives e.g. [7]. The valuation equation of the Black Scholes model is

$$rS \frac{\partial C(S, t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + \frac{\partial C(S, t)}{\partial t} = rC(S, t), \quad (1)$$

This is the fundamental PDE for asset pricing and is referred to as the Black Scholes equation in the sequel.

3. Transformation to the Heat Equation

The transformation of the Black-Scholes equation to the heat equation has been well researched. We make the following transformations:-

$$y = \frac{2}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right) \ln S - \frac{2}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right)^2 t - \frac{2}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right) \ln S_0 + \frac{2}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right)^2 t_0, \quad (2)$$

$$\tau = - \left[\frac{2}{\sigma} \left(r - \frac{1}{2} \sigma^2 \right) \right]^2 t + \frac{2}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right)^2 t_0 \quad (3)$$

$$v = C(S, t) e^{rt_0 + \left[\frac{1}{2\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right)^2 - \frac{1}{2} \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right)^2 - r \right] t} S^{\left[\frac{r}{\sigma^2} - \frac{1}{2} - \frac{1}{\sigma^2} \left(r - \frac{1}{2} \sigma^2 \right) \right]} \quad (4)$$

On implementing these transformations the Black-Scholes equation gets transformed to the heat equation $\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2}$ as can be seen by explicit calculations.

The fundamental solution of the heat equation is given by $v = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{y^2}{4\tau}\right)$ and that of the Black Scholes eq. (1) is obtained by substituting back the transformations (2-4) and we obtain

$$C = \frac{1}{\sigma S_0 \sqrt{2\pi(t_0-t)}} \exp\left\{-\frac{(\ln S - \ln S_0)^2}{2\sigma^2(t_0-t)} - \left[\frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + r\right](t_0-t) - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)(\ln S - \ln S_0)\right\} \quad (5)$$

4. Construction of the Symmetry Group[8-12]

The Black Scholes equation (1) is a partial differential equation in two independent variables viz. the stock price S and time t and one dependent variable in the price of the derivative C . Let us consider the following invertible transformations of the three variables S , t , and C

$$\bar{t} = f(t, S, C, a), \quad \bar{S} = g(t, S, C, a) \quad \text{and} \quad \bar{C} = h(t, S, C, a) \quad (6)$$

where a is a continuous parameter.

The transformations of eq. (6) will constitute symmetry transformations if eq. (1) retains its structure in the new variables \bar{t} , \bar{S} and \bar{C} and the set of all such transformations constitutes the symmetry group G of the Black Scholes equation.

The generator of the symmetry group G is given by the vector field:-

$$X = \xi^0(t, S, C) \frac{\partial}{\partial t} + \xi^1(t, S, C) \frac{\partial}{\partial S} + \eta(t, S, C) \frac{\partial}{\partial C} \quad (7)$$

where $\xi^0(t, S, C)$, $\xi^1(t, S, C)$, $\eta(t, S, C)$ are the parameters of the infinitesimal transformations:-

$$\bar{t} \approx t + a\xi^0(t, S, C), \quad \bar{S} \approx S + a\xi^1(t, S, C) \quad \text{and} \quad \bar{C} \approx C + a\eta(t, S, C) \quad (8)$$

are obtained by solving the following equations:-

$$\frac{d\bar{t}}{da} = \xi^0(\bar{t}, \bar{S}, \bar{C}), \quad \frac{d\bar{S}}{da} = \xi^1(\bar{t}, \bar{S}, \bar{C}) \quad \text{and} \quad \frac{d\bar{C}}{da} = \eta(\bar{t}, \bar{S}, \bar{C}) \quad (9)$$

with the initial conditions $\bar{t}|_{a=0} = t$, $\bar{S}|_{a=0} = S$ and $\bar{C}|_{a=0} = C$.

The transformations represented by eq. (6) would form a symmetry group if $\bar{C} = \bar{C}(\bar{S}, \bar{t})$ satisfies the eq. $\frac{\partial \bar{C}}{\partial t} = -\frac{1}{2}\sigma^2 \bar{S}^2 \frac{\partial^2 \bar{C}}{\partial \bar{S}^2} - r\bar{S} \frac{\partial \bar{C}}{\partial \bar{S}} + r\bar{C}$ whenever $C = C(S, t)$ satisfies eq. (1).

Our objective here is to determine all possible coefficient functions ξ^0, ξ^1, η such that we are able to obtain the symmetry group of eq. (1) by the process of exponentiation. For this purpose we need to obtain the second prolongation of the vector field X of eq. (7). In terms of the various partial derivatives, this is given by:-

$$pr^{(2)}X = X + \eta^S \frac{\partial}{\partial C_S} + \eta^t \frac{\partial}{\partial C_t} + \eta^{SS} \frac{\partial}{\partial C_{SS}} + \eta^{St} \frac{\partial}{\partial C_{St}} + \eta^{tt} \frac{\partial}{\partial C_{tt}}$$

where

$$\begin{aligned} \eta^{SS} &= D_S^2 (\eta - \xi^1 C_S - \xi^0 C_t) + \xi^1 C_{SSS} + \xi^0 c_{SSS} = D_S^2 \eta - C_S D_S^2 \xi^1 - C_t D_S^2 \xi^0 - 2C_{SS} D_S \xi^1 - 2C_{St} D_S \xi^0 \\ &= C_{SS} + (2C_{SC} - \xi_{SS}^1) C_S - \xi_{SS}^0 C_t + (\eta_{CC} - 2\xi_{SC}^1) C_S^2 - 2\xi_{SC}^0 C_S C_t - \xi_{CC}^1 C_S^3 - \xi_{CC}^0 C_S^2 C_t \\ &\quad + (\eta_C - 2\xi_S^1) C_{SS} - 2\xi_S^0 C_{St} - 3\xi_C^1 C_S C_{SS} - \xi_C^0 C_t C_{SS} - 2\xi_C^0 C_S C_{St} \end{aligned}$$

and similar expressions hold for η^{St} and η^{tt} .

The differentials of $\bar{C} = \bar{C}(\bar{S}, \bar{t})$ with respect to \bar{S}, \bar{t} can be expressed in terms of those of $C = C(S, t)$ with respect to S, t through the so called prolongation formulae:-

$$\frac{\partial \bar{C}}{\partial \bar{t}} \approx \frac{\partial C}{\partial t} + a \left[D_t(\eta) - \frac{\partial C}{\partial t} D_t(\xi^0) - \frac{\partial C}{\partial S} D_t(\xi^1) \right] \quad (10)$$

$$\frac{\partial \bar{C}}{\partial \bar{S}} \approx \frac{\partial C}{\partial S} + a \left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] \quad (11)$$

$$\frac{\partial^2 \bar{C}}{\partial \bar{S}^2} \approx \frac{\partial^2 C}{\partial S^2} + a \left\{ D_S \left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] - \frac{\partial^2 C}{\partial S^2} D_S(\xi^1) - \frac{\partial^2 C}{\partial S \partial t} D_S(\xi^0) \right\} \quad (12)$$

where

$$D_t = \frac{\partial}{\partial t} + \frac{\partial C}{\partial t} \frac{\partial}{\partial C} + \frac{\partial}{\partial C_t} \frac{\partial C_t}{\partial t} \frac{\partial}{\partial C_t} + \frac{\partial C_S}{\partial t} \frac{\partial}{\partial C_S} + \dots \quad (13)$$

and

$$D_S = \frac{\partial}{\partial S} + \frac{\partial C}{\partial S} \frac{\partial}{\partial C} + \frac{\partial C_t}{\partial S} \frac{\partial}{\partial C_t} + \frac{\partial C_S}{\partial S} \frac{\partial}{\partial S}. \quad (14)$$

Using eqs.(8, 10-12), we obtain

$$\frac{\partial \bar{C}}{\partial \bar{t}} + \frac{1}{2} \sigma^2 \bar{S}^2 \frac{\partial^2 \bar{C}}{\partial \bar{S}^2} + r \bar{S} \frac{\partial C}{\partial S} - r \bar{C} \approx \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C + a \Gamma \quad (15)$$

where

$$\begin{aligned} \Gamma &= \left[D_t(\eta) - \frac{\partial C}{\partial t} D_t(\xi^0) - \frac{\partial C}{\partial S} D_t(\xi^1) \right] + \\ &\quad \frac{1}{2} \sigma^2 S^2 \left\{ D_S \left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] - \frac{\partial^2 C}{\partial S^2} D_S(\xi^1) - \frac{\partial^2 C}{\partial S \partial t} D_S(\xi^0) \right\} + \\ &\quad r S \frac{\partial C}{\partial S} \left[D_S(\eta) - \frac{\partial C}{\partial t} D_S(\xi^0) - \frac{\partial C}{\partial S} D_S(\xi^1) \right] - r \eta + \sigma^2 S \frac{\partial^2 C}{\partial S^2} \xi^1 + r \frac{\partial C}{\partial S} \xi^1 \end{aligned} \quad (16)$$

Hence, the determining equation for the problem under reference is of the form $\Gamma = 0$ with Γ being given by eq. (16).

Using eqs. (13-14, 16) and equating to zero, the coefficients' of the various monomials of the first and second order partial derivatives of C , we obtain the following equations for the symmetry group of the Black Scholes equation.

$$\xi_C^0 = 0 \quad (17)$$

$$\xi_S^0 = 0 \tag{18}$$

$$\xi_{CC}^0 = 0 \tag{19}$$

$$-\xi_C^1 + \frac{1}{2}\sigma^2 S^2 \xi_{SC}^0 = 0 \tag{20}$$

$$-S\xi_S^1 + \xi^1 + \frac{1}{2}rSC\xi_C^0 + \frac{1}{2}S\xi_t^0 + \frac{1}{4}\sigma^2 S^3 \xi_{CC}^0 = 0 \tag{21}$$

$$\xi_{CC}^1 - rS\xi_{CC}^0 = 0 \tag{22}$$

$$\frac{1}{2}\eta_{CC} - \xi_{SC}^1 + rS\xi_{SC}^0 - \frac{1}{2}rC\xi_{CC}^0 = 0 \tag{23}$$

$$-\xi_t^1 + \sigma^2 S^2 \eta_{SC} - \frac{1}{2}\sigma^2 S^2 \xi_{SS}^1 - rS\xi_S^1 + r\xi^1 - r^2 SC\xi_C^0 + rS\xi_t^0 + r^2 S^2 \xi_S^0 - rC\xi_C^1 - \sigma^2 rS^2 C\xi_{SC}^0 + \frac{1}{2}r\sigma^2 S^3 \xi_{SS}^0 = 0 \tag{24}$$

$$(\eta_t + \frac{1}{2}\sigma^2 S^2 \eta_{SS} + rS\eta_S - r\eta) - (\xi_t^0 + \frac{1}{2}\sigma^2 S^2 \xi_{SS}^0 + rS\xi_S^0 - r\xi^0) rC - r^2 C\xi^0 - r^2 C^2 \xi_C^0 + rC\eta_C = 0 \tag{25}$$

Eqs. (17-18) require that ξ^0 be a function of t only. Hence, eq. (20) reduces to $\xi_C^1 = 0$ which implies that ξ^1 does not depend on C . Further, eq. (21) becomes $-S\xi_S^1 + \xi^1 + \frac{1}{2}S\xi_t^0 = 0$ which has the solution

$$\xi^1(S, t) = \frac{1}{2}\xi_t^0(t) S \ln S + M(t) S \tag{26}$$

Then eq. (23) yields $\frac{1}{2}\eta_{CC} = 0$ which mandates that $\eta(t, S, C)$ is a linear function of C and hence can be written as

$$\eta(t, S, C) = \alpha(t, S) C + \beta(t, S) \tag{27}$$

With the above constraints for ξ^0 we can write eq. (25) as

$$-\xi_t^1 + \sigma^2 S^2 \eta_{SC} - \frac{1}{2}\sigma^2 S^2 \xi_{SS}^1 - rS\xi_S^1 + r\xi^1 + rS\xi_t^0 = 0 \tag{28}$$

Using eqs. (26-27), eq. (28) reduces to

$$In S \xi_{tt}^0 - \left(r - \frac{1}{2}\sigma^2 \right) \xi_t^0 + 2M_t(t) - 2\sigma^2 S \alpha_S(S, t) = 0 \tag{29}$$

with the solution

$$\alpha(S, t) = \frac{1}{2\sigma^2} \left[\frac{1}{2} (In S)^2 \xi_{ttt}^0 - \left(r - \frac{1}{2}\sigma^2 \right) In S \xi_t^0 + 2M_t(t) In S + N(t) \right] \tag{30}$$

Using eqs. (25), (27) we find that $\beta(S, t)$ must be a solution of the Black Scholes equation while $\alpha(S, t)$ must satisfy

$$\alpha_t + \frac{1}{2}\sigma^2 S^2 \alpha_{SS} + rS\alpha_S - r\xi_t^0 = 0 \tag{31}$$

Eqs. (30-31) yield the following:-

$$\xi_{ttt}^0 = 0 \quad \text{so that} \quad \xi^0 = Pt^2 + Qt + R \tag{32}$$

and

$$M_{tt} = 0 \quad \text{so that} \quad M = Ut + V \quad (33)$$

We finally end up with the following solutions for ξ^0, ξ^1, η :-

$$\xi^0 = Pt^2 + Qt + R \quad (34)$$

$$\xi^1 = \frac{1}{2}(2Pt + Q) \operatorname{In} S + Ut + V$$

$$\eta = \frac{1}{2\sigma^2} \left\{ \begin{array}{l} P(\operatorname{In} S)^2 - (r - \frac{1}{2}\sigma^2)(2Pt + Q)\operatorname{In} S + 2U \operatorname{In} S + [(r - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r]Pt^2 + \\ 2\sigma^2 \left[\frac{1}{2\sigma^2} (r - \frac{1}{2}\sigma^2)^2 Q - \frac{1}{2}P + rQ - \frac{1}{\sigma^2} (r - \frac{1}{2}\sigma^2)U \right] t + W \end{array} \right\}_{C+\beta(S,t)} \quad (35)$$

where P, Q, R, U, V, W are arbitrary constants. On substituting these expressions for ξ^0, ξ^1, η in eq. (7), we obtain the expressions for the six generators from the coefficients of these constants as follows:-

$$X_1 = \frac{\partial}{\partial t} \quad (36)$$

$$X_2 = S \frac{\partial}{\partial S} \quad (37)$$

$$X_3 = t \frac{\partial}{\partial t} + \frac{1}{2} S \operatorname{In} S \frac{\partial}{\partial S} - \frac{1}{2\sigma^2} \left(r - \frac{1}{2}\sigma^2 \right) (\operatorname{In} S) C \frac{\partial}{\partial C} + \frac{1}{2\sigma^2} \left(r - \frac{1}{2}\sigma^2 \right)^2 t C \frac{\partial}{\partial C} + rt C \frac{\partial}{\partial C} \quad (38)$$

$$X_4 = t S \frac{\partial}{\partial S} + \frac{1}{\sigma^2} (\operatorname{In} S) C \frac{\partial}{\partial C} - \frac{1}{\sigma^2} \left(r - \frac{1}{2}\sigma^2 \right) t C \frac{\partial}{\partial C} \quad (39)$$

$$X_5 = t^2 \frac{\partial}{\partial t} + (\operatorname{In} S) t S \frac{\partial}{\partial S} + \left\{ \frac{1}{2\sigma^2} (\operatorname{In} S)^2 - \frac{1}{\sigma^2} (r - \frac{1}{2}\sigma^2) (\operatorname{In} S) t + \left[\frac{1}{2\sigma^2} (r - \frac{1}{2}\sigma^2)^2 + r \right] t^2 - \frac{1}{2} t \right\} C \frac{\partial}{\partial C} \quad (40)$$

$$X_6 = C \frac{\partial}{\partial C} \quad X_\beta = \beta(S, t) \frac{\partial}{\partial C} \quad (41)$$

Using eq. (39), we can present eq.(38) in a simplified form as:-

$$X_3 = t \frac{\partial}{\partial t} + \frac{1}{2} (\operatorname{In} S) S \frac{\partial}{\partial S} + \frac{1}{2} \left(r - \frac{1}{2}\sigma^2 \right) t S \frac{\partial}{\partial S} + rt C \frac{\partial}{\partial C} \quad (42)$$

The one-parameter groups G_i corresponding to each of the above generators are given by the usual process of exponentiation e.g.

$$G_1 : (t + \varepsilon, S, C) \quad (43)$$

$$G_2 : (t, \varepsilon S, C), \varepsilon \neq 0 \quad (44)$$

$$G_3 : \left(\varepsilon^2 t, e^{(r - \frac{1}{2}\sigma^2)(\varepsilon^2 - \varepsilon)} t S^\varepsilon, e^{r(\varepsilon^2 - 1)} t C \right), \varepsilon \neq 0 \quad (45)$$

$$G_4 : \left(t, e^{\varepsilon \sigma^2 t} S, e^{\left[\frac{1}{2} \varepsilon^2 \sigma^2 - \varepsilon (r - \frac{1}{2}\sigma^2) \right] t} S^\varepsilon C \right) \quad (46)$$

$$G_5 : \left(\frac{t}{1 - 2\varepsilon \sigma^2 t}, S^{(1 - 2\varepsilon \sigma^2 t)^{-1}}, (1 - 2\varepsilon \sigma^2 t)^{\frac{1}{2}} e^{\left\{ \frac{\varepsilon \left[(\operatorname{In} S - (r - \frac{1}{2}\sigma^2)t \right)^2 + 2r\sigma^2 t^2 \right]}{1 - 2\varepsilon \sigma^2 t} \right\}} C \right) \quad (47)$$

$$G_6 : (t, S, \varepsilon C), \varepsilon \neq 0 \quad (48)$$

$$G_\beta : (t, S, C + \beta(S, t)) \quad (49)$$

We obtain the most general one-parameter symmetry group of the Black Scholes equation as a general linear combination $\sum_{i=1}^6 c_i X_i + X_\beta$ of the generators given by eqs. (36-42). We can also represent an arbitrary group transformation as the composition of transformations in the aforesaid one parameter subgroups.

Since each group G_i is a symmetry group, if $C = C(S, t)$ is a solution of the Black Scholes equation, then so are the functions:-

$$C^{(1)}(S, t) = C(t - \varepsilon, S) \quad (50)$$

$$C^{(2)}(S, t) = C(t, \varepsilon^{-1} S), \varepsilon \neq 0 \quad (51)$$

$$C^{(3)}(S, t) = e^{(1-\varepsilon^{-2})rt} C \left[e^{(\varepsilon^{-2}-\varepsilon^{-1})(r-\frac{1}{2}\sigma^2)t} S^{\varepsilon^{-1}}, \varepsilon^{-2} t \right] \quad (52)$$

$$C^{(4)}(S, t) = e^{-[\frac{1}{2}\varepsilon^2\sigma^2 + \varepsilon(r-\frac{1}{2}\sigma^2)]t} S^\varepsilon C \left[S e^{-\varepsilon\sigma^2 t}, t \right] \quad (53)$$

$$C^{(5)}(S, t) = [1 + 2\varepsilon\sigma^2 t]^{-\frac{1}{2}} e^{\varepsilon \left\{ \frac{[\log S - (r-\frac{1}{2}\sigma^2)t]^2 + 2r\sigma^2 t^2}{1+2\varepsilon\sigma^2 t} \right\}} C \left(\frac{t}{1+2\varepsilon\sigma^2 t}, S^{\frac{t}{1+2\varepsilon\sigma^2 t}} \right) \quad (54)$$

$$C^{(6)}(S, t) = \varepsilon C(S, t), \varepsilon \neq 0 \quad (55)$$

$$C^{(\beta)}(S, t) = C(S, t) + \varepsilon \beta(S, t) \quad (56)$$

Here ε is any real number and $\beta(S, t)$ any other solution to the Black Scholes equation. It is seen from the symmetry group G_6 and G_β that the solutions of the Black Scholes equation are linear and we can add two solutions and multiply them with a constant. The group G_1 shows time invariance of the solutions. The symmetry group G_2 reflects the scaling symmetry with respect to S .

5. Structure of the Lie Algebra $\Lambda = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$ [12-15]

We now explore the structure of the finite dimensional Lie algebra generated by $\Lambda = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$. The commutator table of Λ is given by:-

TABLE 1

| | X_1 | X_2 | X_3 | X_4 | X_5 | X_6 |
|-------|----------------------------------|--------------------------|-------------------------------|-------------------------------|--------------------------------|-------|
| X_1 | 0 | 0 | $X_1 + \frac{K}{2}X_2 + rX_6$ | $X_2 - \frac{K}{\sigma^2}X_6$ | $2X_3 - KX_4 - \frac{1}{2}X_6$ | 0 |
| X_2 | 0 | 0 | $\frac{1}{2}X_2$ | $\frac{1}{\sigma^2}X_6$ | X_4 | 0 |
| X_3 | $-(X_1 + \frac{K}{2}X_2 + rX_6)$ | $-\frac{1}{2}X_2$ | 0 | $\frac{1}{2}X_4$ | X_5 | 0 |
| X_4 | $-X_2 + \frac{K}{\sigma^2}X_6$ | $-\frac{1}{\sigma^2}X_6$ | $-\frac{1}{2}X_4$ | 0 | 0 | 0 |
| X_5 | $-2X_3 + KX_4 + \frac{1}{2}X_6$ | $-X_4$ | $-X_5$ | 0 | 0 | 0 |
| X_6 | 0 | 0 | 0 | 0 | 0 | 0 |

where $K = r - \frac{1}{2}\sigma^2$. Further,

$$[X_1, X_\beta] = X_{\beta t}, \quad [X_2, X_\beta] = X_{S\beta S}, \quad [X_3, X_\beta] = X_{t\beta t + \frac{1}{2}S(\ln S)\beta S + \frac{1}{2\sigma^2}(r - \frac{1}{2}\sigma^2)\beta \ln S - \frac{1}{2\sigma^2}(r - \frac{1}{2}\sigma^2)^2\beta t - r\beta t}$$

$$[X_4, X_\beta] = X_{tS\beta S - \frac{1}{2}\beta \ln S + \frac{1}{\sigma^2}(r - \frac{1}{2}\sigma^2)\beta t}, \quad [X_5, X_\beta] = X_{t^2\beta t + tS(\ln S)\beta S - \frac{1}{2\sigma^2}\beta(\ln S)^2 + \frac{1}{\sigma^2}(r - \frac{1}{2}\sigma^2)\beta t \ln S - \left[\frac{1}{2\sigma^2}(r - \frac{1}{2}\sigma^2)^2 + r\right]\beta t^2 + \frac{1}{2}\beta t}$$

$$[X_6, X_\beta] = X_{-\beta}, \quad [X_\beta, X_\beta] = 0, \quad \text{where } X_\gamma = \gamma \frac{\partial}{\partial C}$$

From table 1, the following readily follow:-

(1) the centralizers of the various elements X_i are:-

$$\chi(X_1) = \langle X_1, X_2, X_6 \rangle, \quad \chi(X_2) = \langle X_1, X_2, X_6 \rangle, \quad \chi(X_3) = \langle X_3, X_6 \rangle, \quad \chi(X_4) = \langle X_4, X_5, X_6 \rangle, \\ \chi(X_5) = \langle X_4, X_5, X_6 \rangle, \quad \chi(X_6) = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle.$$

(2) the centre of Λ is $\chi(\Lambda) = \bigcap_{i=1}^6 \chi(X_i) = \langle X_6 \rangle$.

(3) $[X_1, \Lambda] = \langle X_1, X_2, X_3, X_4, X_6 \rangle, [X_2, \Lambda] = \langle X_2, X_4, X_6 \rangle, [X_3, \Lambda] = \langle X_1, X_2, X_4, X_5, X_6 \rangle, \\ [X_4, \Lambda] = \langle X_2, X_4, X_6 \rangle, [X_5, \Lambda] = \langle X_3, X_4, X_5, X_6 \rangle, [X_6, \Lambda] = 0.$

(4) $U = \langle X_2, X_4, X_6 \rangle$ is a two sided ideal of Λ since $\langle [U, \Lambda] \rangle = \langle [\Lambda, U] \rangle = U$. It is also an invariant subalgebra of Λ .

(5) the Lie algebra Λ is not solvable, since $[\Lambda, \Lambda] = \Lambda$ and hence the derived series of Λ is stationary. However, for the subalgebra U , we have, $[U, U] = \langle X_6 \rangle, [U^{(2)}, U^{(2)}] =$

$[X_6, X_6] = 0$, so that U is solvable. Being the maximal ideal, it is, therefore, the radical of Λ . Also, $V = \langle X_1, X_3, X_5 \rangle$ is a semisimple and simple subalgebra.

- (6) in view of (e), the Lie algebra Λ admits the Levi decomposition $\Lambda = U \oplus V$
- (7) the adjoint representations of the various elements can be trivially written from the commutator table and, in the ordering $\langle X_1, X_3, X_5, X_2, X_4, X_6 \rangle$ take the form:-

$$\begin{aligned}
 X_1 &= \left(a_{21} = -1, a_{24} = -\frac{K}{2}, a_{26} = -r, a_{32} = -2, a_{35} = K, a_{36} = \frac{1}{2}, a_{54} = -1, a_{56} = \frac{K}{\sigma^2} \right); \\
 X_2 &= \left(a_{24} = -\frac{1}{2}, a_{35} = -1, a_{56} = -\frac{1}{\sigma^2} \right); \\
 X_3 &= \left(a_{11} = 1, a_{14} = \frac{K}{2}, a_{16} = r, a_{33} = -1, a_{44} = \frac{1}{2}, a_{55} = -\frac{1}{2} \right); \\
 X_4 &= \left(a_{14} = 1, a_{16} = -\frac{K}{\sigma^2}, a_{25} = \frac{1}{2}, a_{46} = \frac{1}{\sigma^2} \right); \\
 X_5 &= \left(a_{12} = 2, a_{15} = -K, a_{16} = -\frac{1}{2}, a_{23} = 1, a_{45} = 1 \right); \\
 X_6 &= O_{6 \times 6}
 \end{aligned}
 \tag{57}$$

The non-specified elements are 0's in the above matrices.

- (8) the action, defined by $\varphi_{ij} = (e^{\varepsilon \text{adj } X_i}) X_j$, of the adjoints of the various generators X_i on the algebra Λ is summarized below (These constitute the inner automorphism group of the Lie algebra Λ):-

TABLE 2

| $j \rightarrow$ $i \downarrow$ | X_1 | X_2 | X_3 | X_4 | X_5 | X_6 |
|-----------------------------------|---|--|--------------------------|--|---|---|
| X_1 | $X_1 - \varepsilon X_3 + \varepsilon^2 X_5$ | $X_2 - \frac{\varepsilon K}{2} X_3 - \varepsilon X_4$ | $X_3 - 2\varepsilon X_5$ | $X_4 + \varepsilon K X_5$ | X_5 | $-\varepsilon r X_3 + \frac{\varepsilon K}{\sigma^2} X_4 + \left[\frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} \left(2r + \frac{K^2}{\sigma^2} \right) \right] X_5 + X_6$ |
| X_2 | X_1 | $X_2 - \frac{\varepsilon}{2} X_3$ | X_3 | $X_4 - \varepsilon X_5$ | X_5 | $-\frac{\varepsilon}{\sigma^2} X_4 + \frac{\varepsilon^2}{2\sigma^2} X_5 + X_6$ |
| X_3 | $2X_2$ | $e^{\frac{\varepsilon}{2}} \left(e^{\frac{\varepsilon}{2}} - 1 \right) K X_1 + e^{\frac{\varepsilon}{2}} X_2$ | X_3 | $e^{-\frac{\varepsilon}{2}} X_4$ | $e^{-\varepsilon} X_5$ | X_6 |
| X_4 | X_1 | $\varepsilon X_1 + X_2$ | X_3 | $\frac{\varepsilon}{2} X_3 + X_4$ | X_5 | $\left(\frac{\varepsilon^2}{2\sigma^2} - \frac{\varepsilon K}{\sigma^2} \right) X_1 + \frac{\varepsilon}{\sigma^2} X_2 + X_6$ |
| X_5 | X_1 | X_2 | $2\varepsilon X_1 + X_3$ | $-\varepsilon K X_1 + \varepsilon X_2 + X_4$ | $\varepsilon^2 X_1 + \varepsilon X_3 + X_5$ | $-\frac{1}{2} \varepsilon X_1 + X_6$ |
| X_6 | X_1 | X_2 | X_3 | X_4 | X_5 | X_6 |

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