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Existence of Yang–Mills Theory with Vacuum Vector and Mass Gap

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Abstract: This paper shows that quantum theory describing particles in finite expanding space–time exhibits natural ultra–violet and infra–red cutoffs as well as posesses a mass gap and a vacuum vector. Having ultra–violet and infra–red cutoffs, all renormalization issues disappear. This shows that Yang–Mills theory exists for any simple compact gauge group and has a mass gap and a vacuum vector.

Keywords: Finite Expanding Space–time; Yang–Mills fields; Renormalization; Regularization; Mass Gap; Vacuum Vector
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1. Introduction

All the particle interactions – beside gravitational interactions – are today decribable by Standard model. Main theoretical tool for Standard model is Yang–Mills action. This action is able to describe all the possible particle interactions experimentally observed so far – including strong nuclear interactions. The bad part is that there are some issues regarding quantum theory in general.

First issue is that any higher order correction in the perturbation calculations is singular – there are infra–red and ultra–violet catastrophes in calculations. Many mathematical methods have been devised over last half a century in order to cure singularities, but problem of renormalization still remains.

Second issue is that in order to have confined particles, spectrum should exhibit a mass gap. Since the only sound theoretical action is the Yang–Mills action, we are naturally trying to formulate the Yang–Mills action in such a way to produce a mass gap[1].

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2. Finite Space–time

In order to cure these two problems – renormalization and existence of a mass gap – we can hit the problem directly. One may try to find new renormalization schemes, but one may also try to render renormalization obsolete. If there are natural ultra–violet and infra–red cutoffs, then there would be no renormalization at all for there would be no singularities. Luckily, when one manages to solve the renormalization problem, the mass gap shows right away. The unique idea behind both renormalization and a mass gap is simple – physical space–time is bounded.

We should notice one thing in general – space–time is finite and expanding. This simple fact has enormous impact on quantum theory in general. This paper will show that the fact of space–time expanding ensures natural cutoffs as well as the existence of mass gap as soon as basic axioms of quantum theory are fullfilled.

Let us begin by defining three axioms,

Axiom $\mathcal{A}$: Space–time is finite.

Axiom $\mathcal{B}$: Space–time is expanding.

Axiom $\mathcal{C}$: Space–time is connected.

Axiom $\mathcal{B}$ is obvious from the Hubble expansion. If the mass density of Universe is finite, then Axiom $\mathcal{A}$ follows from observing that distances between galaxies are finite. Otherwise, there would be no observable Hubble expansion. And Axiom $\mathcal{C}$ is to be interpreted topologically. For instance, suppose that there exist two regions in space–time and suppose that there exist no continuous path beginning at any point from one region and ending at any point in other region. Suppose that we, observers, live in one of these regions. Since two regions are disconnected, there is no information that we may possibly have about the other region being disconnected from us. This region would not exist for us. It would be unphysical to consider disconnected region for there is nothing to consider. There is no physical information whatsoever coming from disconnected region to us. So we do not have to bother about disconnected parts of the Universe nor even consider them as parts of the same Universe. This is the meaning of Axiom $\mathcal{C}$.

When attempting to interpret wave functions in finite expanding space–time we are to make sense of the normalizing integral of probability density. The main difference that shows when embedding wave function in finite expanding space–time in contrast to the infinite space–time is that boundaries of space–time are finite and not fixed. Having finite space–time expand, the normalization integral becomes under–defined. The reason for this is that probability density $\bar{\phi}\phi$ does not depend on time for conserved systems. In order to have probability conserved – and therefore energy, charge, ...– normalization factor should be time–dependent. Hence in finite expanding space–time normalization becomes

$$N(t) \int_{U(t)} \bar{\phi}\phi d^3x = 1 \tag{1}$$

with $U(t)$ being finite time–dependent 3–volume of Universe and with $N(t)$ being time–dependent normalization factor that serves to interpret $N(t)\bar{\phi}\phi$ as a probability density function in finite expanding space–time. We notice that we do not define probability density $N(t)\bar{\phi}\phi$ outside the 3–volume of physical space–time $U(t)$ following statement of Axiom $\mathcal{C}$.

Since space–time being finite, the lattice calculations give infra–red impulse cutoff[2]. I refer to sentence "In Fourier space, the space–time lattice leads to finite domains for the values of energies and momenta (the Brillouin zones), so that all ultra–violet divergences disappear. If we also wish to ensure the absence of infra–red divergences, we must replace the infinite volume of space and time by a finite box. This is often required if complications arise due to divergent contributions of soft virtual particles, typically photons". We notice that for infinitely dense lattice ultra–violet divergences reappear, but if finite box of integration stays finite, infra–red divergencies stay absent. Since Fourier integral being defined pairwise for pairs (x^μ, p^μ) and for each μ, and since all of x^μ being bounded in finite space–time, we conclude that neither 4–impulse nor any of its components ever vanish.

We conclude that in finite space–time 4–impulse never vanishes for any particle. Therefore there exists energy $\Delta > 0$ that depends on the size of space–time, such that any particle's energy E obeys $E^2 \geq \Delta^2$. We notice that finite space–time produces a mass gap Δ.

3. Gravitational Limitations for Impulse and Position for Massive Particles

We next argue on physical meaning of probability density inside the gravitational horizon around point–like particle. The point is that although the true gravitational singularity lies at the origin where the point–like particle we choose to observe is situated, the region inside the horizon is unaccessible to any measurement. We know that we cannot measure anything outside the space–time, so we need not bother to assign any probability to finding a particle outside the physical boundaries of space–time. We know for sure that particle is inside space–time. Likewise, we know that we cannot measure anything inside the horizon, so we need not bother to assign any probability to finding a particle outside the physical boundaries of space–time – a horizon in this case. There is no probability of finding a particle inside horizon for there is no particle nor any finding for observer outside if particle happens to be inside the horizon. There is tunneling, of course, but it will show automatically because of the Heisenberg uncertainty principle and because of raging fluctuations in impulse on the horizon.

The crucial argument is as follows – moving particle creates horizon that grows with particle's speed. In finite space–time this would produce a catastrophe. For if horizon grows large enough, it could consume entire space–time. There would be no–one left to measure anything. This would happen even if particle is not idealized as a point–like object. This effect occurs for particle velocities a bit below velocity of light, as calculated in this paper. This issue of a horizon expanding with particle's velocity presents no problem in infinite space–time since particle should be moving at the speed of light in order to consume entire infinite space–time. In finite space–time horizon growing with velocity presents problem since it may consume entire space–time if particle producing it moves with velocity below the velocity of light.

We are ready to calculate the ultra–violet cutoffs. To do so, let a not necessarily point–like free particle of any rest mass m move uniformly with velocity u. It produces a gravitational field around itself. We can employ Schwarzschild metric for still particle in cartesian representation

$$\delta s^2 = \frac{\left(1-\frac{a}{r}\right)^2}{\left(1+\frac{a}{r}\right)^2}\delta t^2 - \left(1+\frac{a}{r}\right)^4 \sum_{i=1}^{3} \delta x^i \delta x^i \tag{2}$$

with $a = \frac{Gm}{2c^2}$ and give it a boost in, say, direction x, by applying Lorenz transformation upon metric (2). The result for g'_{00} reads

$$g'_{00} = g_{00}\left(\frac{\partial t}{\partial t'}\right)^2 + g_{11}\left(\frac{\partial x}{\partial t'}\right)^2 = \frac{\left(1-\frac{a}{r'}\right)^2}{\left(1-u^2\right)\left(1+\frac{a}{r'}\right)^2} - \frac{u^2\left(1+\frac{a}{r'}\right)^4}{\left(1-u^2\right)} \tag{3}$$

This metric tensor component vanishes for

$$r' = \sqrt{\frac{(x+ut)^2}{1-u^2} + y^2 + z^2} = a\frac{1+u^2}{1-u^2} \tag{4}$$

If $r' = 2R$, with R being the radius of physical space, then there is no space–time at all whatever the particle's location, and for a not necessarily point–like particle of rest-mass m we find

$$u = c\frac{1-\frac{Gm}{4Rc^2}}{\left(1+\frac{Gm}{4Rc^2}\right)^3} \tag{5}$$

This defines ultra–violet cutoff for any free particle.

For free electron the upper limit for velocity u is

$$u = c\left(1 - 2\cdot 10^{-86}\right)(m/s)$$

in coordinates that span metric with temporal metric tensor component (3).

We conclude that in finite expanding space–time there exist both infra–red and ultra–violet cutoffs for 4-impulse for any elementary particle.

The conclusion is that particles cannot have infinite impulse – there would be no–one to calculate it. Having this ultra–violet cutoff along with the infra–red one, we conclude that there are no singularities left, that energy is bounded both below and above, thus producing a mass gap and leading us to proving the existence of a vacuum vector.

4. Yang–Mills Existence

Our attention should be on Yang–Mills quantum theory now. The reason for this is simple. Namely, only interactions described via compact group representations produce hamiltonians bounded from below[3] in infinite static space–time. The exact sentence I refer to in [3] is "Nonabelian gauge theory must be based on a compact group, because otherwise some of the terms in $\mathcal{L}_{kin}$ would have the wrong sign, leading to a hamiltonian that is unbounded below" with $\mathcal{L}_{kin} = -\frac{1}{4}F^{c\mu\nu}F^{c}_{\mu\nu} = -\frac{1}{2}Tr\,(F^{\mu\nu}F_{\mu\nu})$. Hence the importance of Yang–Mills action in infinite static space–time.

Quantum Yang–Mills theory therefore has energy bounded below for any compact gauge group G as soon as free particle states energies are bounded below. Free particle energies are indeed bounded in finite space–time and therefore, in finite space–time, Yang–Mills hamiltonian H has spectrum bounded below. Let us denote this energy minimum of Yang–Mills hamiltonian H by E_m and let us denote the lowest energy state by ϕ_m. By shifting original hamiltonian H by $-E_m$, the new hamiltonian $H' = H - E_m$ has its minimum at $E = 0$ and the first state ϕ_m is the vacuum vector. We now notice that in finite space–time any hamiltonian's spectrum is not supported in region $(-\Delta, \Delta)$, with $\Delta > 0$ depending upon the size of the Universe – in other words, depending upon the size of the finite space–time. Therefore for quantum Yang–Mills theory over compact gauge groups G there is always a vacuum vector ϕ_m and mass gap Δ.

In finite expanding space–time the issue of renormalizability disappears completely, since in finite expanding space–time there are infra–red and ultra–violet cutoffs. Since there exist only ultra–violet and infra–red catastrophes for Yang–Mills action in infinite static space–time, there are no "green" catastrophes, we no longer need to pay attention to renormalizability as soon as we perform calculations in finite expanding space–time. We conclude that Yang–Mills problem is no problem at all in finite expanding space–time. There obviously always exists a solution to any Yang–Mills type of action.

Conclusions

We conclude that for any simple compact gauge group G in finite expanding space–time Yang–Mills theory with mass gap and vacuum vector exists.

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