On Conformal d’Alembert-Like Equations

E. Capelas de Oliveira*a and R. da Rochab†

*aDepartment of Physics
University of Bologna
Via Irnerio, 46 I-40126 Bologna, Italy
bCentro de Matemática, Computação e Cognição
Universidade Federal do ABC
09210-170, Santo André, SP, Brazil

Received 27 May 2008, Accepted 20 June 2008, Published 30 June 2008

Abstract: Using conformal coordinates associated with projective conformal relativity we obtain a conformal Klein-Gordon partial differential equation. As a particular case we present and discuss a conformal ‘radial’ d’Alembert-like equation. As a by-product we show that this ‘radial’ equation can be identified with a one-dimensional Schrödinger-like equation in which the potential is exactly the second Pöschl-Teller potential.

Keywords: Projective Relativity; Hyperspherical Universes; Conformal Klein-Gordon Differential Equation; d’Alembert-like Equation; Second Pöschl-Teller Potential

PACS (2006): 03.30.+p; 03.50.z; 03.50.De; 03.50.Kk; 04.20.q; 04.30.w; 98.80.k

1. Introduction

After one of the most important of Einstein’s papers [1] concerning Special Relativity was published, several alternative theories were proposed. Among them, some different interpretations and particular generalizations have been presented. In this paper we are interested in one such theory, namely the theory of hyperspherical universes, developed by Arcidiacono[2] several years ago and, more specifically, in the so-called conformal case.

When we write Maxwell equations in six dimensions, with six projective coordinates (we have, in these coordinates, a Pythagorean metric) a natural problem arises, namely, to provide a physical version of the formalism, i.e. to ascribe a physical meaning to the coordinates. For this theory, there are two possible different physical interpretations: a

* Permanent Address: Department of Applied Mathematics, Imecc, University of Campinas, 13083-970, Campinas, SP, Brazil. E-mail: capelas@ime.unicamp.br
† roldao.rocha@ufabc.edu.br
bitemporal interpretation and a biprojective interpretation. In the first case (bitempo-
ral) we introduce a new universal constant $c'$ and the coordinate $x_5 = ic't'$ where $t'$ is
interpreted as a second time; we thus obtain in cosmic scale the so-called multitemporal
relativity, proposed by Kalitzen[3]. The set of Maxwell equations obtained in this theory
generalizes the equations of the unitary theory of electromagnetism and gravitation, as
proposed by Corben[4].

On the other hand (our second, biprojective case) we can interpret the extra coor-
dinate, $x_5$, as a second projective coordinate. We then obtain the so-called conformal
projective relativity, proposed by Arcidiacono[5, 6], which extends in cosmic scale the
theory proposed by Ingraham[7], but with a different physical interpretation. In this
theory we have another universal constant, $r_0$, which can be taken as $r/r_0 = N$, where
$r$ is the radius of the hypersphere and $N$ is the cosmological number appearing in the
Eddington-Dirac theory[8].

Here we consider only the second alternative, i.e., the biprojective interpretation.
With this aim we introduce a projective space $P_5$ tangent to the hypersphere $S^4$. We
then introduce six projective coordinates $\pi_a$, with $a = 0, 1, \ldots, 5$ and normalized as

$$\pi^2 + \pi_0^2 - \pi_5^2 = r^2,$$

where $\pi^2 = x_i x^i$, $i = 1, \ldots, 4$ and $r$ is the radius of the hypersphere. These coordinates
allow us to construct the conformal projective relativity, using a six-dimensional tensor
formalism.

This paper is organized as follows: in Section 1 we present a review of the so-
called theory of hyperspherical universes, proposed by Arcidiacono, considering only
the six-dimensional case, in which conformal projective relativity appears as a partic-
ular case. The choice of convenient coordinates and the link between the derivatives in
these two formulations (a geometric version, six-dimensional, and a physical version, the
five-dimensional conformal version) are also presented. After this review, we discuss in
Section 2 a Klein-Gordon partial differential equation written in conformal coordinates.
In Section 3, we show that a conformal ‘radial’ d’Alembert-like equation, can be led into
a Schrödinger differential equation in which the associated potential is exactly a second
Pöschl-Teller potential.

2. Hyperspherical Universes

In 1952 Fantappiè proposed the so-called theory of the hyperspherical universes\(^1\). This
theory is based on group theory and on the hypothesis that the universe is endowed with
unique physical laws, valid for all observers. As particular cases, Arcidiacono[2] studied
a limitation of that theory, i.e., he considered hyperspherical universes with $3, 4, \ldots, n$
dimensions where motions are given by $n(n+1)/2$-parameter rotation group in spaces with
$4, 5, \ldots, (n + 1)$ dimensions, respectively. Those models of hyperspherical $S^3, S^4, \ldots, S^n$

\(^1\) See the appendix.
universes, can be interpreted as successive physical improvements, because any one of them (after $S^4$) contains its precedents and is contained in its successors.

After 1955 Arcidiacono studied the case $n = 4$, special projective relativity, based on the de Sitter hyperspherical universe with a group (the so-called Fantappiè-de Sitter group) of ten parameters. This theory is an improvement (in a unique way) of Einstein’s special relativity theory and provides a new group-theoretical version of the big-bang cosmology. As a by-product of special projective relativity one can recover several results, for example, Kinematic Relativity, proposed by Milne[9]; Stationary Cosmology, proposed by Bondi-Gold[10] and Plasma Cosmology, proposed by Alfvèn[11].

Moreover, if we consider a universe $S^4$ as globally hyperspherical but endowed with a locally variable curvature, we obtain the so-called general projective relativity which was proposed and studied by Arcidiacono after 1964. This theory allows us to recover several results as particular cases, for example, the unitary theories proposed by Weyl[12], Straneo[13], Kaluza-Klein[14, 15], Veblen[17] and Jordan-Thiry[18] and some generalizations of the gravitational field, as those proposed by Brans-Dicke[19], Rosen[20] and Sciama[21].

In this paper we are interested only in the case $n = 5$, i.e., conformal projective relativity based on the hyperspherical universe $S^4$ and its associated rotation group, with fifteen parameters, which contains the accelerated motions. We remember that, whereas for $n = 4$ we have a unitary theory (a magnetohydrodynamic field), for $n = 5$ we have another unitary theory, i.e., the magnetohydrodynamics and Newton’s gravitation. We also present the relations between Cartesian, projective and conformal coordinates and the link involving derivatives in the six- and five-dimensional formulations.

### 2.1 Conformal Coordinates

We use the notation $x_i$, ($i = 1, 2, 3, 4$) and $x_5$ for conformal coordinates and $\bar{x}_a$, ($a = 0, 1, 2, 3, 4, 5$) for projective coordinates. The relations between these coordinates are

$$x_i = r_0 \frac{\bar{x}_i}{\bar{x}_0 + \bar{x}_5} \quad \text{and} \quad x_5 = r_0 \frac{r}{\bar{x}_0 + \bar{x}_5},$$

which satisfy the condition

$$x_5^2 - x^2 = r_0^2 \frac{\bar{x}_0 - \bar{x}_5}{\bar{x}_0 + \bar{x}_5},$$

where $x^2 = x_i x^i$, and $r_0$ and $r$ are constants. After these considerations, the transformations of the so-called conformal projective group are obtained using the quadratic form in projective coordinates

$$\bar{x}^2 + \bar{x}_0^2 - \bar{x}_5^2 = r^2,$$

decomposing the elements of the six-dimensional rotation group (with fifteen parameters) in fifteen simple rotations $(\bar{x}_a, \bar{x}_b)[22]$. 
2.2 Connection Between Derivatives

Our main objective is to write down a differential equation, more precisely a Klein-Gordon-like equation, associated with conformal coordinates. We first obtain the relation between the six projective derivatives $\frac{\partial}{\partial x^a} \equiv \partial / \partial x_a$ and the five-dimensional derivatives $\partial_i = \partial / \partial x_i$ and $\partial_5 = \partial / \partial x_5$. We can then write the differential equations in the projective formalism, with six dimensions, in physical, i.e., conformal coordinates, with five dimensions.\(^2\)

Taking $\phi = \phi(x_i, x_5)$, a scalar field, and using the chain rule we can write

$$
\partial_i \phi = \left[ (\partial_i x_k) \partial_k + (\partial_i x_5) \partial_5 + (\partial_i x_0) \partial_0 \right] \phi
$$

$$
\partial_5 \phi = \left[ (\partial_5 x_k) \partial_k + (\partial_5 x_5) \partial_5 + (\partial_5 x_0) \partial_0 \right] \phi
$$

with $\phi = \phi(x_1, x_5, x_0)$ and $i, k = 1, 2, 3, 4$.

From now on we take $r = 1 = r_0$. We consider $\phi(\vec{x}_a)$ a homogeneous function with degree $N$ in all six projective coordinates $\vec{x}_a$. Using Euler’s theorem associated with homogeneous function, we get

$$
(\partial_i \vec{x}_i + \partial_5 \vec{x}_5 + \partial_0 \vec{x}_0) \phi = N \phi
$$

where $\vec{x}_a = \partial / \partial x_a$ and $N$ is the degree of homogeneity of the function.

Then, the link between the derivatives can be written as follows\(^2\)

$$
\overline{\partial}_0 \phi = N \frac{A^+}{x_5} \phi + B^- \partial_5 \phi - x_5 x_i \partial_i \phi
$$

$$
\overline{\partial}_5 \phi = -N \frac{A^-}{x_5} \phi + B^+ \partial_5 \phi - x_5 x_i \partial_i \phi
$$

$$
\overline{\partial}_i \phi = N \frac{x_i}{x_5} \phi + x_i \partial_5 \phi + x_5 \partial_i \phi
$$

where we have introduced a convenient notation

$$
2A^\pm = 1 \mp x^2 \pm x_5^2 \quad \text{and} \quad 2B^\pm = 1 \pm x^2 \pm x_5^2.
$$

We observe that for $x_5 = 0$ and considering $\overline{\partial}_5 \phi = 0$ we obtain

$$
\overline{\partial}_i \phi = A \partial_i \phi + \frac{N}{A^+} x_i \phi
$$

$$
\overline{\partial}_0 \phi = -Ax_i \partial_i \phi + \frac{N}{A} \phi
$$

\(^2\) As we already know, in five dimensions we must impose a condition on space in order to account for the fact that we are aware of only four dimensions. We have the same situation here, i.e., we must impose an additional condition.
where $A^2 = 1 + x^2$. These expressions are the same expressions obtained in special projective relativity\cite{25, 26, 27} and provide the link between the five projective derivatives and the four derivatives in Cartesian coordinates, i.e., the relation between five-dimensional (de Sitter) universe and four-dimensional (Minkowski) universe.

3. Conformal Klein-Gordon Equation

In this section we use the previous results to calculate the so-called generalized Klein-Gordon differential equation

$$\frac{\partial^2}{\partial x_a^2} \Phi + m^2 \Phi = 0$$

where $m^2$ is a constant and $a = 0, 1, \ldots, 5$. Introducing projective coordinates (in this case we have a Pythagorean metric) we obtain

$$\frac{\partial^2 U}{\partial x_i^2} + \frac{\partial^2 U}{\partial x_0^2} - \frac{\partial^2 U}{\partial x_5^2} + m^2 U = 0$$

where $i = 1, 2, 3, 4$ and $U = U(x_i, x_0, x_5)$. Using the relations between projective and conformal coordinates and the link (involving the derivatives) in the two formulations we can write

$$\left[ x_5^2 \left( \Box - \frac{\partial^2}{\partial x_5^2} \right) + 3x_5 \frac{\partial}{\partial x_5} + N(N + 5) + m^2 \right] u(x_i, x_5) = 0$$

where $N$ and $m^2$ are constants, $\Box$ is the Daulenberg operator given by

$$\Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

and $\Delta$ is the Laplacian operator. This partial differential equation is the so-called Klein-Gordon differential equation written in conformal coordinates or a conformal Klein-Gordon equation.

The case $m^2 = 0$ transforms this equation in the so-called generalized d’Alembert differential equation. Another way to obtain this differential equation is to consider the conformal metric in cartesian coordinates, which furnishes the so-called Beltrami metric\cite{2} where the d’Alembert equation appears naturally. This equation can also be obtained by means of the second order Casimir invariant operator\footnote{Invariant operators associated with dynamic groups furnish mass formulas, energy spectra and, in general, characterize specific properties of physical systems.} associated with the conformal group.

To solve the conformal Klein-Gordon equation, we first introduce the spherical coordinates $(r, \theta, \phi)$ and get

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial u}{\partial x_5} + \frac{\Lambda}{x_5^2} u = 0,$$

\footnote{Hereafter we consider $m = m_0 c / \hbar$ where $m_0$, $c$ and $\hbar$ have the usual meanings.}
where we introduced $x_4 = i ct$ and defined the operator$^5$

$$
\mathcal{L} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
$$

involving only the angular part. In this partial differential equation we have $u = u(r, \theta, \phi, t, x_5)$ with $\Lambda = N(N + 5) + m^2$.

Using the method of separation of variables we can eliminate the temporal and angular parts, writing

$$
u = u(r, \theta, \phi, t, x_5) = A e^{i n \theta} Y_{\ell m}(\theta, \phi) f(r, x_5),$$

where $A$ is an arbitrary constant, $n > 0$, $\ell = 0, 1, \ldots$ and $m = 0, \pm 1, \ldots$ with $-\ell \leq m \leq \ell$ and $Y_{\ell m}(\theta, \phi)$ are the spherical harmonics, we get the following partial differential equation

$$
\frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} - \frac{\partial^2 f}{\partial x_5^2} + \frac{3}{x_5} \frac{\partial f}{\partial x_5} + \frac{\Lambda}{x_5^2} f + \left[n^2 - \ell(\ell + 1) \right] f = 0
$$

with $f = f(r, x_5)$. If we impose a regular solution at the origin ($r \to 0$), the solution of this partial differential equation can be obtained in terms of a product of two Bessel functions$^{[22]}$.

4. A d’Alembert-Like Equation

In this section we present and discuss a partial differential equation which can be identified to a d’Alembert-like equation, which we call a conformal ‘radial’ d’Alembert equation. We firstly introduce a convenient new set of coordinates, then we use separation of variables and obtain two ordinary differential equations. One of them can be identified as an ordinary differential equation whose solution is a generalization of Newton’s law of gravitation; the other one is identified with an ordinary differential equation similar to a one-dimensional Schrödinger differential equation with a potential equal to the second Pöschl-Teller potential.

We introduce the following change of independent variables

$$
r = \rho \cosh \xi,
$$

$$
x_5 = \rho \sinh \xi,
$$

with $\rho > 0$ and $\xi \geq 0$, in the separated Klein-Gordon equation, obtained in the previous section, and after another separation of variables we can write a pair of ordinary differential equations, namely,

$$
\rho^2 \frac{d^2 U}{d\rho^2} - p(p + 1) U = 0,
$$

$^5$ Here $r$ is a coordinate and should not be confused with the radius of the hypersphere. Besides, it is always possible to define a Wick-rotation$^{[24]}$ of the time coordinate, i.e., $ct \mapsto i ct$. 
where $U = U(\rho)$ and

$$
\frac{d^2V}{d\xi^2} + (2 \tanh \xi - 3 \coth \xi) \frac{dV}{d\xi} + \left[ \frac{\ell(\ell + 1)}{\cosh^2 \xi} - \frac{\Lambda}{\sinh^2 \xi} - p(p + 1) \right] V = 0
$$

where $V = V(\xi)$ and $p$ is a separation constant.

We first discuss the differential equation in the variable $\rho$. Its general solution is given by

$$
U(\rho) = C_1 \rho^{-p} + C_2 \rho^{p+1}
$$

where $C_1$ and $C_2$ are arbitrary constants.

If we consider the case $p = 1$, introducing the notation $C_1 = gM$ with $g$ and $M$ having the usual meanings, we get

$$
U(\rho) = \frac{gM}{\rho} + C_2 \rho^2
$$

i.e., a gravitational potential which can be interpreted as a sum of a Kepler-like potential and a harmonic oscillator potential, giving rise to the gravitational force

$$
f(\rho) = \nabla U = -\frac{gM}{\rho^2 - x_5^2} + 2C_2(x^2 - x_5^2)^{1/2},
$$

with a singularity at $x = x_5$. We note that for $C_2 = 0$ we obtain an expression analogous to Newton’s law of gravitation.

Secondly, the equation in the variable $\xi$. To solve this ordinary differential equation we first introduce the change of dependent variable

$$
V(\xi) = \sinh^{1/2} \xi \tan \xi F(\xi)
$$

and obtain

$$
-\frac{d^2}{d\xi^2} F(\xi) + \left[ \frac{\mu(\mu - 1)}{\sinh^2 \xi} - \frac{\ell(\ell + 1)}{\cosh^2 \xi} + \left( p + \frac{1}{2} \right)^2 \right] F(\xi) = 0,
$$

(1)

where the parameter $\mu$ is given by a root of the following algebraic equation $\mu(\mu - 1) = N^2 + 5N + 15/4 + m^2$.

The differential equation above can be identified with a Schrödinger-like differential equation in which the associated potential is given by

$$
\mathcal{V}_\mu(\xi) = \frac{\mu(\mu - 1)}{\sinh^2 \xi} - \frac{\ell(\ell + 1)}{\cosh^2 \xi},
$$

which is exactly the second Pöschl-Teller potential with energy $E$ given by $E_p = -(p + 1/2)^2 < 0$. The solution of this ordinary differential equation is well known and can be expressed in terms of the hypergeometric function. An algebraic treatment can be found in [28, 29].

We note that the first Pöschl-Teller potential is connected with the study of a Dirac particle on central backgrounds associated with an anti-de Sitter oscillator, i.e., the transformed radial wave functions satisfy the second-order Schrödinger differential equation whose potential is exactly the first Pöschl-Teller potential[30].

Finally, a particular case of Eq.(1), i.e., the case $\mu = 0$, is related to the anti-de Sitter static frame as shown recently by da Rocha and Capelas de Oliveira[31].
Concluding Remarks

In this paper we discussed the calculation of a conformal d’Alembert-like equation. We used the methodology of projective relativity to obtain a conformal Klein-Gordon differential equation and, after the separation of variables, we got another partial differential equation in only two independent variables, the so-called conformal d’Alembert differential equation. Another separation of variables led to an ordinary differential equation which generalizes Newton’s law of gravitation. Finally, we showed that the remaining differential equation, a ‘radial’ differential equation, is transformed into a one-dimensional Schrödinger differential equation with an associated potential that can be identified exactly with the second Pöschl-Teller potential.

From supersymmetric quantum mechanics with periodic potentials, it can be seen that the most general periodic potentials which can be analytically solved involve Jacobi’s elliptic functions, which in various limits become Pöschl-Teller potentials arising in the context of Kaluza-Klein spectrum\cite{14}. Kaluza-Klein modes of the graviton have been widely investigated\cite{32, 33, 34, 35}, since the original formulation of Randall and Sundrum necessarily has a continuum of Kaluza-Klein modes without any mass gap, arising from a periodic system of 3-branes. The methods and equations developed here can shed some new light in the calculation of mass gaps from a distribution of $D$-branes\cite{35} in the context of five-dimensional supergravity.

A natural continuation of this calculation is to prove that all ‘radial’ problems associated with an equation resulting from a problem involving a light cone can be led into a Schrödinger-like differential equation in which the potential is exactly the Pöschl-Teller potential\cite{36}.

Acknowledgment

ECO is grateful to FAPESP for financial support (06/52475-8,08/50603-5). ECO is also grateful to Prof. F. Mainardi for the invitation to visit Department of Physics, University of Bologna, Italy. We are also grateful to Dr. Geraldo Agosti for interesting and useful discussions and Prof. W. A. Rodrigues Jr., for several suggestions about the paper.

A Hyperspherical Universe Models

In this appendix we briefly summarize the idea of hyperspherical universes as originally proposed by Fantappié\cite{37} and developed by Arcidiacono\cite{2}.

The main motivation behind those models is to consider seriously the premise that a Universe must be a harmonic and well ordered system of laws, and that this statement is to be expressed mathematically by using group theory in an appropriate way. Taking into account that Galilean Relativity, which uses the Galileo group as invariance group of physical laws, has been perfectioned into Special Relativity, which uses the Poincaré group as invariance group of physical laws, Fantappié asked himself in which way it would
be possible to perfect Special Relativity into a kind of final relativity. His answer to this question was very simple indeed. He realized that the Poincaré group is the contraction of the 10-parameter Lie group known today as de Sitter group (but which should be called, as in the text, the Fantappié-de Sitter group) which can be made to act projectively on a flat 4-dimensional (Minkowski) spacetime. Next Fantappié asked whether there were other spacetime manifolds where the group could act naturally and serve as invariance group of physical laws. The answer is positive, and Fantappié found that the natural manifold is a hyperspherical universe, which he called $S^4$, of constant curvature radius. Of course, the previously known models for the universe (not based on General Relativity) are all particular cases of this one, corresponding to some group contractions involving the velocity of light and/or the universe radius, or both. Fantappié did not stop there. He proposed that the hyperspherical universe model $S^4$ was only an approximation for truth in the sense that it was embedded in a hyperspherical universe model $S^5$ where the conformal group (a 15-parameter Lie group) acts naturally as invariance group of physical laws. By its turn, $S^5$ may be generalized into $S^6$ and so on. At each generalization new fundamental physical constants make their natural appearance as a kind of group parameter whose contraction produces the group used in the previous universe model. It is clear that the extra dimensions in each Universe model must be interpreted, in an appropriate way (something that is also necessary in modern Kaluza-Klein type theories), and it is at this point that his mathematical skills give us useful physical hints. In particular, Arcidiacono, one of his students, showed that the hyperspherical universe models $S^4$ and $S^5$ contain many aspects of several proposed unified theories. Of course, we do not have space here even to start discussing the many beautiful results found by Arcidiacono and we invite the reader to consult Arcidiacono[2] for more details. However, we would like to emphasize here that many ideas proposed by him are worth to be more developed since, in particular, it seems that in his work there is the seed of a simple solution for the problem of dark energy and dark matter, an issue that we shall discuss elsewhere. Finally, we mention that recently Chiatti[38] has discussed a comparison of this theory with recent cosmological evidences.

References


