EJTP **5, No. 18** (2008) 1–20 *Electronic Journal of Theoretical Physics*

Liénard-Wiechert Electromagnetic Field

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Received 18 October 2006, Accepted 19 November 2006, Published 30 June 2008

Abstract: The electromagnetic field generated by a charge in arbitrary motion in Minkowski space is briefly studied. Particularly important is the deduction of the superpotential for the radiative part of Maxwell tensor.

Keywords: Classical Charged Particles; Liénard-Wiechert Field; Classical Field Theory
PACS (2006): 03.50.De; 42.50.Dv; 41.20.q; 41.60.m; 03.50.z; 11.10.z

1. Introduction

A charge in arbitrary motion in special relativity generates the Liénard-Wiechert retarded potential A_r and its corresponding Faraday tensor F_{rs}, of fundamental importance in point particle electrodynamics. Accordingly, we will dedicate Section I to the study of scalar and vectorial quantities associated to the world line of the charge, with special emphasis in retarded distance and the light cone: the trajectory's kinematics forms a powerful platform for the analysis of the electromagnetic field. Additionally, the valuable Fermi's triad is introduced.

In Section II we consider general aspects regarding 4-potential and Faraday tensor, bringing into Synge classification [1] and an attractive theorem of Stachel [2]. Section III concerns to the Liénard-Wiechert case, obtaining Teitelboim [3], Miglietta [4] and Teitelboim [5] decompositions of A_r and F_{cb}, respectively. We also deduce Plebañski's interesting result [6]: Fij is the antisymmetric product of two gradients.

Section IV deals with retarded Maxwell tensor, its study is channeled through the algebraic-differential properties of its $\underset{B}{\mathrm{T}_{\mathrm{ic}}}$ and radiative $\underset{R}{\mathrm{T}_{\mathrm{ab}}}$ parts.

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A thoughtful analysis of Weert superpotential structure is made for $\underset{B}{T}_{rs}$, and it is shown the non-local superpotential for $\underset{R}{T}_{ij}$.

In whole our study the Synge article [7] is fundamental for the mathematical aspects of point particles electrodynamics.

2. Kinematics of Relativistic Particles

In special relativity, a "particle" means a timelike world line, see Fig. 1, whose unitary tangent vector is the 4-velocity.

$$v^r = dx^r/d\tau \tag{1a}$$

where the proper time t is defined by

$$d\tau^2 = g_{ab}dx^a dx^b = -dx^2 - dy^2 - dz^2 + dt^2 \tag{1b}$$

which means that the metric is Diag(-1, -1,- 1, 1) and c = light's speed in vacuum=1, then

$$(v^r) = (\gamma\vec{v}, \gamma) \quad \text{with}\vec{v} = (dx/dt, dy/dt, dz/dt) \quad \text{and} \quad \gamma = (1 - \vec{v}^2)^{1/2} \tag{1c}$$

So, out of (1.a,b) we have that:

$$v^r v_r = -1, \quad v^r a_r = 0 \quad \text{with} \quad a^r = dv^r/d\tau = 4 - \text{acceleration}, \tag{1d}$$

this implies the timelike and spacelike nature of v^r and a^r, respectively; in consequence:

$$a^2 \equiv a_r a^r \geq 0 \tag{1e}$$

From (1.d) we obtain:

$$s^r v_r + a^2 = 0, \quad \text{where} \quad s^r = da^r/d\tau = \text{superacceleration} \tag{1f}$$

In Fig 1. we have not indicated this last vector since it might be outside or inside the light cone.

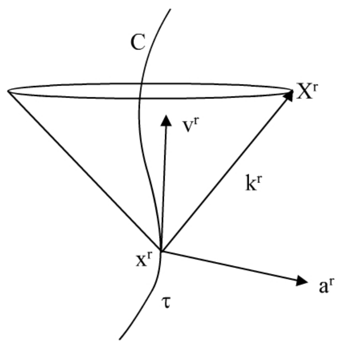

Fig. 1 Timelike trajectory

From an event X^r outside C we trace its null cone's past sheet which intersects to C in the point x^r called "retarded event associated with X^r", so we say that

$$x^r = x^r \left(X^i\right) \tag{2a}$$

because with X^i given, the retarded point over C is automatically determined. This allows to introduce the vector:

$$k^r = X^r - x^r \tag{2b}$$

whose magnitude is zero for resting over the cone:

$$k^r k_r = 0, \tag{2c}$$

so, k^r indicates the propagation direction of the photons emitted by the particle. The null or light type vector (2.b) is truly important in electrodynamics: we could say that studying the Maxwell field is almost equivalent to an analysis of the null cone and its relation to the world line.

From X^r we can build two distances widely used in the study of charges in Minkowski space:

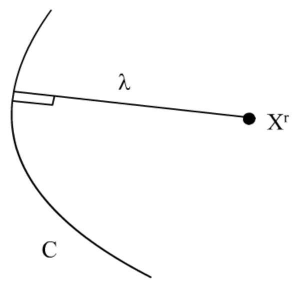

Fig. 2a Instantaneous distance from X^r to C.

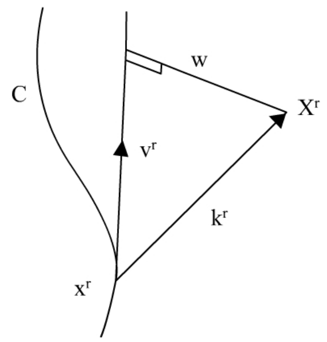

Fig. 2b Retarded distance from X^r to C.

The instantaneous distance, see fig. 2.a, introduced by Dirac [8] is geometrically simpler than retarded distance w, see fig. 2.b, proposed by Bhabha [9] and furthered by Synge [7]; nevertheless, l has the big disadvantage of not involving retarded effects (light cone); for this reason, w has more physical meaning and leads to simpler calculations because it intrinsically takes in account the finite velocity of interaction. Here we will work only with w, its expression is given by:

$$w = -k^r v_r \geq 0. \tag{3a}$$

Bringing to mind that a null vector cannot be orthogonal to a timelike one, (3.a) points that:

$$w = 0 \quad \text{if and only if} \quad k^r = 0, \tag{3b}$$

in other words, the retarded distance is zero only when X^r is over C.

When making calculations, we will need to know how change diverse quantities over C when an external event X^r varies, for this, its enough with having change's law for t because x^r, v^r, a^r, etc. are functions of these parameter:

$$\tau_{,r} = -w^{-1} k_r \quad \text{where} \quad ,r = \partial / \partial X^r, \tag{4}$$

so we have that t, r is a null vector because it is anti-parallel to k^r. Every event X^r over the same cone has an associated unique value of t, that is, the light cone is the t=constant surface, so t, r is the vector normal to the cone even though our Euclidian eyes don't see it like that. Due to (4) it has no sense looking for a unitary normal to the cone. Thanks to (4) it is easy to obtain the useful relationships:

$$x^r_{,j} = -w^{-1} v^r k_j, \quad v^r_{,j} = -w^{-1} a^r k_l, \quad a^r_{,b} = -w^{-1} s^r k_b$$

$$k^r_{,C} = \delta^r_C + w^{-1} v^r k_C, w_{,C} = -v_C + B k_c$$

$$\text{with} \quad B = w^{-1} \left(1 - W\right), \quad W = -k^r a_r$$

$$W_{,b} = W_b = -a_b + w^{-1} k^r s_r k_b \tag{5}$$

$$B_{,C} = w^{-1} \left[U_C - \left(B^2 + w^{-1} k^r s_r \right) k_c \right], U_C = B v_C + a_C$$

$$U^C k_c = -1, U^C v_C = -B, U^C a_C = a^2$$

$$U^C U_C = a^2 - B^2, U^c w_{,c} = 0, U^C_{,C} = 0.$$

In relativity, a spatial triad of vectors is also important in each point of the curve because this triad is a local frame of reference for an observer mounted in the particle. See Fig. 3:

$$\left(e^r_{(a)} e_{(b)r} \right) = Diag \left(1, 1, 1, -1\right) \tag{6a}$$

This tetrad forms a base for each vector associated to world line, in particular for null vector (2.b):

$$k^r = b^\sigma e^r_{(\sigma)} + b^4 e^r_{(4)}; \tag{6b}$$

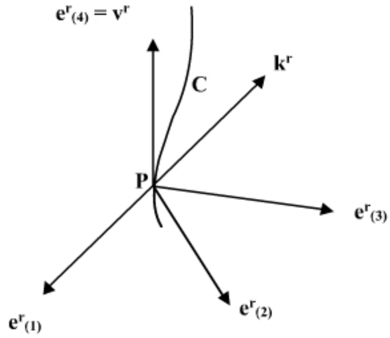

Fig. 3 Orthonormal tetrad

from now on the Greek indexes will only take values 1, 2, and 3. Expansion (6.b) can be written like:

$$k^r = M^r + b^4 v^r \quad \text{with} \quad M^r = b^\sigma e^r_{(\sigma)}, M^r v_r = 0, \tag{6c}$$

M^r is spacelike type because it is a lineal combination of the three spacelike vectors of the tetrad, see Fig. 4:

If $M = (M^r M_r)^{1/2}$ is the magnitude of M^r, then

$$M^r = M p^r \quad \text{with} \quad p^r p_r = 1 \tag{6d}$$

and by (6.c):

$$p_r v^r = 0, \tag{6e}$$

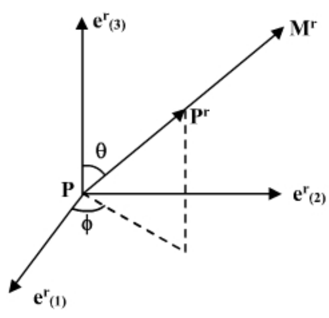

Fig. 4b Spatial triad

so p^r is a spacelike unitary vector. From (2.c, 3.a) its plain that $M = b^4 = w$, as a consequence (6.b,...,e) implies the important Synge [7] - Teitelboim [3] decomposition for k^r:

$$k^r = w\left(p^r + v^r\right), \quad p^r v_r = 0, \quad p_r k^r = w, \tag{7}$$

which is shown in the following two figures:

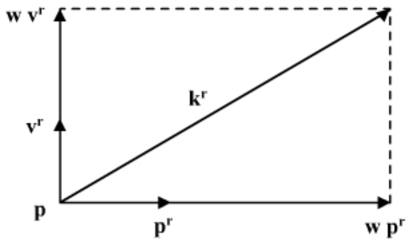

Fig. 5a Spatial and timelike components of k^r.

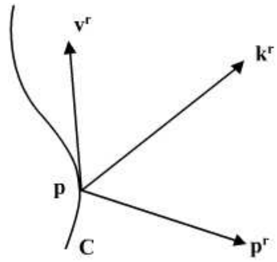

Fig. 5b Null vector k^r splitting

The unitary vector p^r only depends of the spatial triad; so it can be written with the common spherical coordinates, see Fig. 4:

$$p^r = \sin\theta\cos\phi e^r_{(1)} + \sin\theta\sin\phi e^r_{(2)} + \cos\theta e^r_{(3)} = p^{(\sigma)} e^r_{(\sigma)} \tag{8a}$$

$$\therefore p^{(\gamma)} = e^{(\gamma)}_r p^r$$

where we have employed the dual base e^r_γ, defined by:

$$e^{(\gamma)r} e_{(\sigma)r} = \delta^\gamma_\sigma \tag{8b}$$

Vector p^r doesn't have to be necessarily orthogonal to 4-acceleration a^r.

The triad e^r_σ is arbitrary except by the orthonormality conditions (6.a); nevertheless, some triads may be more convenient than others in some set calculations. For our theoretical purposes, the Fermi triad [10] is very important, it satisfies over C the transport law (which we will use in this work):

$$de^r_{(\sigma)}/d\tau = e^b_{(\sigma)} a_b v^r = a_{(\sigma)} v^r \tag{9a}$$

This type of transport has been very fundamental in gravitation, for example Pirani [11] and Synge [12]; but, in electrodynamics we shall show its participation in the deduction of the superpotential for the radiative part of Maxwell tensor, see Section IV. In (9.a) we have used the notation:

$$a_{(\sigma)} = a^r e_{(\sigma)r} \tag{9b}$$

because a^r is spacelike type; to remember that the triad is only defined over C.

To end this Section, we give some useful expressions:

$$w_{,b}k^b = w, \quad w_{,b}v^b = W, \quad w_{,b}a^b = -WB$$

$$w^{,b}w_{,b} = 1 - 2W, \quad W = -wp^r a_r = -wp^{(\sigma)}a_{(\sigma)}$$

$$W_{,c}k^c = W, \quad W_{,c}w^{,c} = WB + k^r s_r, \quad k^r_{,a}v^a = 0$$

$$B_{,c}k^c = -w^{-1}, \quad w_{,c}p^c = wB, \quad w^{-1,a}_{,a} = 0$$

$$w^{,a}_{,a} = 2w^{-1}\left(1 - 2W\right), \quad k^r_{,a}p_r = p_a, \quad U^r p_r = -w^{-1}W \tag{10}$$

$$p^r_{,a} = w^{-1}\left[\delta^r_a + w^{-1}v_a k^r + \left(a^r + w^{-1}v^r - w^{-1}Bk^r\right)k_a\right]$$

$$p^r_{,a}k^r = -w^{-1}Wk_a, p^r_{,a}w_{,r} = w^{-2}W^2 k_a, p_{(\sigma),r}k^r = 0$$

$$p^r_{,a}k^a = 0, \quad p^r_{,r} = w^{-1}\left(2 - W\right), \quad p_{(\sigma),r} = p_{a,r}e^a_{(\sigma)}$$

The relations (5,10) are the basic formulary for any calculation in the electrodynamics of classical charged particles.

3. 4-Potential and Faraday Tensor

In this Section we consider the algebraical and differential properties satisfied by the electromagnetic field in vacuum. Faraday tensor is given by:

$$F_{rc} = -F_{cr} = A_{c,r} - A_{r,c} \tag{11a}$$

in terms of the 4-potential A^b. From (11.a) it is clear the fulfillment of the cyclic relationship:

$$F_{br,c} + F_{rc,b} + F_{cb,r} = 0, \quad \text{in other words} \quad {}^*F^{rc}_{,c} = 0 \tag{11b}$$

where we have employed the dual tensor:

$${}^*F^{rc} = -{}^*F^{cr} = \frac{1}{2}\varepsilon^{rcab}F_{ab} \tag{11c}$$

being ε^{ijkr} the Levi-Civita antisymmetric symbols. In free space we have the remaining Maxwell equation:

$$F^{rc}_{,c} = 0 \tag{11d}$$

which in turn leads to a differential equation for the 4-potential:

$$A^{r;c}_{,c} - \left(A^{c}_{,c}\right)^{,r} = 0 \tag{11e}$$

In (11.a) we have full freedom to add to A_r an arbitrary gradient without modifying Faraday tensor; then without lack of generality we can always demand:

$$A^{c}_{,c} = 0 \quad \text{Lorenz condition,} \tag{11f}$$

simplifying (11.e):

$$A^{r;c}_{,c} = 0 \quad \text{Wave equation} \tag{11g}$$

So, from the mathematical point of view, the problem consists in solving (11.g) with the restriction, which matches to solving (11.b,d), in other words:

$$\begin{array}{ll} \nabla \cdot \vec{B} = 0, & \nabla \times \vec{E} = -\partial \vec{B}/\partial t \\ \nabla \cdot \vec{E} = 0, & \nabla \times \vec{B} = \partial \vec{E}/\partial t \end{array} \tag{12}$$

in the MKS system; to remember that $c = (\varepsilon_0 \mu_0)^{-1/2} = 1$.

Faraday matrix representation turns like:

$$\left[F^{ab}\right] = \begin{bmatrix} 0 & B_z & -B_y & -E_x \\ -B_z & 0 & B_x & -E_y \\ B_y & -B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{bmatrix} \begin{array}{l} a : rows \\ \\ \\ b : columns \end{array} \tag{13a}$$

and with (11.c) an associated matrix for the dual tensor can be constructed:

$$\left[{}^{*}F^{ab}\right] = \begin{bmatrix} 0 & E_z & -E_y & B_x \\ -E_z & 0 & E_x & B_y \\ E_y & -E_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{bmatrix} \tag{13b}$$

Note that (13.b) is obtained if we do the following changes to (13.a):

$$\vec{B} \to \vec{E}, \quad \vec{E} \to -\vec{B} \tag{13c}$$

so it may come to mind that * executes the operation (13.c); then it is clear that $^{**}F^{ar} = -F^{ar}$. Comparing (11.a, 13.a) we obtain the relationship of the electric and magnetic fields with the 4-potential:

$$\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t, \vec{B} = \nabla \times \vec{A} \tag{14}$$

because $(A^r) = \left(\vec{A}, \phi\right)$,

where $\vec{A}$ and ϕ are the magnetic and electric potentials, respectively. We are placing emphasis in f as a scalar, but not as an invariant: the electromagnetic field only possesses two Lorentz invariants, namely

$$F_1 \equiv F_{ab}F^{ab} = 2\left(\tilde{B}^2 - E^2\right), \quad F_2 \equiv^* F_{ab}F^{ab} = 4\vec{E}\cdot\vec{B} \tag{15}$$

with $E = \left|\vec{E}\right|$ and $\tilde{B} = \left|\vec{B}\right|$.

Just like Weyl tensor invariants allow to establish the Petrov classification [13] for the gravitational field, the quantities (15) lead to the Synge [1] – Piña [14] classification for the Faraday tensor:

$$\begin{array}{llll} \text{Type A:} & F_2 \neq 0 & & \\ \text{Type B:} & F_1 < 0, & F_2 = 0 & \\ \text{Type C:} & F_1 = 0, & F_2 = 0 & \text{Null field} \\ \text{Type D:} & F_1 > 0, & F_2 = 0 & \end{array} \tag{16}$$

A point with a null field means that in such event $\vec{E}\perp\vec{B}$ and $E = \tilde{B}$. Non-null field implies a different type of C. The classification (16) is algebraic but the type of electromagnetic field may change from one point to another.

Furthermore, we will see that the field that produces a relativistic charge is B type, which tends to type C(plane wave) towards infinity.

There are very important identities for Maxwell field:

$$F^{ar}F_{br} - {}^*F^{ar*}F_{br} = (F_1/2)\,\delta^a_b \tag{17a}$$

$$^*F^{ar}F_{br} = (F_2/4)\,\delta^a_b \tag{17b}$$

which do not have an specific name, are well known and can be found in Rainich [15], Plebañski [16], Wheeler [17] pp. 239, Penney [18] and Piña [14]. Expressions (17) correspond to Lanczos identities [18] between the Riemann tensor and its double dual. If (17.a) is multiplied by F_{ia} or $^*F_{ia}$ and (17.b) is employed, results in valuable identities in the calculation of an antisymmetric matrix's exponential function [14, 20]:

$$\begin{aligned} F_{ia}F^{ar}F_{rb} &= (F_1/2)\,F_{bi} + (F_2/2)\,{}^*F_{bi} \\ {}^*F_{ia}{}^*F^{ar*}F_{rb} &= (F_2/4)\,F_{bi} - (F_1/4)\,{}^*F_{bi} \end{aligned} \tag{18}$$

From (13.a) it is simple to show that:

$$\det\left(F^{ab}\right) = (1/16)\left(F_2\right)^2 = \left(\vec{E}\cdot\vec{B}\right)^2, \tag{19}$$

which is a particular case of the following theorem - see Drazin [21]:

"The $\det \underset{\sim}{F}$, with antisymmetric $\underset{\sim}{F}_{n\times n}$ and even n, is the square of a rational polynomial in F^{ij}"

Now we mention the interesting and useful Stachel theorem [2] page 1261:

' If $\underset{\sim}{F}$ satisfies:

$$F_{ar} = -F_{ra}, \quad F_{ar,c} + F_{rc,a} + F_{ca,r} = 0, \tag{20a}$$

$$\det\left(F_{ab}\right) = 0$$

then there exist functions β and ψ such that:

$$F_{ar} = \beta_{,a}\psi_{,r} - \beta_{,r}\psi_{,a}\text{'}. \tag{20b}$$

That is, the conditions (20.a) reduce $\underset{\sim}{F}$ to an antisymetric product of gradients. If we extend the Stachel result to Maxwell field, then the first two conditions (20.a) will be immediately verified, thereby:

$$\text{'If Faraday tensor fulfills} \quad F_2 = 0, \quad \text{then it is of the form'}. \tag{20c}$$

Liénard Wiechert solution will satisfy (20.c), it will permit to write $\underset{\sim}{F}$ in the form of Plebañski [6]. In general, an electromagnetic field with a different type of A has the structure (20.b). The results (20) are valid in presence of curvature because its differential expressions remain undisturbed if covariant derivatives are used instead of partial ones:

$$^*F^{ar}_{\ ,r} = {^*F^{ar}_{\ ;r}} = 0, \quad \beta_{,r} = \beta_{;r}$$

In the following Section we will employ the exposed material in (11,. . . ,20) in the analysis of the field produced by a point charge with a relativistic trajectory.

4. Liénard-Wiechert Field

Solution of (11.f,g) for a particle in arbitrary motion in Minkowski space was obtained by Liénard and Wiechert; the corresponding potential carries their names and is given by

$$A^r\left(X^b\right) = qw^{-1}v^r, \quad q = \text{charge}/4\pi\varepsilon_0 \quad \text{Retarded potential} \tag{21a}$$

which is fundamental in everything that follows; by the use of (11.a) it is simple to calculate the associated Faraday tensor :

$$F_{rb} = qw^{-2}\left(U_r k_b - U_b k_r\right) = qw^{-2}U_r \times k_b \tag{21b}$$

where $\times$ means the antisymetric product. This notation is due to Lowry [22] and will make such expressions to be very compact. From (11.c, 21.b) it is clear that:

$$^*F_{mn} = -qw^{-2}\varepsilon_{mnab}U^a k^b \tag{21c}$$

therefore

$$F_1 = -2q^2w^{-4} < 0, \quad F_2 = 0 \tag{21d}$$

in other words, the electric and magnetic fields generated by the charge satisfy:

$$\tilde{B} < E, \quad \vec{E} \cdot \vec{B} = 0 \tag{21e}$$

in consequence $\underset{\sim}{F}$ is type B. Note that in the asymptotic region ($w \to \infty$), the invariant F_1 tends to zero, which means that $\underset{\sim}{F}$ is close to the null case (type C) far away from the charge.

With (21.d), (20.c) is valid, so the Stachel theorem [2] implies that (21.b) can be reduced to (20.b). This is easy to do because from (4,5) the following relationships are available:

$$k_r = -w\tau_{,r}, \quad U_r = wB_{,r} + \left(B^2 + w^{-1}k^c s_c\right) k_r$$

which substituted in (21.b) imply:

$$F_{rc} = -qB_{,r}x\tau_{,c} = q\left(\tau_{,r}B_{,c} - \tau_{,c}B_{,r}\right) \tag{22}$$

which has the form (21.b), meaning that t and B correspond to the functions b and y. Expression (22) was first obtained by Plebañski [6].

Now we will consider the eigenvalue problem of $\underset{\sim}{F}$. For this purpose we stem from (21.b), and due to

$$U_r k^r = -1, \quad k_r k^r = 0$$

then immediately we have one of the two null proper vectors of a non-null field (different type from C) [23]:

$$F_{rm}k^m = qw^{-2}k_r, \quad \text{proper value} = qw^{-2}. \tag{23a}$$

This suggests that U_r may be an eigenvector, but it isn't:

$$F_{rb}U^b = -qw^{-2}U_r - qw^{-2}\left(a^2 - B^2\right) k_r, \tag{23b}$$

nevertheless, if we multiply (23.a) by $\frac{1}{2}\left(a^2 - B^2\right)$ and add the resulting equation with (23.b) we obtain the other null proper vector [24, 25] :

$$F_{rm}\eta^m = -qw^{-2}\eta_r, \quad \text{proper value} = -qw^{-2}$$

with

$$\eta^r = U^r + \tfrac{1}{2}\left(a^2 - B^2\right)k^r, \quad \eta^r\eta_r = 0; \tag{23c}$$

it's not usual to find explicitly η^r in the literature. It is clear that these two proper vectors are independent because:

$$k^r\eta_r = 1. \tag{23d}$$

To remember that two null vectors ξ^r and γ^r are proportional if and only if $\xi^r\gamma_r = 0$, so (23.d) implies the non-parallelism of such proper vectors.

With (10,21.b) it is possible to prove that:

$$F_r^b p_{c,b} = qw^{-4}Wv_ck_r, \quad F_r^b p_{(\sigma),b} = 0, \tag{24}$$

of great importance in the next Section in the deduction of the radiative superpotential.

Teitelboim started an era in electrodynamics by employing only retarded fields and studying Faraday and Maxwell tensors near and away of a point charge. This type of analysis is generated by substituting (5,7) in (21.b) to obtain the decomposition:

$$F_{rb} = \underset{(-1)^{rb}}{F} + \underset{(-2)^{rb}}{F}, \tag{25a}$$

where:

$$\underset{(-1)^{rb}}{F} = qw^{-2}\left(w^{-1}Wv_r + a_r\right) \times k_b \tag{25b}$$

$$= qw^{-1}\left(a_cp^cv_r \times p_b + a_r \times v_b + a_r \times p_b\right) \tag{25c}$$

$$\underset{(-2)^{rb}}{F} = qw^{-3}v_r \times k_b = qw^{-2}\left(v_r \times p_b\right) \tag{25d}$$

meaning that, $\underset{(-i)^{rb}}{F}, i = 1, 2$ varies like w^{-i}: the dependence in w^{-i} is clear because the parenthesis in (25.c,d) are independent of the retarded distance, their terms are functions of x^r which remains stationary when we move away over the light cone. Thus $\underset{(-1)^{rb}}{F}$ and $\underset{(-2)^{rb}}{F}$ are dominant away (w>>1) and near (w<<1), respectively, then $\underset{(-1)^{ij}}{F}$ being the responsible of Larmor formula which provides the radiation speed towards infinitum. Note that (25.b,c) depend on the acceleration of the particle, which is an expected result because the Schild theorem [26]:

$$\text{“Radiation exists if and only if } a^r \neq 0\text{”}. \tag{26}$$

Schild was the first author to give a covariant definition of radiation even though some of his ideas were already implicit in Synge [27], Appendix A, whose 1st edition was made in 1955. We will call $\underset{(-1)^{ij}}{F}$ the radiative part of F_{ab} because it is a null field in classification (16):

$$\underset{(-1)^{ra}}{F}\ \underset{(-1)}{F^{ra}} = \ \underset{(-1)^{ra}}{{}^*F}\ \underset{(-1)}{F^{ra}} = 0, \tag{27a}$$

which doesn't happen with $\underset{(-2)^{ij}}{F}$:

$$\underset{(-2)^{ra}}{F}\ \underset{(-2)}{F^{ra}} = 2q^2 w^{-4} < 0, \qquad \underset{(-2)^{ra}}{{}^*F}\ \underset{(-2)}{F^{ra}} = 0 \tag{27b}$$

which belongs to type B. This portion will be designated as the bounded part of F_{mni}, besides

$$\underset{(-1)^{ab}}{F}\ \underset{(-2)}{F^{ab}} = 0, \qquad \underset{(-1)^{ab}}{{}^*F}\ \underset{(-2)}{F^{ab}} = \underset{(-1)^{ab}}{F}\ \underset{(-2)}{{}^*F^{ab}} = 0 \tag{27c}$$

The relations (27) can be found in Weert [28], page 465.

It is possible to write (25.a) in the form:

$$F_{ab} = w^{-1} N_{ab} + w^{-2} M_{ab} \quad \text{such that} \quad N_{ab} = w \underset{(-1)^{ab}}{F}, \quad M_{ab} = w^2 \underset{(-2)^{ab}}{F} \tag{28}$$

with the properties:

$$N_{ab}\xi^b = 0, \quad M_{ab}\xi^b = q\xi_a, \quad \xi_r \xi^r = 0, \quad \xi_r = w^{-1} k_r = -\tau_{,r}$$

so we see that (28) are coherent with (1.1, 2, 3) of Goldberg-Kerr theorem [29] for the asymptotic behavior of the electromagnetic field.

Teitelboim's decomposition (25.a) is fundamental in everything that follows, and the interesting is that (7) generates of natural manner such splitting:

$$v^r = w^{-1} k^r - p^r$$

which substituting in (21.a) gives

$$A^r = \underset{1}{A^r} + \underset{2}{A^r} \quad \text{with} \quad \underset{1}{A^r} = -q w^{-1} p^r, \quad \underset{2}{A^r} = q w^{-2} k^r \tag{29a}$$

This partition of Liénard-Wiechert 4-potential is found in the Teitelboim [3] well-known article, however, it was also published by Migglietta [4] not-knowing the Ref.[3]. Expressions (29.b) are simpler than Migglietta's (2.3, 3.2). Placing (29.a) into (11.a) we obtain the matching Faraday's tensor decomposition (25.a) with:

$$\underset{(-i)^{bc}}{F} = \underset{i^{c,b}}{A} - \underset{i^{b,c}}{A}, \quad i = 1, 2 \tag{29b}$$

which means that each part of F_{mn} has its own 4-potential. At last, it can be verified that (29.a) does not satisfy the Lorenz condition(11.f):

$$\underset{1}{A^r_{,r}} = -\underset{2}{A^r_{,r}} = -q w^{-2} \neq 0 \tag{29c}$$

5. Energy-momentum Tensor

Now we will consider the Maxwell tensor T_{rb} through which an electromagnetic field's content of energy-momentum is quantified:

$$T_{ab} = \frac{1}{2}\left(F_{ac}F_b^c + {}^*F_{ac}{}^*F_b^c\right), \tag{30a}$$

which satisfies

$$T_{ab} = T_{ba} \qquad \text{Symmetry} \tag{30b}$$

$$T_r^r = 0 \qquad \text{Null trace} \tag{30c}$$

$$T_{ac}T_b^c = \frac{1}{4}\left(T_{mn}T^{mn}\right)g_{ab} \qquad \text{Rainich identity} \tag{30d}$$

Symmetry (30.b) is a property of every energy tensor, (30.c) tells us that the field is made of particles with null mass at rest, photons in this case; (30.d) was obtained by Rainich [15].

If we employ (17.a) in the second term of (30.a) we obtain an alternative expression for Maxwell tensor:

$$T_{ab} = F_{ac}F_b^c - \left(F_1/4\right)g_{ab}; \tag{30e}$$

by substitution of (21.d, 25) in (30.e) it results in the important Teitelboim splitting [5]:

$$T_{ab} = \underset{(-2)^{ab}}{T} + \underset{(-3)^{ab}}{T} + \underset{(-4)^{ab}}{T} \tag{31a}$$

where

$$\underset{(-2)^{rn}}{T} = \underset{(-1)^{rc}}{F}\ \underset{(-1)^{n}}{F^c} = q^2w^{-4}\left(a^2 - w^{-2}W^2\right)k_rk_n \tag{31b}$$

$$\underset{(-3)^{rn}}{T} = \underset{(-1)^{rc}}{F}\ \underset{(-2)^{n}}{F^c} + \underset{(-1)^{nc}}{F}\ \underset{(-2)^{r}}{F^c} = q^2w^{-4}\left[k_ra_n + k_na_r + 2w^{-2}Wk_rk_n - w^{-1}W\left(k_rv_n + k_nv_r\right)\right] \tag{31c}$$

$$\underset{(-4)^{rn}}{T} = \underset{(-2)^{rc}}{F}\ \underset{(-2)^{n}}{F^c} - \left(F_1/4\right)g_{rn} = q^2w^{-4}\left[\frac{1}{2}g_{rn} + w^{-1}\left(v_rk_n + v_nk_r\right) - w^{-2}k_rk_n\right] \tag{31d}$$

with the properties:

$$\underset{(-2)^{rn}}{T}k^n = 0, \qquad \underset{(-3)^{rn}}{T}k^n = 0, \qquad \underset{(-4)^{rn}}{T}k^n = -\left(q^2/2\right)w^{-4}k_r \tag{31e}$$

From (31.a,e) it is clear that k^r is a null proper vector of Maxwell tensor:

$$T_{rn}k^n = \left(q^2/2\right) w^{-4}k_r, \tag{32}$$

which was to be expected due to (21.d, 23.a, 30.e):

$$T_{rn}k^n = -qw^{-2}F_{rn}k^n + \left(q^2/2\right) w^{-4}k_r = -\tfrac{1}{2}q^2w^{-4}k_r \quad \text{identical to (32).}$$

The notation $\underset{(-i)^{rn}}{T}$, $i = 2, 3, 4$ evokes that (31.b,c,d) vary like w^{-i}, in consequence:

$$\underset{(-2)^{ab}}{T} \quad \text{dominates when} w \to \infty \quad \text{(away from the charge)} \tag{33}$$

$$\underset{(-3)^{ab}}{T} \quad \text{and} \quad \underset{(-4)^{ab}}{T} \quad \text{dominates when} w \to 0 \quad \text{(close from q)}$$

So Larmor formula comes from $\underset{(-2)^{ab}}{T}$, and $\underset{(-i)^{ab}}{T}$, $i = 3, 4$ are responsible of the singularities in the point charge's position. Thus Teitelboim wrote (31.a) in the form:

$$T_{rn} = \underset{R^{rn}}{T} + \underset{B^{rn}}{T}, \tag{34a}$$

where

$$\underset{R^{rn}}{T} = \text{radiative part} = \underset{(-2)^{rn}}{T} = q^2w^{-4}\left(a^2 - w^{-2}W^2\right) k_rk_n \tag{34b}$$

$$\underset{B^{rn}}{T} = \text{bounded part} = \underset{(-3)^{rn}}{T} + \underset{(-4)^{rn}}{T}$$
$$= q^2w^{-4}\left[\tfrac{1}{2}g_{rn} + (k_ra_n + k_na_r) + B\left(k_rv_n + k_nv_r\right) w^{-2}\left(1 - 2W\right) k_rk_n\right]$$

and proves that such parts are dynamically independent, which means that they verify separately (outside the world line) :

$$\underset{R^{r,n}}{T^n} = 0, \tag{35a}$$

$$\underset{B^{r,n}}{T^n} = 0. \tag{35b}$$

It is simple to obtain the relations:

$$T_{rn}U^n = \lambda U_r, \quad T_{tn}B^{,n} = \lambda B_{,r}, \quad T_{rn}\eta^n = \lambda\eta_r, \quad \lambda = -\left(q^2/2\right) w^{-4} \tag{36}$$

so we have that F_{aj} and T_{rc} have the same null proper vectors, which is a general result, see Synge [27], p. 337. Plebański [6], p. 41, was the first one to observe that $B, \ r$ is a proper vector of T_{ij}. If we substitute (34.b,c) in (34.a) we obtain the Synge [7] compact expression for the energy tensor associated to Liénard-Wiechert retarded potential:

$$T_{rn} = q^2w^{-4}\left[k_rU_n + k_nU_r + \left(a^2 - B^2\right) k_rk_n + \tfrac{1}{2}g_{rn}\right]. \tag{37}$$

Weert [30, 31] attention was to lead towards the fact that (35) are valid *identically,* and therefore he suggested the existence of Superpotentials for the bounded and radiative parts. However, he only obtained successfully the explicit form of the superpotential (which now carries his name) $\underset{B}{K}_{sar}$ which generates the bounded part [32-35] :

$$\underset{B}{T}_{rn} = \underset{B}{T}_{nr} = \underset{B}{K}^{a}_{nr,a}, \tag{38a}$$

$$\underset{B}{K}_{sar} = -\left(q^2/4\right) w^{-4} \left[w^{-1}\left(3-4W\right)\left(v_s \times k_a\right) k_r + 4\left(a_s \times k_a\right) k_r + g_{rs}k_a - g_{ra}k_s\right], \tag{38b}$$

which means that $\underset{B}{T}_{ij}$ is the divergence of $\underset{B}{K}_{sar}$. These idea of the superpotential isn't Weert's original, actually it is quite old and was introduced by Freud [36] constructing the superpotential for the canonical energy-momentum pseudotensor of Einstein [37, 38].

Weert didn't study deeply the algebraic and differential properties of $\underset{B}{K}_{sar}$ which was remedied in [33, 39-41] obtaining a better comprehension of such superpotential structure:

$$\begin{aligned} \underset{B}{K}_{sar} &= -\underset{B}{K}_{sar} && \text{Antisymmetry} \\ \underset{B}{K}^{r}_{sr} &= 0 && \text{Null trace} \\ \underset{B}{K}^{r}_{sa,r} &= 0 && \text{Null divergence} \\ \underset{B}{K}_{sar} + \underset{B}{K}_{ars} + \underset{B}{K}_{rsa} &= 0 && \text{Cyclic} \end{aligned} \tag{39}$$

Surprisingly, (39) is also satisfied in curved spaces (replacing partial derivatives with covariant ones) for the Lanczos spintensor K_{sar} [42], which generates the Weyl conformal tensor in 4 dimensions [43-49]:

$$C_{jrim} = K_{jri;m} - K_{jrm;i} + K_{imj;r} - K_{imr;j} + g_{jm}K_{ir} - g_{ij}K_{mr} + g_{ri}K_{mj} - g_{rm}K_{ij} \tag{40a}$$

so that $K_{rj} = K_{jr} = K^{a}_{rj;a}$.

This fact suggests at least two things:

(1) The introduction in electrodynamics of the definition:

"A Minkowski spintensor is that which satisfies (39)", (40b)

so, the Weert superpotential is a Minkowskian spintensor.

(2) To construct an "Electromagnetic Weyl tensor" through prescription (40.a) (in this case $\underset{B}{K}_{rj} = \underset{B}{T}_{rj}$):

$$\underset{B}{C}_{jrim} = \underset{B}{K}_{jri,m} - \underset{B}{K}_{jrm,i} + \underset{B}{K}_{imj,r} - \underset{B}{K}_{imr,j} + g_{jm}\underset{B}{T}_{ir} - g_{ij}\underset{B}{T}_{mr} + g_{ri}\underset{B}{T}_{mj} - g_{rm}\underset{B}{T}_{ij} \tag{40c}$$

The Petrov classification [13] can be applied to $\underset{B}{C}_{jrim}$, see [39, 41], resulting Type II in the Penrose diagram, that is:

$$\text{“The Liénard-Wiechert field is type II”;} \tag{40d}$$

this strengths the analogies found by Newman [50] between Robinson-Trautman metrics (Einstein's equations solution type II) [51] and the electromagnetic field of a point charge. The physical meaning of the Weert superpotential was elucidated in [40].

The idea (40.b) motivates the following question:

Can $\underset{B^{sar}}{K}$ be written like the sum of two or more Minkowskian spintensors?

The answer is affirmative because the terms in (38.b) can be grouped in the form [33]:

$$\underset{B^{sar}}{K} = \underset{B^{sar}}{\tilde{K}} + \underset{B^{sar}}{\bar{K}} \tag{41a}$$

with

$$\underset{B^{scr}}{\tilde{K}} = qw^{-2}\left[qw^{-3}\left(v_s \times k_c\right) - F_{sc}\right]k_r, \tag{41b}$$

$$\underset{B^{sar}}{\bar{K}} = \left(q^2/4\right)w^{-4}\left[3w^{-1}\left(v_a \times k_s\right)k_r + g_{ra}k_s - g_{rs}k_a\right]. \tag{41c}$$

Both parts of $\underset{B^{sar}}{K}$ satisfy (39), so, they are spintensors. By substituting (41.a) in (38.a) we obtain of natural manner the splitting of Lopez [52]:

$$\underset{B^{ra}}{T} = \underset{B^{ra}}{\tilde{T}} + \underset{B^{ra}}{\bar{T}} \tag{42a}$$

where

$$\underset{B^{rs}}{\tilde{T}} = \underset{B^{sr,a}}{\tilde{K}^a}, \quad \underset{B^{rs}}{\bar{T}} = \underset{B^{sr,a}}{\bar{K}^a} \tag{42b}$$

so

$$\underset{B^{r,s}}{\tilde{T}^s} = \underset{B^{r,s}}{\bar{T}^s} = 0. \tag{42c}$$

Decomposition (42.a) is valuable in the study of electromagnetic angular momentum, here it came out as a consequence of the spintensor concept(40.b).

It can be proven that:

$$\underset{B^{sar}}{\bar{K}} = \left(\frac{q^2}{4}w^{-4}D^b_{sar}\right)_{,b} = (41.\text{c})\,, \tag{43a}$$

where D_{ijrm} is a tensor employed by Synge [7] in other context:

$$D_{sarb} = g_{rs}k_a k_b - g_{ab}k_r k_s - g_{ar}k_s k_b - g_{sb}k_a k_r. \tag{43b}$$

Identity (43.a) was obtained by Rowe [53].

Weert didn't study (35.a) : Its analysis was considered in [54-60] to determine a non-local superpotential (it depends on integrals over the world line) for the radiative part:

$$\underset{R^{rs}}{T} = \underset{R^{sr,a}}{K^a} \tag{44a}$$

with

$$\underset{R^{scr}}{K}(X^i) = qF_{sc}[p_{(\sigma)}p_{(\theta)}\left(\int_0^\tau a^{(\sigma)}a^{(\theta)}v_r d\gamma + p_{(\beta)}\int_0^\tau a^{(\sigma)}a^{(\theta)}e_r^{(\beta)}d\gamma\right) -$$
$$- \int_0^\tau a^2 v_r d\gamma - p_{(\sigma)}\int_0^\tau a^2 e_r^{(\sigma)} d\gamma], \quad \sigma,\theta,\beta = 1,2,3 \tag{44b}$$

where $e^{(\sigma)}$ is the Fermi triad and τ is the proper time in the retarded point associated to X^i. Trying out (44.a) brings into relevance the identities (24); the integrals in (44.b) indicate the non-local character of radiative superpotential; besides, if the 4-acceleration a^r is annulated, then $\underset{R^{ijc}}{K} = 0$ which was to be expected due to (26). When obtaining (44) it is basic the transport (9.a); never before had been shown the great value of Fermi triad in electrodynamics.

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