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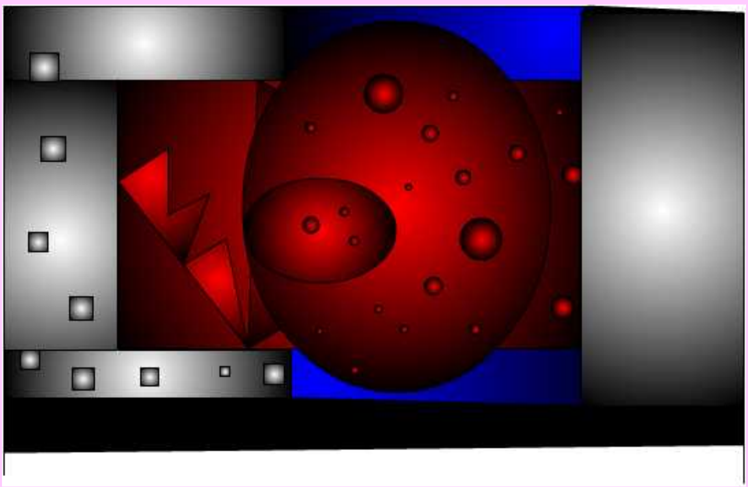

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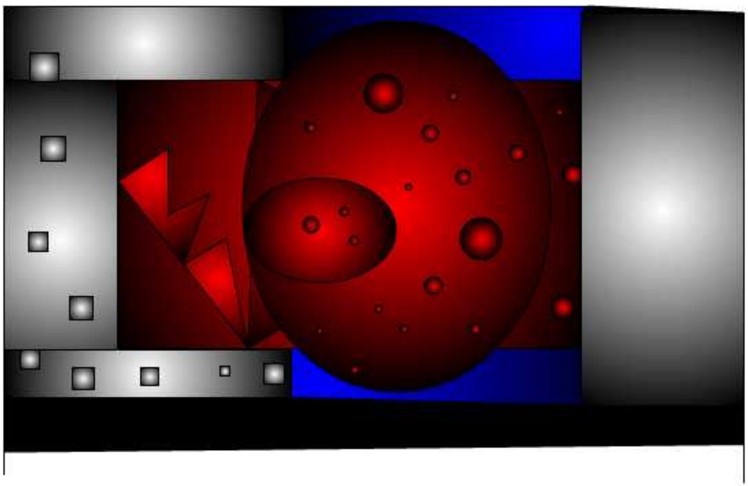

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Liénard-Wiechert Electromagnetic Field

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Abstract: The electromagnetic field generated by a charge in arbitrary motion in Minkowski space is briefly studied. Particularly important is the deduction of the superpotential for the radiative part of Maxwell tensor.

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1. Introduction

A charge in arbitrary motion in special relativity generates the Liénard-Wiechert retarded potential A_r and its corresponding Faraday tensor F_{rs}, of fundamental importance in point particle electrodynamics. Accordingly, we will dedicate Section I to the study of scalar and vectorial quantities associated to the world line of the charge, with special emphasis in retarded distance and the light cone: the trajectory's kinematics forms a powerful platform for the analysis of the electromagnetic field. Additionally, the valuable Fermi's triad is introduced.

In Section II we consider general aspects regarding 4-potential and Faraday tensor, bringing into Synge classification [1] and an attractive theorem of Stachel [2]. Section III concerns to the Liénard-Wiechert case, obtaining Teitelboim [3], Miglietta [4] and Teitelboim [5] decompositions of A_r and F_{cb}, respectively. We also deduce Plebañski's interesting result [6]: Fij is the antisymmetric product of two gradients.

Section IV deals with retarded Maxwell tensor, its study is channeled through the algebraic-differential properties of its $\underset{\mathrm{B}}{\mathrm{T}_{\mathrm{ic}}}$ and radiative $\underset{\mathrm{R}}{\mathrm{T}_{\mathrm{ab}}}$ parts.

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A thoughtful analysis of Weert superpotential structure is made for $\underset{B}{T_{rs}}$, and it is shown the non-local superpotential for $\underset{R}{T_{ij}}$.

In whole our study the Synge article [7] is fundamental for the mathematical aspects of point particles electrodynamics.

2. Kinematics of Relativistic Particles

In special relativity, a "particle" means a timelike world line, see Fig. 1, whose unitary tangent vector is the 4-velocity.

$$v^r = dx^r/d\tau \tag{1a}$$

where the proper time t is defined by

$$d\tau^2 = g_{ab}dx^a dx^b = -dx^2 - dy^2 - dz^2 + dt^2 \tag{1b}$$

which means that the metric is Diag(-1, -1,- 1, 1) and c = light's speed in vacuum=1, then

$$(v^r) = (\gamma\vec{v}, \gamma) \quad \text{with} \vec{v} = (dx/dt, dy/dt, dz/dt) \quad \text{and} \quad \gamma = (1 - \vec{v}^2)^{1/2} \tag{1c}$$

So, out of (1.a,b) we have that:

$$v^r v_r = -1, \quad v^r a_r = 0 \quad \text{with} \quad a^r = dv^r/d\tau = 4 - \text{acceleration}, \tag{1d}$$

this implies the timelike and spacelike nature of v^r and a^r, respectively; in consequence:

$$a^2 \equiv a_r a^r \geq 0 \tag{1e}$$

From (1.d) we obtain:

$$s^r v_r + a^2 = 0, \quad \text{where} \quad s^r = da^r/d\tau = \text{superacceleration} \tag{1f}$$

In Fig 1. we have not indicated this last vector since it might be outside or inside the light cone.

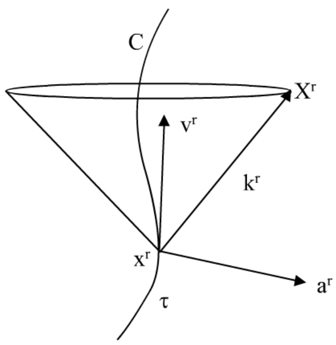

Fig. 1 Timelike trajectory

From an event X^r outside C we trace its null cone's past sheet which intersects to C in the point x^r called "retarded event associated with X^r", so we say that

$$x^r = x^r \left(X^i \right) \tag{2a}$$

because with X^i given, the retarded point over C is automatically determined. This allows to introduce the vector:

$$k^r = X^r - x^r \tag{2b}$$

whose magnitude is zero for resting over the cone:

$$k^r k_r = 0, \tag{2c}$$

so, k^r indicates the propagation direction of the photons emitted by the particle. The null or light type vector (2.b) is truly important in electrodynamics: we could say that studying the Maxwell field is almost equivalent to an analysis of the null cone and its relation to the world line.

From X^r we can build two distances widely used in the study of charges in Minkowski space:

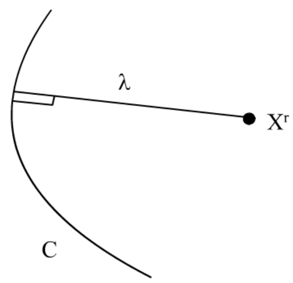

Fig. 2a Instantaneous distance from X^r to C.

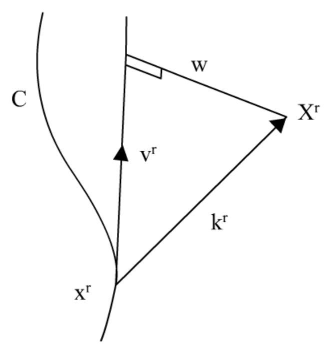

Fig. 2b Retarded distance from X^r to C.

The instantaneous distance, see fig. 2.a, introduced by Dirac [8] is geometrically simpler than retarded distance w, see fig. 2.b, proposed by Bhabha [9] and furthered by Synge [7]; nevertheless, l has the big disadvantage of not involving retarded effects (light cone); for this reason, w has more physical meaning and leads to simpler calculations because it intrinsically takes in account the finite velocity of interaction. Here we will work only with w, its expression is given by:

$$w = -k^r v_r \geq 0. \tag{3a}$$

Bringing to mind that a null vector cannot be orthogonal to a timelike one, (3.a) points that:

$$w = 0 \quad \text{if and only if} \quad k^r = 0, \tag{3b}$$

in other words, the retarded distance is zero only when X^r is over C.

When making calculations, we will need to know how change diverse quantities over C when an external event X^r varies, for this, its enough with having change's law for t because x^r, v^r, a^r, etc. are functions of these parameter:

$$\tau_{,r} = -w^{-1} k_r \quad \text{where} \quad ,r = \partial / \partial X^r, \tag{4}$$

so we have that t, r is a null vector because it is anti-parallel to k^r. Every event X^r over the same cone has an associated unique value of t, that is, the light cone is the t=constant surface, so t, r is the vector normal to the cone even though our Euclidian eyes don't see it like that. Due to (4) it has no sense looking for a unitary normal to the cone. Thanks to (4) it is easy to obtain the useful relationships:

$$x^r_{,j} = -w^{-1} v^r k_j, \quad v^r_{,j} = -w^{-1} a^r k_l, \quad a^r_{,b} = -w^{-1} s^r k_b$$

$$k^r_{,C} = \delta^r_C + w^{-1} v^r k_C, w_{,C} = -v_C + B k_c$$

$$\text{with} \quad B = w^{-1}\left(1 - W\right), \quad W = -k^r a_r$$

$$W_{,b} = W_b = -a_b + w^{-1} k^r s_r k_b \tag{5}$$

$$B_{,C} = w^{-1}\left[U_C - \left(B^2 + w^{-1} k^r s_r\right) k_c\right], U_C = B v_C + a_C$$

$$U^C k_c = -1, U^C v_C = -B, U^C a_C = a^2$$

$$U^C U_C = a^2 - B^2, U^c w_{,c} = 0, U^C_{,C} = 0.$$

In relativity, a spatial triad of vectors is also important in each point of the curve because this triad is a local frame of reference for an observer mounted in the particle. See Fig. 3:

$$\left(e^r_{(a)} e_{(b)r}\right) = Diag\left(1, 1, 1, -1\right) \tag{6a}$$

This tetrad forms a base for each vector associated to world line, in particular for null vector (2.b):

$$k^r = b^\sigma e^r_{(\sigma)} + b^4 e^r_{(4)}; \tag{6b}$$

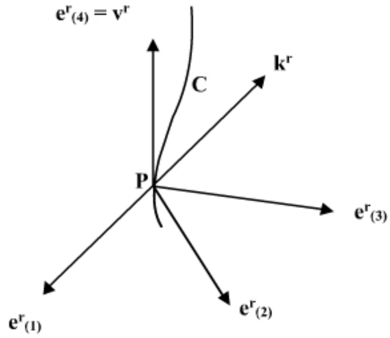

Fig. 3 Orthonormal tetrad

from now on the Greek indexes will only take values 1, 2, and 3. Expansion (6.b) can be written like:

$$k^r = M^r + b^4 v^r \quad \text{with} \quad M^r = b^\sigma e^r_{(\sigma)}, M^r v_r = 0, \tag{6c}$$

M^r is spacelike type because it is a lineal combination of the three spacelike vectors of the tetrad, see Fig. 4:

If $M = (M^r M_r)^{1/2}$ is the magnitude of M^r, then

$$M^r = M p^r \quad \text{with} \quad p^r p_r = 1 \tag{6d}$$

and by (6.c):

$$p_r v^r = 0, \tag{6e}$$

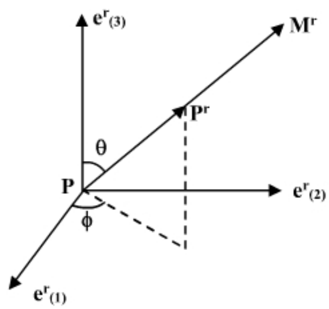

Fig. 4b Spatial triad

so p^r is a spacelike unitary vector. From (2.c, 3.a) its plain that $M = b^4 = w$, as a consequence (6.b,...,e) implies the important Synge [7] - Teitelboim [3] decomposition for k^r:

$$k^r = w\left(p^r + v^r\right), \quad p^r v_r = 0, \quad p_r k^r = w, \tag{7}$$

which is shown in the following two figures:

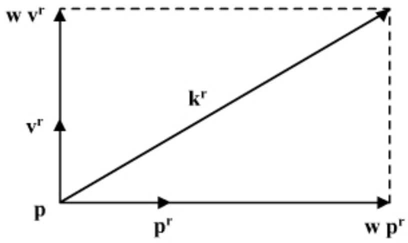

Fig. 5a Spatial and timelike components of k^r.

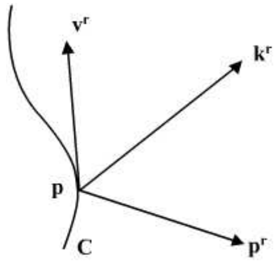

Fig. 5b Null vector k^r splitting

The unitary vector p^r only depends of the spatial triad; so it can be written with the common spherical coordinates, see Fig. 4:

$$p^r = \sin\theta\cos\phi e^r_{(1)} + \sin\theta\sin\phi e^r_{(2)} + \cos\theta e^r_{(3)} = p^{(\sigma)} e^r_{(\sigma)} \tag{8a}$$

$$\therefore p^{(\gamma)} = e^{(\gamma)}_r p^r$$

where we have employed the dual base e^r_γ, defined by:

$$e^{(\gamma)r} e_{(\sigma)r} = \delta^\gamma_\sigma \tag{8b}$$

Vector p^r doesn't have to be necessarily orthogonal to 4-acceleration a^r.

The triad e^r_σ is arbitrary except by the orthonormality conditions (6.a); nevertheless, some triads may be more convenient than others in some set calculations. For our theoretical purposes, the Fermi triad [10] is very important, it satisfies over C the transport law (which we will use in this work):

$$de^r_{(\sigma)}/d\tau = e^b_{(\sigma)} a_b v^r = a_{(\sigma)} v^r \tag{9a}$$

This type of transport has been very fundamental in gravitation, for example Pirani [11] and Synge [12]; but, in electrodynamics we shall show its participation in the deduction of the superpotential for the radiative part of Maxwell tensor, see Section IV. In (9.a) we have used the notation:

$$a_{(\sigma)} = a^r e_{(\sigma)r} \tag{9b}$$

because a^r is spacelike type; to remember that the triad is only defined over C.

To end this Section, we give some useful expressions:

$$w_{,b}k^b = w, \quad w_{,b}v^b = W, \quad w_{,b}a^b = -WB$$

$$w^{,b}w_{,b} = 1 - 2W, \quad W = -wp^r a_r = -wp^{(\sigma)}a_{(\sigma)}$$

$$W_{,c}k^c = W, \quad W_{,c}w^{,c} = WB + k^r s_r, \quad k^r_{,a}v^a = 0$$

$$B_{,c}k^c = -w^{-1}, \quad w_{,c}p^c = wB, \quad w^{-1,a}_{,a} = 0$$

$$w^{,a}_{,a} = 2w^{-1}\left(1 - 2W\right), \quad k^r_{,a}p_r = p_a, \quad U^r p_r = -w^{-1}W \tag{10}$$

$$p^r_{,a} = w^{-1}\left[\delta^r_a + w^{-1}v_a k^r + \left(a^r + w^{-1}v^r - w^{-1}Bk^r\right)k_a\right]$$

$$p^r_{,a}k^r = -w^{-1}Wk_a, p^r_{,a}w_{,r} = w^{-2}W^2k_a, p_{(\sigma),r}k^r = 0$$

$$p^r_{,a}k^a = 0, \quad p^r_{,r} = w^{-1}\left(2 - W\right), \quad p_{(\sigma),r} = p_{a,r}e^a_{(\sigma)}$$

The relations (5,10) are the basic formulary for any calculation in the electrodynamics of classical charged particles.

3. 4-Potential and Faraday Tensor

In this Section we consider the algebraical and differential properties satisfied by the electromagnetic field in vacuum. Faraday tensor is given by:

$$F_{rc} = -F_{cr} = A_{c,r} - A_{r,c} \tag{11a}$$

in terms of the 4-potential A^b. From (11.a) it is clear the fulfillment of the cyclic relationship:

$$F_{br,c} + F_{rc,b} + F_{cb,r} = 0, \quad \text{in other words} \quad {}^*F^{rc}_{,c} = 0 \tag{11b}$$

where we have employed the dual tensor:

$${}^*F^{rc} = -{}^*F^{cr} = \frac{1}{2}\varepsilon^{rcab}F_{ab} \tag{11c}$$

being ε^{ijkr} the Levi-Civita antisymmetric symbols. In free space we have the remaining Maxwell equation:

$$F^{rc}_{,c} = 0 \tag{11d}$$

which in turn leads to a differential equation for the 4-potential:

$$A^{r;c}_{,c} - \left(A^{c}_{,c}\right)^{,r} = 0 \tag{11e}$$

In (11.a) we have full freedom to add to A_r an arbitrary gradient without modifying Faraday tensor; then without lack of generality we can always demand:

$$A^{c}_{,c} = 0 \quad \text{Lorenz condition,} \tag{11f}$$

simplifying (11.e):

$$A^{r;c}_{,c} = 0 \quad \text{Wave equation} \tag{11g}$$

So, from the mathematical point of view, the problem consists in solving (11.g) with the restriction, which matches to solving (11.b,d), in other words:

$$\begin{array}{ll} \nabla \cdot \vec{B} = 0, & \nabla \times \vec{E} = -\partial \vec{B}/\partial t \\ \nabla \cdot \vec{E} = 0, & \nabla \times \vec{B} = \partial \vec{E}/\partial t \end{array} \tag{12}$$

in the MKS system; to remember that $c = (\varepsilon_0 \mu_0)^{-1/2} = 1$.
Faraday matrix representation turns like:

$$\left[F^{ab}\right] = \left[\begin{array}{cccc} 0 & B_z & -B_y & -E_x \\ -B_z & 0 & B_x & -E_y \\ B_y & -B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{array}\right] \begin{array}{l} a : rows \\ \\ \\ b : columns \end{array} \tag{13a}$$

and with (11.c) an associated matrix for the dual tensor can be constructed:

$$\left[{}^{*}F^{ab}\right] = \left[\begin{array}{cccc} 0 & E_z & -E_y & B_x \\ -E_z & 0 & E_x & B_y \\ E_y & -E_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{array}\right] \tag{13b}$$

Note that (13.b) is obtained if we do the following changes to (13.a):

$$\vec{B} \rightarrow \vec{E}, \quad \vec{E} \rightarrow -\vec{B} \tag{13c}$$

so it may come to mind that * executes the operation (13.c); then it is clear that $^{**}F^{ar} = -F^{ar}$. Comparing (11.a, 13.a) we obtain the relationship of the electric and magnetic fields with the 4-potential:

$$\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t, \vec{B} = \nabla \times \vec{A} \tag{14}$$

because $(A^r) = \left(\vec{A}, \phi\right)$,

where $\vec{A}$ and ϕ are the magnetic and electric potentials, respectively. We are placing emphasis in f as a scalar, but not as an invariant: the electromagnetic field only possesses two Lorentz invariants, namely

$$F_1 \equiv F_{ab}F^{ab} = 2\left(\tilde{B}^2 - E^2\right), \quad F_2 \equiv^* F_{ab}F^{ab} = 4\vec{E}\cdot\vec{B} \tag{15}$$

with $E = \left|\vec{E}\right|$ and $\tilde{B} = \left|\vec{B}\right|$.

Just like Weyl tensor invariants allow to establish the Petrov classification [13] for the gravitational field, the quantities (15) lead to the Synge [1] – Piña [14] classification for the Faraday tensor:

$$\begin{array}{llll} \text{Type A:} & F_{\ 2} \neq 0 & & \\ \text{Type B:} & F_{\ 1} < 0, & F_2 = 0 & \\ \text{Type C:} & F_{\ 1} = 0, & F_2 = 0 & \text{Null field} \\ \text{Type D:} & F_{\ 1} > 0, & F_2 = 0 & \end{array} \tag{16}$$

A point with a null field means that in such event $\vec{E}\perp\vec{B}$ and $E = \tilde{B}$. Non-null field implies a different type of C. The classification (16) is algebraic but the type of electromagnetic field may change from one point to another.

Furthermore, we will see that the field that produces a relativistic charge is B type, which tends to type C(plane wave) towards infinity.

There are very important identities for Maxwell field:

$$F^{ar}F_{br} - {}^*F^{ar*}F_{br} = (F_1/2)\,\delta^a_b \tag{17a}$$

$$^*F^{ar}F_{br} = (F_2/4)\,\delta^a_b \tag{17b}$$

which do not have an specific name, are well known and can be found in Rainich [15], Plebañski [16], Wheeler [17] pp. 239, Penney [18] and Piña [14]. Expressions (17) correspond to Lanczos identities [18] between the Riemann tensor and its double dual. If (17.a) is multiplied by F_{ia} or $^*F_{ia}$ and (17.b) is employed, results in valuable identities in the calculation of an antisymmetric matrix's exponential function [14, 20]:

$$\begin{aligned} F_{ia}F^{ar}F_{rb} &= (F_1/2)\,F_{bi} + (F_2/2)\,{}^*F_{bi} \\ {}^*F_{ia}{}^*F^{ar*}F_{rb} &= (F_2/4)\,F_{bi} - (F_1/4)\,{}^*F_{bi} \end{aligned} \tag{18}$$

From (13.a) it is simple to show that:

$$\det\left(F^{ab}\right) = (1/16)\,(F_2)^2 = \left(\vec{E}\cdot\vec{B}\right)^2, \tag{19}$$

which is a particular case of the following theorem - see Drazin [21]:

"The $\det \underset{\sim}{F}$, with antisymmetric $\underset{\sim}{F}_{n\times n}$ and even n, is the square of a rational polynomial in F^{ij}"

Now we mention the interesting and useful Stachel theorem [2] page 1261:

' If $\underset{\sim}{F}$ satisfies:

$$F_{ar} = -F_{ra}, \quad F_{ar,c} + F_{rc,a} + F_{ca,r} = 0, \tag{20a}$$

$$\det\left(F_{ab}\right) = 0$$

then there exist functions β and ψ such that:

$$F_{ar} = \beta_{,a}\psi_{,r} - \beta_{,r}\psi_{,a}\text{'}. \tag{20b}$$

That is, the conditions (20.a) reduce $\underset{\sim}{F}$ to an antisymetric product of gradients. If we extend the Stachel result to Maxwell field, then the first two conditions (20.a) will be immediately verified, thereby:

$$\text{'If Faraday tensor fulfills} \quad F_2 = 0, \quad \text{then it is of the form'}. \tag{20c}$$

Liénard Wiechert solution will satisfy (20.c), it will permit to write $\underset{\sim}{F}$ in the form of Plebañski [6]. In general, an electromagnetic field with a different type of A has the structure (20.b). The results (20) are valid in presence of curvature because its differential expressions remain undisturbed if covariant derivatives are used instead of partial ones:

$$^*F^{ar}_{,r} = {}^*F^{ar}_{;r} = 0, \quad \beta_{,r} = \beta_{;r}$$

In the following Section we will employ the exposed material in (11,. . . ,20) in the analysis of the field produced by a point charge with a relativistic trajectory.

4. Liénard-Wiechert Field

Solution of (11.f,g) for a particle in arbitrary motion in Minkowski space was obtained by Liénard and Wiechert; the corresponding potential carries their names and is given by

$$A^r\left(X^b\right) = qw^{-1}v^r, \quad q = \text{charge}/4\pi\varepsilon_0 \quad \text{Retarded potential} \tag{21a}$$

which is fundamental in everything that follows; by the use of (11.a) it is simple to calculate the associated Faraday tensor :

$$F_{rb} = qw^{-2}\left(U_r k_b - U_b k_r\right) = qw^{-2}U_r \times k_b \tag{21b}$$

where $\times$ means the antisymetric product. This notation is due to Lowry [22] and will make such expressions to be very compact. From (11.c, 21.b) it is clear that:

$$^{*}F_{mn} = -qw^{-2}\varepsilon_{mnab}U^{a}k^{b} \tag{21c}$$

therefore

$$F_1 = -2q^2w^{-4} < 0, \quad F_2 = 0 \tag{21d}$$

in other words, the electric and magnetic fields generated by the charge satisfy:

$$\tilde{B} < E, \quad \vec{E}\cdot\vec{B} = 0 \tag{21e}$$

in consequence $\underset{\sim}{F}$ is type B. Note that in the asymptotic region ($w \to \infty$), the invariant F_1 tends to zero, which means that $\underset{\sim}{F}$ is close to the null case (type C) far away from the charge.

With (21.d), (20.c) is valid, so the Stachel theorem [2] implies that (21.b) can be reduced to (20.b). This is easy to do because from (4,5) the following relationships are available:

$$k_r = -w\tau_{,r}, \quad U_r = wB_{,r} + \left(B^2 + w^{-1}k^c s_c\right)k_r$$

which substituted in (21.b) imply:

$$F_{rc} = -qB_{,r}x\tau_{,c} = q\left(\tau_{,r}B_{,c} - \tau_{,c}B_{,r}\right) \tag{22}$$

which has the form (21.b), meaning that t and B correspond to the functions b and y. Expression (22) was first obtained by Plebañski [6].

Now we will consider the eigenvalue problem of $\underset{\sim}{F}$. For this purpose we stem from (21.b), and due to

$$U_rk^r = -1, \quad k_rk^r = 0$$

then immediately we have one of the two null proper vectors of a non-null field (different type from C) [23]:

$$F_{rm}k^m = qw^{-2}k_r, \quad \text{proper value} = qw^{-2}. \tag{23a}$$

This suggests that U_r may be an eigenvector, but it isn't:

$$F_{rb}U^b = -qw^{-2}U_r - qw^{-2}\left(a^2 - B^2\right)k_r, \tag{23b}$$

nevertheless, if we multiply (23.a) by $\frac{1}{2}\left(a^2 - B^2\right)$ and add the resulting equation with (23.b) we obtain the other null proper vector [24, 25] :

$$F_{rm}\eta^m = -qw^{-2}\eta_r, \quad \text{proper value} = -qw^{-2}$$

with

$$\eta^r = U^r + \tfrac{1}{2}\left(a^2 - B^2\right) k^r, \quad \eta^r \eta_r = 0; \tag{23c}$$

it's not usual to find explicitly η^r in the literature. It is clear that these two proper vectors are independent because:

$$k^r \eta_r = 1. \tag{23d}$$

To remember that two null vectors ξ^r and γ^r are proportional if and only if $\xi^r \gamma_r = 0$, so (23.d) implies the non-parallelism of such proper vectors.

With (10,21.b) it is possible to prove that:

$$F^b_r p_{c,b} = qw^{-4} W v_c k_r, \quad F^b_r p_{(\sigma),b} = 0, \tag{24}$$

of great importance in the next Section in the deduction of the radiative superpotential.

Teitelboim started an era in electrodynamics by employing only retarded fields and studying Faraday and Maxwell tensors near and away of a point charge. This type of analysis is generated by substituting (5,7) in (21.b) to obtain the decomposition:

$$F_{rb} = \underset{(-1)^{rb}}{F} + \underset{(-2)^{rb}}{F}, \tag{25a}$$

where:

$$\underset{(-1)^{rb}}{F} = qw^{-2}\left(w^{-1} W v_r + a_r\right) \times k_b \tag{25b}$$

$$= qw^{-1}\left(a_c p^c v_r \times p_b + a_r \times v_b + a_r \times p_b\right) \tag{25c}$$

$$\underset{(-2)^{rb}}{F} = qw^{-3} v_r \times k_b = qw^{-2}\left(v_r \times p_b\right) \tag{25d}$$

meaning that, $\underset{(-i)^{rb}}{F}, i = 1, 2$ varies like w^{-i}: the dependence in w^{-i} is clear because the parenthesis in (25.c,d) are independent of the retarded distance, their terms are functions of x^r which remains stationary when we move away over the light cone. Thus $\underset{(-1)^{rb}}{F}$ and $\underset{(-2)^{rb}}{F}$ are dominant away (w>>1) and near (w<<1), respectively, then $\underset{(-1)^{ij}}{F}$ being the responsible of Larmor formula which provides the radiation speed towards infinitum. Note that (25.b,c) depend on the acceleration of the particle, which is an expected result because the Schild theorem [26]:

$$\text{“Radiation exists if and only if } a^r \neq 0\text{”.} \tag{26}$$

Schild was the first author to give a covariant definition of radiation even though some of his ideas were already implicit in Synge [27], Appendix A, whose 1st edition was made in 1955. We will call $\underset{(-1)^{ij}}{F}$ the radiative part of F_{ab} because it is a null field in classification (16):

$$\underset{(-1)^{ra}}{F}\ \underset{(-1)}{F^{ra}} = \ {}^*\underset{(-1)^{ra}}{F}\ \underset{(-1)}{F^{ra}} = 0, \tag{27a}$$

which doesn't happen with $\underset{(-2)^{ij}}{F}$:

$$\underset{(-2)^{ra}}{F}\ \underset{(-2)}{F^{ra}} = 2q^2w^{-4} < 0, \qquad {}^*\underset{(-2)^{ra}}{F}\ \underset{(-2)}{F^{ra}} = 0 \tag{27b}$$

which belongs to type B. This portion will be designated as the bounded part of F_{mni}, besides

$$\underset{(-1)^{ab}}{F}\ \underset{(-2)}{F^{ab}} = 0, \qquad {}^*\underset{(-1)^{ab}}{F}\ \underset{(-2)}{F^{ab}} = \underset{(-1)^{ab}}{F}\ {}^*\underset{(-2)}{F^{ab}} = 0 \tag{27c}$$

The relations (27) can be found in Weert [28], page 465.

It is possible to write (25.a) in the form:

$$F_{ab} = w^{-1}N_{ab} + w^{-2}M_{ab} \quad \text{such that} \quad N_{ab} = w \underset{(-1)^{ab}}{F}, \quad M_{ab} = w^2 \underset{(-2)^{ab}}{F} \tag{28}$$

with the properties:

$$N_{ab}\xi^b = 0, \quad M_{ab}\xi^b = q\xi_a, \quad \xi_r\xi^r = 0, \quad \xi_r = w^{-1}k_r = -\tau_{,r}$$

so we see that (28) are coherent with (1.1, 2, 3) of Goldberg-Kerr theorem [29] for the asymptotic behavior of the electromagnetic field.

Teitelboim's decomposition (25.a) is fundamental in everything that follows, and the interesting is that (7) generates of natural manner such splitting:

$$v^r = w^{-1}k^r - p^r$$

which substituting in (21.a) gives

$$A^r = \underset{1}{A^r} + \underset{2}{A^r} \quad \text{with} \quad \underset{1}{A^r} = -qw^{-1}p^r, \quad \underset{2}{A^r} = qw^{-2}k^r \tag{29a}$$

This partition of Liénard-Wiechert 4-potential is found in the Teitelboim [3] well-known article, however, it was also published by Migglietta [4] not-knowing the Ref.[3]. Expressions (29.b) are simpler than Migglietta's (2.3, 3.2). Placing (29.a) into (11.a) we obtain the matching Faraday's tensor decomposition (25.a) with:

$$\underset{(-i)^{bc}}{F} = \underset{i^{c,b}}{A} - \underset{i^{b,c}}{A}, \quad i = 1, 2 \tag{29b}$$

which means that each part of F_{mn} has its own 4-potential. At last, it can be verified that (29.a) does not satisfy the Lorenz condition(11.f):

$$\underset{1}{A^r_{,r}} = -\underset{2}{A^r_{,r}} = -qw^{-2} \neq 0 \tag{29c}$$

5. Energy-momentum Tensor

Now we will consider the Maxwell tensor T_{rb} through which an electromagnetic field's content of energy-momentum is quantified:

$$T_{ab} = \tfrac{1}{2}\left(F_{ac}F^{c}_{b} + {}^{*}F_{ac}{}^{*}F^{c}_{b}\right), \tag{30a}$$

which satisfies

$$T_{ab} = T_{ba} \qquad \text{Symmetry} \tag{30b}$$

$$T^{r}_{r} = 0 \qquad \text{Null trace} \tag{30c}$$

$$T_{ac}T^{c}_{b} = \tfrac{1}{4}\left(T_{mn}T^{mn}\right)g_{ab} \qquad \text{Rainich identity} \tag{30d}$$

Symmetry (30.b) is a property of every energy tensor, (30.c) tells us that the field is made of particles with null mass at rest, photons in this case; (30.d) was obtained by Rainich [15].

If we employ (17.a) in the second term of (30.a) we obtain an alternative expression for Maxwell tensor:

$$T_{ab} = F_{ac}F^{c}_{b} - (F_1/4)\,g_{ab}; \tag{30e}$$

by substitution of (21.d, 25) in (30.e) it results in the important Teitelboim splitting [5]:

$$T_{ab} = \underset{(-2)^{ab}}{T} + \underset{(-3)^{ab}}{T} + \underset{(-4)^{ab}}{T} \tag{31a}$$

where

$$\underset{(-2)^{rn}}{T} = \underset{(-1)^{rc}}{F}\;\underset{(-1)^{n}}{F^{c}} = q^2 w^{-4}\left(a^2 - w^{-2}W^2\right)k_r k_n \tag{31b}$$

$$\underset{(-3)^{rn}}{T} = \underset{(-1)^{rc}}{F}\;\underset{(-2)^{n}}{F^{c}} + \underset{(-1)^{nc}}{F}\;\underset{(-2)^{r}}{F^{c}} = q^2 w^{-4}\left[k_r a_n + k_n a_r + 2w^{-2}W k_r k_n - w^{-1}W\left(k_r v_n + k_n v_r\right)\right] \tag{31c}$$

$$\underset{(-4)^{rn}}{T} = \underset{(-2)^{rc}}{F}\;\underset{(-2)^{n}}{F^{c}} - (F_1/4)\,g_{rn} = q^2 w^{-4}\left[\tfrac{1}{2}g_{rn} + w^{-1}\left(v_r k_n + v_n k_r\right) - w^{-2}k_r k_n\right] \tag{31d}$$

with the properties:

$$\underset{(-2)^{rn}}{T}\,k^n = 0, \qquad \underset{(-3)^{rn}}{T}\,k^n = 0, \qquad \underset{(-4)^{rn}}{T}\,k^n = -\left(q^2/2\right)w^{-4}k_r \tag{31e}$$

From (31.a,e) it is clear that k^r is a null proper vector of Maxwell tensor:

$$T_{rn}k^n = \left(q^2/2\right) w^{-4} k_r, \tag{32}$$

which was to be expected due to (21.d, 23.a, 30.e):

$$T_{rn}k^n = -qw^{-2}F_{rn}k^n + \left(q^2/2\right) w^{-4}k_r = -\tfrac{1}{2}q^2w^{-4}k_r \quad \text{identical to (32).}$$

The notation $\underset{(-i)^{rn}}{T}$, $i = 2, 3, 4$ evokes that (31.b,c,d) vary like w^{-i}, in consequence:

$$\underset{(-2)^{ab}}{T} \quad \text{dominates when} w \to \infty \quad \text{(away from the charge)} \tag{33}$$

$$\underset{(-3)^{ab}}{T} \quad \text{and} \quad \underset{(-4)^{ab}}{T} \quad \text{dominates when} w \to 0 \quad \text{(close from q)}$$

So Larmor formula comes from $\underset{(-2)^{ab}}{T}$, and $\underset{(-i)^{ab}}{T}$, $i = 3, 4$ are responsible of the singularities in the point charge's position. Thus Teitelboim wrote (31.a) in the form:

$$T_{rn} = \underset{R^{rn}}{T} + \underset{B^{rn}}{T}, \tag{34a}$$

where

$$\underset{R^{rn}}{T} = \text{radiative part} = \underset{(-2)^{rn}}{T} = q^2w^{-4}\left(a^2 - w^{-2}W^2\right) k_r k_n \tag{34b}$$

$$\begin{aligned}\underset{B^{rn}}{T} = \text{bounded part} &= \underset{(-3)^{rn}}{T} + \underset{(-4)^{rn}}{T} \\ &= q^2w^{-4}\left[\tfrac{1}{2}g_{rn} + (k_r a_n + k_n a_r) + B\left(k_r v_n + k_n v_r\right) w^{-2}\left(1 - 2W\right) k_r k_n\right]\end{aligned}$$

and proves that such parts are dynamically independent, which means that they verify separately (outside the world line) :

$$\underset{R^{r,n}}{T^n} = 0, \tag{35a}$$

$$\underset{B^{r,n}}{T^n} = 0. \tag{35b}$$

It is simple to obtain the relations:

$$T_{rn}U^n = \lambda U_r, \quad T_{tn}B^{,n} = \lambda B_{,r}, \quad T_{rn}\eta^n = \lambda \eta_r, \quad \lambda = -\left(q^2/2\right) w^{-4} \tag{36}$$

so we have that F_{aj} and T_{rc} have the same null proper vectors, which is a general result, see Synge [27], p. 337. Plebański [6], p. 41, was the first one to observe that $B,\ r$ is a proper vector of T_{ij}. If we substitute (34.b,c) in (34.a) we obtain the Synge [7] compact expression for the energy tensor associated to Liénard-Wiechert retarded potential:

$$T_{rn} = q^2w^{-4}\left[k_r U_n + k_n U_r + \left(a^2 - B^2\right) k_r k_n + \tfrac{1}{2}g_{rn}\right]. \tag{37}$$

Weert [30, 31] attention was to lead towards the fact that (35) are valid *identically,* and therefore he suggested the existence of Superpotentials for the bounded and radiative parts. However, he only obtained successfully the explicit form of the superpotential (which now carries his name) $\underset{B^{sar}}{K}$ which generates the bounded part [32-35] :

$$\underset{B^{rn}}{T} = \underset{B^{nr}}{T} = \underset{B^{nr,a}}{K^{a}}, \tag{38a}$$

$$\underset{B^{sar}}{K} = -\left(q^2/4\right) w^{-4} \left[w^{-1} \left(3 - 4W\right) \left(v_s \times k_a\right) k_r + 4 \left(a_s \times k_a\right) k_r + g_{rs} k_a - g_{ra} k_s\right], \tag{38b}$$

which means that $\underset{B^{ij}}{T}$ is the divergence of $\underset{B^{sar}}{K}$. These idea of the superpotential isn't Weert's original, actually it is quite old and was introduced by Freud [36] constructing the superpotential for the canonical energy-momentum pseudotensor of Einstein [37, 38].

Weert didn't study deeply the algebraic and differential properties of $\underset{B^{sar}}{K}$ which was remedied in [33, 39-41] obtaining a better comprehension of such superpotential structure:

$$\begin{aligned} \underset{B^{sar}}{K} &= -\underset{B^{sar}}{K} && \text{Antisymmetry} \\ \underset{B^{sr}}{K^{r}} &= 0 && \text{Null trace} \\ \underset{B^{sa,r}}{K^{r}} &= 0 && \text{Null divergence} \\ \underset{B^{sar}}{K} + \underset{B^{ars}}{K} + \underset{B^{rsa}}{K} &= 0 && \text{Cyclic} \end{aligned} \tag{39}$$

Surprisingly, (39) is also satisfied in curved spaces (replacing partial derivatives with covariant ones) for the Lanczos spintensor K_{sar} [42], which generates the Weyl conformal tensor in 4 dimensions [43-49]:

$$C_{jrim} = K_{jri;m} - K_{jrm;i} + K_{imj;r} - K_{imr;j} + g_{jm}K_{ir} - g_{ij}K_{mr} + g_{ri}K_{mj} - g_{rm}K_{ij} \tag{40a}$$

so that $K_{rj} = K_{jr} = K^{a}_{rj;a}$.

This fact suggests at least two things:

(1) The introduction in electrodynamics of the definition:

$$\text{“A Minkowski spintensor is that which satisfies (39)”}, \tag{40b}$$

so, the Weert superpotential is a Minkowskian spintensor.

(2) To construct an “Electromagnetic Weyl tensor” through prescription (40.a) (in this case $\underset{B^{rj}}{K} = \underset{B^{rj}}{T}$):

$$\underset{B^{jrim}}{C} = \underset{B^{jri,m}}{K} - \underset{B^{jrm,i}}{K} + \underset{B^{imj,r}}{K} - \underset{B^{imr,j}}{K} + g_{jm}\underset{B^{ir}}{T} - g_{ij}\underset{B^{mr}}{T} + g_{ri}\underset{B^{mj}}{T} - g_{rm}\underset{B^{ij}}{T} \tag{40c}$$

The Petrov classification [13] can be applied to $\underset{B^{jrim}}{C}$, see [39, 41], resulting Type II in the Penrose diagram, that is:

$$\text{“The Liénard-Wiechert field is type II”;} \tag{40d}$$

this strengths the analogies found by Newman [50] between Robinson-Trautman metrics (Einstein's equations solution type II) [51] and the electromagnetic field of a point charge. The physical meaning of the Weert superpotential was elucidated in [40].

The idea (40.b) motivates the following question:

Can $\underset{B^{sar}}{K}$ be written like the sum of two or more Minkowskian spintensors?

The answer is affirmative because the terms in (38.b) can be grouped in the form [33]:

$$\underset{B^{sar}}{K} = \underset{B^{sar}}{\tilde{K}} + \underset{B^{sar}}{\bar{K}} \tag{41a}$$

with

$$\underset{B^{scr}}{\tilde{K}} = qw^{-2}\left[qw^{-3}\left(v_s \times k_c\right) - F_{sc}\right]k_r, \tag{41b}$$

$$\underset{B^{sar}}{\bar{K}} = \left(q^2/4\right)w^{-4}\left[3w^{-1}\left(v_a \times k_s\right)k_r + g_{ra}k_s - g_{rs}k_a\right]. \tag{41c}$$

Both parts of $\underset{B^{sar}}{K}$ satisfy (39), so, they are spintensors. By substituting (41.a) in (38.a) we obtain of natural manner the splitting of Lopez [52]:

$$\underset{B^{ra}}{T} = \underset{B^{ra}}{\tilde{T}} + \underset{B^{ra}}{\bar{T}} \tag{42a}$$

where

$$\underset{B^{rs}}{\tilde{T}} = \underset{B^{sr,a}}{\tilde{K}^{a}}, \quad \underset{B^{rs}}{\bar{T}} = \underset{B^{sr,a}}{\bar{K}^{a}} \tag{42b}$$

so

$$\underset{B^{r,s}}{\tilde{T}^{s}} = \underset{B^{r,s}}{\bar{T}^{s}} = 0. \tag{42c}$$

Decomposition (42.a) is valuable in the study of electromagnetic angular momentum, here it came out as a consequence of the spintensor concept(40.b).

It can be proven that:

$$\underset{B^{sar}}{\bar{K}} = \left(\frac{q^2}{4}w^{-4}D^{b}_{sar}\right)_{,b} = (41.\text{c}), \tag{43a}$$

where D_{ijrm} is a tensor employed by Synge [7] in other context:

$$D_{sarb} = g_{rs}k_a k_b - g_{ab}k_r k_s - g_{ar}k_s k_b - g_{sb}k_a k_r. \tag{43b}$$

Identity (43.a) was obtained by Rowe [53].

Weert didn't study (35.a) : Its analysis was considered in [54-60] to determine a non-local superpotential (it depends on integrals over the world line) for the radiative part:

$$\underset{R^{rs}}{T} = \underset{R^{sr,a}}{K^{a}} \tag{44a}$$

with

$$\underset{R^{scr}}{K}(X^i) = qF_{sc}[p_{(\sigma)}p_{(\theta)}\left(\int_0^\tau a^{(\sigma)}a^{(\theta)}v_r d\gamma + p_{(\beta)}\int_0^\tau a^{(\sigma)}a^{(\theta)}e_r^{(\beta)}d\gamma\right) - \tag{44b}$$

$$-\int_0^\tau a^2 v_r d\gamma - p_{(\sigma)}\int_0^\tau a^2 e_r^{(\sigma)} d\gamma], \quad \sigma,\theta,\beta = 1,2,3$$

where $e^{(\sigma)}$ is the Fermi triad and τ is the proper time in the retarded point associated to X^i. Trying out (44.a) brings into relevance the identities (24); the integrals in (44.b) indicate the non-local character of radiative superpotential; besides, if the 4-acceleration a^r is annulated, then $\underset{R^{ijc}}{K} = 0$ which was to be expected due to (26). When obtaining (44) it is basic the transport (9.a); never before had been shown the great value of Fermi triad in electrodynamics.

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On Conformal d'Alembert-Like Equations

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Abstract: Using conformal coordinates associated with projective conformal relativity we obtain a conformal Klein-Gordon partial differential equation. As a particular case we present and discuss a conformal 'radial' d'Alembert-like equation. As a by-product we show that this 'radial' equation can be identified with a one-dimensional Schrödinger-like equation in which the potential is exactly the second Pöschl-Teller potential.

Keywords: Projective Relativity; Hyperspherical Universes; Conformal Klein-Gordon Differential Equation; d'Alembert-like Equation; Second Pöschl-Teller Potential
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1. Introduction

After one of the most important of Einstein's papers [1] concerning Special Relativity was published, several alternative theories were proposed. Among them, some different interpretations and particular generalizations have been presented. In this paper we are interested in one such theory, namely the theory of hyperspherical universes, developed by Arcidiacono[2] several years ago and, more specifically, in the so-called conformal case.

When we write Maxwell equations in six dimensions, with six projective coordinates (we have, in these coordinates, a Pythagorean metric) a natural problem arises, namely, to provide a physical version of the formalism, i.e. to ascribe a physical meaning to the coordinates. For this theory, there are two possible different physical interpretations: a

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bitemporal interpretation and a biprojective interpretation. In the first case (bitemporal) we introduce a new universal constant c' and the coordinate $x_5 = ic't'$ where t' is interpreted as a second time; we thus obtain in cosmic scale the so-called multitemporal relativity, proposed by Kalitzen[3]. The set of Maxwell equations obtained in this theory generalizes the equations of the unitary theory of electromagnetism and gravitation, as proposed by Corben[4].

On the other hand (our second, biprojective case) we can interpret the extra coordinate, x_5, as a second projective coordinate. We then obtain the so-called conformal projective relativity, proposed by Arcidiacono[5, 6], which extends in cosmic scale the theory proposed by Ingraham[7], but with a different physical interpretation. In this theory we have another universal constant, r_0, which can be taken as $r/r_0 = N$, where r is the radius of the hypersphere and N is the cosmological number appearing in the Eddington-Dirac theory[8].

Here we consider only the second alternative, i.e., the biprojective interpretation. With this aim we introduce a projective space P_5 tangent to the hypersphere S^4. We then introduce six projective coordinates $\overline{x}_a$, with $a = 0, 1, \ldots, 5$ and normalized as

$$\overline{x}^2 + \overline{x}_0^2 - \overline{x}_5^2 = r^2,$$

where $\overline{x}^2 = x_i x^i$, $i = 1, \ldots, 4$ and r is the radius of the hypersphere. These coordinates allow us to construct the conformal projective relativity, using a six-dimensional tensor formalism.

This paper is organized as follows: in Section 1 we present a review of the so-called theory of hyperspherical universes, proposed by Arcidiacono, considering only the six-dimensional case, in which conformal projective relativity appears as a particular case. The choice of convenient coordinates and the link between the derivatives in these two formulations (a geometric version, six-dimensional, and a physical version, the five-dimensional conformal version) are also presented. After this review, we discuss in Section 2 a Klein-Gordon partial differential equation written in conformal coordinates. In Section 3, we show that a conformal 'radial' d'Alembert-like equation, can be led into a Schrödinger differential equation in which the associated potential is exactly a second Pöschl-Teller potential.

2. Hyperspherical Universes

In 1952 Fantappié proposed the so-called theory of the hyperspherical universes[1]. This theory is based on group theory and on the hypothesis that the universe is endowed with unique physical laws, valid for all observers. As particular cases, Arcidiacono[2] studied a limitation of that theory, i.e., he considered hyperspherical universes with $3, 4, \ldots, n$ dimensions where motions are given by $n(n+1)/2$-parameter rotation group in spaces with $4, 5, \ldots, (n + 1)$ dimensions, respectively. Those models of hyperspherical $S^3, S^4, \ldots, S^n$

[1] See the appendix.

universes, can be interpreted as successive physical improvements, because any one of them (after S^4) contains its precedents and is contained in its successors.

After 1955 Arcidiacono studied the case $n = 4$, special projective relativity, based on the de Sitter hyperspherical universe with a group (the so-called Fantappié-de Sitter group) of ten parameters. This theory is an improvement (in a unique way) of Einstein's special relativity theory and provides a new group-theoretical version of the big-bang cosmology. As a by-product of special projective relativity one can recover several results, for example, Kinematic Relativity, proposed by Milne[9]; Stationary Cosmology, proposed by Bondi-Gold[10] and Plasma Cosmology, proposed by Alfvèn[11].

Moreover, if we consider a universe S^4 as globally hyperspherical but endowed with a locally variable curvature, we obtain the so-called general projective relativity which was proposed and studied by Arcidiacono after 1964. This theory allows us to recover several results as particular cases, for example, the unitary theories proposed by Weyl[12], Straneo[13], Kaluza-Klein[14, 15], Veblen[17] and Jordan-Thiry[18] and some generalizations of the gravitational field, as those proposed by Brans-Dicke[19], Rosen[20] and Sciama[21].

In this paper we are interested only in the case $n = 5$, i.e., conformal projective relativity based on the hyperspherical universe S^4 and its associated rotation group, with fifteen parameters, which contains the accelerated motions. We remember that, whereas for $n = 4$ we have a unitary theory (a magnetohydrodynamic field), for $n = 5$ we have another unitary theory, i.e., the magnetohydrodynamics and Newton's gravitation. We also present the relations between Cartesian, projective and conformal coordinates and the link involving derivatives in the six- and five-dimensional formulations.

2.1 Conformal Coordinates

We use the notation x_i, $(i = 1, 2, 3, 4)$ and x_5 for conformal coordinates and $\overline{x}_a$, $(a = 0, 1, 2, 3, 4, 5)$ for projective coordinates. The relations between these coordinates are

$$x_i = r_0 \frac{\overline{x}_i}{\overline{x}_0 + \overline{x}_5} \qquad \text{and} \qquad x_5 = r_0 \frac{r}{\overline{x}_0 + \overline{x}_5},$$

which satisfy the condition

$$x_5^2 - x^2 = r_0^2 \frac{\overline{x}_0 - \overline{x}_5}{\overline{x}_0 + \overline{x}_5},$$

where $x^2 = x_i x^i$, and r_0 and r are constants. After these considerations, the transformations of the so-called conformal projective group are obtained using the quadratic form in projective coordinates

$$\overline{x}^2 + \overline{x}_0^2 - \overline{x}_5^2 = r^2,$$

decomposing the elements of the six-dimensional rotation group (with fifteen parameters) in fifteen simple rotations $(\overline{x}_a, \overline{x}_b)$[22].

2.2 Connection Between Derivatives

Our main objective is to write down a differential equation, more precisely a Klein-Gordon-like equation, associated with conformal coordinates. We first obtain the relation between the six projective derivatives $\overline{\partial}_a \equiv \partial/\partial\overline{x}_a$ and the five-dimensional derivatives $\partial_i = \partial/\partial x_i$ and $\partial_5 = \partial/\partial x_5$. We can then write the differential equations in the projective formalism, with six dimensions, in physical, i.e., conformal coordinates, with five dimensions.[2]

Taking $\phi = \phi(x_i, x_5)$, a scalar field, and using the chain rule we can write

$$\partial_i\phi = \left[(\overline{\partial}_i\overline{x}_k)\overline{\partial}_k + (\overline{\partial}_i\overline{x}_5)\overline{\partial}_5 + (\overline{\partial}_i\overline{x}_0)\overline{\partial}_0\right]\overline{\phi}$$

$$\partial_5\phi = \left[(\overline{\partial}_5\overline{x}_k)\overline{\partial}_k + (\overline{\partial}_5\overline{x}_5)\overline{\partial}_5 + (\overline{\partial}_5\overline{x}_0)\overline{\partial}_0\right]\overline{\phi}$$

with $\overline{\phi} = \overline{\phi}(\overline{x}_i, \overline{x}_5, \overline{x}_0)$ and $i, k = 1, 2, 3, 4$.

From now on we take $r = 1 = r_0$. We consider $\overline{\phi}(\overline{x}_a)$ a homogeneous function with degree N in all six projective coordinates $\overline{x}_a$. Using Euler's theorem associated with homogeneous function, we get

$$\left(\overline{x}_i\overline{\partial}_i + \overline{x}_5\overline{\partial}_5 + \overline{x}_0\overline{\partial}_0\right)\phi = N\phi$$

where $\overline{\partial}_a = \partial/\partial\overline{x}_a$ and N is the degree of homogeneity of the function.

Then, the link between the derivatives can be written as follows[23]

$$\overline{\partial_0}\,\overline{\phi} = N\frac{A^+}{x_5}\phi + B^-\partial_5\phi - x_5x_i\partial_i\phi$$

$$\overline{\partial_5}\,\overline{\phi} = -N\frac{A^-}{x_5}\phi - B^+\partial_5\phi - x_5x_i\partial_i\phi$$

$$\overline{\partial_i}\,\overline{\phi} = N\frac{x_i}{x_5}\phi + x_i\partial_5\phi + x_5\partial_i\phi$$

where we have introduced a convenient notation

$$2A^\pm = 1 \mp x^2 \pm x_5^2 \qquad \text{and} \qquad 2B^\pm = 1 \pm x^2 \pm x_5^2.$$

We observe that for $\overline{x}_5 = 0$ and considering $\overline{\partial}_5\overline{\phi} = 0$ we obtain

$$\overline{\partial}_i\overline{\phi} = A\partial_i\phi + \frac{N}{A}x_i\phi$$

$$\overline{\partial}_0\overline{\phi} = -Ax_i\partial_i\phi + \frac{N}{A}\phi$$

[2] As we already know, in five dimensions we must impose a condition on space in order to account for the fact that we are aware of only four dimensions. We have the same situation here, i.e., we must impose an additional condition.

where $A^2 = 1+x^2$. These expressions are the same expressions obtained in special projective relativity[25, 26, 27] and provide the link between the five projective derivatives and the four derivatives in Cartesian coordinates, i.e., the relation between five-dimensional (de Sitter) universe and four-dimensional (Minkowski) universe.

3. Conformal Klein-Gordon Equation

In this section we use the previous results to calculate the so-called generalized Klein-Gordon differential equation

$$\frac{\partial^2}{\partial \overline{x}_a^2}\Phi + \mathsf{m}^2\Phi = 0$$

where m^2 is a constant and $a = 0, 1, \ldots, 5$. Introducing projective coordinates (in this case we have a Pythagorean metric) we obtain[3]

$$\frac{\partial^2 U}{\partial \overline{x}_i^2} + \frac{\partial^2 U}{\partial \overline{x}_0^2} - \frac{\partial^2 U}{\partial \overline{x}_5^2} + \mathsf{m}^2 U = 0$$

where $i = 1, 2, 3, 4$ and $U = U(\overline{x}_i, \overline{x}_0, \overline{x}_5)$.

Using the relations between projective and conformal coordinates and the link (involving the derivatives) in the two formulations we can write

$$\left[x_5^2\left(\Box - \frac{\partial^2}{\partial x_5^2}\right) + 3x_5\frac{\partial}{\partial x_5} + N(N+5) + \mathsf{m}^2\right] u(x_i, x_5) = 0$$

where N and m^2 are constants, $\Box$ is the Dalembertian operator given by

$$\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

and Δ is the Laplacian operator. This partial differential equation is the so-called Klein-Gordon differential equation written in conformal coordinates or a conformal Klein-Gordon equation.

The case $\mathsf{m}^2 = 0$ transforms this equation in the so-called generalized d'Alembert differential equation. Another way to obtain this differential equation is to consider the conformal metric in cartesian coordinates, which furnishes the so-called Beltrami metric[2] where the d'Alembert equation appears naturally. This equation can also be obtained by means of the second order Casimir invariant operator[4] associated with the conformal group.

To solve the conformal Klein-Gordon equation, we first introduce the spherical coordinates (r, θ, ϕ) and get

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\mathcal{L}u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_5^2} + \frac{3}{x_5}\frac{\partial u}{\partial x_5} + \frac{\Lambda}{x_5^2}u = 0,$$

[3] Hereafter we consider $\mathsf{m} = m_0 c/\hbar$ where m_0, c and $\hbar$ have the usual meanings.

[4] Invariant operators associated with dynamic groups furnish mass formulas, energy spectra and, in general, characterize specific properties of physical systems.

where we introduced $x_4 = ict$ and defined the operator[5]

$$\mathcal{L} \equiv \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

involving only the angular part. In this partial differential equation we have $u = u(r,\theta,\phi,t,x_5)$ with $\Lambda = N(N+5) + \mathsf{m}^2$.

Using the method of separation of variables we can eliminate the temporal and angular parts, writing

$$u = u(r,\theta,\phi,t,x_5) = A\,\mathrm{e}^{inct}Y_{\ell m}(\theta,\phi)\,f(r,x_5),$$

where A is an arbitrary constant, $n > 0$, $\ell = 0, 1, \ldots$ and $m = 0, \pm 1, \ldots$ with $-\ell \leq m \leq \ell$ and $Y_{\ell m}(\theta,\phi)$ are the spherical harmonics, we get the following partial differential equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} - \frac{\partial^2 f}{\partial x_5^2} + \frac{3}{x_5}\frac{\partial f}{\partial x_5} + \frac{\Lambda}{x_5^2}f + \left[n^2 - \frac{\ell(\ell+1)}{r^2}\right]f = 0$$

with $f = f(r,x_5)$. If we impose a regular solution at the origin ($r \to 0$), the solution of this partial differential equation can be obtained in terms of a product of two Bessel functions[22].

4. A d'Alembert-Like Equation

In this section we present and discuss a partial differential equation which can be identified to a d'Alembert-like equation, which we call a conformal 'radial' d'Alembert equation. We firstly introduce a convenient new set of coordinates, then we use separation of variables and obtain two ordinary differential equations. One of them can be identified as an ordinary differential equation whose solution is a generalization of Newton's law of gravitation; the other one is identified with an ordinary differential equation similar to a one-dimensional Schrödinger differential equation with a potential equal to the second Pöschl-Teller potential.

We introduce the following change of independent variables

$$\begin{aligned} r &= \rho\cosh\xi, \\ x_5 &= \rho\sinh\xi, \end{aligned}$$

with $\rho > 0$ and $\xi \geq 0$, in the separated Klein-Gordon equation, obtained in the previous section, and after another separation of variables we can write a pair of ordinary differential equations, namely,

$$\rho^2\frac{d^2U}{d\rho^2} - p(p+1)U = 0,$$

[5] Here r is a coordinate and should not be confused with the radius of the hypersphere. Besides, it is always possible to define a Wick-rotation[24] of the time coordinate, i.e., $ct \mapsto ict$.

where $U = U(\rho)$ and

$$\frac{d^2V}{d\xi^2} + (2\tanh\xi - 3\coth\xi)\frac{dV}{d\xi} + \left[\frac{\ell(\ell+1)}{\cosh^2\xi} - \frac{\Lambda}{\sinh^2\xi} - p(p+1)\right] V = 0$$

where $V = V(\xi)$ and p is a separation constant.

We first discuss the differential equation in the variable ρ. Its general solution is given by

$$U(\rho) = C_1\rho^{-p} + C_2\rho^{p+1}$$

where C_1 and C_2 are arbitrary constants.

If we consider the case $p = 1$, introducing the notation $C_1 = gM$ with g and M having the usual meanings, we get

$$U(\rho) = \frac{gM}{\rho} + C_2\,\rho^2$$

i.e., a gravitational potential which can be interpreted as a sum of a Kepler-like potential and a harmonic oscillator potential, giving rise to the gravitational force

$$f(\rho) = \nabla U = -\frac{gM}{x^2 - x_5^2} + 2C_2(x^2 - x_5^2)^{1/2},$$

with a singularity at $x = x_5$. We note that for $C_2 = 0$ we obtain an expression analogous to Newton's law of gravitation.

Secondly, the equation in the variable ξ. To solve this ordinary differential equation we first introduce the change of dependent variable

$$V(\xi) = \sinh^{\frac{1}{2}}\xi\tanh\xi\, F(\xi)$$

and obtain

$$-\frac{d^2}{d\xi^2}F(\xi) + \left[\frac{\mu(\mu-1)}{\sinh^2\xi} - \frac{\ell(\ell+1)}{\cosh^2\xi} + \left(p + \frac{1}{2}\right)^2\right] F(\xi) = 0, \tag{1}$$

where the parameter μ is given by a root of the following algebraic equation $\mu(\mu-1) = N^2 + 5N + 15/4 + \mathsf{m}^2$.

The differential equation above can be identified with a Schrödinger-like differential equation in which the associated potential is given by

$$\mathcal{V}_{\mu\ell}(\xi) = \frac{\mu(\mu-1)}{\sinh^2\xi} - \frac{\ell(\ell+1)}{\cosh^2\xi},$$

which is exactly the second Pöschl-Teller potential with energy E given by $E_p = -(p + 1/2)^2 < 0$. The solution of this ordinary differential equation is well known and can be expressed in terms of the hypergeometric function. An algebraic treatment can be found in [28, 29].

We note that the first Pöschl-Teller potential is connected with the study of a Dirac particle on central backgrounds associated with an anti-de Sitter oscillator, i.e., the transformed radial wave functions satisfy the second-order Schrödinger differential equation whose potential is exactly the first Pöschl-Teller potential[30].

Finally, a particular case of Eq.(1), i.e., the case $\mu = 0$, is related to the anti-de Sitter static frame as shown recently by da Rocha and Capelas de Oliveira[31].

Concluding Remarks

In this paper we discussed the calculation of a conformal d'Alembert-like equation.We used the methodology of projective relativity to obtain a conformal Klein-Gordon differential equation and, after the separation of variables, we got another partial differential equation in only two independent variables, the so-called conformal d'Alembert differential equation. Another separation of variables led to an ordinary differential equation which generalizes Newton's law of gravitation. Finally, we showed that the remaining differential equation, a 'radial' differential equation, is transformed into a one-dimensional Schrödinger differential equation with an associated potential that can be identified exactly with the second Pöschl-Teller potential.

From supersymmetric quantum mechanics with periodic potentials, it can be seen that the most general periodic potentials which can be analytically solved involve Jacobi's elliptic functions, which in various limits become Pöschl-Teller potentials arising in the context of Kaluza-Klein spectrum[14]. Kaluza-Klein modes of the graviton have been widely investigated [32, 33, 34, 35], since the original formulation of Randall and Sundrum necessarily has a continuum of Kaluza-Klein modes without any mass gap, arising from a periodic system of 3-branes. The methods and equations developed here can shed some new light in the calculation of mass gaps from a distribution of D-branes[35] in the context of five-dimensional supergravity.

A natural continuation of this calculation is to prove that all 'radial' problems associated with an equation resulting from a problem involving a light cone can be led into a Schrödinger-like differential equation in which the potential is exactly the Pöschl-Teller potential[36].

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A Hyperspherical Universe Models

In this appendix we briefly summarize the idea of hyperspherical universes as originally proposed by Fantappié [37] and developed by Arcidiacono[2].

The main motivation behind those models is to consider seriously the premise that a Universe must be a harmonic and well ordered system of laws, and that this statement is to be expressed mathematically by using group theory in an appropriate way. Taking into account that Galilean Relativity, which uses the Galileo group as invariance group of physical laws, has been perfectioned into Special Relativity, which uses the Poincaré group as invariance group of physical laws, Fantappié asked himself in which way it would be possible to perfect Special Relativity into a kind of *final* relativity. His answer to this question was very simple indeed. He realized that the Poincaré group is the contraction of the 10-parameter Lie group known today as de Sitter group (but which should be called, as in the text, the Fantappié-de Sitter group) which can be made to act projectively on a flat 4-dimensional (Minkowski) spacetime. Next Fantappié asked whether there were other spacetime manifolds where the group could act naturally and serve as invariance group of physical laws. The answer is positive, and Fantappié found that the natural manifold is a hyperspherical universe, which he called S^4, of constant curvature radius. Of course, the previously known models for the universe (not based on General Relativity) are all particular cases of this one, corresponding to some group contractions involving the velocity of light and/or the universe radius, or both. Fantappié did not stop there. He proposed that the hyperspherical universe model S^4 was only an approximation for truth in the sense that it was embedded in a hyperspherical universe model S^5 where the conformal group (a 15-parameter Lie group) acts naturally as invariance group of physical laws. By its turn, S^5 may be generalized into S^6 and so on. At each generalization new fundamental physical constants make their natural appearance as a kind of group parameter whose contraction produces the group used in the previous universe model. It is clear that the extra dimensions in each Universe model must be interpreted, in an appropriate way (something that is also necessary in modern Kaluza-Klein type theories), and it is at this point that his mathematical skills give us useful physical hints. In particular, Arcidiacono, one of his students, showed that the hyperspherical universe models S^4 and S^5 contain many aspects of several proposed unified theories. Of course, we do not have space here even to start discussing the many beautiful results found by Arcidiacono and we invite the reader to consult Arcidiacono[2] for more details. However, we would like to emphasize here that many ideas proposed by him are worth to be more developed since, in particular, it seems that in his work there is the seed of a simple solution for the problem of dark energy and dark matter, an issue that we shall discuss elsewhere. Finally, we mention that recently Chiatti[38] has discussed a comparison of this theory with recent cosmological evidences.

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Existence of Yang–Mills Theory with Vacuum Vector and Mass Gap

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Abstract: This paper shows that quantum theory describing particles in finite expanding space–time exhibits natural ultra–violet and infra–red cutoffs as well as posesses a mass gap and a vacuum vector. Having ultra–violet and infra–red cutoffs, all renormalization issues disappear. This shows that Yang–Mills theory exists for any simple compact gauge group and has a mass gap and a vacuum vector.

Keywords: Finite Expanding Space–time; Yang–Mills fields; Renormalization; Regularization; Mass Gap; Vacuum Vector
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1. Introduction

All the particle interactions – beside gravitational interactions – are today decribable by Standard model. Main theoretical tool for Standard model is Yang–Mills action. This action is able to describe all the possible particle interactions experimentally observed so far – including strong nuclear interactions. The bad part is that there are some issues regarding quantum theory in general.

First issue is that any higher order correction in the perturbation calculations is singular – there are infra–red and ultra–violet catastrophes in calculations. Many mathematical methods have been devised over last half a century in order to cure singularities, but problem of renormalization still remains.

Second issue is that in order to have confined particles, spectrum should exhibit a mass gap. Since the only sound theoretical action is the Yang–Mills action, we are naturally trying to formulate the Yang–Mills action in such a way to produce a mass gap[1].

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2. Finite Space–time

In order to cure these two problems – renormalization and existence of a mass gap – we can hit the problem directly. One may try to find new renormalization schemes, but one may also try to render renormalization obsolete. If there are natural ultra–violet and infra–red cutoffs, then there would be no renormalization at all for there would be no singularities. Luckily, when one manages to solve the renormalization problem, the mass gap shows right away. The unique idea behind both renormalization and a mass gap is simple – physical space–time is bounded.

We should notice one thing in general – space–time is finite and expanding. This simple fact has enormous impact on quantum theory in general. This paper will show that the fact of space–time expanding ensures natural cutoffs as well as the existence of mass gap as soon as basic axioms of quantum theory are fullfilled.

Let us begin by defining three axioms,

Axiom $\mathcal{A}$: Space–time is finite.
Axiom $\mathcal{B}$: Space–time is expanding.
Axiom $\mathcal{C}$: Space–time is connected.

Axiom $\mathcal{B}$ is obvious from the Hubble expansion. If the mass density of Universe is finite, then Axiom $\mathcal{A}$ follows from observing that distances between galaxies are finite. Otherwise, there would be no observable Hubble expansion. And Axiom $\mathcal{C}$ is to be interpreted topologically. For instance, suppose that there exist two regions in space–time and suppose that there exist no continuous path beginning at any point from one region and ending at any point in other region. Suppose that we, observers, live in one of these regions. Since two regions are disconnected, there is no information that we may possibly have about the other region being disconnected from us. This region would not exist for us. It would be unphysical to consider disconnected region for there is nothing to consider. There is no physical information whatsoever coming from disconnected region to us. So we do not have to bother about disconnected parts of the Universe nor even consider them as parts of the same Universe. This is the meaning of Axiom $\mathcal{C}$.

When attempting to interpret wave functions in finite expanding space–time we are to make sense of the normalizing integral of probability density. The main difference that shows when embedding wave function in finite expanding space–time in contrast to the infinite space–time is that boundaries of space–time are finite and not fixed. Having finite space–time expand, the normalization integral becomes under–defined. The reason for this is that probability density $\bar{\phi}\phi$ does not depend on time for conserved systems. In order to have probability conserved – and therefore energy, charge, ... – normalization factor should be time–dependent. Hence in finite expanding space–time normalization becomes

$$N(t)\int\limits_{U(t)}\bar{\phi}\phi d^3x = 1 \tag{1}$$

with $U(t)$ being finite time–dependent 3–volume of Universe and with $N(t)$ being time–dependent normalization factor that serves to interpret $N(t)\bar{\phi}\phi$ as a probability density function in finite expanding space–time. We notice that we do not define probability density $N(t)\bar{\phi}\phi$ outside the 3–volume of physical space–time $U(t)$ following statement of Axiom $\mathcal{C}$.

Since space–time being finite, the lattice calculations give infra–red impulse cutoff[2]. I refer to sentence "In Fourier space, the space–time lattice leads to finite domains for the values of energies and momenta (the Brillouin zones), so that all ultra–violet divergences disappear. If we also wish to ensure the absence of infra–red divergences, we must replace the infinite volume of space and time by a finite box. This is often required if complications arise due to divergent contributions of soft virtual particles, typically photons". We notice that for infinitely dense lattice ultra–violet divergences reappear, but if finite box of integration stays finite, infra–red divergencies stay absent. Since Fourier integral being defined pairwise for pairs (x^μ, p^μ) and for each μ, and since all of x^μ being bounded in finite space–time, we conclude that neither 4–impulse nor any of its components ever vanish.

We conclude that in finite space–time 4–impulse never vanishes for any particle. Therefore there exists energy $\Delta > 0$ that depends on the size of space–time, such that any particle's energy E obeys $E^2 \geq \Delta^2$. We notice that finite space–time produces a mass gap Δ.

3. Gravitational Limitations for Impulse and Position for Massive Particles

We next argue on physical meaning of probability density inside the gravitational horizon around point–like particle. The point is that although the true gravitational singularity lies at the origin where the point–like particle we choose to observe is situated, the region inside the horizon is unaccessible to any measurement. We know that we cannot measure anything outside the space–time, so we need not bother to assign any probability to finding a particle outside the physical boundaries of space–time. We know for sure that particle is inside space–time. Likewise, we know that we cannot measure anything inside the horizon, so we need not bother to assign any probability to finding a particle outside the physical boundaries of space–time – a horizon in this case. There is no probability of finding a particle inside horizon for there is no particle nor any finding for observer outside if particle happens to be inside the horizon. There is tunneling, of course, but it will show automatically because of the Heisenberg uncertainty principle and because of raging fluctuations in impulse on the horizon.

The crucial argument is as follows – moving particle creates horizon that grows with particle's speed. In finite space–time this would produce a catastrophe. For if horizon grows large enough, it could consume entire space–time. There would be no–one left to measure anything. This would happen even if particle is not idealized as a point–like object. This effect occurs for particle velocities a bit below velocity of light, as calculated in this paper. This issue of a horizon expanding with particle's velocity presents no problem in infinite space–time since particle should be moving at the speed of light in order to consume entire infinite space–time. In finite space–time horizon growing with velocity presents problem since it may consume entire space–time if particle producing it moves with velocity below the velocity of light.

We are ready to calculate the ultra–violet cutoffs. To do so, let a not necessarily point–like free particle of any rest mass m move uniformly with velocity u. It produces a gravitational field around itself. We can employ Schwarzschild metric for still particle in cartesian representation

$$\delta s^2 = \frac{\left(1-\frac{a}{r}\right)^2}{\left(1+\frac{a}{r}\right)^2}\delta t^2 - \left(1+\frac{a}{r}\right)^4 \sum_{i=1}^{3} \delta x^i \delta x^i \tag{2}$$

with $a = \frac{Gm}{2c^2}$ and give it a boost in, say, direction x, by applying Lorenz transformation upon metric (2). The result for g'_{00} reads

$$g'_{00} = g_{00}\left(\frac{\partial t}{\partial t'}\right)^2 + g_{11}\left(\frac{\partial x}{\partial t'}\right)^2 = \frac{\left(1-\frac{a}{r'}\right)^2}{\left(1-u^2\right)\left(1+\frac{a}{r'}\right)^2} - \frac{u^2\left(1+\frac{a}{r'}\right)^4}{\left(1-u^2\right)} \tag{3}$$

This metric tensor component vanishes for

$$r' = \sqrt{\frac{(x+ut)^2}{1-u^2} + y^2 + z^2} = a\frac{1+u^2}{1-u^2} \tag{4}$$

If $r' = 2R$, with R being the radius of physical space, then there is no space–time at all whatever the particle's location, and for a not necessarily point–like particle of rest-mass m we find

$$u = c\frac{1-\frac{Gm}{4Rc^2}}{\left(1+\frac{Gm}{4Rc^2}\right)^3} \tag{5}$$

This defines ultra–violet cutoff for any free particle.

For free electron the upper limit for velocity u is

$$u = c\left(1 - 2\cdot 10^{-86}\right)(m/s)$$

in coordinates that span metric with temporal metric tensor component (3).

We conclude that in finite expanding space–time there exist both infra–red and ultra–violet cutoffs for 4-impulse for any elementary particle.

The conclusion is that particles cannot have infinite impulse – there would be no–one to calculate it. Having this ultra–violet cutoff along with the infra–red one, we conclude that there are no singularities left, that energy is bounded both below and above, thus producing a mass gap and leading us to proving the existence of a vacuum vector.

4. Yang–Mills Existence

Our attention should be on Yang–Mills quantum theory now. The reason for this is simple. Namely, only interactions described via compact group representations produce hamiltonians bounded from below[3] in infinite static space–time. The exact sentence I refer to in [3] is "Nonabelian gauge theory must be based on a compact group, because otherwise some of the terms in $\mathcal{L}_{kin}$ would have the wrong sign, leading to a hamiltonian that is unbounded below" with $\mathcal{L}_{kin} = -\frac{1}{4}F^{c\mu\nu}F^{c}_{\mu\nu} = -\frac{1}{2}Tr\,(F^{\mu\nu}F_{\mu\nu})$. Hence the importance of Yang–Mills action in infinite static space–time.

Quantum Yang–Mills theory therefore has energy bounded below for any compact gauge group G as soon as free particle states energies are bounded below. Free particle energies are indeed bounded in finite space–time and therefore, in finite space–time, Yang–Mills hamiltonian H has spectrum bounded below. Let us denote this energy minimum of Yang–Mills hamiltonian H by E_m and let us denote the lowest energy state by ϕ_m. By shifting original hamiltonian H by $-E_m$, the new hamiltonian $H' = H - E_m$ has its minimum at $E = 0$ and the first state ϕ_m is the vacuum vector. We now notice that in finite space–time any hamiltonian's spectrum is not supported in region $(-\Delta, \Delta)$, with $\Delta > 0$ depending upon the size of the Universe – in other words, depending upon the size of the finite space–time. Therefore for quantum Yang–Mills theory over compact gauge groups G there is always a vacuum vector ϕ_m and mass gap Δ.

In finite expanding space–time the issue of renormalizability disappears completely, since in finite expanding space–time there are infra–red and ultra–violet cutoffs. Since there exist only ultra–violet and infra–red catastrophes for Yang–Mills action in infinite static space–time, there are no "green" catastrophes, we no longer need to pay attention to renormalizability as soon as we perform calculations in finite expanding space–time. We conclude that Yang–Mills problem is no problem at all in finite expanding space–time. There obviously always exists a solution to any Yang–Mills type of action.

Conclusions

We conclude that for any simple compact gauge group G in finite expanding space–time Yang–Mills theory with mass gap and vacuum vector exists.

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One-parameter Potential from Darboux Theorem

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Abstract: We consider the stationary one-dimensional Schrödinger equation with potential $u(x;i) = \sum_{j=-2}^{2} f_j(i)x^j$, where the coefficients $f_j(i)$ are functions of a discrete parameter i. We establish the most general form of the coefficients $f_j(i)$ and obtain the ladder operators for the solution of Schrödinger equation by a Darboux transform. Generally speaking, the Darboux transform is obtained through a so-called superpotential $W(x)$, which is derived from a Riccati equation. We first propose a convenient *ansatz* for the function $W'(x)$ and then yield a set of nine difference equations for the coefficients $f_j(i)$. This set of difference equations establishes the explicit form of the coefficients $f_j(i)$, in the potential $u(x;i)$. Our results are consistent with some well-known quantum potentials in special cases.

Keywords: Darboux Theorem; Ladder Operator; Superpotential; Riccati Equation
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1. Introduction

The classical Darboux theorem was formulated in the 19th century [1-2], and define basically a mapping between solutions of a pair of second-order differential equations of the same form. This mapping called Darboux transformation, is functionally parametrized by a pair of solutions of the differential equation and the transform vanishes if the solutions coincide. The Darboux transformations are very closely related to the SUSYQM, intertwining operators and inverse scattering techniques as well as other topics [3].

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In this work by applying the Darboux theorem to the Schrödinger equation with the potential $u(x;i) = \sum_{j=-2}^{2} f_j(i)x^j$ dependent of a discrete parameter i, different to principal quantum number n since this potential is superposition of some important potentials in quantum mechanics [4], we obtain the explicit form of the coefficients $f_j(i)$ such that the ladder operators for the solution of Schrödinger equation are identified by a Darboux transform.

This paper is organized as follows. In Section 2 we briefly review the Darboux theorem. Section 3 is devoted to showing how from a convenient *ansatz* for the function involved in the Darboux transform to obtain a complicated coupling equation for the coefficients $f_j(i)$. A set of nine difference equations for coefficients $f_j(i)$ is presented. The ladder operators for corresponding solution are constructed by the Darboux transform technique. In Section 4, some particular i-dependent potentials and their applications in quantum mechanics are discussed. Finally, in Section 5 we give our conclusions.

2. Darboux Theorem

This theorem addresses that if the $W(x)$ is an arbitrary function, $\psi_n(x)$ a solution of eigenvalues equation

$$\frac{d^2}{dx^2}\psi_n(x) - 2u(x)\psi_n(x) = -2\lambda_n\psi_n(x), \tag{2.1}$$

then the function $\varphi_n(x)$ defined as

$$\varphi_n(x) \equiv B\psi_n(x), \quad B \equiv \left(\frac{d}{dx} - W(x)\right), \tag{2.2}$$

satisfies the following differential equation

$$\varphi_n''(x) - 2\left[u(x) - W'(x)\right]\varphi_n(x) = -2\lambda_n\varphi_n(x), \tag{2.3}$$

only if the expression

$$W^2(x) + W'(x) - 2u(x) \equiv k, \tag{2.4}$$

is independent of variable x. The prime denotes the derivative with respect to x. By taking into Eq.(2.4) account, Eq.(2.3) can be rewritten as

$$\varphi_n''(x) - \left[W^2(x) - W'(x)\right]\varphi_n(x) = -2\left(\lambda_n + \frac{k}{2}\right)\varphi_n(x). \tag{2.5}$$

As a result, it is shown that the inverse transformation of Eq.(2.2) becomes[1]

$$\left(\frac{d}{dx} + W(x)\right)\varphi_n(x) = -2\left(\lambda_n + \frac{k}{2}\right)\psi_n(x). \tag{2.6}$$

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[1] It should be noted that the classical Darboux theorem can also be established if a solution $\psi_m(x)$ of Eq.(2.1) is known *a priory*. Actually, it is shown from the definition $W(x) = \psi_m'(x)/\psi_m(x)$ that $k = -2\lambda_m$ in Eq.(2.4). If taking $\psi_m(x) \equiv e^{v(x)}$, Eq.(2.1) is thus transformed to a Riccati equation $w^2(x) + w'(x) - 2u(x) = -2\lambda_n, w \equiv v'$. Nevertheless, throughout this paper the function $W(x)$ is not connected *a priory* with some eigenfunction of Eq.(2.1), as made in the classical Darboux transform.

3. Ladder Operators for One-parameter Potential

It is well known that in quantum mechanics the function $u(x)$ given in Eq.(2.1) may be dependent of a parameter i, but independent of the principal quantum number n. If so, the $u(x)$ and $\psi_n(x)$ shown above should be denoted as $u(x;i)$ and $\psi_n^{(i)}(x)$ for clearness.

Let us consider the following *ansatz* for $W'(x)$

$$W'(x) = U(x) + \Lambda, \tag{3.1}$$

where

$$U(x) \equiv u(x;i) - u(x;i+\beta), \qquad \Lambda \equiv \lambda_{n+\alpha} - \lambda_n, \quad \alpha, \beta \in \text{integer}. \tag{3.2}$$

If this *ansatz* satisfies Eq.(2.4), then substituting it into Eq.(2.5) leads to

$$\varphi_n''(x) - 2u(x;i+\beta)\varphi_n(x) = -2\lambda_{n+\alpha}\varphi_n(x). \tag{3.3}$$

Consequently, the function $\varphi_n(x)$ must be identified as eigenfunction $\psi_{n+\alpha}^{i+\beta}(x)$, and the operator B given in Eq.(2.2) becomes a ladder operator acting on the parameters i and n of function $\psi_n^{(i)}(x)$, with steps β and α, respectively, *i.e.*

$$\psi_{n+\alpha}^{(i+\beta)}(x) = \left(\frac{d}{dx} - W(x)\right)\psi_n^{(i)}(x). \tag{3.4}$$

The different $u(x;i)$ corresponds to different ladder operators. For potential application in physics, we first consider a general potential with the form

$$u(x;i) = \sum_{j=-q}^{n} f_j(i)x^j, \tag{3.5}$$

where the coefficients $f_j(i)$ are to be determined. By integrating Eq.(3.1) we get

$$W(x) = \int U(x)dx + \int \Lambda dx = \sum_{\substack{j=-q \\ j\neq -1}}^{n} \frac{F_j(i)}{j+1}x^{j+1} + F_{-1}(i)\ln x + \Lambda x + C, \tag{3.6}$$

where

$$U(x) = \sum_{j=-q}^{n} F_j(i)x^j, \quad F_j(i) \equiv f_j(i) - f_j(i+\beta), \quad C = \text{constant}. \tag{3.7}$$

If $f_{-1}(i)$ taken as the numerical value $\mathfrak{f}_{-1}$, then the logarithmic term in Eq.(3.6) drops out due to Eq.(3.7). By using Eqs.(3.1), (3.2), (3.5) and (3.6), we are able to explicitly express Eq. (2.4) as follows:

$$\begin{aligned}
&\left(\sum_{\substack{j=-q\\ j\neq -1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)^2+\Lambda^2x^2+C^2\\
&+2\left(\sum_{\substack{j=-q\\ j\neq -1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)\Lambda x+2\Lambda Cx+2\left(\sum_{\substack{j=-q\\ j\neq -1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)C\\
&+\sum_{\substack{j=-q\\ j\neq -1}}^{n}F_j(i)x^j+\Lambda-2\sum_{j=-q}^{n}f_j(i)x^j=k.
\end{aligned} \tag{3.8}$$

This is a rather complicated coupling equation for the coefficients $f_j(i)$. For clearness, in this work we attempt to study the particular case

$$u(x;i)=\sum_{\substack{j=-2\\ j\neq -1}}^{2}f_j(i)x^j+\mathfrak{f}_{-1}x^{-1}. \tag{3.9}$$

In this case, Eq.(3.8) yields immediately a set of nine difference equations for $f_j(i)$ by equating the coefficients of x^j

$$(F_{-2})^2+F_{-2}-2f_{-2}(i)=0, \tag{3.10}$$

$$-F_{-2}C-\mathfrak{f}_{-1}=0, \tag{3.11}$$

$$-2F_{-2}F_0+C^2-2F_{-2}\Lambda+F_0+\Lambda-2f_0(i)=k, \tag{3.12}$$

$$-F_{-2}F_1+2\Lambda C+2F_0C+F_1-2f_1(i)=0, \tag{3.13}$$

$$(F_0)^2-\frac{2}{3}F_{-2}F_2+\Lambda^2+2F_0\Lambda+F_1C+F_2-2f_2(i)=0, \tag{3.14}$$

$$F_0F_1+F_1\Lambda+\frac{2}{3}F_2C=0, \tag{3.15}$$

$$\frac{(F_1)^2}{4}+\frac{2}{3}F_0F_2+\frac{2}{3}F_2\Lambda=0, \tag{3.16}$$

$$\frac{1}{3}F_1F_2=0, \tag{3.17}$$

$$\frac{(F_2)^2}{9}=0. \tag{3.18}$$

From the last four equations we deduce $F_2=F_1=0$, or equivalently by Eq.(3.7) $f_2(i)$ and $f_1(i)$ are the constants $\mathfrak{f}_2$ and $\mathfrak{f}_1$, respectively. The equations (3.10)-(3.14) are reduced to

$$(F_{-2})^2+F_{-2}-2f_{-2}(i)=0, \tag{3.19}$$

$$F_{-2}C=-\mathfrak{f}_{-1}, \tag{3.20}$$

$$(-2F_{-2}+1)(F_0+\Lambda)+C^2-2f_0(i)=k, \tag{3.21}$$

$$(F_0 + \Lambda)\, C = \mathfrak{f}_1, \tag{3.22}$$

$$(F_0 + \Lambda)^2 = 2\mathfrak{f}_2. \tag{3.23}$$

It is shown from Eq.(3.19) that $f_{-2}(i)$ must be zero if it is any constant function and from Eq.(3.20) that the coefficient $\mathfrak{f}_{-1} = 0$. Thus, the term $f_{-2}(i)$ is in general zero or equal to

$$f_{-2}^{\pm}(i;\beta) = \frac{1}{2}\left(\frac{i}{\beta}\right)^2 + b_{\pm}\left(\frac{i}{\beta}\right) + c, \quad \beta \neq 0, \tag{3.24}$$

where

$$b_{\pm} = \pm\sqrt{\frac{1}{4} + 2c}; \quad c \geq -\frac{1}{8}, \tag{3.25}$$

since this is the solution of difference equation (3.19) [5]. Note that

$$f_{-2}^{+}(i;-\beta) = f_{-2}^{-}(i;\beta), \tag{3.26}$$

so if choosing

$$f_{-2}(i) \equiv f_{-2}^{+}(i;\beta), \tag{3.27}$$

then we must consider the case $-\beta$. To do this we write the difference F_{-2} in Eq.(3.7) as

$$F_{-2}^{\pm} = f_{-2}(i) - f_{-2}(i \pm \beta) = \mp\frac{1}{2}\left(\frac{2i}{\beta} \pm 1\right) \mp b_{+}. \tag{3.28}$$

Additionally, equation (3.23) implies that F_0 is independent of the parameter i, because as indicated in Eq.(3.2), Λ does not depend on i. For this reason, $f_0(i)$ has the form

$$f_0(i) = -\frac{d}{\beta}i + \mathfrak{f}_0; \quad F_0 \equiv d. \tag{3.29}$$

The solution of Eq.(3.23) is

$$\Lambda_{\pm} = -d \pm \sqrt{2\mathfrak{f}_2}; \quad \mathfrak{f}_2 = 0 \text{ or } \mathfrak{f}_2 > \frac{d^2}{2}. \tag{3.30}$$

Hence, the second difference equation (3.2) implies

$$\Lambda_{+} = \lambda_{n+\alpha} - \lambda_n, \quad \Lambda_{-} = \lambda_{n+\widetilde{\alpha}} - \lambda_n, \tag{3.31}$$

from which we deduce that

$$\lambda_n = \frac{\Lambda_{+}}{\alpha}n + \lambda_0, \tag{3.32}$$

and

$$\widetilde{\alpha} = \frac{\Lambda_{-}}{\Lambda_{+}}\alpha. \tag{3.33}$$

As shown above, we have considered Eq.(2.1) with given $u(x;i)$ in Eq.(3.9) and found that the operators defined in Eq.(3.4) are nothing but the ladder operators of the function $\psi_n^{(i)}(x)$ if and only if $u(x;i)$ has the following form

$$u(x;i) = f_0(i) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2, \tag{3.34}$$

or

$$u(x;i) = f_{-2}(i)x^{-2} + \mathfrak{f}_{-1}x^{-1} + f_0(i) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2, \tag{3.35}$$

where $\mathfrak{f}_{-1}$, $\mathfrak{f}_1$, $\mathfrak{f}_2$ are real numbers, $f_{-2}(i)$ and $f_0(i)$ are the functions of the parameter i, given in Eq.(3.27) and Eq.(3.29), such that they make Eqs.(3.19)-(3.23) consistent. Moreover, Eqs.(3.24) and (3.30) imply, in principle, that there are four ladder operators contained in Eq.(3.4). To clarify this, we consider specific examples below.

4. Discussions and Applications

4.1 Potential with $f_{-2}(i) = 0$

Substituting Eq.(3.30) into Eq.(3.22) yields

$$C_{\pm} = \frac{\mathfrak{f}_1}{\Lambda_{\pm} + d} = \frac{\mathfrak{f}_1}{\pm\sqrt{2\mathfrak{f}_2}}; \quad \mathfrak{f}_2 \neq 0. \tag{4.1}$$

Considering this and substituting Eq.(3.30) into Eq.(3.21) allow us to obtain

$$k_{\pm} = -2f_0(i) + \frac{(\mathfrak{f}_1)^2}{2\mathfrak{f}_2} \pm \sqrt{2\mathfrak{f}_2}. \tag{4.2}$$

As a result, if taking $u(x;i)$ as the form of Eq.(3.34), the function

$$\varphi_{n(\pm)}(x) = \left(\frac{d}{dx} - W_{\pm}(x)\right)\psi_n^{(i)}(x), \tag{4.3}$$

with (see Eq.(3.6))

$$W_{\pm}(x) = \pm\sqrt{2\mathfrak{f}_2}x + C_{\pm}, \tag{4.4}$$

satisfies Eq. (2.5)

$$\begin{aligned} &\varphi''_{n(\pm)}(x) - 2\left[\frac{(\mathfrak{f}_1)^2}{4\mathfrak{f}_2} + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2 \mp \frac{\sqrt{2\mathfrak{f}_2}}{2}\right]\varphi_{n(\pm)}(x) = \\ &-2\left(\lambda_n - f_0(i) + \frac{(\mathfrak{f}_1)^2}{4\mathfrak{f}_2} \pm \frac{\sqrt{2\mathfrak{f}_2}}{2}\right)\varphi_{n(\pm)}(x), \quad \mathfrak{f}_2 \neq 0, \end{aligned} \tag{4.5}$$

which can be rearranged as

$$\varphi''_{n(\pm)}(x) - 2\left[f_0(i \pm \beta) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2\right]\varphi_{n(\pm)}(x) = -2(\lambda_n \pm \Lambda_+)\varphi_{n(\pm)}(x). \tag{4.6}$$

This can be directly identified as

$$\varphi_{n(\pm)}(x) = \psi^{(i)}_{\lambda_n \pm \sqrt{2f_2}}(x). \tag{4.7}$$

In particular, when $f_0(i) = 0$, $\mathfrak{f}_1 = 0$ (or 1), and $\mathfrak{f}_2 = 1/2$, equation (2.1) is the Schrödinger equation for a simple one-dimensional harmonic oscillator. The substitution of Eqs.(4.1) and (4.4) into Eq.(4.3) leads to well-known raising and lowering operators of this system.

4.2 Potential with $f_{-2}(i) \neq 0$ and $\mathfrak{f}_{-1} = 0$

Due to Eq. (3.28), we have for Eq. (3.20)

$$F^{\pm}_{-2}C = -\mathfrak{f}_{-1}. \tag{4.8}$$

By taking into Eq.(3.30) account, Eq.(3.22) becomes

$$\pm\sqrt{2\mathfrak{f}_2}C = \mathfrak{f}_1. \tag{4.9}$$

At first glance, these equations sound paradoxical, since Eq.(4.8) implies that C is function of parameter i, while Eq.(4.9) shows that C is independent of parameter i. In such a case, the only constraint imposed is the consistence of these equations. For example, if $\mathfrak{f}_{-1} = 0$ then $C = 0$ and $\mathfrak{f}_1 = 0$, Eq.(3.21) can be written as

$$k_{+\pm} = +\sqrt{2\mathfrak{f}_2}(1 - 2F^{\pm}_{-2}) - 2f_0(i), \tag{4.10}$$

$$k_{-\pm} = -\sqrt{2\mathfrak{f}_2}(1 - 2F^{\pm}_{-2}) - 2f_0(i). \tag{4.11}$$

Therefore, by taking

$$u(x;i) = f_{-2}(i)x^{-2} + f_0(i) + \mathfrak{f}_2 x^2, \tag{4.12}$$

in Eq.(2.1), the functions

$$\varphi^{(i)}_{n(+\pm)}(x) = \left(\frac{d}{dx} - W_{+\pm}(x)\right)\psi^{(i)}_n(x), \tag{4.13}$$

$$\varphi^{(i)}_{n(-\pm)}(x) = \left(\frac{d}{dx} - W_{-\pm}(x)\right)\psi^{(i)}_n(x), \tag{4.14}$$

with (see Eq.(3.6))

$$W_{+\pm}(x) = -F^{\pm}_{-2}x^{-1} + \sqrt{2\mathfrak{f}_2}x, \tag{4.15}$$

$$W_{-\pm}(x) = -F^{\pm}_{-2}x^{-1} - \sqrt{2\mathfrak{f}_2}x, \tag{4.16}$$

satisfy (see Eq.(2.5))

$$\varphi^{(i)\prime\prime}_{n(+\pm)}(x) - \left[W^2_{+\pm}(x) - W'_{+\pm}(x)\right]\varphi^{(i)}_{n(+\pm)}(x) = -2\left(\lambda_n + \frac{k_{+\pm}}{2}\right)\varphi^{(i)}_{n(+\pm)}(x), \tag{4.17}$$

$$\varphi^{(i)\prime\prime}_{n(-\pm)}(x) - \left[W^2_{-\pm}(x) - W'_{-\pm}(x)\right]\varphi^{(i)}_{n(-\pm)}(x) = -2\left(\lambda_n + \frac{k_{-\pm}}{2}\right)\varphi^{(i)}_{n(-\pm)}(x), \tag{4.18}$$

or explicitly

$$\varphi^{(i)\prime\prime}_{n(++)}(x) - 2(\Delta_{+\beta})\varphi^{(i)}_{n(++)}(x) = -2\left(\lambda_n + \Lambda_+\right)\varphi^{(i)}_{n(++)}(x), \tag{4.19}$$

$$\varphi^{(i)\prime\prime}_{n(+-)}(x) - 2(\Delta_{-\beta})\varphi^{(i)}_{n(+-)}(x) = -2\left(\lambda_n - \Lambda_-\right)\varphi^{(i)}_{n(+-)}(x), \tag{4.20}$$

$$\varphi^{(i)\prime\prime}_{n(-+)}(x) - 2(\Delta_{+\beta})\varphi^{(i)}_{n(-+)}(x) = -2\left(\lambda_n + \Lambda_-\right)\varphi^{(i)}_{n(-+)}(x), \tag{4.21}$$

$$\varphi^{(i)\prime\prime}_{n(--)}(x) - 2(\Delta_{-\beta})\varphi^{(i)}_{n(--)}(x) = -2\left(\lambda_n - \Lambda_+\right)\varphi^{(i)}_{n(--)}(x), \tag{4.22}$$

where

$$F^{\pm}_{-2}(F^{\pm}_{-2}-1)=2f_{-2}(i\pm\beta),\qquad \Delta_{\pm\beta}=f_{-2}(i\pm\beta)x^{-2}+\mathfrak{f}_2x^2+f_0(i\pm\beta) \tag{4.23}$$

are used. We immediately identify that

$$\varphi^{(i)}_{n(+\pm)}(x)=\psi^{(i\pm\beta)}_{\lambda_n\pm\Lambda_\pm}(x), \tag{4.24}$$

$$\varphi^{(i)}_{n(-\pm)}(x)=\psi^{(i\pm\beta)}_{\lambda_n\pm\Lambda_\mp}(x). \tag{4.25}$$

It should be noted that there are two cases $\mathfrak{f}_2=0$ and $\mathfrak{f}_2\neq 0$ for the potential with $f_{-2}(i)\neq 0$ and $\mathfrak{f}_{-1}=0$. The results for these two special cases are given in Appendixes A and B, respectively.

4.3 Potential with $f_{-2}(i)\neq 0$ and $\mathfrak{f}_{-1}\neq 0$

It is shown from Eqs.(3.28) and (4.8) that C is a general function of the parameter i. However, this and Eq.(4.9) imply that $\mathfrak{f}_1=\mathfrak{f}_2=0$. By using Eq.(4.8) and Eq.(3.30) in Eqs.(3.21) and (3.6), we arrive at

$$k_\pm=\left(\frac{\mathfrak{f}_{-1}}{F^{\pm}_{-2}}\right)^2-2f_0(i), \tag{4.26}$$

$$W_\pm=-F^{\pm}_{-2}x^{-1}-\frac{\mathfrak{f}_{-1}}{F^{\pm}_{-2}}. \tag{4.27}$$

respectively. These equations reduce, for instance to

$$k_\pm=\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)^{-2}-2f_0(i), \tag{4.28}$$

$$W_\pm=-\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)x^{-1}+\frac{1}{\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)}, \tag{4.29}$$

when

$$u(x;i)=\frac{1}{2}i(i-1)x^{-2}-x^{-1}+f_0(i). \tag{4.30}$$

Generally speaking, from Eq.(4.29) we can obtain the radial ladder operators of N-dimensional hydrogen atom, which has the effective potential [8-10]

$$V_{eff}(r)=\frac{L(L-1)}{2r^2}-\frac{1}{r};\quad L\equiv l+\frac{1}{2}(N-1). \tag{4.31}$$

Conclusions

In this work we have carried out the one-dimensional Schrödinger equation with potential function (3.9) and the coefficient $\mathfrak{f}_{-1}$ independent of the parameter i by applying the Darboux theorem. By studying Eqs.(3.19)-(3.23) we have established the coefficients

$f_{-2}(i)$ and $f_0(i)$ through Eq.(3.27) and Eq.(3.29), respectively. The remained coefficients must be constants $\mathfrak{f}_1$ and $\mathfrak{f}_2$. However, the consistence of those difference equations allows us to consider only the possibilities for $u(x;i)$ expressed in Eq.(3.34), Eq.(4.12) and potential $u(x;i) = f_{-2}(i)x^{-2} + \mathfrak{f}_{-1}x^{-1} + f_0(i)$. For example, the coefficient $f_{-2}(i)$ allows the coefficient $\mathfrak{f}_{-1}$ to appear in $u(x;i)$, but this can not be compatible with the inclusion of linear and quadratic terms.

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Appendix A: Potential with $f_{-2}(i) \neq 0$, $\mathfrak{f}_{-1} = 0$ but $\mathfrak{f}_2 = 0$

In this appendix, we are going to show the special results for those obtained in subsection 4.2 in the case of $\mathfrak{f}_2 = 0$. On the other hand, we shall show the results for two typical examples in quantum mechanics.

When $\mathfrak{f}_2 = 0$, the obtained results (3.30), (4.10), (4.11), (4.15), (4.16), (4.13), (4.14) are now reduced to

$$\Lambda_\pm = -d, \quad k_{+\pm} = k_{-\pm} = -2f_0(i) \equiv k_\pm, \tag{A.1}$$

$$W_{+\pm}(x) = W_{-\pm}(x) = -F_{-2}^{\pm}x^{-1} \equiv W_\pm(x), \tag{A.2}$$

$$\varphi_{n(+\pm)}^{(i)}(x) = \varphi_{n(-\pm)}^{(i)}(x) \equiv \varphi_{n(\pm)}^{(i)}(x), \tag{A.3}$$

and those (4.19)-(4.22) are simplified as

$$\varphi_{n(+)}^{(i)\prime\prime}(x) - 2\left[f_{-2}(i+\beta)x^{-2} + f_0(i+\beta)\right]\varphi_{n(+)}^{(i)}(x) = -2\left(\lambda_n - d\right)\varphi_{n(+)}^{(i)}(x), \tag{A.4}$$

$$\varphi_{n(-)}^{(i)\prime\prime}(x) - 2\left[f_{-2}(i-\beta)x^{-2} + f_0(i-\beta)\right]\varphi_{n(-)}^{(i)}(x) = -2\left(\lambda_n + d\right)\varphi_{n(-)}^{(i)}(x), \tag{A.5}$$

The corresponding (4.24) and (4.25) now become

$$\varphi_{n(+)}^{(i)}(x) = \psi_{\lambda_n - d}^{(i+\beta)}(x), \tag{A.6}$$

$$\varphi_{n(-)}^{(i)}(x) = \psi_{\lambda_n + d}^{(i-\beta)}(x), \tag{A.7}$$

where $\psi_{\lambda_n}^{(i)}(x)$ satisfies

$$\frac{d^2}{dx^2}\psi_{\lambda_n}^{(i)}(x) - 2\left[f_{-2}(i)x^{-2} + f_0(i)\right]\psi_{\lambda_n}^{(i)}(x) = -2\lambda_n\psi_{\lambda_n}^{(i)}(x). \tag{A.8}$$

Let us illustrate two typical examples. First, we study

$$u(x;i) = \frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i). \tag{A.9}$$

The coefficient $f_{-2}(i)$ has the form of Eq.(3.27). For $\beta = 1$ and $c = -1/8$, Eq.(3.28) is simplified as

$$F_{-2}^{\pm} = \mp\frac{1}{2}\left(2i \pm 1\right). \tag{A.10}$$

It should be noted from Eq.(A.1) $k_\pm = -2f_0(i)$ and $\Lambda_\pm = -d$. By substituting $F^\pm_{-2}$ into Eq.(A.2), we get

$$W_\pm(x) = \pm\frac{1}{2}\left(2i \pm 1\right) x^{-1}. \tag{A.11}$$

So, if the function $\psi_n^{(i)}(x)$ fulfills

$$\psi_n^{(i)\prime\prime}(x) - 2\left[\frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i)\right]\psi_n^{(i)}(x) = -2\lambda_n\psi_n^{(i)}(x), \tag{A.12}$$

then

$$\psi^{(i\pm1)}_{\lambda_n\pm d}(x) = \left(\frac{d}{dx} \mp \frac{1}{2}\left(2i \pm 1\right) x^{-1}\right)\psi_n^{(i)}(x). \tag{A.13}$$

As a consequence, when $f_0(i) = 0$, we have $k_\pm = 0$ and $\Lambda_\pm = 0$. However, based on Eq.(3.32) we have $\lambda_n \equiv \lambda_0$. Equation (2.1) can be written as

$$\frac{d^2}{dx^2}\psi^{(i)}(x) - \left(i^2 - \frac{1}{4}\right)x^{-2}\psi^{(i)}(x) = -2\lambda\psi^{(i)}(x), \quad \lambda \equiv \lambda_0. \tag{A.14}$$

For special case $\lambda = 1/2$, its solution is given by

$$\psi^{(i)}(x) = x^{1/2} J_i(x), \tag{A.15}$$

where $J_i(x)$ is the Bessel function of the first kind [6]. Generally, we are able to obtain the ladder operators acting on the order of the Bessel function from the operators given in Eq.(A.13).

Second, let us consider another case of i-dependent potential

$$u(x;i) = \frac{i(i-1)}{2}x^{-2} + f_0(i). \tag{A.16}$$

In comparison with Eq.(3.27), we have $\beta = -1$, $c = 0$. Eq.(3.28) now becomes

$$F^\pm_{-2} = \mp\frac{1}{2}\left(-2i \pm 1\right) \mp \frac{1}{2} = \pm i - \frac{1}{2} \mp \frac{1}{2}. \tag{A.17}$$

Again, by Eq.(A.1) we have $k_\pm = -2f_0(i)$ and $\Lambda_\pm = -d$. From Eqs.(A.2), (A.6) and (A.7) we can say that

$$\psi_n^{(i\mp1)}(x) = \left[\frac{d}{dx} + \left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1}\right]\psi_n^{(i)}(x), \tag{A.18}$$

where $\psi_n^{(i)}(x)$ is solution of the following differential equation

$$\psi_n^{(i)\prime\prime}(x) - 2\left[\frac{1}{2}i\left(i-1\right)x^{-2} + f_0(i)\right]\psi_n^{(i)}(x) = -2\lambda_n\psi_n^{(i)}(x). \tag{A.19}$$

A special case occurs for $f_0(i) = 0$. In this case we have $k_\pm = 0$ and $\Lambda_\pm = 0$. The function $\psi_n^{(i)}(x)$ satisfying Eq.(2.1) for any $\lambda_n \equiv \lambda_0$ must be identified as a spherical

Bessel (Neumann) function [6]. Accordingly, the operators given in Eq.(A.18) correspond to the ladder operators acting on the order of the spherical Bessel (Neumann) function.

Appendix B: Potential with $f_{-2}(i) \neq 0$, $\mathfrak{f}_{-1} = 0$ but $\mathfrak{f}_2 \neq 0$

Similarly, in this appendix we are going to show the special results for those obtained in subsection 4.2 in the case of $\mathfrak{f}_2 \neq 0$. Let us consider the following potential

$$u(x;i) = \frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i) + \frac{x^2}{2}. \tag{B.1}$$

From Eq.(3.30), one has

$$\Lambda_{\pm} = -d \pm 1. \tag{B.2}$$

By substituting Eq.(A.10) into Eqs.(4.10) and (4.11) we have

$$k_{+\pm} = 2\left(\pm i + 1 - f_0(i)\right), \tag{B.3}$$

$$k_{-\pm} = -2\left(\pm i + 1 - f_0(i)\right). \tag{B.4}$$

As a result, the superpotentials (4.15) and (4.16) are simplified as

$$W_{+\pm} = \left(\pm i + \frac{1}{2}\right)x^{-1} + x, \tag{B.5}$$

$$W_{-\pm} = \left(\pm i + \frac{1}{2}\right)x^{-1} - x. \tag{B.6}$$

By the way we are going to mention another potential with the form

$$u(x;i) = \frac{i(i-1)}{2}x^{-2} + f_0(i) + \frac{x^2}{2}. \tag{B.7}$$

Likewise, we have $\Lambda_{\pm} = -d \pm 1$. Substituting Eq.(A.17) into Eqs.(4.10) and (4.11) leads to

$$k_{+\pm} = (\mp 2i + 2 \pm 1) - 2f_0(i), \tag{B.8}$$

$$k_{-\pm} = -\left(\mp 2i + 2 \pm 1\right) - 2f_0(i). \tag{B.9}$$

The corresponding superpotentials (4.15) and (4.16) are now expressed as

$$W_{+\pm}(x) = -\left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1} + x, \tag{B.10}$$

$$W_{-\pm}(x) = -\left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1} - x. \tag{B.11}$$

Formally, these four operators correspond to radial ladder operators of N-dimensional isotropic harmonic oscillator defined by following effective potential

$$V_{eff}(r) = \frac{L(L-1)}{2r^2} + \frac{r^2}{2}; \quad L \equiv l + \frac{1}{2}(N-1), \tag{B.12}$$

in which l is the quantum orbital number [7-9].

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Group Properties of the Black Scholes Equation & its Solutions

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Abstract: Several techniques of fundamental physics like quantum mechanics, field theory and related tools of non-commutative probability, gauge theory, path integral etc. are being applied for pricing of contemporary financial products and for explaining various phenomena of financial markets like stock price patterns, critical crashes etc.. The cardinal contribution of physicists to the world of finance came from Fischer Black & Myron Scholes through the option pricing formula which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton and which constitutes the cornerstone of contemporary valuation theory. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions. In this paper, we apply the well-entrenched group theoretic methods to obtain various solutions of the Black Scholes equation for the pricing of contingent claims. We also examine the infinitesimal symmetries of the said equation and explore group transformation properties. The structure of the Lie algebra of the Black Scholes equation is also studied.

Keywords: Econophysics; Financial Markets; Black Scholes; Stock Price Patterns
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1. Introduction

The origin of the association between physics and finance, though, can be traced way back to the seminal works of Pareto [1] and Batchlier [2], the former being instrumental in establishing empirically that the distribution of wealth in several nations follows a power law with an exponent of 1.5, while the latter pioneered the modeling of speculative prices by the random walk and Brownian motion. The cardinal contribution of physicists to the

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world of finance came from Fischer Black & Myron Scholes through the option pricing formula [3] which bears their epitaph and which won them the Nobel Prize for economics in 1997 together with Robert Merton [4]. They obtained closed form expressions for the pricing of financial derivatives by converting the problem to a heat equation and then solving it for specific boundary conditions.

The theory of stochastic processes constitutes the "golden thread" that unites the disciplines of physics and finance. Modeling of non relativistic quantum mechanics as energy conserving diffusion processes is, by now, well known [5]. Unification of the general theory of relativity and quantum mechanics to enable a consistent theory of quantum gravity has also been attempted on "stochastic spaces" [6]. Time evolution of stock prices has been, by suitable algebraic manipulations, shown to be equivalent to a diffusion process [7].

2. The Black Scholes Model

The Black Scholes valuation theory constitutes the cornerstone of modern finance. The model, as initially propounded, envisaged the formulation of a partial differential equation for the pricing of an European call option by creating a portfolio that exactly replicated the payoff of the option and the value of whose constituents was known. The European call option is a financial contingent claim that entails a right (but not an obligation) to the holder of the option to buy one unit of the underlying asset at a future date (called the exercise date or maturity date) at a price (called the exercise price). The option contract, therefore, has a terminal payoff of $\max\left[S\left(T\right)-E,0\right]=\left[S\left(T\right)-E\right]^{+}$ where$S\left(T\right)$is the stock price on the exercise date and $E$is the exercise price.

The theory behind this valuation methodology is well disseminated and can be found in any text on financial derivatives e.g. [7]. The valuation equation of the Black Scholes model is

$$rS\frac{\partial C\left(S,t\right)}{\partial S}+\frac{1}{2}\sigma^{2}S^{2}\frac{\partial^{2}C\left(S,t\right)}{\partial S^{2}}+\frac{\partial C\left(S,t\right)}{\partial t}=rC\left(S,t\right), \tag{1}$$

This is the fundamental PDE for asset pricing and is referred to as the Black Scholes equation in the sequel.

3. Transformation to the Heat Equation

The transformation of the Black-Scholes equation to the heat equation has been well researched. We make the following transformations:-

$$y=\frac{2}{\sigma^{2}}\left(r-\frac{1}{2}\sigma^{2}\right)\ln S-\frac{2}{\sigma^{2}}\left(r-\frac{1}{2}\sigma^{2}\right)^{2}t-\frac{2}{\sigma^{2}}\left(r-\frac{1}{2}\sigma^{2}\right)\ln S_{0}+\frac{2}{\sigma^{2}}\left(r-\frac{1}{2}\sigma^{2}\right)^{2}t_{0}, \tag{2}$$

$$\tau=-\left[\frac{2}{\sigma}\left(r-\frac{1}{2}\sigma^{2}\right)\right]^{2}t+\frac{2}{\sigma^{2}}\left(r-\frac{1}{2}\sigma^{2}\right)^{2}t_{0} \tag{3}$$

$$v=C\left(S,t\right)e^{rt_{0}+\left[\frac{1}{2\sigma^{2}}\left(r-\frac{1}{2}\sigma\right)^{2}-\frac{1}{2}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^{2}-r\right]t}S^{\left[\frac{r}{\sigma^{2}}-\frac{1}{2}-\frac{1}{\sigma^{2}}\left(r-\frac{1}{2}\sigma\right)\right]} \tag{4}$$

On implementing these transformations the Black-Scholes equation gets transformed to the heat equation $\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial y^2}$as can be seen by explicit calculations.

The fundamental solution of the heat equation is given by $v = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{y^2}{4\tau}\right)$and that of the Black Scholes eq. (1) is obtained by substituting back the transformations (2-4) and we obtain

$$C=\frac{1}{\sigma S_0\sqrt{2\pi(t_0-t)}}\exp\left\{-\frac{(\ln S-\ln S_0)^2}{2\sigma^2(t_0-t)}-\left[\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2+r\right](t_0-t)-\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)(\ln S-\ln S_0)\right\} \tag{5}$$

4. Construction of the Symmetry Group[8-12]

The Black Scholes equation (1) is a partial differential equation in two independent variables viz. the stock price S and time $t$and one dependent variable in the price of the derivative C. Let us consider the following invertible transformations of the three variables S, t, and C

$$\bar{t} = f(t, S, C, a), \quad \bar{S} = g(t, S, C, a) \quad \text{and} \quad \bar{C} = h(t, S, C, a) \tag{6}$$

where a is a continuous parameter.

The transformations of eq. (6) will constitute symmetry transformations if eq. (1) retains its structure in the new variables $\bar{t}$, $\bar{S}$and $\bar{C}$and the set of all such transformations constitutes the symmetry group G of the Black Scholes equation.

The generator of the symmetry group $G$is given by the vector field:-

$$X = \xi^0(t, S, C)\frac{\partial}{\partial t} + \xi^1(t, S, C)\frac{\partial}{\partial S} + \eta(t, S, C)\frac{\partial}{\partial C} \tag{7}$$

where $\xi^0(t, S, C)$, $\xi^1(t, S, C)$, $\eta(t, S, C)$ are the parameters of the infinitesimal transformations:-

$$\bar{t} \approx t + a\xi^0(t, S, C), \quad \bar{S} \approx S + a\xi^1(t, S, C) \quad \text{and} \quad \bar{C} \approx C + a\eta(t, S, C) \tag{8}$$

are obtained by solving the following equations:-

$$\frac{d\bar{t}}{da} = \xi^0\left(\bar{t}, \bar{S}, \bar{C}\right), \quad \frac{d\bar{S}}{da} = \xi^1\left(\bar{t}, \bar{S}, \bar{C}\right) \quad \text{and} \quad \frac{d\bar{C}}{da} = \eta\left(\bar{t}, \bar{S}, \bar{C}\right) \tag{9}$$

with the initial conditions $\bar{t}\big|_{a=0} = t$, $\bar{S}\big|_{a=0} = S$ and $\bar{C}\big|_{a=0} = C$.

The transformations represented by eq. (6) would form a symmetry group if $\bar{C} = \bar{C}\left(\bar{S}, \bar{t}\right)$ satisfies the eq. $\frac{\partial \bar{C}}{\partial t} = -\frac{1}{2}\sigma^2\bar{S}^2\frac{\partial^2\bar{C}}{\partial \bar{S}^2} - r\bar{S}\frac{\partial \bar{C}}{\partial \bar{S}} + r\bar{C}$ whenever $C = C(S, t)$ satisfies eq. (1).

Our objective here is to determine all possible coefficient functions ξ^0, ξ^1, ηsuch that we are able to obtain the symmetry group of eq. (1) by the process of exponentiation. For this purpose we need to obtain the second prolongation of the vector field $X$of eq. (7). In terms of the various partial derivatives, this is given by:-

$$pr^{(2)}X = X + \eta^S\frac{\partial}{\partial C_S} + \eta^t\frac{\partial}{\partial C_t} + \eta^{SS}\frac{\partial}{\partial C_{SS}} + \eta^{St}\frac{\partial}{\partial C_{St}} + \eta^{tt}\frac{\partial}{\partial C_{tt}}$$

where

$$\eta^{SS}=D_S^2\left(\eta-\xi^1C_S-\xi^0C_t\right)+\xi^1C_{SSS}+\xi^0c_{SSt}=D_S^2\eta-C_SD_S^2\xi^1-C_tD_S^2\xi^0-2C_{SS}D_S\xi^1-2C_{St}D_S\xi^0$$

$$=C_{SS}+\left(2C_{SC}-\xi^1_{SS}\right)C_S-\xi^0_{SS}C_t+\left(\eta_{CC}-2\xi^1_{SC}\right)C_S^2-2\xi^0_{SC}C_SC_t-\xi^1_{CC}C_S^3-\xi^0_{CC}C_S^2C_t$$

$$+\left(\eta_C-2\xi^1_S\right)C_{SS}-2\xi^0_SC_{St}-3\xi^1_CC_SC_{SS}-\xi^0_CC_tC_{SS}-2\xi^0_CC_SC_{St}$$

and similar expressions hold for $\eta^{St}$and η^{tt}.

The differentials of $\bar{C}=\bar{C}\left(\bar{S},\bar{t}\right)$ with respect to $\bar{S},\bar{t}$can be expressed in terms of those of $C=C\left(S,t\right)$ with respect to S,tthrough the so called prolongation formulae:-

$$\frac{\partial\bar{C}}{\partial\bar{t}}\approx\frac{\partial C}{\partial t}+a\left[D_t\left(\eta\right)-\frac{\partial C}{\partial t}D_t\left(\xi^0\right)-\frac{\partial C}{\partial S}D_t\left(\xi^1\right)\right]\tag{10}$$

$$\frac{\partial\bar{C}}{\partial\bar{S}}\approx\frac{\partial C}{\partial t}+a\left[D_S\left(\eta\right)-\frac{\partial C}{\partial t}D_S\left(\xi^0\right)-\frac{\partial C}{\partial S}D_S\left(\xi^1\right)\right]\tag{11}$$

$$\frac{\partial^2\bar{C}}{\partial S^2}\approx\frac{\partial^2 C}{\partial S^2}+a\left\{D_S\left[D_S\left(\eta\right)-\frac{\partial C}{\partial t}D_S\left(\xi^0\right)-\frac{\partial C}{\partial S}D_S\left(\xi^1\right)\right]-\frac{\partial^2 C}{\partial S^2}D_S\left(\xi^1\right)-\frac{\partial^2 C}{\partial S\partial t}D_S\left(\xi^0\right)\right\}\tag{12}$$

where

$$D_t=\frac{\partial}{\partial t}+\frac{\partial C}{\partial t}\frac{\partial}{\partial C}+\frac{\partial}{\partial C_t}\frac{\partial C_t}{\partial t}\frac{\partial}{\partial C_t}+\frac{\partial C_S}{\partial t}\frac{\partial}{\partial C_S}+...\tag{13}$$

and

$$D_S=\frac{\partial}{\partial S}+\frac{\partial C}{\partial S}\frac{\partial}{\partial C}+\frac{\partial C_t}{\partial S}\frac{\partial}{\partial C_t}+\frac{\partial C_S}{\partial S}\frac{\partial}{\partial S}.\tag{14}$$

Using eqs.(8, 10-12), we obtain

$$\frac{\partial\bar{C}}{\partial\bar{t}}+\frac{1}{2}\sigma^2\bar{S}^2\frac{\partial^2\bar{C}}{\partial\bar{S}^2}+r\bar{S}\frac{\partial C}{\partial S}-r\bar{C}\approx\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC+a\Gamma\tag{15}$$

where

$$\begin{aligned}\Gamma=&\left[D_t(\eta)-\frac{\partial C}{\partial t}D_t(\xi^0)-\frac{\partial C}{\partial S}D_t(\xi^1)\right]+\\&\frac{1}{2}\sigma^2S^2\left\{D_s\left[D_s(\eta)-\frac{\partial C}{\partial t}D_s(\xi^0)-\frac{\partial C}{\partial S}D_t(\xi^1)\right]-\frac{\partial^2 C}{\partial S^2}D_s(\xi^1)-\frac{\partial^2 C}{\partial S\partial t}D_s(\xi^0)\right\}+\\&rS\frac{\partial C}{\partial S}\left[D_s(\eta)-\frac{\partial C}{\partial t}D_s(\xi^0)-\frac{\partial C}{\partial S}D_s(\xi^1)\right]-r\eta+\sigma^2S\frac{\partial^2 C}{\partial S^2}\xi^1+r\frac{\partial C}{\partial S}\xi^1\end{aligned}\tag{16}$$

Hence, the determining equation for the problem under reference is of the form $\Gamma=0$with Γbeing given by eq. (16).

Using eqs. (13-14, 16) and equating to zero, the coefficients' of the various monomials of the first and second order partial derivatives of C, we obtain the following equations for the symmetry group of the Black Scholes equation.

$$\xi^0_C=0\tag{17}$$

$$\xi^0_S = 0 \tag{18}$$

$$\xi^0_{CC} = 0 \tag{19}$$

$$-\xi^1_C + \frac{1}{2}\sigma^2 S^2 \xi^0_{SC} = 0 \tag{20}$$

$$-S\xi^1_S + \xi^1 + \frac{1}{2}rSC\xi^0_C + \frac{1}{2}S\xi^0_t + \frac{1}{4}\sigma^2 S^3 \xi^0_{CC} = 0 \tag{21}$$

$$\xi^1_{CC} - rS\xi^0_{CC} = 0 \tag{22}$$

$$\frac{1}{2}\eta_{CC} - \xi^1_{SC} + rS\xi^0_{SC} - \frac{1}{2}rC\xi^0_{CC} = 0 \tag{23}$$

$$\begin{aligned} &-\xi^1_t + \sigma^2 S^2 \eta_{SC} - \tfrac{1}{2}\sigma^2 S^2 \xi^1_{SS} - rS\xi^1_S + r\xi^1 - r^2 SC\xi^0_C + rS\xi^0_t + \\ &r^2 S^2 \xi^0_S - rC\xi^1_C - \sigma^2 r S^2 C\xi^0_{SC} + \tfrac{1}{2}r\sigma^2 S^3 \xi^0_{SS} = 0 \end{aligned} \tag{24}$$

$$\left(\eta_t + \tfrac{1}{2}\sigma^2 S^2 \eta_{SS} + rS\eta_S - r\eta\right) - \left(\xi^0_t + \tfrac{1}{2}\sigma^2 S^2 \xi^0_{SS} + rS\xi^0_S - r\xi^0\right) rC - r^2 C\xi^0 - r^2 C^2 \xi^0_C + rC\eta_C = 0 \tag{25}$$

Eqs. (17-18) require that ξ^0 be a function of t only. Hence, eq. (20) reduces to $\xi^1_C = 0$ which implies that ξ^1 does not depend on C. Further, eq. (21) becomes $-S\xi^1_S + \xi^1 + \frac{1}{2}S\xi^0_t = 0$ which has the solution

$$\xi^1(S,t) = \frac{1}{2}\xi^0_t(t)\, S \ln S + M(t)\, S \tag{26}$$

Then eq. (23) yields $\frac{1}{2}\eta_{CC} = 0$which mandates that $\eta(t,S,C)$is a linear function of $C$and hence can be written as

$$\eta(t,S,C) = \alpha(t,S)\, C + \beta(t,S) \tag{27}$$

With the above constraints for ξ^0 we can write eq. (25) as

$$-\xi^1_t + \sigma^2 S^2 \eta_{SC} - \frac{1}{2}\sigma^2 S^2 \xi^1_{SS} - rS\xi^1_S + r\xi^1 + rS\xi^0_t = 0 \tag{28}$$

Using eqs. (26-27), eq. (28) reduces to

$$In\, S\xi^0_{tt} - \left(r - \frac{1}{2}\sigma^2\right)\xi^0_t + 2M_t(t) - 2\sigma^2 S\alpha_S(S,t) = 0 \tag{29}$$

with the solution

$$\alpha(S,t) = \frac{1}{2\sigma^2}\left[\frac{1}{2}(In\, S)^2 \xi^0_{ttt} - \left(r - \frac{1}{2}\sigma^2\right) In\, S\xi^0_t + 2M_t(t)\, In\, S + N(t)\right] \tag{30}$$

Using eqs. (25), (27) we find that $\beta(S,t)$must be a solution of the Black Scholes equation while $\alpha(S,t)$must satisfy

$$\alpha_t + \frac{1}{2}\sigma^2 S^2 \alpha_{SS} + rS\alpha_S - r\xi^0_t = 0 \tag{31}$$

Eqs. (30-31) yield the following:-

$$\xi^0_{ttt} = 0 \quad \text{so that} \quad \xi^0 = Pt^2 + Qt + R \tag{32}$$

and

$$M_{tt} = 0 \quad \text{so that} \quad M = Ut + V \tag{33}$$

We finally end up with the following solutions for ξ^0, ξ^1, η:-

$$\xi^0 = Pt^2 + Qt + R \tag{34}$$

$$\xi^1 = \frac{1}{2}\left(2Pt + Q\right) S\, In\, S + Ut + V$$

$$\eta = \frac{1}{2\sigma^2}\left\{\begin{array}{l} P(In\, S)^2 - \left(r - \frac{1}{2}\sigma^2\right)(2Pt+Q) In\, S + 2U\, In\, S + \left[\left(r - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r\right] Pt^2 + \\ 2\sigma^2\left[\frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 Q - \frac{1}{2}P + rQ - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)U\right] t + W \end{array}\right\} C + \beta(S,t) \tag{35}$$

where P, Q, R, U, V, W are arbitrary constants. On substituting these expressions for ξ^0, ξ^1, η in eq. (7), we obtain the expressions for the six generators from the coefficients of these constants as follows:-

$$X_1 = \frac{\partial}{\partial t} \tag{36}$$

$$X_2 = S\frac{\partial}{\partial S} \tag{37}$$

$$X_3 = t\frac{\partial}{\partial t} + \frac{1}{2} S\, In\, S\frac{\partial}{\partial S} - \frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)\left(In\, S\right) C\frac{\partial}{\partial C} + \frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 tC\frac{\partial}{\partial C} + rtC\frac{\partial}{\partial C} \tag{38}$$

$$X_4 = tS\frac{\partial}{\partial S} + \frac{1}{\sigma^2}\left(In\, S\right) C\frac{\partial}{\partial C} - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right) tC\frac{\partial}{\partial C} \tag{39}$$

$$X_5 = t^2\frac{\partial}{\partial t} + (In\, S)tS\frac{\partial}{\partial S} + \left\{\frac{1}{2\sigma^2}(In\, S)^2 - \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)(In\, S)t + \left[\frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + r\right]t^2 - \frac{1}{2}t\right\} C\frac{\partial}{\partial C} \tag{40}$$

$$X_6 = C\frac{\partial}{\partial C} \quad X_\beta = \beta\left(S, t\right)\frac{\partial}{\partial C} \tag{41}$$

Using eq. (39), we can present eq.(38) in a simplified form as:-

$$X_3 = t\frac{\partial}{\partial t} + \frac{1}{2}\left(In\, S\right) S\frac{\partial}{\partial S} + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right) tS\frac{\partial}{\partial S} + rtC\frac{\partial}{\partial C} \tag{42}$$

The one-parameter groups G_i corresponding to each of the above generators are given by the usual process of exponentiation e.g.

$$G_1 : \ \left(t + \varepsilon, S, C\right) \tag{43}$$

$$G_2 : \ \left(t, \varepsilon S, C\right), \varepsilon \neq 0 \tag{44}$$

$$G_3 : \ \left(\varepsilon^2 t, e^{\left(r - \frac{1}{2}\sigma^2\right)\left(\varepsilon^2 - \varepsilon\right)t} S^\varepsilon, e^{r\left(\varepsilon^2 - 1\right)t} C\right), \varepsilon \neq 0 \tag{45}$$

$$G_4 : \ \left(t, e^{\varepsilon\sigma^2 t} S, e^{\left[\frac{1}{2}\varepsilon^2\sigma^2 - \varepsilon\left(r - \frac{1}{2}\sigma^2\right)\right]t} S^\varepsilon C\right) \tag{46}$$

$$G_5 : \left(\frac{t}{1 - 2\varepsilon\sigma^2 t}, S^{\left(1 - 2\varepsilon\sigma^2 t\right)^{-1} t}, \left(1 - 2\varepsilon\sigma^2 t\right)^{\frac{1}{2}} e^{\left\{\frac{\varepsilon\left[\left(In\, S - \left(r - \frac{1}{2}\sigma^2\right)t\right)^2 + 2r\sigma^2 t^2\right]}{1 - 2\varepsilon\sigma^2 t}\right\}} C\right) \tag{47}$$

$$G_6 : \ (t, S, \varepsilon C), \varepsilon \neq 0 \tag{48}$$

$$G_\beta : \ (t, S, C + \beta(S, t)) \tag{49}$$

We obtain the most general one-parameter symmetry group of the Black Scholes equation as a general linear combination $\sum_{i=1}^{6} c_i X_i + X_\beta$of the generators given by eqs. (36-42). We can also represent an arbitrary group transformation $g$as the composition of transformations in the aforesaid one parameter subgroups.

Since each group $G_i$is a symmetry group, if $C = C(S, t)$ is a solution of the Black Scholes equation, then so are the functions:-

$$C^{(1)}(S, t) = C(t - \in, S) \tag{50}$$

$$C^{(2)}(S, t) = C\left(t, \in^{-1} S\right), \in \neq 0 \tag{51}$$

$$C^{(3)}(S, t) = e^{\left(1-\in^{-2}\right)rt} C\left[e^{\left(\in^{-2}-\in^{-1}\right)\left(r-\frac{1}{2}\sigma^2\right)t} S^{\in^{-1}}, \in^{-2} t\right] \tag{52}$$

$$C^{(4)}(S, t) = e^{-\left[\frac{1}{2}\in^2\sigma^2 + \in\left(r-\frac{1}{2}\sigma^2\right)\right]t} S^{\in} C\left[S e^{-\in\sigma^2 t}, t\right] \tag{53}$$

$$C^{(5)}(S, t) = \left[1 + 2 \in \sigma^2 t\right]^{-\frac{1}{2}} e^{\varepsilon\left\{\frac{\left[\log S - \left(r-\frac{1}{2}\sigma^2\right)t\right]^2 + 2r\sigma^2 t^2}{1+2\in\sigma^2 t}\right\}} C\left(\frac{t}{1 + 2 \in \sigma^2 t}, S^{\frac{t}{1+2\in\sigma^2 t}}\right) \tag{54}$$

$$C^{(6)}(S, t) = \in C(S, t), \in \neq 0 \tag{55}$$

$$C^{(\beta)}(S, t) = C(S, t) + \in \beta(S, t) \tag{56}$$

Here $\in$is any real number and $\beta(S, t)$ any other solution to the Black Scholes equation. It is seen from the symmetry group G_6 and $G_\beta$that the solutions of the Black Scholes equation are linear and we can add two solutions and multiply them with a constant. The group G_1 shows time invariance of the solutions. The symmetry group G_2 reflects the scaling symmetry with respect to S.

5. Structure of the Lie Algebra $\Lambda = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$ [12-15]

We now explore the structure of the finite dimensional Lie algebra generated by $\Lambda = \langle X_1, X_2, X_3, X_4, X_5, X_6 \rangle$. The commutator table of Λ is given by:-

TABLE 1

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	$X_1+\frac{\mathrm{K}}{2}X_2+rX_6$	$X_2-\frac{\mathrm{K}}{\sigma^2}X_6$	$2X_3-\mathrm{K}X_4-\frac{1}{2}X_6$	0
X_2	0	0	$\frac{1}{2}X_2$	$\frac{1}{\sigma^2}X_6$	X_4	0
X_3	$-\left(X_1+\frac{\mathrm{K}}{2}X_2+rX_6\right)$	$-\frac{1}{2}X_2$	0	$\frac{1}{2}X_4$	X_5	0
X_4	$-X_2+\frac{\mathrm{K}}{\sigma^2}X_6$	$-\frac{1}{\sigma^2}X_6$	$-\frac{1}{2}X_4$	0	0	0
X_5	$-2X_3+\mathrm{K}X_4+\frac{1}{2}X_6$	$-X_4$	$-X_5$	0	0	0
X_6	0	0	0	0	0	0

where $\mathrm{K} = r - \frac{1}{2}\sigma^2$. Further,

$$\left[X_1,X_\beta\right]=X_{\beta_t},\quad \left[X_2,X_\beta\right]=X_{S\beta_S},\quad \left[X_3,X_\beta\right]=X_{t\beta_t+\frac{1}{2}S(\ln S)\beta_S+\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta\ln S-\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2\beta t-r\beta t}$$

$$\left[X_4,X_\beta\right]=X_{tS\beta_S-\frac{1}{\sigma^2}\beta\ln S+\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta t},\left[X_5,X_\beta\right]=X_{t^2\beta_t+tS(\ln S)\beta_S-\frac{1}{2\sigma^2}\beta(\ln S)^2+\frac{1}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)\beta t\ln S-\left[\frac{1}{2\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2+r\right]\beta t^2+\frac{1}{2}\beta t}$$

$[X_6, X_\beta] = X_{-\beta}\,[X_\beta, X_\beta] = 0$, where $X_\gamma = \gamma\frac{\partial}{\partial C}$

From table 1, the following readily follow:-

(1) the centralizers of the various elements $X_i$are:-

$$\chi(X_1) = \langle X_1, X_2, X_6\rangle, \chi(X_2) = \langle X_1, X_2, X_6\rangle, \chi(X_3) = \langle X_3, X_6\rangle, \chi(X_4) = \langle X_4, X_5, X_6\rangle,$$

$$\chi(X_5) = \langle X_4, X_5, X_6\rangle, \chi(X_6) = \langle X_1, X_2, X_3, X_4, X_5, X_6\rangle.$$

(2) the centre of $\Lambda$is $\chi(\Lambda) = \bigcap_{i=1}^{6}\chi(X_i) = \langle X_6\rangle$.

(3) $[X_1, \Lambda] = \langle X_1, X_2, X_3, X_4, X_6\rangle$,$[X_2, \Lambda] = \langle X_2, X_4, X_6\rangle$,$[X_3, \Lambda] = \langle X_1, X_2, X_4, X_5, X_6\rangle$,

$$[X_4, \Lambda] = \langle X_2, X_4, X_6\rangle, [X_5, \Lambda] = \langle X_3, X_4, X_5, X_6\rangle, [X_6, \Lambda] = 0.$$

(4) $U = \langle X_2, X_4, X_6\rangle$is a two sided ideal of Λsince $\langle[U, \Lambda]\rangle = \langle[\Lambda, U]\rangle = U$.It is also an invariant subalgebra of Λ.

(5) the Lie algebra $\Lambda$is not solvable, since $[\Lambda, \Lambda] = \Lambda$and hence the derived series of $\Lambda$is stationary. However, for the subalgebra U, we have, $[U, U] = \langle X_6\rangle$,$\left[U^{(2)}, U^{(2)}\right] =$

$[X_6, X_6] = 0$, so that $U$is solvable. Being the maximal ideal, it is, therefore, the radical of Λ. Also, $V = \langle X_1, X_3, X_5\rangle$is a semisimple and simple subalgebra.

(6) in view of (e), the Lie algebra Λadmits the Levi decomposition $\Lambda = U \oplus V$

(7) the adjoint representations of the various elements can be trivially written from the commutator table and, in the ordering $\langle X_1, X_3, X_5, X_2, X_4, X_6\rangle$ take the form:-

$$\begin{aligned} X_1 &= \left(a_{21} = -1, a_{24} = -\frac{\mathrm{K}}{2}, a_{26} = -r, a_{32} = -2, a_{35} = \mathrm{K}, a_{36} = \frac{1}{2}, a_{54} = -1, a_{56} = \frac{\mathrm{K}}{\sigma^2}\right); \\ X_2 &= \left(a_{24} = -\frac{1}{2}, a_{35} = -1, a_{56} = -\frac{1}{\sigma^2}\right); \end{aligned} \tag{57}$$

$$\begin{aligned} X_3 &= \left(a_{11} = 1, a_{14} = \frac{\mathrm{K}}{2}, a_{16} = r, a_{33} = -1, a_{44} = \frac{1}{2}, a_{55} = -\frac{1}{2}\right); \\ X_4 &= \left(a_{14} = 1, a_{16} = -\frac{\mathrm{K}}{\sigma^2}, a_{25} = \frac{1}{2}, a_{46} = \frac{1}{\sigma^2}\right); \\ X_5 &= \left(a_{12} = 2, a_{15} = -\mathrm{K}, a_{16} = -\frac{1}{2}, a_{23} = 1, a_{45} = 1\right); \\ X_6 &= \mathrm{O}_{6\times 6} \end{aligned} \tag{58}$$

The non-specified elements are $0's$ in the above matrices.

(8) the action, defined by $\varphi_{ij} = \left(e^{\varepsilon adj\ X_i}\right) X_j$, of the adjoints of the various generators $X_i$on the algebra $\Lambda$is summarized below (These constitute the inner automorphism group of the Lie algebra Λ):-

TABLE 2

$j \rightarrow$ $i \downarrow$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	$X_1-\varepsilon X_3$ $+\varepsilon^2 X_5$	$X_2-\frac{\varepsilon\mathrm{K}}{2}X_3-\varepsilon X_4$	X_3- $2\varepsilon X_5$	$X_4+\varepsilon\mathrm{K}X_5$	X_5	$-\varepsilon r X_3+\frac{\varepsilon\mathrm{K}}{\sigma^2}X_4+$ $\left[\frac{\varepsilon}{2}+\frac{\varepsilon^2}{2}\left(2r+\frac{\mathrm{K}^2}{\sigma^2}\right)\right]X_5+X_6$
X_2	X_1	$X_2-\frac{\varepsilon}{2}X_3$	X_3	$X_4-\varepsilon X_5$	X_5	$-\frac{\varepsilon}{\sigma^2}X_4+\frac{\varepsilon^2}{2\sigma^2}X_5$ $+X_6$
X_3	$2X_2$	$e^{\frac{\varepsilon}{2}}\left(e^{\frac{\varepsilon}{2}}-1\right)\mathrm{K}X_1+e^{\frac{\varepsilon}{2}}X_2$	X_3	$e^{-\frac{\varepsilon}{2}}X_4$	$e^{-\varepsilon}X_5$	X_6
X_4	X_1	εX_1+X_2	X_3	$\frac{\varepsilon}{2}X_3+X_4$	X_5	$\left(\frac{\varepsilon^2}{2\sigma^2}-\frac{\varepsilon\mathrm{K}}{\sigma^2}\right)X_1+$ $\frac{\varepsilon}{\sigma^2}X_2+X_6$
X_5	X_1	X_2	$2\varepsilon X_1+$ X_3	$-\varepsilon\mathrm{K}X_1+$ εX_2+X_4	$\varepsilon^2 X_1+$ εX_3 $+X_5$	$-\frac{1}{2}\varepsilon X_1+X_6$
X_6	X_1	X_2	X_3	X_4	X_5	X_6

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Physical Invariants of Intelligence

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Abstract: The objective of this work is to extend the physical invariants of biosignature (from disorder to order) to invariants of intelligent behavior: *from disorder to order via phase transition*. The approach is based upon the extension of the physics' First Principles that includes behavior of living systems. The new architecture consists of motor dynamics simulating actual behavior of the object, and mental dynamics representing evolution of the corresponding knowledge-base and incorporating it in the form of information flows into the motor dynamics. Due to feedback from mental dynamics, the motor dynamics attains quantum-like properties: its trajectory splits into a family of different trajectories, and each of those trajectories can be chosen with the probability prescribed by the mental dynamics. Intelligence is considered as a tool to preserve and improve survivability of Livings. From the viewpoint of mathematical formalism, it can be associated with the capability to make decisions that *control* the motor dynamics via a feedback from the *mental* dynamics by providing a quantum-like collapse of a random motion into an appropriate deterministic state. Special attention is focused on data-driven discovery of the underlying physical model displaying an intelligent behavior within the proposed formalism.

Keywords: Biophysics; Dynamical Model of Livings; Biological Models; Intelligent and Extraterrestrial Life
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1. Introduction

1.1 Motivation, definition and general remarks

Recent advances in astrophysics, and in particular, discovery of a solar system similar to ours, revitalized the interest to extraterrestrial life. A panel of scientists convened by America's leading scientific advisory group says the hunt for extraterrestrial life should be

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greatly expanded to include what they call "weird life": organisms that lack DNA or other molecules found in life as we know it. That raises the following question: what are physical invariants of Living's behavior, and in particular, of *intelligent* one that would not depend upon the specific features of Life composition? The concept of intelligence has many different definitions depending upon the context in which it is used. The well-established one is the following: intelligence is a general mental capability that involves the ability to reason, plan, solve problems, think abstractly, comprehend ideas and language, and learn. Although this definition sounds perfect, it is inconvenient for mathematical formalization that would express it in a short first-principle-like statement. In order to create such a definition, we will link intelligence to the model of Livings proposed in our earlier publications. One of the main objectives of this model is to *extend* the First Principles of classical physics to include phenomenological behavior of living systems, i.e. to develop a new mathematical formalism within the framework of classical dynamics that would allow one to capture the specific properties of natural or artificial living systems such as formation of the collective mind based upon abstract images of the selves and non-selves, exploitation of this collective mind for communications and predictions of future expected characteristics of evolution, as well as for making decisions and implementing the corresponding corrections if the expected scenario is different from the originally planned one. The approach is based upon our previous publications (M. Zak, 1999a, 2003, 2004, 2005a, 2006a, 2007a, 2007b and 2007c) that postulate that even a primitive living species possesses additional non-Newtonian properties which are not included in the laws of Newtonian or statistical mechanics. These properties follow from a privileged ability of living systems to possess a self-image (a concept introduced in psychology) and to interact with it. The proposed mathematical formalism is quantum-inspired: it is based upon coupling the classical dynamical system representing the motor dynamics with the corresponding Liouville equation describing the evolution of initial uncertainties in terms of the probability density and representing the mental dynamics. (Compare with the Madelung equation that couples the Hamilton-Jacobi and Liouville equations via the quantum potential.)The coupling is implemented by the information-based supervising forces that can be associated with the self-awareness. These forces fundamentally change the pattern of the probability evolution, and therefore, leading to a major departure of the behavior of living systems from the patterns of both Newtonian and statistical mechanics. Further extension, analysis, interpretation, and application of this approach to complexity in Livings and emergent intelligence have been addressed in the papers referenced above.

In the next introductory sub-sections we will briefly review the model of Livings.

2. Dynamical Model

Without going into mathematical details of the dynamical model of Livings, we will illustrate its performance by the Figure 1. (For mathematical details, see Zak,M., 2007c; however, a minimum of mathematical details necessary for a formulation of physical invariants of intelligence as well as for decoding of a time-series-like signals will be introduced in the next sections).

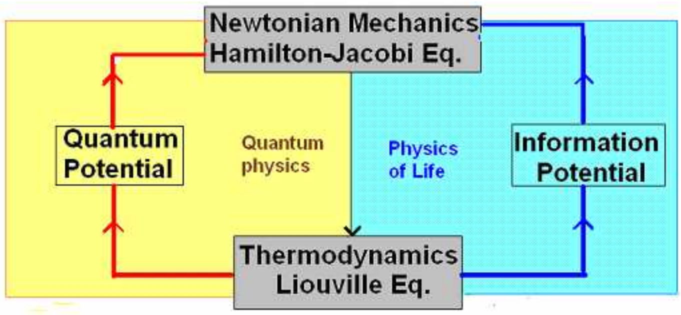

Figure 1 . Classical Physics, Quantum physics, and Physics of Life

The model is represented by a system of nonlinear ODE and a nonlinear parabolic PDE coupled in a master-slave fashion. The coupling is implemented by a feedback that includes the first gradient of the probability density, and that converts the first order PDE (the Liouville equation) to the second order PDE (the Fokker-Planck equation). Its solution, in addition to positive diffusion, can display negative diffusion as well, and that is the major departure from the classical Fokker-Planck equation. The nonlinearity is generated by a feedback from the PDE to the ODE. As a result of the nonlinearity, the solutions to PDE can have attractors (static, periodic, or chaotic) in *probability* space. The multi-attractor limit sets allow one to introduce an extension of neural nets that can converge to a prescribed type of a stochastic process in the same way in which a regular neural net converges to a prescribed deterministic attractor. The solution to ODE represents another major departure from classical ODE: due to violation of Lipchitz conditions at states where the probability density has a sharp value, the solution loses its uniqueness and becomes random. However, this randomness is controlled by the PDE in such a way that each random sample occurs with the corresponding probability, (see Fig.2).

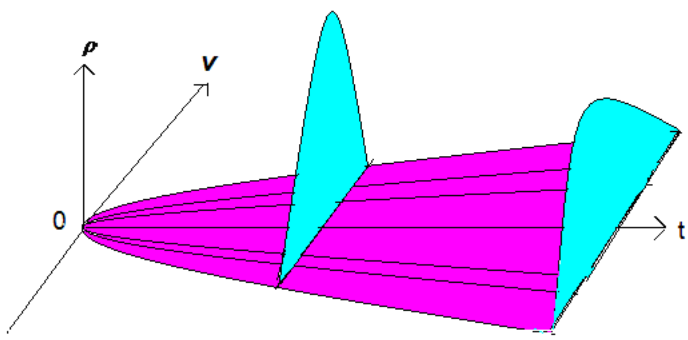

Figure 2. Stochastic process and probability density

The model represents a fundamental departure from both Newtonian and statistical mechanics. In particular, negative diffusion cannot occur in isolated systems without help of the Maxwell sorting demon that is strictly forbidden in statistical mechanics. The only conclusion to be made is that the model is non-Newtonian, although it is fully consistent with the theory of differential equations and stochastic processes. Strictly speaking, it is a matter of definition weather the model represents an isolated or an open system since the additional energy applied via the information potential is generated by the system "itself" out of components of the probability density. In terms of a *topology* of its dynamical structure, the proposed model links to quantum mechanics: if the information potential is replaced by the quantum potential, the model turns into the Madelung equations that are equivalent to the Schrödinger equation. The system of ODE describes a mechanical motion of the system driven by *information* forces. Due to specific properties of these forces, this motion acquires characteristics similar to those of quantum mechanics. These properties are discussed below.

α.*Superposition.* In quantum mechanics, any observable quantity corresponds to an eigenstate of a Hermitian linear operator. The linear combination of two or more eigenstates results in quantum superposition of two or more values of the quantity. If the quantity is measured, the projection postulate states that the state will be randomly collapsed onto one of the values in the superposition (with a probability proportional to the square of the amplitude of that eigenstate in the linear combination). Let us compare the behavior of the model of Livings from that viewpoint, Fig. 3.

As follows from Fig. 3, all the particular solutions intersect at the same point $x = 0$ at $t = 0$, and that leads to non-uniqueness of the solution due to violation of the Lipcshitz condition. Therefore, the same initial condition $x = 0$ at $t = 0$ yields infinite number of different solutions forming a family; each solution of this family appears with a certain probability guided by the corresponding Fokker-Planck equation. For instance, in case of the solution plotted in Fig. 2, the "winner" solution is $v \equiv 0$ since it passes through the maxima of the probability density. However, with lower probabilities, other solutions of the same family can appear as well. Obviously, this is a non-classical effect. Qualitatively, this property is similar to those of quantum mechanics: the system keeps all the solutions simultaneously and displays each of them "by a chance", while that chance is controlled by the evolution of probability density. It should be emphasized that the choice of displaying a certain solution is made by the Livings model only once, at $t = 0$, i.e. when it departs from the deterministic to a random state; since than, it stays with this solution as long as the Liouville feedback is present.

b. Decoherence. In quantum mechanics, decoherence is the process by which quantum systems in complex environments exhibit classical behavior. It occurs when a system interacts with its environment in such a way that different portions of its wavefunction can no longer interfere with each other.

Qualitatively similar effects are displayed by the model of Livings. In order to illustrate that, let us turn to Fig.3, and notice that the system makes a choice of the particular solution only once i.e. when it departs from the deterministic to a random state; since

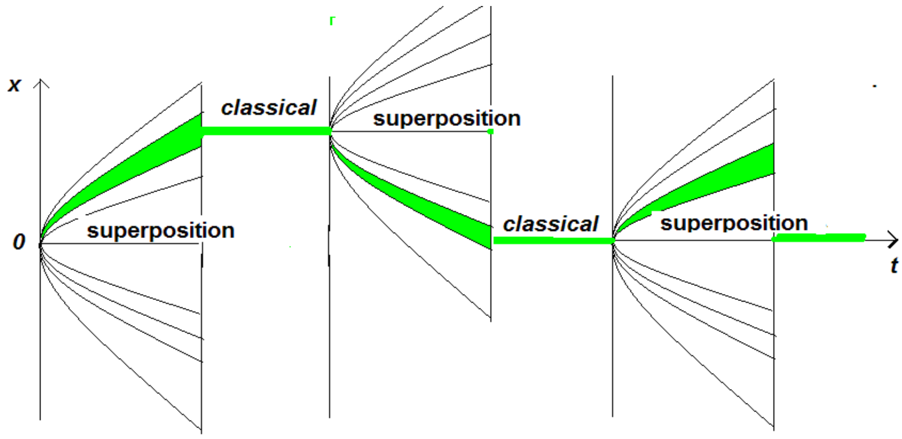

Figure 3. Switching between superposition and classical states

then, it stays with this solution as long as the Liouville feedback is present. However, as soon as this feedback disappears, the system becomes classical, i.e. fully deterministic, while the deterministic solution is a continuation of the corresponding "chosen" random solution.

Modified versions of such quantum properties as uncertainty and entanglement are also described in the referenced papers.

c. Negative diffusion. In addition to quantum-like properties, the model under consideration exhibits a very special and unique capability of progressive evolution – from disorder to order – using only "internal resources". The best illustration to that is a negative diffusion which can be organized by such a Liouville feedback that converts the Liouville equation into the Fokker-Planck equation with a negative diffusion coefficient. However, the linear version of this equation is ill-posed, Zak, M., 2005, and an appropriate nonlinearity for elimination of ill-posedness is introduced. Negative diffusion will play a central role in our approach to data-driven discovery of an intelligent life in Space, Fig. 4.

The model illuminates the "border line" between living and non-living systems. The model introduces a biological particle that, in addition to Newtonian properties, possesses the ability to process information. The probability density can be associated with the *self-image* of the biological particle as a member of the class to which this particle belongs, while its ability to convert the density into the information force - with the *self-awareness* (both these concepts are adopted from psychology). Continuing this line of associations, the equation of motion can be identified with a motor dynamics, while the evolution of density –with a mental dynamics. Actually the mental dynamics plays the role of the Maxwell sorting demon: it rearranges the probability distribution by creating the information potential and converting it into a force that is applied to the particle. One should notice that mental dynamics describes evolution of the whole class of state variables (differed from each other only by initial conditions), and that can be associated with the ability to generalize that is a privilege of living systems.

Continuing our biologically inspired interpretation, it should be recalled that the

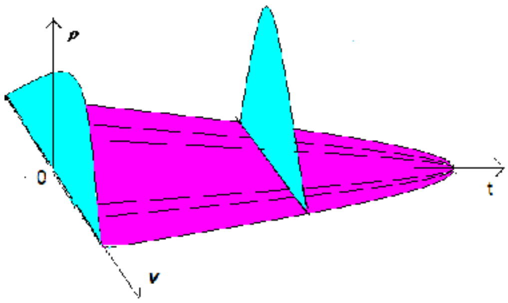

Fig. 4 Negative diffusion

second law of thermodynamics states that the entropy of an isolated system can only increase. This law has a clear probabilistic interpretation: increase of entropy corresponds to the passage of the system from less probable to more probable states, while the highest probability of the most disordered state (that is the state with the highest entropy) follows from a simple combinatorial analysis. However, this statement is correct only if there is no Maxwell' sorting demon, i.e., nobody inside the system is rearranging the probability distributions. But this is precisely what the Liouville feedback is doing: it takes the probability density ρ from the mental dynamics, creates functions of this density, converts them into a force and applies this force to the equation of motor dynamics. As already mentioned above, because of that property of the model, the evolution of the probability density becomes nonlinear, and the entropy may decrease "against the second law of thermodynamics", Fig5.

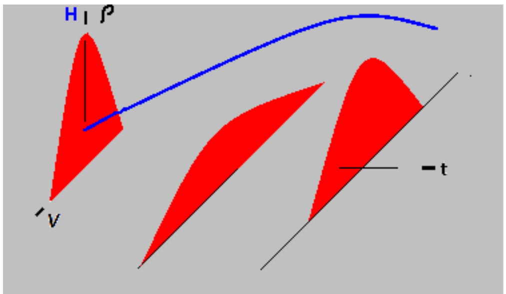

Figure 5. Living system: deviation from thermodynamics

Obviously the last statement should not be taken literary; indeed, the proposed model captures only those aspects of the living systems that are associated with their behavior, and in particular, with their motor-mental dynamics, since other properties are beyond the dynamical formalism. Therefore, such physiological processes that are needed for the metabolism are not included into the model. That is why this model is in a formal disagreement with the second law of thermodynamics while the living systems are not. In order to further illustrate the connection between the life-nonlife discrimination and the second law of thermodynamics, consider a small physical particle in a state of random migration due to thermal energy, and compare its diffusion i.e. physical random walk, with a biological random walk performed by a bacterium. The fundamental difference between these two types of motions (that may be indistinguishable in physical space) can be detected in probability space: the probability density evolution of the physical particle is always linear and it has only one attractor: a stationary stochastic process where the motion is trapped. On the contrary, a *typical* probability density evolution of a biological particle is nonlinear: it can have many different attractors, but eventually each attractor can be departed from without any "help" from outside.

That is how H. Berg, 1983, describes the random walk of an E. coli bacterium:" If a cell can diffuse this well by working at the limit imposed by rotational Brownian movement, why does it bother to tumble? The answer is that the tumble provides the cell with a mechanism for *biasing its random walk*. When it swims in a spatial gradient of a chemical attractant or repellent and it happens to run in a favorable direction, the probability of tumbling is reduced. As a result, favorable runs are extended, and the cell diffuses with drift". Berg argues that the cell analyzes its sensory cue and generates the bias *internally*, by changing the way in which it rotates its flagella. This description demonstrates that actually a bacterium interacts with the medium, i.e., it is not isolated, and that reconciles its behavior with the second law of thermodynamics. However, since these interactions are beyond the dynamical world, they are incorporated into the proposed model via the self-supervised forces that result from the interactions of a biological particle with "itself," and that formally "violates" the second law of thermodynamics. Thus, the proposed model offers a unified description of the progressive evolution of living systems. Based upon this model, one can formulate and implement the principle of maximum increase of complexity that governs the large-time-scale evolution of living systems.

3. Formulation of a Physical Invariant of Intelligence

Following our referenced publications, and remaining within the framework of dynamical formalism discussed above, we will associate life with the inequality that holds during, at least, some time interval

$$\frac{dH}{dt} < 0 \tag{1}$$

where

$$H(t) = -\int_V \rho(V,t)\ln\rho(V,t)dV \tag{2}$$

Here H is entropy of the particle, V is the particle velocity, and ρ is the probability density characterizing the velocity distribution. Obviously, the condition (1) is only sufficient, but not necessary since even a living particle may choose not to exercise its privilege to decrease disorder. We will introduce, as a measure of survivability, the strength of the random force that, being applied to a particle, nullifies the inequality (1). The capabilities to support and improve the survivability we will associate with the decision making process that implements intelligence.

The mechanism for decision making process can be represented by a collapse of a random process into a deterministic state such that the entropy suffers a sharp drop (see

Fig. 8):

$$[H(t_0 + \Delta t) - H(t_0)] \to [H(t_0)] < 0 \quad \text{at} \quad \Delta t \to 0 \tag{3}$$

Thus, the property (3) can be considered as an invariant of intelligent behavior of Livings regardless of an origin and composition of Life. However, in terms of detection of intelligence based upon a signal generated by an underlying system, the situation is more complex. The problem is in the fact that the condition (3) is necessary but *not sufficient* for detection of intelligence. Indeed, in physics, entropy is discontinuous at phase transition temperatures, and in *isolated* systems this discontinuity leads to increase of entropy by a jump. Therefore, the criterion (3) allows one to distinguish between *isolated* physical and intelligent living systems, and that is the limitation of our approach since *open* physical system, in principle, can display the property (3) as well. Such systems are usually associated with self-organization in physics and chemistry via decrease of entropy through exchange mass and energy with environment. However, the probability of such an exchange has never been evaluated, and the underlying dynamical process has never been described *quantitatively*. It should be recalled that even the second law of thermodynamics that forbids decrease of entropy in isolated systems, in principle, can be violated since it is based upon a simple combinatorics, namely: the number of complexions in which N molecules can be arranged in a prescribed order is much smaller than those in a total disorder; therefore, with a vanishingly small probability, entropy of isolated systems, in principle, can decrease. That chain of argumentations devaluate importance of the limitation of the proposed approach mentioned above.

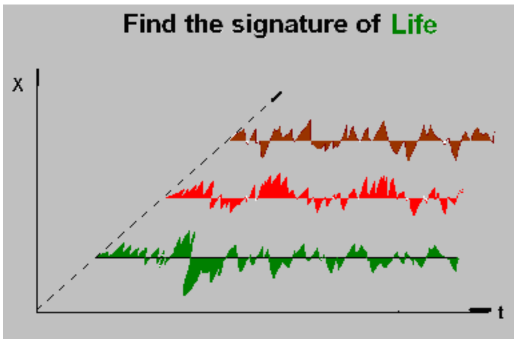

Figure 6 Which particle is alive?

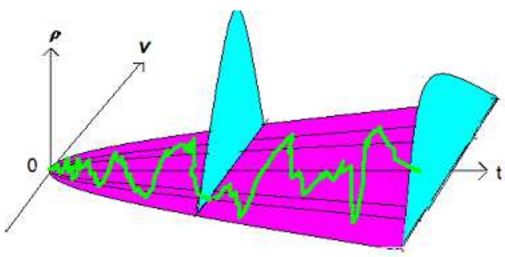

Figure 7 . Noice-contaminated trajectory

4. Data-driven Discovery of Intelligent Life

The model of Livings under consideration is equipped by a set of parameters that control the properties of the solutions discussed above. The only realistic way to reconstruct these parameters for an object to be discovered is to solve the inverse problem: given time series of sensor data describing dynamics of an unknown object find the parameters of the underlying dynamical model of this object within the formalism discussed above. As soon as such a model is reconstructed, one can predict future object behavior by running the model ahead of actual time as well as analyze a hypothetical (never observed) object behavior by appropriate changes of the model parameters. But the most important novelty of the proposed approach is the capability to detect Life that occurs if, at least, some of "non-Newtonian" parameters are present. The methodology of such an inverse problem is demonstrated by M. Zak, 2007c. However, implementation of this methodology requires availability of multi-dimensional time series for many independent runs, and that is unrealistic in case of search for Life in Space: at best, we can relay only upon one particular signal. Turning to the Fig. 3, we can observe a trajectory marked in green out of the whole family of possible trajectories. Obviously this information is hardly sufficient for the reconstruction of all the parameters of the underlying dynamical model. Therefore our goal here is more modest: to detect an intelligent Life in terms of "Yes or No", see Fig.6, and for that we need a physical invariant of intelligence that would not depend upon an origin of Life or its composition.

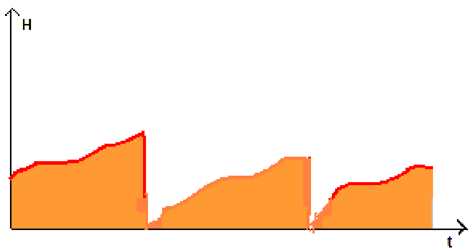

Fig. 8 Sharp drops of entropy during decision making process.

The approach is based upon the comparison of the observed signal in the form of time series and a typical signal produced by a hypothetical living system that, being described by the model discussed above, in ideal case is supposed to have the form of a trajectory marked in green in Fig. 3. However, strictly speaking, an actual realization of the trajectory may be affected by a non-Lipcshitz-originated instability at $t = 0$; as a result, small initial errors may grow exponentially, and the motion will be randomly deviated from the theoretical trajectory in such a way that a moving particle visits all the possible trajectories with the probability prescribed by the Liouville equation, Fig.7.

The basic idea of the proposed methodology is the following: find such an operator that being applied to the time series of the signal under consideration filters out non-stationary components thereby exposing only the stationary part of the signal; then, based upon the type of this operator, one can make the conclusion about existence of sharp drops (down to zero) of entropy. A mathematical structure of the filter operator can be based upon the underlying model of Livings described above. The mathematical representation of this model will be given below.

The proposed model that describes *mechanical behavior* of Livings qualitatively has been illustrated in the previous sections. Now we will turn to brief discussion of its mathematical formalism. The model can be presented in the following compressed invariant form

$$\dot{v} = -\zeta\alpha \bullet \nabla_v \ln \rho, \tag{4}$$

$$\dot{\rho} = \zeta\nabla_V^2 \rho \bullet \bullet \alpha, \tag{5}$$

where ν is velocity vector, ρ is probability density, ζ is universal constant, and α *(D,w)* is a tensor co-axial with the tensor of the variances D that may depend upon these variances. Eq. (1) represents the second Newton's law in which the physical forces are replaced by information forces via the gradient of the information potential$\Pi = \zeta \ln \rho$, while the constant ζ connects the information and inertial forces formally replacing the Planck constant in the Madelung equations of quantum mechanics. Eq. (5) represents the continuity of the probability density (the Liouville equation), and unlike the classical case, it is non-linear because of dependence of the tensor α upon the components of the density D. This model is equipped by a set of parameters w that control the properties of the solutions discussed above. We will start our discussion with solution of the following inverse problem: given time series of sensor data describing dynamics of an unknown object, find the parameters of the underlying dynamical model of this object within the formalism of Eqs(4) and (5). As soon as such a model is reconstructed, one can predict future object behavior by running the model ahead of actual time as well as analyze a hypothetical (never observed) object behavior by appropriate changes of the model parameters. But the most important novelty of the proposed approach is the capability to detect Life that occurs if, at least, some of "non-Newtonian" parameters are present. The methodology of such an inverse problem is demonstrated by M. Zak, 2007c and illustrated in Figure 9. Here we will give a brief description of this methodology.

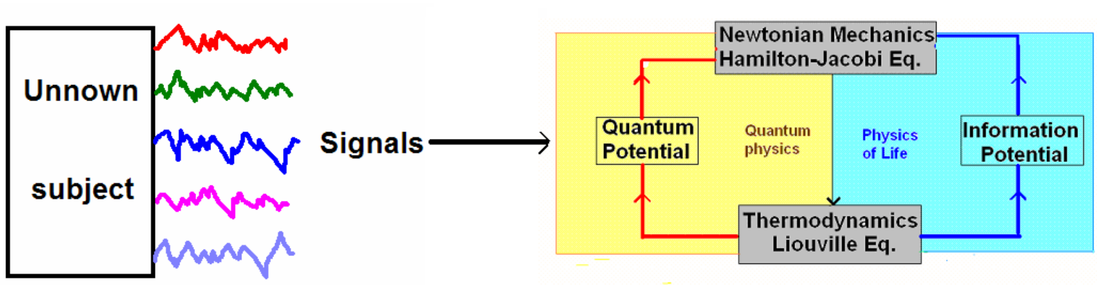

Fig. 9 Data-driven discovery of biological models

We will first assume that the experimental data are available in the form of time series for the state variables in the form

$$v_i = v_i(t, C_i), \quad C_i = 1, 2, ...m_i, \tag{6}$$

Here each function at a fixed C_i describes a sample of the stochastic process associated with the variablev_i, while the family of these curves at $C_i = 1, 2, ...m_i$, approximates the whole i^{th} ensemble. Omitting details of extracting the correlation moments $D_{ij} = D_{ij}(t)$from the functions (6), we assume that these moments as well as their time derivatives are reconstructed in the form of time series. Then, substitutingD_{ij} and $\dot{D}_{ij}$ into Eq. (4) that follows from Eq. (6)

$$\dot{D}_{ij} = -\frac{1}{2}\zeta w_{ijks} \tanh \tilde{D}_{ks} \quad i = 1, 2, ...n, \tag{7}$$

for times $t_1, ...t_q$, one arrives at a linear system of algebraic equation with respect to the constant parameters w_{ijks}, c_{ij} and ζ that, for compression, can be denoted and enumerated as W_i

$$.\sum_{i=1}^{m} A_i W_i = B. \quad i = 1, 2, ...m = 2^{2n} + 2^n \tag{8}$$

where m is the number of the parameters defining the model, and

$$A_i = A_i(D_{ks}, \dot{D}_{ks}), \quad B = B(D_{ks}, \quad \dot{D}_{ks}), \tag{9}$$

are the coefficients at the parameters W$_i$ and the free term, respectively. Introducing values of A_i and B at the points $t = t_j, j = 1, ...q$, and denoting them as A_{ij}, and B_j , one obtains a linear system of $2^n q$algebraic equations

$$\sum_{i=1}^{m} A_{ij} W_i = B_j, \quad j = 1, 2, ...q \tag{10}$$

with respect to m unknown parameters. It is reasonable to assume that

$$2^n q \geq m, \quad \text{i.e.} \quad q \geq 2^n + 1 \tag{11}$$

so the system becomes over-determined. The best-fit solution is found via pseudo-inverse of the matrix

$$A = \{A_{ij}\}, \quad \text{i.e.} \quad W = A * \bar{B} \tag{12}$$

Here

$$A^* = (A^T A)^{-1} A^T, \quad \text{and} \quad \bar{B} = \{B_j\} \tag{13}$$

If at least some of the parameters W found from Eq. (12) are not zero, the system is biological rather than physical, or it may be physical, but manipulated by a human. Based upon the discovered model, one can find sharp drops of entropy if such drops exist.

Thus, the basic idea of the proposed methodology is the following: in order to reconstruct the model of a stochastic process, one has to have an ensemble of samples, unless the stochastic process is stationary. Since the ensembles are not always available, and the stochastic processes of interest are not necessarily stationary, an alternative approach should be developed. We propose a methodology based upon the decomposition of a given signal into stationary and non-stationary components with further filtering-out the non-stationary one. The criterion of stationarity is associated with the area enveloped by the autocorrelation function: the smaller the area the closer the process to a stationary one, Fig. 10. After the stationary component of the sought stochastic process is found, one can reconstruct the underlying mathematical model, or, at least, its fragments.

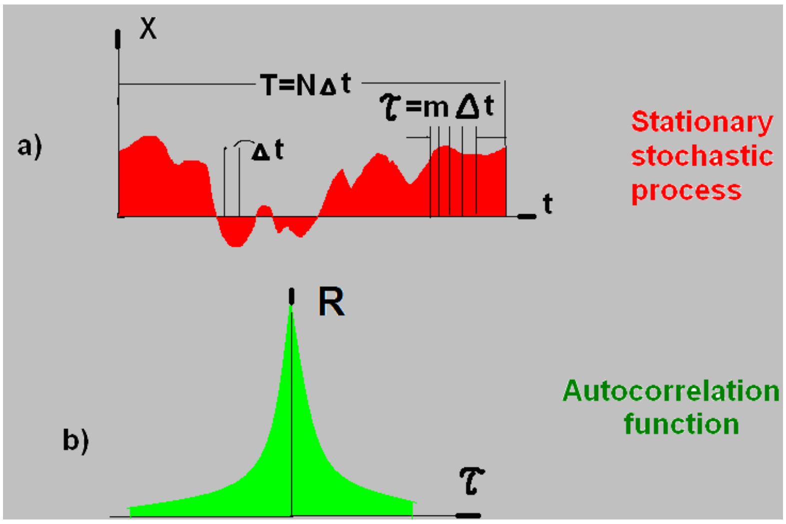

Fig. 10 Criterion of stationarity of a stochastic process

5. Filter-operators

In this section we will introduce operators that filter out non-stationarities of a stochastic process associated with sharp drops of its entropy. For that purpose, we have to go into some mathematical details of the underlying mathematical model Eqs (4,5) in its simplest form.

a. Destabilizing effect of Liouville feedback. We will start with derivation of an auxiliary result that illuminates departure from Newtonian dynamics. For mathematical clarity, we will consider here a one-dimensional motion of a unit mass under action of a force $f$depending upon the *velocity* v and time t

$$\dot{v} = f(v, t), \tag{14}$$

If initial conditions are not deterministic, and their probability density is given in the form

$$\rho_0 = \rho_0(V), \quad \text{where} \quad \rho \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \rho dV = 1 \tag{15}$$

while ρ is a *single- valued* function, then the evolution of this density is expressed by the corresponding Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial v}(\rho f) = 0 \tag{16}$$

The solution of this equation subject to initial conditions and normalization constraints (16) determines probability density as a function of V and$t$: $\rho = \rho(V, t)$.

Let us now specify the force f as a feedback from the Liouville equation

$$f(v, t) = \varphi[\rho(v, t)] \tag{17}$$

and analyze the motion after substituting the force (17) into Eq.(14)

$$\dot{v} = \varphi[\rho(v, t)], \tag{18}$$

Substituting the force $f$from Eq. (4) into Eq. (3), one arrives at the *nonlinear* equation for evolution of the probability density

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial V}\{\rho\varphi[\rho(V,t)]\}=0 \tag{19}$$

Let us now demonstrate the destabilizing effect of the feedback (4). For that purpose, it should be noted that the derivative $\partial\rho/\partial v$must change its sign, at least once, within the interval$-\infty<v<\infty$, in order to satisfy the normalization constraint (15).

But since

$$Sign\frac{\partial\dot{v}}{\partial v}=Sign\frac{d\varphi}{d\rho}Sign\frac{\partial\rho}{\partial v} \tag{20}$$

there will be regions of v where the motion is unstable, and this instability generates randomness with the probability distribution guided by the Liouville equation (19). It should be noticed that the condition (20) may lead to exponential or polynomial growth of v (in the last case the motion is called neutrally stable, however, as will be shown below, it causes the emergence of randomness as well if prior to the polynomial growth, the Lipcshitz condition is violated).

b. Emergence of randomness. In order to illustrate mathematical aspects of the concepts of Liouville feedback, as well as associated with it instability and randomness let us take the feedback (19) in the form

$$f=-\sigma^2\frac{\partial}{\partial v}\ln\rho \tag{21}$$

to obtain the following equation of motion

$$\dot{v}=-\sigma^2\frac{\partial}{\partial v}\ln\rho, \tag{22}$$

This equation should be complemented by the corresponding Liouville equation (in this particular case, the Liouville equation takes the form of the Fokker-Planck equation)

$$\frac{\partial\rho}{\partial t}=\sigma^2\frac{\partial^2\rho}{\partial V^2} \tag{23}$$

Here v stands for a particle velocity, and $\sigma^2$is the constant diffusion coefficient.

The solution of Eq. (23) subject to the sharp initial condition is

$$\rho=\frac{1}{2\sigma\sqrt{\pi t}}\exp(-\frac{V^2}{4\sigma^2 t}) \tag{24}$$

Substituting this solution into Eq. (22) at $V=v$ one arrives at the differential equation with respect to $v(t)$

$$\dot{v}=\frac{v}{2t} \tag{25}$$

and therefore,

$$v=C\sqrt{t} \tag{26}$$

where C is an arbitrary constant. Since $v = 0$ at $t = 0$ for any value of C, the solution (26) is consistent with the sharp initial condition for the solution (24) of the corresponding Liouvile equation (23). The solution (13) describes the simplest irreversible motion: it is characterized by the "beginning of time" where all the trajectories intersect (that results from the violation of Lipschitz condition at $t = 0$), while the backward motion obtained by replacement of t with $(-t)$ leads to imaginary values of velocities. One can notice that the probability density (24) possesses the same properties, Fig. 2.

For a fixed C, the solution (26) is *unstable* since

$$\frac{d\dot{v}}{dv} = \frac{1}{2t} > 0 \tag{27}$$

and therefore, an initial error always grows generating *randomness.* Initially, at $t = 0$, this growth is of infinite rate since the Lipschitz condition at this point is violated

$$\frac{d\dot{v}}{dv} \to \infty \quad \text{at} \quad t \to 0 \tag{28}$$

This type of instability has been introduced and analyzed by (Zak, M., 1992).

Considering first Eq. (26) at fixed C as a sample of the underlying stochastic process (24), and then varying C, one arrives at the whole ensemble characterizing that process.

$$D_v \propto v^2 \propto t \tag{29}$$

As follows from Fig. 2, the solution (26) describes the behavior of Livings *after* a sharp drop of entropy. In order to describe its behavior *before* such a drop we should turn to the effect of negative diffusion.

c. From disorder to order. Negative diffusion is another non-trivial property of systems with the Liouville feedback. In order to demonstrate that, let us modify Eq. (22) as following

$$\dot{v} = \sigma^2 \sqrt{D} \frac{\partial}{\partial v} \ln \rho \tag{30}$$

Then the corresponding Liouville equation is

$$\frac{\partial \rho}{\partial t} = -\sigma^2 \sqrt{D} \frac{\partial^2 \rho}{\partial V^2} \tag{31}$$

Multiplying Eq.(31) by V^2 , then integrating it with respect to V over the whole space, one arrives at ODE for the variance D(t)

$$\dot{D} = -2\sigma^2 \sqrt{D} \tag{32}$$

The solution to this equation for $t \leq 0$ and $D = 0$ at $t = 0$ is

$$\sqrt{D} = -\sigma^2 t \tag{33}$$

It has a terminal attractor at $t = 0$ that is approached in a finite time and that cannot be overcome by the solution (33), Zak, M., 1992. Now the solution to the nonlinear version of the Fokker-Planck equation in (31) can be approximated by the first term in the Gram-Charlier series represented by the normal distribution with the variance D. For the case close to a sharp initial value at $V = 0$

$$\rho = \frac{1}{2\sigma\sqrt{D\pi t}} \exp(-\frac{V^2}{4\sigma^2 Dt}) \tag{34}$$

Substituting Eg. (34) (with reference to Eq. (33)) into Eq. (30), one obtains the ODE for the family of trajectories representing the stochastic process under consideration

$$\dot{v} = \frac{v}{2\sigma^2 t^2} \quad \text{for} \quad t \leq 0 \tag{35}$$

whence for $t \leq 0$ and $v = 0$ at $t = 0$

$$v = \exp(-\frac{1}{2\sigma^2 t}) \tag{36}$$

This solution has the same terminal attractor at t = 0 as the solution (33) does.

The family of trajectories representing the underlying stochastic process before and after the critical point of the phase transition, i.e. of the sharp drop of the entropy, is shown in Fig. 8. It should be noticed that actual trajectory may not display any singularities at the critical point, and therefore, we will need to deal with the statistical invariants of this process, and in particular, with its variance.

d. Coordinate transformation. Let us now change the coordinate v to x such that the new stochastic process is stationary. Since in our approximation, the stochastic process is defined by its variance, the condition for its stationarity can be express by the requirement that

$$\frac{\partial D_x}{\partial t} \equiv 0, \quad \text{i.e.} \quad D_x = const. \tag{37}$$

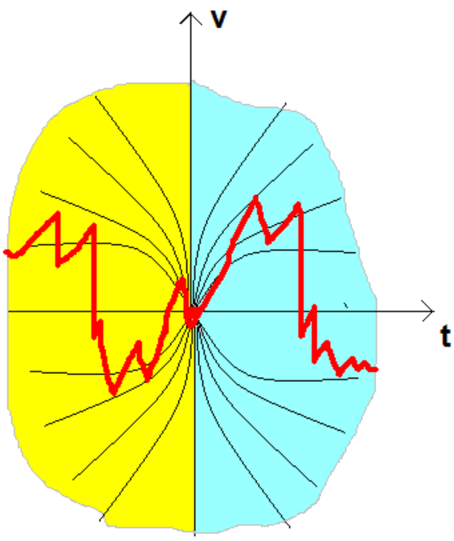

Fig. 11 Neighborhood of trajectory around of the point of sharp drop of entropy.

The connection between the variances in the old and in the new variables can be approximated as

$$D_v \approx (\frac{dv}{dx})^2 D_x \tag{38}$$

whence

$$\frac{dv}{dx} = \sqrt{\frac{D_v}{D_x}} \tag{39}$$

Let us now select a point on the time series representing the given signal and verify whether this is a critical point where a phase transition occurs, (see Figs. 8 and11). As follows from Eq. (33), at the left side of its neighborhood,$D_v \propto t^2$, and therefore.

$$(\frac{dv}{dx})_L \propto t \tag{40}$$

Similarly, as follows from Eq. (29), on the right side of the neighborhood, $D_v \propto t$, and therefore,

$$(\frac{dv}{dx})_R \propto \sqrt{t} \tag{41}$$

6. Methodology for Detection of Intelligent Life

The process of verification of the criticality of the selected point can be started as following:

1. Choose $x = v = 0$ at the selected point.

2. Move to the right of the selected point incrementally and using Eg. (41), find the trajectory in x,t coordinates in the right-hand neighborhood

$$\Delta x_i = \frac{\Delta v_i}{\sqrt{t}}, \quad i = 1, 2..n \tag{42}$$

3. Move to the left of the selected point incrementally and using Eq. (40), find the trajectory in x,t coordinates in the left-hand neighborhood

$$\Delta x_i = \frac{\Delta v_i}{|t|}, \quad i = 1, 2, ..m \tag{43}$$

4. Find the autocorrelation function R between $x_n$and x_m

$$R = \frac{1}{C_0 N}\sum_{t=1}^{n+m}(x_t - \mu)(x_{t+l} - \mu) \tag{44}$$

in which

$$\mu = \frac{1}{n+m}\sum_{t=1}^{n+m} x_t, \quad C_0 = \frac{1}{n+m}\sum_{t=1}^{n+m}(x_t - \mu) \tag{45}$$

5. Check for an exponential decay of the autocorrelation function (44) that is the most reliable criterion of the stationarity of the underlying stochastic process. A quantitative criterion of such decay can be the area enveloped by the correlation function, Park, H., &Zak, M., (see Fig. 10).

6. If the decay is exponential, than the selected point is critical, i.e. its neighborhood has the structure of the family of trajectories (in the original coordinates v, t) that corresponds to a sharp drop of entropy, (see Figs. 8 and 11), and that manifests behavior of an intelligent Living as it was defined above.

7. If the decay is not exponential, and therefore, the area enveloped by the autocorrelation function is larger than some pre-assigned value, than the selected point in not critical.

7. Discussion and Conclusion

The interest in discovery of extraterrestrial life, and in particular, in an intelligent life, started many centuries ago, and it is associated with names of the great ancient philosophers (Aristotle, Ptolemy, etc). Italian physicist Enrico Fermi suggested in the 1950s that if technologically advanced civilizations are common in the universe, then they should be detectable in one way or another. This is known as a Fermi paradox. (According to those who were there, Fermi either asked "Where are they?" or "Where is everybody?" Jones, E.). But is it possible that extraterrestrial intelligence is so different from those we know that it is unrecognizable for our sensors? In order to deal with this argument, we introduced a definition of intelligence that is based strictly upon the laws of physics assuming that these laws are universal. This definition is actually consists of two sequential steps. The first step deals with any type of life that may include not only intelligent, but a primitive one as well. But since we have to prepare ourselves to decoding signals, we concentrate our definition only upon *kinematics* of behavior of Livings disregarding such component of life as metabolism, reproduction, etc. The main challenge at this step is to discriminate between motions of Livings and motions of non-Livings. This problem have been solved by extension of Newton's laws via an appropriate coupling the equations of motion and the corresponding Liouville equation by a feedback representing information forces. Existing of this feedback provides a possibility for isolated systems to decrease their entropy, and that is a mathematical signature of Life. The next step was to isolate an intelligent life from a primitive one. For that purpose, it has been postulated that the global objective of Livings is improvement of their survivability, while the capability to support and improve the survivability has been associated with the decision making process that implements intelligence. The mechanism for decision making process has been represented by a collapse of a random process into a deterministic state such that the entropy suffers a sharp drop. Actually this property has been considered as an invariant of intelligent behavior of Livings regardless of an origin and composition of Life.

Thus,the objective of this work is to extend the physical invariants of biosignature (from disorder to order) to invariants of intelligent behavior: *from disorder to order via phase transition*. The approach is based upon the extension of the physics' First Principles that includes behavior of living systems. The new architecture consists of motor dynamics simulating actual behavior of the object, and mental dynamics representing evolution of the corresponding knowledge-base and incorporating it in the form of information flows into the motor dynamics. Due to feedback from mental dynamics, the motor dynamics attains quantum-like properties: its trajectory splits into a family of different trajectories, and each of those trajectories can be chosen with the probability prescribed by the mental dynamics. Intelligence is considered as a tool to preserve and improve survivability of Livings. From the viewpoint of mathematical formalism, it can be associated with the capability to make decisions that *control* the motor dynamics via a feedback from the *mental* dynamics by providing a quantum-like collapse of a random motion into an appropriate deterministic state. Special attention is focused on data-driven discovery of the underlying physical model displaying an intelligent behavior within the proposed formalism. Possible applications of the results presented in the paper are search for extraterrestrial intelligence as well as detection of physical object manipulated by human.

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The Numbers Universe: An Outline of the Dirac/Eddington Numbers as Scaling Factors for Fractal, Black Hole Universes

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Abstract: The large number coincidences that fascinated theorists such as Eddington and Dirac are shown here to be a specific example of a general set of scaling factors defining universes in which fundamental forces are equated. The numbers have prescriptive power and they are therefore correct and exact *a priori*. The universes thus defined exhibit a fractal structure centred on the Planck/Stoney scale with some formal resemblance to black holes and with properties analogous to Hawking radiation. The problematic case of emerging and evaporating universes is briefly considered in the context of quantum gravity. Historically, the large numbers are associated with the mass of a charged particle and the mass of the universe. This paper demonstrates that the numbers are properly understood in the context of four masses including a non-zero mass derived from Hubble's Constant and the Planck or Stoney mass.

Keywords: Large Numbers; Planck, Stoney; Fractal; Black Hole; Hawking Radiation; Dirac; Eddington; Linde; Nottale; El Naschie
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1. Introduction

According to Dirac, as reported by Gamow [1], *an elegant theory must be correct.* The large numbers noticed by Dirac, Eddington and others [2,3,4,5,6,7,8], suggest the possibility of an elegant theory that derives physical laws from numerical relations. According to Dirac's particular interpretation of the large numbers, for instance, gravity's strength is inversely proportional to the universe's age. Any such large variation in gravity has since been ruled out by a wealth of carefully analyzed geophysical and astrophysical data

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[9,10,11,12,13] yet such is the elegance of his theory that it continues to inspire theorists even today (as for example [14,15,16,17,18]).

It will be shown in this paper that the 'large' numbers are scaling factors that equate an idealized electromagnetic particle with the Planck/Stoney mass, and the Planck/Stoney mass with the total mass of all the electromagnetic particles in the universe. A *Numbers Universe* (the kind of universe defined by these scaling factors) could comprise as many or as few particles as imagination permits and the numbers therefore are not necessarily large. Irrespective of its size, a Numbers Universe is fractal in structure - the universe and every particle in it are rescaled forms of the Planck/Stoney mass. The fractal structure of the universe is a topic that has excited a considerable amount of interest and speculation recently (e.g. [19]), particularly however in the context of unification physics [20, 21,22,23,24] and even with some explicit reference to the Large Numbers Hypothesis [25][26]. However, in this author's opinion, the fractal quality of a Numbers Universe is not fully understood without reference to a fourth mass that emerges from a triad of larger masses comprising the idealized charged particle, the Planck/Stoney mass and the universe itself. The fourth mass and its associated energy, derived from Hubble's Constant, have found a significant role in a variety of theories, including Hawking's theory of radiating black holes, yet few theorists fully understand the intellectual scaffolding that supports it and which may be dubbed the 'Numbers Universe'. This paper will address that shortcoming.

The paper is divided into three main sections. The Introduction includes subsections dealing with definitions of some key terms, including a revised electrical charge (for convenient comparison of electromagnetic and gravitational quantities), the Stoney scale and the Large Numbers themselves. The middle section is a study of the Numbers Universe, particularly the fractal relations of the four masses. It also considers deflationary and inflationary scenarios associated with the sequencing of numbers. The third and final section is a brief discussion of speculative issues.

1.1 Unified Dimensions

The 'large' numbers of Dirac, Eddington *et al.* are ratios of various electromagnetic and gravitational quantities dimensioned in force, mass, length and time (see section 1.3). There is no ratio of charges except perhaps by implication in the ratio of forces. In a Numbers Universe, therefore, electrical charge might best be understood in electrostatic units (esu), as a compound of dimensions associated with force, rather than in SI, where charge is formally a separate and unique dimension measured in Coulombs. However, there are advantages in retaining the SI context, or at least some key elements of it. A compromise between SI and esu is convenient and it is simply achieved by defining charge according to Ampere's Law while setting the permeability of free space equal to a dimensionless unity [27]. This compromise assigns electric charge a compound of dimensions conventionally associated with force, somewhat in the esu manner, without however sacrificing the general SI context. The SI electrostatic force can then be rephrased thus:

$$\begin{aligned} e^2/4\pi\varepsilon_0 &= c^2 e_R^2, \\ e^2 &= 2.567 \times 10^{-38} C^2, \\ e_R^2 &= 2.567 \times 10^{-45} kg.m, \end{aligned} \tag{1}$$

where e is the elementary charge measured in Coulombs, ε_0 is the permittivity of free space, c is the speed of light in a vacuum and e_R is a revised charge. This revision not only allows for an easier comparison of electromagnetic and gravitational phenomena, it also simplifies calculations for some purely electromagnetic identities. For instance, the electromagnetic radius r_E of any mass (conventionally the 'classical radius' in the case of the electron) is calculated in SI and in revised units thus:

$$r_E = e^2/4\pi\varepsilon_0 m_E c^2 = e_R^2/m_E, \tag{2}$$

where m_E is any charged mass. Moreover, the relativistic nature of the magnetic force is made explicit in revised units:

$$e\vec{v} \times \vec{B} = \frac{c^2 e_R^2}{r^2} \times \frac{e_R^2}{m_E r} = \frac{v^2 e_R^2}{r^2}, \tag{3}$$

where $B$is the magnetic field, m_E is any charged mass, v is its speed as determined by the electrostatic force and r is its distance from another charge. The above three equations will help the reader interpret other equations in this paper.

1.2 Stoney Scale

The Stoney scale equates the electrostatic force with the self-gravitation of the Stoney mass, which may be considered a smaller version of the Planck mass:

$$c^2 e_R^2 = G m_S^2 = \alpha G m_{Pl}^2 = \alpha c\hbar, \tag{4}$$

where $G$is the Gravitational Constant, m_S is the Stoney mass, α is the Fine Structure Constant, m_{Pl} is the Planck mass and the crossed h is the reduced Planck's Constant . Some theorists have interpreted Dirac's and Eddington's large numbers in the context of the Planck scale and indeed some findings in this paper have been anticipated in a purely Planck context by other authors [26]. However, the 'large' numbers known to Dirac, Eddington *et al.* are formally Stoney numbers, being factors that equate electromagnetic and gravitational phenomena, whereas the Planck scale is conventionally the scale of unification for all forces. In this paper, the Stoney scale is retained as the scale of the Numbers Universe, partly because of its historical significance, but also because this limited or specialized form of the Planck scale is still relevant for certain theoretical tasks and deserves to be better recognized. The relative strengths of the electrostatic and gravitational forces can then be expressed simply as a mass ratio:

$$\frac{c^2 e_R^2}{G m_E^2} = \frac{m_S^2}{m_E^2} = N = n^2, \tag{5}$$

where N is not necessarily a large number since it depends on the size of the charged mass m_E.The square root form n is often a more useful quantity.

1.3 'Large' Numbers

The Numbers Universe is defined by ratios representing differences in force, length, time and mass. In the context of the 'real' universe, they are conventionally understood as approximations because exact calculations are beyond the practical capabilities of scientific observation:

$$c^2 e_R^2 / G m_E^2 = N \approx 10^{40}, \tag{6}$$

$$R_U / r_E = N \approx 10^{40}, \tag{7}$$

$$\frac{R_U/c}{r_E/c} = N \approx 10^{40}, \tag{8}$$

$$M_U / m_E = N^2 \approx 10^{80}, \tag{9}$$

where m_E is the mass of a typical electromagnetic particle such as the electron or proton, r_E is its electromagnetic radius as defined in (2), and where R_U and M_U are the radius and mass of the universe. Variations in these identities are often seen, such as substituting a particle's Compton wavelength for its electromagnetic radius and equating m_E with the root mean square of two different charged masses. In fact m_E could even be regarded as an ideal particle that emerges from whatever parameters the theorist considers important.

Expressed as a set of 'rubbery' approximations based at least partly on unmeasured and unmeasurable quantities, the numbers are practically useless. Dirac however identified R_U with the radius of an expanding universe and, by equating (6) and (7), arrived at an interesting conclusion:

$$G = \frac{c^2 e_R^2 r_E}{m_E^2 R_U}. \tag{10}$$

Dirac boldly suggested that gravity weakens as the universe expands since G is inversely proportional to R_U. However, as already noted, the hypothesis is not supported by scientific analysis and, moreover, there are other terms in the equation that could be used to offset changes in R_U - in particular, the mass m_E.

2. The Numbers Universe

If we assume that $N$is the exact same number for all four equations (6)-(9) those equations and all their terms can then be deduced from each other. For example:

$$M_U = N^2 m_E = \frac{c^2 e_R^2}{G m_E^2} \times \frac{c^2 e_R^2}{G m_E^2} \times m_E. \tag{11}$$

Rearranging and cancelling some terms:

$$\frac{G M_U}{c^2} = \frac{c^2 e_R^2}{G m_E^2} \times \frac{e_R^2}{m_E} = N r_E = R_U. \tag{12}$$

Thus R_U is half a Schwarzschild radius (or 'gravitational radius') and we need only know the exact value for one of the variable terms N, M_U, R_U or m_E in order to know the exact values for all of them (uncertainties in the value of G are a different issue and may be considered trivial in the circumstances). The Numbers Universe could thus be the exact size we choose and every adjustment in R_U is simply offset by an adjustment in m_E. Interpreting equations (6)-(9) within the context established by (5):

$$M_U = n^3 m_S = n^4 m_E. \tag{13}$$

According to this relation, the Stoney mass is the whole universe when $N = n$=1. In this context, the Numbers Universe seems to be an enlarged form of the Stoney mass and m_E seems to be a reduced form of the Stoney mass - the more the Stoney mass is subdivided, the greater the universe becomes as a whole, somewhat in the fractal manner of an organism growing by the subdivision of its cells.

Physicists have long wondered why gravity is so weak at the electromagnetic scale. According to some physicists (e.g.[28], the significant fact is not the weakness of gravity but rather the tiny mass of charged particles. According to (13), however, the mass of the universe is equally significant in accounting for the relative strength of gravity and it is relevant to ask - why is the universe so massive? The answer to this question is perhaps best found in the 'anthropic argument' [29][20], according to which the large numbers are fairly representative of a universe that is able to support life. Paraphrasing the 'anthropic argument', we might say the Numbers Universe is scaled according to the biological needs of numerate beings - or perhaps according to the intellectual needs of beings clever enough to use very big numbers!

While a conventional system of units such as SI is quite appropriate for our universe it would not be appropriate for all Numbers Universes, some of which might comprise only a handful of large particles while others might comprise an almost infinite number of almost zero mass particles. The only appropriate units of measurement for all of these universes are of course the natural units derived from the Stoney scale. In that case, m_S is an invariant unit of mass and any change in n involves a change in mass for M_U and m_E. Thus the factor n^4 is a product of two factors - the factor n, which is the number of particles needed to offset changes in the mass of m_E relative to the mass of m_S, and the factor n^3, which is the number of Stoney masses in the universe. There is however another fundamental mass in the Numbers Universe, and the triad in (13) is in fact better understood as a tetrad.

2.1 Minimum Energy or Non-zero Mass

The Numbers Universe does not make itself known to theorists by means of numbers alone. Some theorists (e.g.[30]) have been intrigued by this relation:

$$\sqrt[3]{\frac{\hbar^2 H_0}{Gc}} \approx m_E, \tag{14}$$

where H_0 is Hubble's constant ($H_0 = c/R_U$). This particular relation emerges from a Planck-scale Numbers Universe and it implicitly derives the electromagnetic mass m_E

from a cubed mass product featuring the squared Planck mass and a minimum mass, here to be denoted m_ω:

$$\sqrt[3]{\frac{\hbar H_0}{c^2}\frac{c\hbar}{G}} = \sqrt[3]{m_\omega m_{Pl}^2} = m_E. \tag{15}$$

Phrased in Stoney terms and with adjusted values for m_E and m_ω:

$$\sqrt[3]{\frac{\alpha\hbar H_0}{c^2}\frac{c^2 e_R^2}{G}} = \sqrt[3]{m_\omega m_S^2} = m_E, \tag{16}$$

$$M_U = n^6 m_\omega, \tag{17}$$

$$m_\omega = \frac{m_E^3}{m_S^2} = \frac{m_S^2}{M_U}, \tag{18}$$

where m_ω is the smallest mass in a Numbers Universe. The mass m_ω can be derived from the following physical relations, one electromagnetic and the other gravitational:

$$m_\omega = \frac{e_R^2}{R_U}, \tag{19}$$

$$\frac{GM_U m_\omega}{R_U} = \alpha\hbar H_0. \tag{20}$$

The mass m_ω however can also be derived from physical relations that seem baffling and paradoxical :

$$m_\omega c^2 = \frac{c^2 e_R^2}{R_U} = \frac{Gm_E^2}{r_E} = m_E v_\omega^2. \tag{21}$$

Here v_ω is the speed of the charged mass m_E at the edge of the universe in an electromagnetic field originating in the centre of the universe, and it is also the speed of the same mass self-gravitating around its own electromagnetic radius. These relations are mathematical ideals based on the paradoxical assumption that m_E/R_U is not affected by the gravitational mass of the universe and that m_E/r_E is not affected by the electrostatic force, since in both these cases the speed of m_E should in fact be the speed of light.

It is possible that there are some real world phenomena that might resemble the mathematical ideals expressed in (21). Since the Numbers universe is predicated on the realistic assumption that the fundamental forces are in fact different manifestations of the same force ($N = n$=1), there is nothing absurd in the additional assumption that those ideals have some parallel or analogous manifestation in the real universe. We might for instance interpret (21) in the context of quantum entanglement, the kind of 'spooky action at a distance' considered by Einstein, Podolsky and Rosen [31]. In that case, the charged mass m_E could be considered a single particle in two different places, responding only to gravity at the electromagnetic boundary r_E and responding only to the electrostatic force at the gravitational boundary R_U . Some such bizarre particle might be necessary for the unification of the fundamental forces.

Rearranging and cancelling terms in (21) leads to another intriguing relation:

$$\frac{Gm_E^2}{R_U^2} = \frac{v_\omega^2 e_R^2}{R_U^2}. \tag{22}$$

The self-gravitation of the charge particle m_E is here equal to the magnetic force at the boundary of the universe. This idealized relation suggests the possibility that a particle's own self gravitational field and its magnetic field might substitute for each other should either be negated or cancelled out. The nearest real-world analogy to the de facto continuation of an excluded magnetic field is the Ahronov-Bohm effect [32] a gravitational analogue of which has in fact already begun to be developed in the context of large number coincidences [33][34][35].

The minimum energy and its associated non-zero mass m_ω have fascinated theorists for many years, usually however without any reference to a Numbers Universe and always in the Plank context. Walter Nernst, for instance, associated the minimum energy with a mechanism for tired light and constant entropy in a steady state universe [36](see also [37]). The non-zero mass of a photon is a feature common to the Einstein, de Broglie and Vigier theories of light, for which an overview and a quite comprehensive list of references is supplied by Vigier [38]. For other identifications, such as with gravitons and the energy associated with the non-zero conductance of the energy vacuum, see for example Kropotkin [39]. The minimum energy has a logical if not necessarily a physical significance and it often features in scientific theories of an 'alternative' or 'fringe' variety. However, it is also familiar to mainstream science, particularly in the form of Hawking radiation, as discussed in the next section.

2.2 Decreasing Numbers Universes

The dynamics and structures of black holes are a focus of ongoing debate among theoretical physicists (e.g.[40][41]). Whether or not black holes exist in physical fact they are a theoretical 'mineshaft' for speculative workers in unification physics. Thus for example Steven Hawking has combined the quantum theory of particle/anti-particle pairs with the gravitational theory of black holes in order to remove the singularity from space-time through evaporation [42] apparently with a view to its ultimate removal from scientific theory as well [43a]. The singularity is inconsistent with the quantum Heisenberg principle of uncertainty and Hawking has sought to replace it with a Planck scaled region where time becomes a fourth spatial dimension [44a]. Any distinction between 'real' and 'imaginary' time is dismissed by Hawking as irrelevant: *"... a scientific theory is just a mathematical model we make to describe our observations: it exists only in our minds. So it is meaningless to ask: Which is real, 'real' or 'imaginary' time? It is simply a matter of which is the more useful description."* [43b] The Numbers Universe would be unthinkable without that sort of rational expediency.

A quantum of Hawking radiation can be defined thus:

$$k_B T_H = \frac{c^3 \hbar}{8\pi G M}, \tag{23}$$

where M is the mass of the black hole, k_B is Boltzmann's constant, T_H is the Hawking temperature and the $k_B T_H$ product is the energy of a particle radiated by the black hole.

The Hawking formula has already appeared implicitly in (14)-(21) but in a Stoney rather than Planck context. It can for example be recovered thus:

$$\frac{Gm_E^2}{r_E} = \frac{c^2 e_R^2}{R_U} = \frac{c^4 e_R^2}{GM_U} = \frac{\alpha c^3 \hbar}{GM_U} = m_\omega c^2 = m_E v_\omega^2. \tag{24}$$

The particle radiated by the black hole is analogous to the non-zero mass that emerges from the electrostatic force at the gravitational boundary of the Numbers Universe, and which also emerges from the self-gravitation of the m_E particle at the electromagnetic radius r_E. In Hawking's theory, a particle/anti-particle pair originating outside the black hole is torn apart such that one particle falls towards the singularity while the other is either radiated away or orbits at the boundary. In an evaporating Numbers Universe, on the other hand, all particles must surely originate internally and we can only speculate about their final destination. As the Numbers Universe diminishes, the numbers that define it also diminish, a process that leads to fewer but larger charged particles, presumably ending with a single Planck/Stoney mass - otherwise the evaporating Numbers Universe would begin radiating particles more massive than itself (which is perhaps a novel definition of the inflationary universe!)

Hawking's work with evaporating black holes allows the Numbers Universe to develop according to physical principles, whether decreasing like a black hole or even in reverse as a kind of inflationary universe. As a mathematical fiction, the changing Numbers Universe can be defined by any sequence of numbers we choose, in either ascending or descending order or even alternately ascending and descending. If there is to be any resemblance to physical reality, however, the numbers must choose themselves and the sequencing of numbers must proceed at some naturally determined rate. The rate of increase/decrease for a Numbers Universe is conceptually tied to the duration of a Hawking black hole, which can be calculated as follows:

$$t = M^3/k, \tag{25}$$

$$k = \hbar c^4/G^2 \pi 15360,$$

where t is the duration of the black hole and k is a constant. The time t approximates to Planck/Stoney time when the mass M is the Planck/Stoney mass.

2.3 Increasing Numbers Universes

An increasing Numbers Universe does not increase in volume in the way that our universe is thought to expand from an initial Big Bang. It resembles an inflationary universe, increasing in volume while increasing in total mass and in the total number of charged particles. The standard models of cosmology and particle physics have settled on a set of phase transitions that appears to be supported by observational data (e.g.[45,46,47,48]) and these models are not easily or naturally formulated in the context of a time-reversed Hawking black hole. Indeed, according to the duration given in (25), a Numbers Universe as massive as our universe must be considerably older than conventional scientific estimates allow. Moreover, concepts such as a radiation dominated universe are not easily interpreted in the context of a Numbers model. However, the Numbers Universe could be well suited to some alternative cosmological models, such as Linde's fractal model of 'eternal inflation' [20]. Further more, our universe *today* approximates well to a Numbers Universe and it might serve adequately as a model for any contemporary phase transitions that might be thought to be occurring now (e.g. [47][17]).

In an inflationary Numbers Universe, particles decrease in size even as they increase in number. If the particle is elementary, the decrease in its size might be understood merely as a change in scale or in energy level without any change in internal structure, and yet according to the standard model even elementary particles come in discrete generations. In an inflationary Numbers Universe that resembles reality therefore, the process of change seems to require a set of phase transitions even for elementary particles, as for example:

$$\frac{c^2 e_R^2}{G m_{E0}^2 / i^2} = i^2 n^2, \qquad (26)$$
$$m_{E0}/i = m_E,$$

where m_{E0} is the mass of an early generation particle, i is a factor that represents a continuing increase in the 'large' number n, and m_E is the mass of a later generation particle. The phase transition requires the earlier mass m_{E0} to remain unchanged until i reaches some critical value, at which point m_{E0} suddenly becomes the smaller, more numerous mass m_E. Until that critical value is reached, what physical change is signified by increases in i? If (26) is sufficient to tell the story every trivial increase in i must represent a variation in one or more of the fundamental physical 'constants' G, c and/or e_R . This variation in constants however can only be temporary otherwise m_E would never emerge. The variability of fundamental physical constants (such as the speed of light in a vacuum) is one of the most hotly discussed topics in contemporary physics (e.g.[49][50][51]). It is a curious fact that the topic is implicit and even unavoidable in the concept of an increasing Numbers Universe.

It is possible of course that the conditions allowing for a phase transition are expressed by some other mathematical relation. For example, trivial changes in i (trivial from the viewpoint of a transition from m_{E0} to m_E) might never the less represent significant changes in the energy associated with the minimum mass m_ω. Indeed, in a universe like ours, the minimum energy is so small that any set of stepwise changes in its non-zero energy level is hardly different to a smooth continuum. We can then express the result of a phase transition thus:

$$M_U = m_S i^3 n^3 = m_{E0} i^3 n^4 = m_E i^4 n^4 = m_\omega i^6 n^6. \qquad (27)$$

Such an equation assumes that the universe and its components are elaborately synchronized, an impossibility in a universe as large as ours if communication is limited to the speed of light. However, such synchronization might be explained as the quantum entanglement of a fractal organization.

3. Discussion: The Quantum Numbers Universe

If the universe can be considered a single quantum particle, it might also be thought to have an internal clock that keeps the same time everywhere:

$$T_S = \frac{r_S}{c} = \frac{Gm_S}{c^3} = \frac{e_R^2}{cm_S} = \frac{R_U}{n^3 c}, \tag{28}$$

where T_S is the Stoney time and r_S is the Stoney length. Other times are synchronized with the Stoney time by means of the factor n. In fact, the large number N, equating gravity and electromagnetism, can be understood as a ratio of electromagnetic times:

$$N = \frac{m_S^2}{m_E^2} = \frac{m_E c^2}{\hbar} \times \frac{\hbar}{m_\omega c^2} = \nu_E T_\omega. \tag{29}$$

Here ν_E is the electromagnetic frequency of m_E and T_ω is the electromagnetic time of m_ω. Expressed in Stoney units for time, which are proportional to mass:

$$\nu_E = m_E/m_S = 1/n, \tag{30}$$

$$T_\omega = (m_S/m_E)^3 = m_S/m_\omega = n^3.$$

The phase transition described by (27) requires ongoing changes in T_ω while ν_E is retarded as ν_{E0} (the frequency of m_{E0}). In effect, m_{E0} behaves as if it were subject to a time delay until it is suddenly updated and revised to m_E. In that case, the Stoney mass is completely synchronized with M_U and with m_ω but only imperfectly synchronized with particles like m_{E0}. Whether space and time are absolute or relative might therefore depend on the degree of synchronicity.

In the quantum universe defined here, gravity might be understood entirely in electromagnetic terms:

$$G = \frac{c^2 e_R^2}{m_E^2 \nu_E T_\omega}. \tag{31}$$

However, according to (19)–(21) m_ω can be derived from either electromagnetic or gravitational relations and therefore G could be expressed as a ratio of electromagnetic and gravitational times, which is probably more consistent with the Stoney scale's role as the mediator between two forces. The opposite case, that the electromagnetic force emerges from gravitational principles, is almost never heard, though the occasional attempt is made [27].

3.1 The Mathematical Universe

In the opinion of Arthur Eddington, the large number ratios "*... are not arbitrary but will ultimately be found to have a theoretical explanation, though I have also heard the contrary view expressed.*" [8a] In Eddington's day, the Planck scale did not seem so significant as it does today and the possibilities offered by fractal self-organization had not yet been conceived. Never the less, in spite of the new relevance of a Numbers Universe, most theorists today would probably still maintain 'the contrary view'. The contrary view is understandable partly as a reluctance to submit empirical science to the *a priori* dictates of mere numbers, and partly because the standard models of cosmology and particle physics do not seem consistent with the idealized parameters offered by those numbers. Never the less, many eminent theorists (e.g.[52,53,54,55,20,44]) have marvelled at the mathematical intelligence that our universe seems to demonstrate and even today highly respected theorists such as Hawking scaffold their theories around concepts that happen also to be key aspects of the Numbers Universe. It is difficult therefore to deny that the Numbers Universe could be a useful tool, if only as a signpost to analogous phenomena in the real universe as it exists today. In that case, it could actually inspire theorists like Hawking instead of just deriving its relevance from them.

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Quantum Analog of the Black- Scholes Formula(market of financial derivatives as a continuous weak measurement)

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Abstract: We analyze the properties of optimum portfolios, the price of which is considered a new quantum variable and derive a quantum analog of the Black-Scholes formula for the price of financial variables in assumption that the market dynamics can by considered as its continuous weak measurement at no-arbitrage condition.

Keywords: Econophysics; Quantum Finance; Financial markets; Classical and Quantum Black-Scoles Formulas
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1. Introduction

Most of the modern market models are based on classical representation of its state. It is assumed that this state is characterized by a set of parameters (in general case by vector $\vec{u}$), and its dynamics is characterized by the trajectory $\vec{u}(t)$. Each of the state parameters has a well determined value, and inaccuracy of its measurement is determined only by imperfection of means of measurement used. At the same time the influence of measurement procedure on the state of the studied object is neglected.

Similar approximation in classical physics proved to be unacceptable for analysis of experiments with micro objects, resulting in the advent of quantum mechanics 100 years ago. Recently a great number of publications (and even books [1]) dedicated to the development and analysis of quantum economic models has appeared. This tendency

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is justified by the fact that the main property of quantum systems – the unforeseeable influence of measurement on system state, is fully realized here. It is due to the fact that traders are guided more by current dynamics of price formation, rather than by its objective factors in choosing the strategy. On the other hand, it is their strategies, which determine the results of "measurement" of assets price.

Several attempts of generalization of the Black-Scholes-Merton formula have been recently made for calculating the "fair" price of financial derivatives using the quantum-mechanical formalism. Let us discuss some of them. In the paper [2] the differential equation of the classical Black-Scholes-Merton model is represented in the form of analogue of the Schrödinger equation. At the same time, in the obtained equation the imaginary unit is not present, thus it cannot be considered a quantum equation. It is virtually the same classical equation, which can give classical solutions in the simplest cases. The advantages of using quantum-mechanical formalism become apparent in consideration of volatility as a random variable, but this generalization actually describes classical (in physical meaning) market states.

The quantum nature of price dynamics can principally occur, in the same way as in physics, due to simultaneous immeasurability of variables characterizing the classical state of the system. For a strict foundation of the necessity of using the quantum formalism in economics (and in particular in the Black-Scholes-Merton model), one must formally determine the measurement procedure for the share price and the price of its financial derivative, and to demonstrate the non-commutativity of these procedures. We are planning to represent the review of results obtained in this sphere and our own substantiation of the inevitability of quantum-mechanical description of market dynamics [3,4] in our next publication.

For the present, we postulate the simultaneous immeasurability of share price and the velocity of its variation, as it has been made i.e. in [5] or [6].

In paper [5] the authors suggest to add to the Brownian motion B_t of the directly observed share price a random process Y_t, related to the influence of factors, which are not simultaneously observed with the factors included in B_t. However, in this case the mechanism of occurrence of both the first and the second random processes is not discussed. Due to this limitation, the model [5] should be considered phenomenological.

We would like to note that the orthodox quantum theory describes the processes of two principally incompatible types. It is the evolution of a closed quantum system, which is described by the action of a unitary operator on the state vector; and the variation procedure (interaction with a classical detector), which is described by the projection operator. In the framework of this formalism we can describe the market dynamics only as a series of instant destructive measurements (collapses of state). It has been shown in [7], that the attempt to describe the continuous observation of the quantum system in the framework of the orthodox approach results in contradictions (Zeno quantum effect).

In the information model of the financial market's state one can assume that the behavior of each trader is determined by the set of possible alternative scenarios of price dynamics. Then the economic analog of the collapse can be treated as a calculation and declaration of price, as a result of which each of the traders obtains objective information. On the basis of this information, a number of possible alternatives are reduced in his mind. Accordingly, the whole market state changes in general. This variation is described in the framework of the information approach of the quantum Bayes rule [8] and is unambiguously determined by the information obtained as a result of the measurement. It has been shown in [9] that the use of quantum Bayes rule is mathematically equivalent to the projective postulate in the orthodox approach; however it allows generalizations for the used operators. In particular, operators corresponding to obtaining various results in the process of measurement of a certain value are not necessarily orthogonal. This corresponds to the case of quantum measurement and allows avoiding contradictions in description of continuous weak quantum measurements.

There are a number of reasons for using this generalization for modeling the dynamics of the financial market.

1. The interval of price declaration in the financial market δt (up to 1 minute) is negligibly small compared to the characteristic time periods of the external factors' effect.

2. The velocity of price information obtained by the traders is finite in the limit of $\delta t \to 0$

Due to this fact the traders only partially "trust" the price information contained in each separate declaration. At the same time the information, sufficient for a substantial variation of a trader's state, is accumulated during a certain period of time, substantially exceeding δt.

3. Simultaneous calculation and declaration of prices corresponding to non-commuting variables (i.e. prices for a share and its futures) is possible.

In the present paper we are discussing the market dynamics as a process of continuous observation of market state variables. For simplicity we shall assume that the market state corresponds to a pure quantum state. This means that all traders possess the same information both on declared prices and on external factors influencing the prices. In terms of quantum mechanics we can say that all the traders in each moment of time are "identically prepared". A significant feature of the quantum-mechanical model of continuous measurement is the existence of two complementary sources of stochasticity of measurement results. The first of them is related to the external factors and is described in the Hamiltonian as a potential component random in time. The second source is related to the principal quantum unpredictability of the observation results. This means that even in a market fully isolated from the influence of external factors the stochasticity will occur as a consequence of influence of the results of continuous market state observation. It is interesting to note that in a mathematical expression this random factor can be represented as an imaginary component in the Hamiltonian, which corresponds to the conclusion made in [5] on the grounds of other considerations.

2. Modification of Classical Derivation of the Black-Scholes Formula

Let us comment the main assumptions used in classical derivation and their quantum interpretation.

1. Like in a classical model we assume ideality of traders, i.e. we consider that they possess full and identical information, on the basis of which they build their quantum strategies. In the classical model this means that all traders trade identical optimum portfolios at identical price. In the quantum model this assumption allows considering the condition of the whole market $|\psi\rangle$ as a pure and not mixed quantum state. Further it will be appropriate considering a more general model, using the apparatus of density matrix.

2. Continuous time corresponds to performance of continuous weak quantum measurements which are not limited to a series of destructing measurements.

3. $r = const$ - means that the market of bonds is considered as a classical factor without account of the feedback (influence of the market of shares and options on the price of bonds is not taken into account). It is equivalent to the classical description of the controlling external influence on the quantum system. The dependency $r(t)$ can be set randomly. Moreover, we can also generalize the model and take account of the influence of "observation" (declared prices) result on this dependency in the framework of a particular classical model of formation of bank interest rate.

4. A priori set law of share price evolution

$$\frac{dS}{dt} = \mu S + \sigma S \cdot R(t), \tag{1}$$

where $R(t)$ is the uncorrelated Gauss stochastic noise with zero main value transforms into a stochastic differential equation of market state dynamics at continuous weak measurement of "optimum" portfolio price.

$$\frac{d\,|\psi\rangle}{dt} = \left[-\frac{i}{\hbar}\hat{H} - K(\hat{A} - c)^2\right]|\psi\rangle + \sqrt{2K}(\hat{A} - c)\,|\psi\rangle\,\frac{dw}{dt}, \tag{2}$$

where $\hat{A}$ is the operator of the measured variable, $c = \langle\psi|\,\hat{A}\,|\psi\rangle$ is its mathematical expectation, H is the Hamilton operator, the function of which we will discuss later, K is the parameter of fuzziness of measurement. In order to retain vector norm $|\psi\rangle$,$(\langle\psi| + \langle d\psi|)\,(|\psi\rangle + |d\psi\rangle) = \langle\psi \mid \psi\rangle$ it is sufficient to take $dw^2 = dt$ [10]. Let us note that we are considering an idealized model, in which the stochastic nature of price dynamics is caused by its constant weak measurement, unlike the classical model, in which it is set a priori. Classical external influences, also causing price fluctuations, are not considered here, though formally they can be accounted as a random component of the Hamilton operator. The result of price measurement is considered its declaration at the current moment of trades. On the one hand, it contains information about the market status, as it is calculated in accordance with certain rules on the basis of submitted bids.

On the other hand, it influences the market state, because on the basis of obtain information the traders submit their bids. As the speed of obtaining the information is limited, in the model of continuous market at decreasing of the time step of discretization, the degree of uncertainty of the obtained results correspondingly increases. It is the main source of stochasticity in the model of continuous weak measurement.

5. Absence of riskless arbitrating capabilities in the classical model of optimum portfolio price is set as $dV/V = rdt$. In the quantum model the same condition must be satisfied for the mathematical price expectation of measured variable A. Otherwise, it will be possible to obtain a (generally) riskless gain by exchanging the corresponding securities for bonds. If the portfolio price is "measured", the operator $\hat{A} \equiv \hat{V}$ corresponds to this procedure, influencing the traders' state in accordance with their strategies and rules of price determination. Then we obtain the condition

$$\frac{d(\ln \langle V \rangle)}{dt} = \frac{1}{\langle V \rangle} \cdot \frac{d \langle V \rangle}{dt} = r, \quad \text{where} \quad \langle \psi | \hat{V} | \psi \rangle = \langle V \rangle \tag{3}$$

At the same time it is however possible to measure directly the speed of changing of the logarithmic price of the optimum portfolio (in practice in means conclusion of contracts with more complex structure, in which the payment is agreed depending on this parameter). In this case we obtain the ratio

$$\left\langle \frac{d \ln V}{dt} \right\rangle = r, \quad \text{and} \quad \hat{A} \equiv \left(\frac{d \ln \hat{V}}{dt} \right) \tag{4}$$

Let us note that in the classical limit both these variants, correspond to the same formula. However, the influence of measurement on market state in the quantum model makes them different. In the present paper we limit ourselves to the consideration of the first variant.

6. In the classical model the condition of optimality of portfolio means that its structure $V_{opt} = -f + (\partial f / \partial S) S$ ensures riskless condition at any related share and financial derivative price variations. With such a structure the risk turns into 0. However, in the quantum mechanical model the function $f(S, t)$ exists only for mathematical expectations of corresponding prices, which are calculated as average quantum mechanical values of results of weak measurements. In this connection the risk value remains non-zero, and the portfolio structure is to minimize it.

As a measure of risk as one of the possibilities we use the value of dispersion of optimum portfolio price distribution (in the classical model it turns into 0). Then

$$\sigma_V = \left\langle V^2 \right\rangle - \langle V \rangle^2 = \left\langle (-f + kS)^2 \right\rangle - \langle (-f + kS) \rangle^2 \to \min . \tag{5}$$

Assuming that it is the continuous function of parameter k, determining the portfolio structure, we can write down the optimality condition in the following form $\partial \sigma_V / \partial k = 0$. From it we can obtain:

$$k_{opt} = \frac{\langle f \cdot S \rangle - \langle f \rangle \cdot \langle S \rangle}{\sigma_S}. \tag{6}$$

The same as in the classical model, the parameter k can depend on time. In the quantum mechanical model the ratio, determining the optimum portfolio structure connects corresponding operators, rather than the measured values of the variables. Due to incommutability of operators $\hat{S}$ and $\hat{f}$:

$$\langle f \cdot S \rangle = \left[\langle \psi | \, \hat{f}\hat{S} \, |\psi\rangle + \langle \psi | \, \hat{S}\hat{f} \, |\psi\rangle \right] /2 \tag{7}$$

3. The Black-Scholes Quantum Formula for a Specific Financial Derivative

The first step in the simplest variant of classical derivation using Ito formula is obtaining of financial derivative price dynamics equation. This allows further excluding the random variable R(t) from classical equations for dynamics S and f.

For dynamics of average values of quantum variables with account of (2) we obtain:

$$\frac{d\langle S \rangle}{dt} = \langle \psi | \, \hat{S}\hat{B} + \hat{B}\hat{S} \, |\psi\rangle + \langle \psi | \, \hat{S}\hat{C} + \hat{C}\hat{S} \, |\psi\rangle \, R(t) \tag{8}$$

and

$$\frac{d\langle f \rangle}{dt} = \langle \psi | \, \hat{f}\hat{B} + \hat{B}\hat{f} \, |\psi\rangle + \langle \psi | \, \hat{f}\hat{C} + \hat{C}\hat{f} \, |\psi\rangle \, R(t) + \left\langle \frac{\partial f}{\partial t} \right\rangle, \tag{9}$$

where

$$\hat{B} = \left[-\frac{i}{\hbar}\hat{H} - k(\hat{V} - c)^2 \right]; \tag{10}$$

$$\hat{C} = \sqrt{2k}(\hat{V} - c) \tag{11}$$

Excluding from them the random factor R(t), we obtain a quantum analog of the classical formula:

$$\frac{d\langle f \rangle}{dt} - \left\langle \frac{\partial f}{\partial t} \right\rangle - \left\langle \hat{f}\hat{B} + \hat{B}\hat{f} \right\rangle = \frac{\left\langle \hat{f}\hat{C} + \hat{C}\hat{f} \right\rangle}{\left\langle \hat{S}\hat{C} + \hat{C}\hat{S} \right\rangle} \cdot \left[\frac{d\langle S \rangle}{dt} - \left\langle \hat{S}\hat{B} + \hat{B}\hat{S} \right\rangle \right] \tag{12}$$

For shortening the notation we shall further in this paper represent the symmetrized product of operators of type $\left(\hat{f}\hat{C} + \hat{C}\hat{f}\right)/2$ as an ordinary product.

For further use it necessary to substitute the expressions for operators $\hat{B}$ and $\hat{C}$ in the decisive form. In this case a significant difference from the classical derivation method occurs. The point is that the share price dynamics in the classical model is considered set a priori, while in the quantum model it depends on the "measured" value. In our case this is price of optimum portfolio. Therefore at substituting of operators we should take account of its structure, set by the quantum expression for k_{opt}. Besides, in cases when a different quantum variable, other than the optimum portfolio price, is being "measured", we should use operator $\hat{A}$ instead of $\hat{V}$.

At the next stage of the classical derivation the condition of no-arbitrage is used, from which an additional connection of variables S and f is obtained. In the quantum case from (3) we obtain

$$\frac{d\langle f\rangle}{dt} - r\langle f\rangle = -k_{opt}\left(\frac{d\langle S\rangle}{dt} - r\langle S\rangle\right) \tag{13}$$

Despite the "similarity" of both formulas (12,13), unlike the classical analog they include different proportionality coefficients between $\frac{d\langle f\rangle}{dt}$ and $\frac{d\langle S\rangle}{dt}$. They become identical and equal $\frac{\partial f}{\partial S}$ either in the classical limit, or at a certain value$c = \langle\psi|\,\hat{V}\,|\psi\rangle$. In this case, the same as in the classical case, we can exclude the total time derivatives and obtain the following system:

$$\left[\begin{array}{l} \frac{\langle \hat{f}\hat{C}\rangle}{\langle \hat{S}\hat{C}\rangle} = k_{opt} \\ \frac{r\langle \hat{f}\rangle - \langle \frac{\partial \hat{f}}{\partial t}\rangle - \langle \hat{f}\hat{B}\rangle}{\langle \hat{f}\hat{C}\rangle} = \frac{r\langle \hat{S}\rangle - \langle \hat{S}\hat{B}\rangle}{\langle \hat{S}\hat{C}\rangle} \end{array}\right. \models \left[\begin{array}{l} \frac{\langle \hat{f}\hat{C}\rangle}{\langle \hat{f}\hat{S}\rangle - \langle \hat{f}\rangle\langle \hat{S}\rangle} = \frac{\langle \hat{S}\hat{C}\rangle}{\langle \hat{S}^2\rangle - \langle \hat{S}\rangle^2} \\ \left\langle \frac{\partial \hat{f}}{\partial t}\right\rangle + r\left\langle \hat{S}\right\rangle \frac{\langle \hat{f}\hat{C}\rangle}{\langle \hat{S}\hat{C}\rangle} + \frac{\langle \hat{f}\hat{B}\rangle\langle \hat{S}\hat{C}\rangle - \langle \hat{S}\hat{B}\rangle\langle \hat{f}\hat{C}\rangle}{\langle \hat{S}\hat{C}\rangle} = r\left\langle \hat{f}\right\rangle \end{array}\right. \tag{14}$$

This system is in fact a quantum analog of the classical formula

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} = rf \tag{15}$$

The additional equation occurs due to the fact that in our case the dynamics of price average value is not set a priori, it is the result of procedure of its continuous measurement and influence of Hamiltonian. However, if we assume that not the price of optimum portfolio in one of the variants (3, 4), but the price of share is measured, then, by substituting S instead of $\hat{V}$ in (10) and (11), we obtain an identity from the first condition, and from the second condition we obtain

$$\left\langle \frac{\partial \hat{f}}{\partial t}\right\rangle + r\left\langle \hat{S}\right\rangle \frac{\left\langle \hat{f}\hat{S}\right\rangle - \left\langle \hat{f}\right\rangle\left\langle \hat{S}\right\rangle}{\left\langle \hat{S}^2\right\rangle - \left\langle \hat{S}\right\rangle^2} + \frac{\left\langle \hat{f}\hat{B}\right\rangle\left\langle \hat{S}\hat{C}\right\rangle - \left\langle \hat{S}\hat{B}\right\rangle\left\langle \hat{f}\hat{C}\right\rangle}{\left\langle \hat{S}\hat{C}\right\rangle} = r\left\langle \hat{f}\right\rangle \tag{16}$$

The result of solution of the obtained system with account of boundary conditions connecting the share prices and the financial derivative in the moment of closing the contract will be a functional dependency of operator$\hat{f}$ on the operator $\hat{S}$ and time t. However, for this purpose it is necessary to explicitly draw out the expression for the Hamiltonian. In a number of works dedicated to the analysis of quantum economic phenomena the Hamiltonian is used without sufficient grounds for the particle in the potential field. In physics, the form of Hamiltonian is determined by the general requirements connected with the homogeneity and isotropy of space and the principle of relativity [11]. We assume that in the same manner in economic models the form of Hamiltonian should also be determined by the type of symmetries set by the formal rules of trading. Our further research will be dedicated to the analysis of these properties and derivation of formulas for the Hamiltonian in various economic systems. Therefore, in the present paper we limited ourselves to the derivation of Black- Scholes quantum formula in the general form.

Let us note that the obtained system of operator equations does not include the market states, described by the wave function, it only includes the individual parameter of measurement weakness k. It follows from the general considerations of invariance of description with respect to the selection of the discretization step δt that $k \approx (\delta t)^{-1/2}$. Besides, the functional dependence $\hat{f}(\hat{S}, t)$ for the operator of the financial derivative price measurement is determined only by the terms and conditions of the contract and by the form of the Hamiltonian.

As a conclusion let us note the specific features of the financial market's dynamics in the framework of the suggested approach. In the discussed examples we assumed the price of optimum portfolio V to be continuously measured.

As this value represents a linear combination of the share price and the price of futures, in the orthodox approach these prices, being non-commuting variables, do not have a defined value. At the same time, in the real market both these prices (and the portfolio price as their linear combination) can be declared simultaneously. In the framework of the theory of continuous weak quantum measurement this contradiction is resolved due to the fact that the operators of weak measurement are not orthogonal.

Probabilistic distribution of the price of optimum portfolio V is unambiguously determined by the value of the wave function at the current moment of time. On receiving a next random value of V_j the market state described by the wave function also changes in a random manner. At the next step we have a new distribution of probabilities for V_j. As a result a random process of portfolio price variation is induced by the continuous weak measurement even without external random factors (their presence can be taken into account by a formal addition of a random additive component to the Hamiltonian). At the same time the new value of k_{opt} must be recalculated at each step. Thus, we obtain a realization of a random process, in which the probabilistic distributions of the share price and the financial derivative at each moment of time eliminate the possibility of arbitraging.

Let us note, that in the process of deriving the quantum Black-Scholes-Merton formula we have excluded the random factor, related to the observation of the market portfolio price, only for the dynamics of the mean values $\langle S \rangle$ and $\langle f \rangle$. In this case the absence of risks for the optimum portfolio means that the variation of the mean quantum-mechanical (expected) price of the portfolio does not depend on the results of random measurement. However, they affect its structure, which must be recalculated at each step as in the classical case. As the measured portfolio price at each step is different from the average, risk cannot be avoided for a specific realization. It can only be minimized for the optimum strategy.

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Faster than Light Quantum Communication I

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Abstract: Faster than light communication might be possible using the collapse of the quantum wave-function without any accompanying paradoxes.

Keywords: Quantum Communication; Faster-than-light Signalling
PACS (2006): 03.67.Hk; 03.65.w

1. Introduction

It has long been wondered if faster than light communication might be possible [1] and the collapse of the quantum wave-function, upon measurement, might be utilized to achieve this, with due concern for any paradoxes that might result.

Firstly, there is no direct violation of special relativity since it is the quantum wave-function that collapses and no energy or matter travels at faster than light speed.

2. Unitary Communicator

The conservation of a particle in quantum theory (unitarity) might suggest a possible mechanism, since the destructive interference in one part of the system will imply a greater probability of locating the particle in another part, no matter how dispersed the system has become.[1]

To try and implement this, imagine a beam splitting mechanism that breaks the beam into two arms that can be widely separated, and then again splits and recombines one of the two resulting arms.

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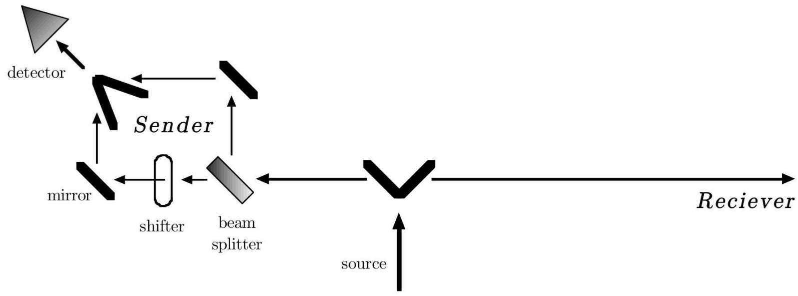
[1] A similar mechanism has been proposed to augment the ability of a quantum computer [2], [3].



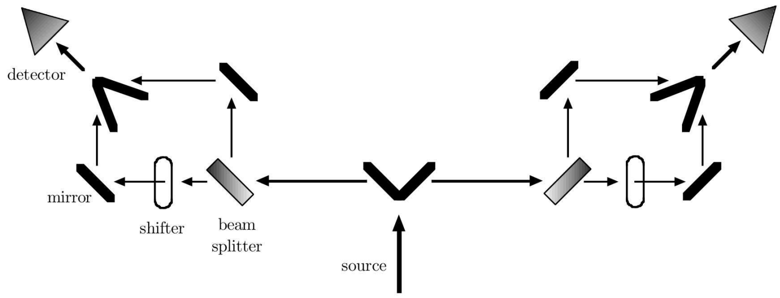
Quantum Transmitter

The recombination can be arranged to constructively, or destructively interfere, depending on a phase shifter in one of the two paths.

If the sender arranges for constructive interference then some of the particles will be 'taken up' by the sender, but none if destructive interference is arranged; in this way the intensity of the receivers beam might be controlled. So a faster than light transmitter of information (but not energy or matter) might be possible.

3. Two-way Communication

The above proposal, for simplicity, was a one way transmission device, but this can be easily duplicated for a full-duplex device or simply extended for half-duplex.



Half-Duplex Quantum Communicator

4. Absence of Paradox

The above proposal would seem more plausible if it can be demonstrated that no paradox arises from its supposed ability to communicate 'instantly' over indefinite distances; namely, that no use can be made of a communication to alter events in the past.

The collapse of the wave-function upon the act of measurement has long been a dilemma [4, 5], and one seeks to explain when and how the reduction occurs, if at all.

A possible clarification to the usual quantum measurement axiom might be that the probabilistic collapse happens when distinguishability occurs, and that 'instantaneous' might make more sense if relative to a preferred frame (a quantum-ether).

4.1 When does the Collapse Occur

Taking as the model of a 'macroscopic' system, the interference of large molecules, which has been performed [6]; if the energy of an outer electron is modified on one path alone (which is not enough to wash out the interference pattern), interference is still lost. This is due to distinguishability, as one would then know which path the object took from the observed state of the outer electron.

If one now goes a step further and argues that distinguishability not only stops interference, but actually triggers the collapse of the wave function (is itself the act of measurement), one may have another view on the question of where the boundary between the quantum and classical worlds occurs. This should not lead to any new prediction, since the quantum effect (interference) is no longer present anyhow.

Further, since the act of distinguishability inevitably involves the interaction with another system, there is no dilemma with momentum/energy non-conservation, as would be the case in proposals invoking spontaneous reduction, such as the GRW model.

4.2 How fast does the Collapse Happen

It is said that the collapse of the wave-function happens 'instantly', but as is well known, relativity does not respect this concept; what is instant in one frame is not in another. It also does not seem reasonable that a moving measuring device would instigate a different collapse from a stationary one, and one way around this dilemma is that there is a preferred frame in which the collapse occurs.

Up until now this was not a pressing issue for, although it would alter the cause and effect ordering for the measuring of an EPR pair, the end result was not influenced by which end made the measurement first. This uneasy state of affairs is brought to a head here, but fortunately the above suggestion that the collapse occurs in some preferred frame also severs to prevent the faster than light proposal from being able to communicate into the past.

Other backward time travel proposals, such as worm holes, have been refuted [7].

Conclusion

Faster than light communication may be possible using the collapse of the wave-function, and without any paradoxical powers accompanying the device.

These proposals for when and how the measurement occurs might clarify, in a natural way, why simple systems such as elementary particles express their quantum nature so easily, and why more structured systems do not. It is not necessary to explain why the hypothesized mechanisms operate, anymore than Newtonian gravity explains why a mass exerts a force or Einstein gravity sees it as warping space-time.

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Reply to 'On a Recent Proposal of Faster than Light Quantum Communication' II

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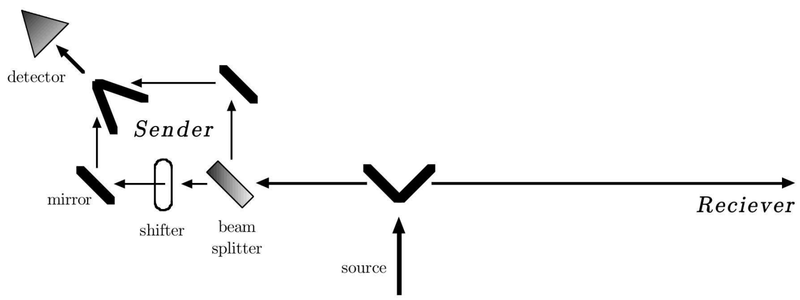
Abstract: In a recent paper [1] the author proposed the possibility of an experiment to perform faster-than-light communication via the collapse of the quantum wave-function. This was analyzed by Bassi and Ghirardi [2], and it is believed that this analysis itself merits a detailed examination.

Keywords: Quantum Communication; Faster-than-light Signalling
PACS (2006): 03.67.Hk; 03.65.w

1. The Original Proposal

The proposed device is based upon the conservation of a particle in quantum theory (unitarity), and is founded upon the following apparatus [1]



Quantum Transmitter

where a beam splitting mechanism breaks a single particle wave-function into two arms that can be widely separated, and then again splits and recombines one of the two resulting arms.

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The recombination can be arranged to constructively, or destructively interfere, depending on a phase shifter in one of the two sub-paths.

If the sender chooses to arrange for constructive interference then some of the particles will be 'taken up' by the sender, but less if destructive interference is arranged, and in this way the intensity of the receiver's beam might be controlled. So a faster-than-light transmitter of information (but not energy or matter) might be possible.

2. The Counter Analysis

The proposal outlined above was recently analyzed by Bassi and Ghirardi [2], and while they seem to agree that the arranged for destructive interference will induce a re-unitarization, they seem convinced that this will only happen for the one arm where interference is being arranged, while the original proposal re-normalized the entire state.

However, a 'localized' re-normalization, rather than avoiding a faster-than-light communicator can itself not only result in one, but has some other possibly unexpected consequences, as detailed below.

2.1 A faster-than-light communicator, when there should be none

All authors agree that the wave function after the second splitting, but before recombination, is described by

$$\frac{|\phi_{s_1}\rangle}{\sqrt{2}} + \frac{|\phi_{s_2}\rangle}{\sqrt{2}} + |\phi_r\rangle \tag{1}$$

where s_1 and s_2 are the two sender's arms and r is the receiver; overall re-normalization of each state is understood, but left out for clarity.

Now suppose that a measurement is made first in one of the sender's sub-arms, and then the other. In the case where the particle was *not* found in this first measurement one would get, for the Bassi Ghirardi approach of re-normalizing the sender's arm alone

$$|\phi_{s_2}\rangle + |\phi_r\rangle \tag{2}$$

as opposed to the original re-normalizing of the entire state, where one would get

$$\frac{|\phi_{s_2}\rangle}{\sqrt{2}} + |\phi_r\rangle \tag{3}$$

These two approaches naturally lead to differing predictions for the probability of finding the particle in the sender's arm after both measurements are made. For the first (component re-normalization) one would get a predicted probability of 1/4 of finding the particle in the first sub-arm and then 1/2 of finding it in the second sub-arm (if it was not found in the first), yielding a total probability of 1/4 + 1/2 (1 - 1/4), an unexpected 62.5% result. On the other hand, for the overall re-normalization, the same calculation 1/4 + 1/3 (1 - 1/4) yields the expected result of 50%.

So the first approach (however justified) would immediately give rise to a faster-than-light communicator, as the sender could then either opt to measure the main arm (with a resulting 50% chance of seeing the particle), or instead perform measurements of the two sub-arms for a predicted 62.5% chance of seeing the particle, so reducing the receiver's intensity if this is done for a beam of particles.

While, for the re-normalization of the entire state, as done for the original proposal, there is no possibility of a faster-than-light communicator in this case, which seems more reasonable.

2.2 Component Re-normalization

Further problems result from this component re-normalization approach, in that if one uses it to analyze the case of interference (in the previous analysis we elected to measure before interference took place), then according to Bassi and Ghirardi's section III an area of destructive interference where the particle is less likely to be found does indeed occur, but is accompanied by a boosting of the wave-function just outside of this area to maintain unitarity (FIG 2b). This in itself would constitute a faster-than-light communicator, albeit over a rather limited range. In the original proposal it is the entire state, not just part, that is boosted by the conservation of unitarity (which is not violated in either work).

On a side note, perfect cancellation is certainly not possible in practice, and only needs to be partial in this application.

2.3 General Proof

While there is reference to proofs of 'full generality' against faster-than-light communication, it is at the same time conceded that such proofs are for two particle entangled states and that the present case does not fall under such 'general' proofs.

2.4 Super-luminal Communication

Section VI (the Conclusions) claims that such a faster-than-light communication device would allow for the synchronizing of clocks and so imply the existence of absolute time. However, in general, since the sender and receiver would have a relative velocity, their respective clocks would be running at differing rates, and so could not run in synchrony.

Conclusion

While the possibility of constructing a faster-than-light communicator is highly unlikely, it is not believed that the work of Bassi and Ghirardi has yet located the flaw in the present proposal. As demonstrated here, their approach of re-normalizing part (and not all) of the state itself gives rise to a faster-than-light communicator, and so cannot constitute a proof against such a proposal.

The mechanism of 'magnification' through re-unitarization (seen in both works) can also be used to pick out desirable components in the case of quantum computation [3, 4], although in that instance the cancellation needs to be near perfect, and achievable in polynomial time (as can be easily demonstrated).

3. Acknowledgements

I would like to thank Professor Giancarlo Ghirardi for extensive discussions on the above issues.

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