

# Hamilton-Jacobi Formulation of A Non-Abelian Yang-Mills Theories

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**Abstract:** A non-Abelian theory of fermions interacting with gauge bosons is treated as a constrained system using the Hamilton-Jacobi approach. The equations of motion are obtained as total differential equations in many variables. The integability conditions are satisfied, and the set of equations of motion is integrable. A comparison with Dirac's method is done.

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## 1. Introduction

The most common method for investigating the Hamiltonian treatment of constrained systems was initiated by Dirac[1]. The main feature of his method is to consider primary constraints first. All constraints are obtained using consistency conditions. Besides, he showed that the number of degrees of freedom of the dynamical system can be reduced. Hence, the equations of motion of constrained system are obtained in terms of arbitrary parameters.

The canonical method (or Güler's method) developed Hamilton-Jacobi formulation to investigate constrained systems [2-3]. The Hamilton-Jacobi treatment of constrained systems leads us to obtain the equations of motion as total differential equations in many variables. These equations are integrable if the corresponding system of partial differential equations is a Jacobi system [3,4,5]. Since there are few physical examples were discussed by using Hamilton-Jacobi approach [6-9], it is still necessary to study more

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of them and compare the results that can be obtained by Dirac's method. In this paper, a non-abelian theory of fermions interacting with gauge bosons will be studied by using both Hamilton-Jacobi formulation and Dirac's method.

A review of the Hamilton-Jacobi approach can be introduced as follows:

If the rank of the Hess matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, 2, \dots, n, \quad (1)$$

is  $(n - r), r < n$ , then the standard definition of a linear momenta

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \quad a = 1, 2, \dots, n - r, \quad (2)$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \quad \mu = n - r + 1, \dots, n, \quad (3)$$

enables us to solve eq.(2) for  $\dot{q}_a$  as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_b) \equiv \omega_a. \quad (4)$$

Substituting eq.(4), into eq.(3), we obtain the constraints as

$$H'_\mu \equiv p_\mu + H_\mu(\tau, q_i, p_a) = 0, \quad (5)$$

where

$$H_\mu = - \left. \frac{\partial L}{\partial \dot{q}_\mu} \right|_{\dot{q}_a \equiv \omega_a}. \quad (6)$$

The usual Hamiltonian  $H_0$  is defined as

$$H_0 = -L + p_a \dot{q}_a - \dot{q}_\mu H_\mu. \quad (7)$$

Like functions  $H_\mu$ , the function  $H_0$  is not an explicit function of the velocities  $\dot{q}_\nu$ . Therefore, the Hamilton-Jacobi function  $S(\tau, q_i)$  should satisfy the following set of Hamilton-Jacobi partial differential equations (HJPDE) simultaneously for an extremum of the function:

$$H'_\alpha \left( t_\beta, q_\alpha, P_i = \frac{\partial S}{\partial q_i}, P_0 = \frac{\partial S}{\partial t_0} \right) = 0, \quad (8)$$

where

$$\alpha, \beta = 0, n - r + 1, \dots, n; \quad a = 1, 2, \dots, n - r, \text{ and}$$

$$H'_\alpha = p_\alpha + H_\alpha. \quad (9)$$

The canonical equations of motion are given as total differential equations in variables  $t_\beta$ ,

$$dq_p = \frac{\partial H'_\alpha}{\partial p_p} dt_\alpha, \quad p = 0, 1, \dots, n; \quad \alpha = 0, n - r + 1, \dots, n, \quad (10)$$

$$dp_a = - \frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad a = 1, \dots, n - r, \quad (11)$$

$$dp_\mu = - \frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \quad \alpha = 0, n - r + 1, \dots, n, \quad (12)$$

$$dZ = \left( -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha \right), \quad (13)$$

where

$$Z \equiv S(t_\alpha, q_a), \quad (14)$$

being the action. Thus, the analysis of a constrained system is reduced to solve equations (10-12) with constraints

$$H'_\alpha(t_\beta, q_a, P_i) = 0, \quad \alpha, \beta = 0, n - r + 1, \dots, n. \quad (15)$$

Since the equations above are total differential equations, integrability conditions should be checked. These equations of motion are integrable [3,4,5] if and only if the variations of  $H'_\alpha$  vanish identically, that is

$$dH'_\alpha = 0. \quad (16)$$

If they do not vanish identically, then we consider them as new constraints. This procedure is repeated until a complete system is obtained.

This paper is arranged as follows: Dirac's method is used in sect.2 and Güler's method in sect.3. The paper closes with a conclusion in sect.4.

## 1.1 Dirac's method

Consider the Lagrangian density for a non-Abelian theory of fermions interacting with gauge bosons as

$$L = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2, \quad (17)$$

where  $\xi$  can be any finite constant.

In Eq.(17)  $F_{\mu\nu}^a$  is given by the formula

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \quad (18)$$

where  $f^{abc}$  are the structure constants of the Lie algebra and  $g$  represents the coupling constant.

The generalized momenta (2) and (3) read as

$$\pi_a^i = \frac{\partial L}{\partial \dot{A}_i^a} = -F_a^{0i}, \quad (19)$$

$$\pi_a^0 = \frac{\partial L}{\partial \dot{A}_0^a} = \frac{1}{\xi} \partial^\mu A_\mu^a, \quad (20)$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = -H_\psi, \quad (21)$$

$$p_{\bar{\psi}} = \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \quad (22)$$

$$p_\mu = \frac{\partial L}{\partial \dot{A}_\mu} = 0 = -H_\mu, \quad (23)$$

where we must call attention to necessity of being careful with the spinor indexes. Considering, as usual  $\psi$  as a column vector and  $\bar{\psi}$  as a row vector implies that  $p_\psi$  will be a row vector while  $p_{\bar{\psi}}$  will be a column vector.

Equations (19) and (20), respectively leads us to express the velocities  $\dot{A}_a^i$  and  $\dot{A}_a^0$  as

$$\dot{A}_a^i = \pi_i^a - \partial_i A_a^0 + g f_{abc} A_b^0 A_c^i, \quad (24)$$

$$\dot{A}_a^0 = \xi \pi_a^0 - \partial_i A_i^a. \quad (25)$$

The Hamiltonian density is given by

$$H_0 = \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a - g f^{abc} \pi_a^i A_0^b A_i^c + \frac{1}{2} \xi \pi_0^a \pi_a^0 - \pi_0^a \partial_i A_i^a + \frac{1}{4} F_a^{ij} F_{ij}^a - \bar{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi. \quad (26)$$

The total Hamiltonian density is constructed as

$$H_T = \frac{1}{2} \pi_i^a \pi_i^a - \pi_i^a \partial_i A_0^a - g f^{abc} \pi_a^i A_0^b A_i^c + \frac{1}{2} \xi \pi_0^a \pi_a^0 - \pi_0^a \partial_i A_i^a + \frac{1}{4} F_a^{ij} F_{ij}^a - \bar{\psi} (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + \lambda_\psi (p_\psi - i \gamma^0 \bar{\psi}) + \lambda_{\bar{\psi}} p_{\bar{\psi}} + \lambda_\mu p_\mu, \quad (27)$$

where  $\lambda_\psi$ ,  $\lambda_{\bar{\psi}}$  and  $\lambda_\mu$  are Lagrange multipliers to be determined. From the consistency conditions, the time derivative of the primary constraints should be zero, that is

$$\dot{H}'_\psi = \{H'_\psi, H_T\} = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m) - i \lambda_{\bar{\psi}} \gamma^0 \approx 0, \quad (28)$$

$$\dot{H}'_{\bar{\psi}} = \{H'_{\bar{\psi}}, H_T\} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi \approx 0, \quad (29)$$

$$\dot{H}'_\mu = \{H'_\mu, H_T\} = \bar{\psi} e \gamma^\mu \psi \approx 0. \quad (30)$$

Relations (28) and (29) fix the multipliers  $\lambda_{\bar{\psi}}$  and  $\lambda_\psi$  respectively as

$$\lambda_{\bar{\psi}} = i \bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m) \gamma^0, \quad (31)$$

$$\lambda_\psi = i \gamma^0 (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi. \quad (32)$$

Eq.(27) lead to the secondary constraints

$$H''_\mu = \bar{\psi} e \gamma^\mu \psi \approx 0. \quad (33)$$

There are no tertiary constraints, since

$$\dot{H}''_\mu = \{H''_\mu, H_T\} = 0. \quad (34)$$

By taking suitable linear combinations of constraints, one has to find the first-class, that is

$$\Phi_1 = H'_\mu = p_\mu, \quad (35)$$

whereas the constraints

$$\Phi_2 = H'_\psi = p_\psi - i \gamma^0 \bar{\psi}, \quad (36)$$

$$\Phi_3 = H'_{\bar{\psi}} = p_{\bar{\psi}}, \quad (37)$$

$$\Phi_4 = H''_\mu = \bar{\psi} e \gamma^\mu \psi = 0, \quad (38)$$

are second-class.

The equations of motion read as

$$\dot{A}_0^a = \{A_0^a, H_T\} = \xi \pi_0^a - \partial^i A_i^a, \quad (39)$$

$$\dot{A}_i^a = \{A_i^a, H_T\} = \pi_i^a - \partial_i A_0^a + g f^{abc} A_0^b A_i^c, \quad (40)$$

$$\dot{\psi} = \{\psi, H_T\} = \lambda_\psi, \quad (41)$$

$$\dot{\bar{\psi}} = \{\bar{\psi}, H_T\} = \lambda_{\bar{\psi}}, \quad (42)$$

$$\dot{A}_\mu = \{A_\mu, H_T\} = \lambda_\mu, \quad (43)$$

$$\dot{\pi}_a^0 = \{\pi_a^0, H_T\} = \partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c, \quad (44)$$

$$\dot{\pi}_a^i = \{\pi_a^i, H_T\} = g f^{abc} \pi_c^i A_0^b - \partial_l (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b, \quad (45)$$

$$\dot{p}_\psi = \{p_\psi, H_T\} = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m), \quad (46)$$

$$\dot{p}_{\bar{\psi}} = \{p_{\bar{\psi}}, H_T\} = (i \gamma^i \partial_i + e \gamma^\mu A_\mu - m) \psi + i \gamma^0 \lambda_\psi, \quad (47)$$

$$\dot{p}_\mu = \{p_\mu, H_T\} = \bar{\psi} e \gamma^\mu \psi. \quad (48)$$

Substituting from Eq. (32) into Eqs. (41) and (47), we get

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0, \quad (49)$$

$$\dot{p}_{\bar{\psi}} = 0, \quad (50)$$

and from Eq.(31) into (42), we have

$$\bar{\psi} (i \overleftarrow{\partial}_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0. \quad (51)$$

We will contact ourselves with a partial gauge fixing by introducing gauge constraints for the first-class primary constraints only, just to fix the multiplier  $\lambda_\mu$  in Eq.(27). Since  $p_\mu$  is vanishing weakly, a gauge choice near at hand would be

$$\phi'_1 = A_\mu = 0. \quad (52)$$

But for this forbids dynamics at all, since the requirement  $\dot{A}_\mu = 0$  implies  $\lambda_\mu = 0$ . In the following section the same system will be discussed using Hamilton-Jacobi approach.

## 1.2 Hamilton-Jacobi method

The set of Hamilton-Jacobi Partial Differential Equations (HJPDE) (8) read as

$$H'_0 = \pi_4^a + H_0 = 0, \quad (53)$$

$$H'_\psi = p_\psi + H_\psi = p_\psi - i\bar{\psi}\gamma^0 = 0, \quad (54)$$

$$H'_{\bar{\psi}} = p_{\bar{\psi}} + H_{\bar{\psi}} = p_{\bar{\psi}} = 0, \quad (55)$$

$$H'_\mu = p_\mu + H_\mu = p_\mu = 0. \quad (56)$$

The equations of motion are obtained as total differential equations as follows:

$$\begin{aligned} dA_a^i &= \frac{\partial H'_0}{\partial \pi_i^a} dt + \frac{\partial H'_\psi}{\partial \pi_i^a} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_i^a} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi_i^a} dA_\mu, \\ &= [\pi_i^a - \partial_i A_0^a + g f^{abc} A_0^b A_i^c] dt, \end{aligned} \quad (57)$$

$$\begin{aligned} dA_a^0 &= \frac{\partial H'_0}{\partial \pi_0^a} dt + \frac{\partial H'_\psi}{\partial \pi_0^a} d\psi + \frac{\partial H'_{\bar{\psi}}}{\partial \pi_0^a} d\bar{\psi} + \frac{\partial H'_\mu}{\partial \pi_0^a} dA_\mu, \\ &= [\xi \pi_0^a - \partial^i A_i^a] dt, \end{aligned} \quad (58)$$

$$\begin{aligned} d\pi_a^i &= -\frac{\partial H'_0}{\partial A_i^a} dt - \frac{\partial H'_\psi}{\partial A_i^a} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_i^a} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_i^a} dA_\mu, \\ &= [g f^{abc} \pi_c^i A_0^b - \partial_l (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b] dt, \end{aligned} \quad (59)$$

$$\begin{aligned} d\pi_a^0 &= -\frac{\partial H'_0}{\partial A_0^a} dt - \frac{\partial H'_\psi}{\partial A_0^a} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_0^a} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_0^a} dA_\mu, \\ &= [\partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c] dt, \end{aligned} \quad (60)$$

$$\begin{aligned} dp_\psi &= -\frac{\partial H'_0}{\partial \psi} dt - \frac{\partial H'_\psi}{\partial \psi} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \psi} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \psi} dA_\mu, \\ &= [-\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e\gamma^\mu A_\mu + m)] dt, \end{aligned} \quad (61)$$

$$\begin{aligned} dp_{\bar{\psi}} &= -\frac{\partial H'_0}{\partial \bar{\psi}} dt - \frac{\partial H'_\psi}{\partial \bar{\psi}} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial \bar{\psi}} d\bar{\psi} - \frac{\partial H'_\mu}{\partial \bar{\psi}} dA_\mu, \\ &= [(i \gamma^i \partial_i + e\gamma^\mu A_\mu - m)\psi] dt + i \gamma^0 d\psi, \end{aligned} \quad (62)$$

$$\begin{aligned} dp_\mu &= -\frac{\partial H'_0}{\partial A_\mu} dt - \frac{\partial H'_\psi}{\partial A_\mu} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial A_\mu} d\bar{\psi} - \frac{\partial H'_\mu}{\partial A_\mu} dA_\mu, \\ &= (\bar{\psi} e \gamma^\mu \psi) dt, \end{aligned} \quad (63)$$

$$d\pi_4^a = -\frac{\partial H'_0}{\partial t} dt - \frac{\partial H'_\psi}{\partial t} d\psi - \frac{\partial H'_{\bar{\psi}}}{\partial t} d\bar{\psi} - \frac{\partial H'_\mu}{\partial t} dA_\mu. \quad (64)$$

The integrability conditions imply that the variation of the constraints  $H'_\psi, H'_{\bar{\psi}}$  and  $H'_\mu$  should be identically zero; that is

$$dH'_\psi = dp_\psi - i d\bar{\psi} \gamma^0 = 0, \quad (65)$$

$$dH'_{\bar{\psi}} = dp_{\bar{\psi}} = 0, \quad (66)$$

$$dH'_\mu = dp_\mu = 0. \quad (67)$$

The vanishing of total differential of  $H'_\mu$  leads to a new constraint

$$H''_\mu = \bar{\psi} e \gamma^\mu \psi. \quad (68)$$

When we taking a gain the total differential of  $H''_\mu$ , we notice that it vanishes identically,

$$dH''_\mu = 0. \quad (69)$$

From Eqs.(57) and (58), respectively we obtain

$$\dot{A}_a^i = \pi_a^i - \partial_i A_0^a + g f^{abc} A_0^b A_i^c, \quad (70)$$

and

$$\dot{A}_0^i = \xi \pi_0^a - \partial^i A_i^a. \quad (71)$$

Substituting from Eqs. (61) and (62) into Eqs. (65), and (66), respectively we get

$$\bar{\psi} (i \overleftarrow{\partial}_\mu \gamma^\mu - e \gamma^\mu A_\mu + m) = 0, \quad (72)$$

$$(i \gamma^\mu \partial_\mu + e \gamma^\mu A_\mu - m) \psi = 0. \quad (73)$$

Also from Eqs. (59-61, 63), we get the following equations of motion:

$$\dot{\pi}_a^i = g f^{abc} \pi_c^i A_0^b - \partial_i (F_a^{li} + \pi_0^a) - F_a^{il} g f^{abc} A_c^b, \quad (74)$$

$$\dot{\pi}_a^0 = \partial_i \pi_a^i + g f^{abc} \pi_b^i A_i^c, \quad (75)$$

$$\dot{p}_\psi = -\bar{\psi} (i \overleftarrow{\partial}_i \gamma^i - e \gamma^\mu A_\mu + m), \quad (76)$$

$$\dot{p}_\mu = \bar{\psi} e \gamma^\mu \psi. \quad (77)$$

Substituting from Eq. (73) into Eq.(62), we have

$$\dot{p}_{\bar{\psi}} = 0. \quad (78)$$

As a comparison between the above two methods, we get that the Hamilton-Jacobi method and Dirac's method give the same equations of motion.

## 2. Conclusion

A non-Abelian theory of fermions interacting with gauge bosons is discussed as constrained system by using both Dirac's and Hamilton-Jacobi methods. In Dirac's method the total Hamiltonian composed by adding the constraints multiplied by Lagrange multipliers to the canonical Hamiltonian. In order to derive the equations of motion, one needs to redefine these unknown multipliers in an arbitrary way. However, in the Hamilton-Jacobi approach (or Güler's method)[2-8], there is no need to introduce Lagrange multipliers to the canonical Hamiltonian. In Hamilton-Jacobi approach it is not necessary to distinguish between first-class and second-class constraints, there is no need to introduce any gauge fixing conditions as in Dirac's approach. Both the consistency conditions and the integrability conditions lead to the same constraints.

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