Noncommutative Geometry constraints and the Standard Renormalization Group approach: Two Doublets Higgs Model as an Example

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Abstract: The Chamssedine-Fröhlich Approach to Noncommutative Geometry (NCG) is extended and applied to the reformulation of the two doublets Higgs model. The Fuzzy mass, coupling and unitarity relations are derived. It is shown that the latter are no more preserved under the renormalization group equations obtained from the standard quantization method. This suggests to look for an appropriate NCG quantization procedure.

Keywords: Noncommutative Geometry; Renormalization Group; Higgs Model

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1. Introduction

In the last decade many interest has been devoted to understand the unsolved problems of the standard model and to find some origins and mechanisms of the various parameters. Many approaches have been developed namely the Connes Noncommutative geometry [1] – [20] formalism. The latter is based on a unitary algebra, where gauge bosons and Higgs are treated in an equal footing. The drawback of this approach, is that we cannot go beyond the standard model [1] – [3]. To overcome this situation, Chamessedine and Fröhlich proposed an extension of the Dirac operator which allows to consider other unified models based on unitary and orthogonal groups $SU(5), SO(10), ..etc.$[21] – [23].
However, the spontaneous breakdown of the symmetry is not well controled to allow getting reasonable mass and mixing angles relations. Moreover, the elimination of the auxilliary unphysical fields undergo through field equations which is in general a complicated procedure [21] – [23]. The goal of this paper is two fold: First to extend the Chamesseddine - Frohlich approach (through the two doublet Higgs model) by including the strong interaction sector in the new mathematical formalism and generalizing the Dirac operator as well as the scalar product. The elimination of the junks forms is done by applying the orthogonality condition before the construction of the action. Second to derive the tree level Fuzzy mass, coupling and unitarity relations and to show that we cannot preserve these constraints under the ordinary renormalization group flow (running masses, couplings etc.).

In section 2, we give the formalism and derive some of the NCG constraints. We present the renormalization group (RG) analysis and show that the corresponding evolution differential equations, once solved will not in general preserve the tree level NCG relations. Finally, in section 3, we draw our conclusions.

2. Formalism

Consider a model consisting of the spectral triplet \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) where \(\mathcal{A}\) and \(\mathcal{D}\) are an involutive algebra of operators and unbounded self-adjoint operators on a Hilbert space \(\mathcal{H}\) respectively. Let \(X\) be a compact Riemannian spin-manifold, and \((\mathcal{A}_1, \mathcal{H}_1, \mathcal{D}_1, \chi)\) the Dirac K- cycle where \(\mathcal{H}_1 \equiv L^2(X, \sqrt{g} dx)\) (\(g\) is the metric) acts on the \(\mathcal{A}_1\) algebra of functions on the \(X\) manifold. Let \((\mathcal{A}_2, \mathcal{H}_2, \mathcal{D}_2)\) another triplet where \(\mathcal{A}_2\) is given by [1], [2], [3], [7]:

\[
\mathcal{A}_2 = M_2(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C})
\]

(1)

where \(M_2(\mathbb{C})\), (resp.\(M_3(\mathbb{C})\)) are \(2 \times 2\), (resp.\(3 \times 3\)) matrices algebra and \(\mathcal{H}_2 = h_{2,1} \oplus h_{2,2} \oplus h_{2,3}\) with \(h_{2,1}, h_{2,2}\) and \(h_{2,3}\) are Hilbert spaces on \(\mathbb{C}^2\), \(\mathbb{C}\) and \(\mathbb{C}^3\) respectively. Then, \(\mathcal{A}\) and \(\mathcal{D}\) are taken to be:

\[
\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2
\]

(2)

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2
\]

(3)

and
\[ \mathcal{D} = \mathcal{D}_1 \otimes 1 \oplus \chi \otimes \mathcal{D}_2 \]  

(4)

Where \( \chi \) is chirality operator \( \chi \) acting on the Hilbert space \( \mathcal{H} \) such that:

\[ \chi = \chi^*, \quad \chi^2 = 1, \quad \chi \mathcal{D} = -\mathcal{D} \chi \]  

(5)

In this case, the Hilbert space \( \mathcal{H} \) can be splitted into two orthogonal subspaces \( \mathcal{H}_L \) and \( \mathcal{H}_R \) such that:

\[ \mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \]  

(6)

and

\[ \mathcal{H}_{L,R} = \frac{1}{2}(1 \mp \chi)\mathcal{H} \]  

(7)

or in the formal form:

\[ \mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}) \]  

(8)

and

\[ \mathcal{H}_R = ((\mathbb{C} + \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}) \]  

(9)

Therefore, the \( \mathcal{A} \) algebra which represents the gauge group \( U(2) \otimes U(1) \otimes U(3) \) can be written in the following form:

\[ \mathcal{A} = C^\infty(\mathbb{R}) \otimes (M_2(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C})) \]  

(10)

where here, \( C^\infty(\mathbb{R}) \) is the algebra of differentiable functions. It is to be noted that, an involutive representation \( \pi \) of this algebra is provided by the map:

\[ \pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}) \]  

(11)

\[ f \rightarrow 1_{4x4} \otimes \text{diag}(f_1, f_2, f_3, f_4, f_5, f_6) \]  

(12)

where \( \mathcal{B}(\mathcal{H}) \) is the algebra of bounded operators in the Hilbert space \( \mathcal{H} \) and to every block diagonal \( f \in \mathcal{A} \), we associate a sixtet \( (f_1, f_2, f_3, f_4, f_5, f_6) \) of matrix-valued functions on \( X \) such that:
Now, regarding the explicit form of the Hermitian Dirac operator $D$, it is given by:

$$D = \begin{pmatrix}
\partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{12} \otimes K_{12} & \ldots & \gamma_5 \otimes M_{16} \otimes K_{16} \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes 1 \otimes 1_N & \ldots & \gamma_5 \otimes M_{26} \otimes K_{26} \\
\ldots & \ldots & \ldots & \ldots \\
\gamma_5 \otimes M_{61} \otimes K_{61} & \gamma_5 \otimes M_{62} \otimes K_{62} & \ldots & \partial \otimes 1 \otimes 1_N
\end{pmatrix}$$

(16)

where $\partial = \gamma^\mu \partial_\mu$. The reality condition of the operator $D$ implies that:

$$M_{mn}^* = M_{nm}, \quad K_{mn}^* = K_{nm}, \quad m \neq n = 1, 2, \ldots, 6 \quad (17)$$

The $M_{mn}$ matrices determine the tree level vacuum expectation values of the higgs fields and control the corresponding symmetry breaking scheme. The $N \times N$, $K_{ij}$ ($i, j = 1, 6$) matrices where $N$ is the fermionic family number, are introduced to include informations about the mixing between the various generations of quarks and leptons.

Now, in order to get the standard model with two Higgs doublets, we have to choose:

$$K_{13} = K_{24} = K \quad (18)$$

$$K_{31} = K_{42} = K^*$$

and take:

$$M_{13} = \begin{pmatrix}
\tilde{M}_u \\
\tilde{M}_d
\end{pmatrix} = S' \otimes \tilde{M}_u \otimes 1_N + S \otimes \tilde{M}_d \otimes 1_N \quad (19)$$

and

$$M_{24} = \begin{pmatrix}
0 \\
\tilde{M}_e
\end{pmatrix} = S \otimes \tilde{M}_e \otimes 1_N \quad (20)$$
where

$$S = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (21)$$

and

$$S' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (22)$$

The remaining $K_{mn}$ and $M_{mn}$ matrices vanish. The $\tilde{M}_u$, $\tilde{M}_d$ and $\tilde{M}_e$ are the Up, Down quarks-like and leptons mass matrices respectively such that:

$$\tilde{M}_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$  \hspace{1cm} (23)$$

$$\tilde{M}_d = C_{CKM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}$$  \hspace{1cm} (24)$$

and

$$\tilde{M}_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}$$  \hspace{1cm} (25)$$

where

$$C_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{tb} & V_{ts} & V_{tb} \end{pmatrix}$$  \hspace{1cm} (26)$$
is the Cabbibo-Kobayashi-Maskawa matrix which is supposed non degenerate and the elements \( m_l, (l = u, c, t, d, s, b, e, \mu, \tau) \) are positive fermions masses.

If we denote by \( \Omega^p(A) \) the algebra of \( p \)-forms where \( \Omega^0(A) = A \), the elements of the canonical universal algebra \( \Omega_D(A) \) can be obtained from those of \( A \) as follows:

\[
\Omega_D(A) := \bigoplus_{p \geq 0} \Omega^p_D(A) = \frac{\pi(\Omega(A))}{\pi(Aux)}
\]

(27)

where:

\[
\Omega^p_D(A) = \left( \frac{\pi(\Omega^p(A))}{\pi(Aux^p)} \right)
\]

(28)

represents the \( p \)-form canonical differential algebra and \( Aux^p \) are the \( p \) junks forms (non dynamical auxiliary fields). It is worth to mention that the scalar product between the elements of this algebra is defined as:

\[
\Omega^p_D(A) \otimes \Omega^l_D(A) \rightarrow IR
\]

\[
(\omega, \eta) \rightarrow \langle \omega, \eta \rangle = \delta_{pl} Tr(\omega^\ast \eta Z)
\]

(29)

Here \( Tr \) stands for trace and the positive definite operator \( Z \) has as expression:

\[
Z = \begin{pmatrix}
    z & 0 & 0 & 0 & 0 \\
    0 & z & 0 & 0 & 0 \\
    0 & 0 & y & 0 & 0 \\
    0 & 0 & y & 0 & 0 \\
    0 & 0 & 0 & z & 0 \\
    0 & 0 & 0 & z & 0
\end{pmatrix}
\]

(30)

with

\[
\begin{align*}
    x & \rightarrow 1_4 \otimes 1_2 \otimes x \otimes 1_N \\
    z & \rightarrow 1_4 \otimes 1_1 \otimes y \otimes 1_N \\
    z & \rightarrow 1_4 \otimes 1_3 \otimes z \otimes 1_N
\end{align*}
\]

(\( x \), \( y \) and \( z \) \( \in M_3(\mathbb{C}) \) and \( 1_D \) are \( DxD \) identity matrices). Notice that \( Z \) verifies the following commutation relations:
\[ [D, Z] = 0 \] \hspace{1cm} (32)

and

\[ [Z, \pi (A)] = 0 \] \hspace{1cm} (33)

which corresponds to the case:

\[ \bar{x} = x_{13}, \quad \bar{y} = y_{13}, \quad \bar{z} = z_{13} \] \hspace{1cm} (34)

with \( x, y \) and \( z \in \mathbb{R} \). Now, the 1-form differential algebra \( \Omega^1_\mathcal{D}(A) \) reads:

\[ \Omega^1_\mathcal{D}(A) \approx \pi (\Omega^1(A)) = \left\{ \pi (\omega) = \pi \left( \sum_1 \alpha^i \delta^i \right) = \sum_1 \pi (\alpha^i) [D, \pi (\beta^i)] \right\} \] \hspace{1cm} (35)

where \( \alpha^i, \beta^i \in \mathcal{A} \). If we denote by \( \omega^m_\mu \) and \( \Phi_{mn} \) (\( m \neq n = 1, 2, \ldots, 6 \)) the vector and scalar fields respectively and \( N_f \) the fermionic family number (in our case \( N_f = 3 \)) such that:

\[ (\omega^m_\mu)^* = -\omega^m_\mu \] \hspace{1cm} (36)

and

\[ \Phi^*_{mn} = \Phi_{mn} \] \hspace{1cm} (37)

Then, the non vanishing elements are given by:

\[ \pi (\omega)_{1i} = \gamma^i \otimes \omega^i_\mu \otimes 1_3 \otimes 1_N, \quad i = 1,6 \]
\[ \pi (\omega)_{13} = \gamma_5 \otimes \Phi_{13} \otimes K_{13} \]
\[ \pi (\omega)_{31} = \gamma_5 \otimes \Phi_{31} \otimes K_{31} \]
\[ \pi (\omega)_{24} = \gamma_5 \otimes \Phi_{24} \otimes K_{24} \]

and

\[ \pi (\omega)_{42} = \gamma_5 \otimes \Phi_{42} \otimes K_{42} \] \hspace{1cm} (39)

with
\[ \Phi_{13} = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 - 1 \end{array} \right) \otimes M_d \otimes 1_3 + \left( \begin{array}{c} \chi_1 \\ \chi_2 - 1 \end{array} \right) \otimes M_u \otimes 1_3 \] (40)

and

\[ \Phi_{24} = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 - 1 \end{array} \right) \otimes M_e \otimes 1 \] (41)

Regarding the 2-form canonical differential algebra \( \Omega^2_D (A) \), it is defined as:

\[ \Omega^2_D (A) = \frac{\pi (\Omega^2 (A))}{Aux^2} \] (42)

Where, the 2-form non physical (junks) differential algebra \( Aux^2 \) can be obtained as:

\[ Aux^2 = \{ \pi (\delta \omega) / \pi (\omega) = 0 \} \] (43)

which implies that:

\[ \omega^m_{\mu} = 0, \ m = 1, 2, ..6 \] (44)

and

\[ \Phi_{mn} = 0, \ m \neq n = 1, 2, ..6 \] (45)

Straightforward calculations using the fact that:

\[ \pi (\delta \omega) = \left\{ \pi (\delta \alpha_i \delta \beta^j) = \sum_i [D, \pi (\alpha^i)] [D, \pi (\beta^j)] \right\} \] (46)

as well as the relations:

\[ | M_{31} |^2 = 1_2 \otimes (\bar{\mu}_u + \bar{\mu}_d) \otimes 1_3 \]

\[ | M_{13} |^2 = 1_2 \otimes (\mu_u + \mu_d) 1_3 + \tau_3 \otimes \frac{1}{2} (\mu_u - \mu_d) \otimes 1_3 + \tau_1 \otimes \mu_{ud} \otimes 1_3 \]

\[ | M_{42} |^2 = 1_2 \otimes \bar{\mu}_e \otimes 1_1 \] (47)
and

\[ | M_{24} |^2 = (1_2 - \tau_3) \otimes \frac{1}{2} \mu_e \otimes 1_1 \]

where

\[ \mu_a = \tilde{M}_a \tilde{M}_a^* \]
\[ \tilde{\mu}_a = \tilde{M}_a^* \tilde{M}_a \]
\[ \mu_{ab} = \tilde{M}_a \tilde{M}_b^* \]

(48)

and

\[ \tilde{\mu}_{ab} = \tilde{M}_b^* \tilde{M}_a \]

(\( \tau_3 \) and \( \tau_1 \) are the Pauli matrices) show that the elements of this algebra have as expressions:

\[
(Aux^2)_{11} = \gamma^\mu \gamma^\nu \otimes 1_3 \otimes 1_N + 1_4 \otimes i \varkappa_2 \otimes \frac{1}{2} (\mu_d - \mu_a) \otimes 1_3 \otimes KK^* \\
+ 1_4 \otimes i \varkappa_3 \otimes \mu_{ad} \otimes 1_3 \otimes KK^*
\]

\[
(Aux^2)_{22} = \gamma^\mu \gamma^\nu \otimes \varkappa_{1\mu\nu} \otimes 1_3 \otimes 1_N + 1_4 \otimes i \varkappa_2 \otimes \mu_e \otimes 1_1 \otimes KK^* \]

(49)

\[
(Aux^2)_{33} = (Aux^2)_{44} = \gamma^\mu \gamma^\nu \otimes \varkappa_{\mu\nu} \otimes 1_3 \otimes 1_N
\]
\[
(Aux^2)_{55} = (Aux^2)_{66} = \gamma^\mu \gamma^\nu \otimes y_{\mu\nu} \otimes 1_3 \otimes 1_N
\]

and

\[
(Aux^2)_{mn} = 0, \quad m \neq n = 1, 6
\]

with

\[
\varkappa_{1\mu\nu} = - \sum_i \alpha_i^1 \partial_\mu \partial_\nu \beta_i^1
\]
\[
\varkappa_{\mu\nu} = - \sum_i \alpha_i^3 \partial_\mu \partial_\nu \beta_i^3
\]
\[
\varkappa_2 = - \sum_i \alpha_i^1 [i \tau_3, \beta_i^1]
\]

(50)
\[ \pi_3 = \sum_i \alpha_i^i [i \tau_1, \beta_i^i] \]

and

\[ y_{\mu\nu} = -\sum_i \alpha_i^5 \partial_\mu \partial_\nu \beta_i^5 \] (51)

Now, taking into account the fact that the elements of \( \pi (\Omega^2 (A)) \) are given by:

\[ \pi (\Omega^2 (A)) = \{ \pi (C) = \pi (\delta \omega) + \pi (\omega^2) : \omega \in \Omega^1 (A) \} \] (52)

the orthogonality condition leads to the following canonical differential algebra \( \Omega^2_D (A) \) elements:

\[
\begin{align*}
\pi (C)_{11} &= \frac{1}{2} \gamma_{\mu\nu} \otimes F_{\mu\nu}^1 \otimes 1_3 \otimes 1_N \\
\pi (C)_{22} &= \frac{1}{2} \gamma_{\mu\nu} \otimes F_{\mu\nu}^2 \otimes 1_3 \otimes 1_N \\
\pi (C)_{33} &= \frac{1}{2} \gamma_{\mu\nu} \otimes F_{\mu\nu}^3 \otimes 1_3 \otimes 1_N \\
&\quad + 14 \otimes (\varphi^* \varphi - 1) \otimes \bar{\mu}_d \otimes 1_3 \otimes K^* K + 14 \otimes (\varphi^* \varphi - 1) \otimes \bar{\mu}_u \otimes 1_3 \otimes K^* K \\
&\quad + 14 \otimes \varphi^* \varphi \otimes \mu_{du} \otimes 1_3 \otimes K^* K + 14 \otimes \varphi^* \varphi \otimes \bar{\mu}_{du} \otimes 1_3 \otimes K^* K \\
&\quad + \gamma^\mu \gamma^\nu \otimes \frac{Tr K^* K}{6N} \left\{ \begin{array}{l}
tr (3\mu_d + \mu_e) (\varphi^* \varphi - 1)_{\mu\nu} + tr (3\mu_u) (\varphi^* \varphi - 1)_{\mu\nu} \\
+ tr (3\mu_{du}) (\varphi^* \varphi)_{\mu\nu} + tr (3\bar{\mu}_{du}) (\varphi^* \varphi)_{\mu\nu}
\end{array} \right\} (53)
\end{align*}
\]

\[
\begin{align*}
\pi (C)_{44} &= \frac{1}{2} \gamma_{\mu\nu} \otimes F_{\mu\nu}^4 \otimes 1_3 \otimes 1_N + 1_4 \otimes (\varphi^* \varphi - 1) \otimes \bar{\mu}_e \otimes 1_1 \otimes K^* K \\
&\quad + \gamma^\mu \gamma^\nu \otimes \frac{Tr K^* K}{6N} \left\{ \begin{array}{l}
tr (3\mu_d + \mu_e) (\varphi^* \varphi - 1)_{\mu\nu} + tr (3\mu_u) (\varphi^* \varphi - 1)_{\mu\nu} \\
+ tr (3\mu_{du}) (\varphi^* \varphi)_{\mu\nu} + tr (3\bar{\mu}_{du}) (\varphi^* \varphi)_{\mu\nu}
\end{array} \right\} (54)
\end{align*}
\]

\[
\begin{align*}
\pi (C)_{55} &= \pi (C)_{66} = \frac{1}{2} \gamma_{\mu\nu} \otimes F_{\mu\nu}^5 \otimes 1_3 \otimes 1_N \\
\pi (C)_{13} &= -\gamma^5 \gamma^\mu \otimes D_\mu \varphi \otimes M_d \otimes 1_3 \otimes K - \gamma^5 \gamma^\mu \otimes D_\mu \varphi \otimes M_u \otimes 1_3 \otimes K \\
\pi (C)_{31} &= -\gamma^5 \gamma^\mu \otimes (D_\mu \varphi)^* \otimes M_d^* \otimes 1_3 \otimes K^* - \gamma^5 \gamma^\mu \otimes (D_\mu \varphi)^* \otimes M_u^* \otimes 1_3 \otimes K^* \\
\pi (C)_{24} &= -\gamma^5 \gamma^\mu \otimes D_\mu \varphi \otimes M_e \otimes 1_1 \otimes K
\end{align*}
\]

and
\[ \pi (C)_{42} = -\gamma^5 \gamma^\mu \otimes (D_\mu \varphi)^* \otimes M^* \otimes 1 \otimes K^* \]  
(54)

where

\[ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \]  
(55)

\[ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \]  
(56)

and

\[ D_\mu = \partial_\mu + \omega^1_\mu - \omega^3_\mu \]  
(57)

The \( F_{\mu\nu} \)'s are the gauge fields strength tensor.

Before we proceed with further details of the calculation, it is worth mentioning that our approach is considered to be an extension of Chamssedine-Frohlich formalism and not that of Conne’s in the sense that like in ours, the former introduce the \( M_{ij} \) and \( K_{ij} \) matrices where in the later are taken to be proportional to the identity matrix (special case). Moreover, the difference between our approach and that of Chamssedine-Frohlich \([20] – [22]\) lies on:

a) The generalization of the scalar product

b) The elimination of the junk forms (or the non-dynamical auxiliary fields) by using the orthogonality condition instead of the field equations which is in general a very complicated procedure.

c) The modification of the algebra \( A \) (see Eq.(10)) for the gauge group \( U(2) \otimes U(1) \otimes U(3) \) instead of \( C^\infty (X) \otimes (M_2 (\mathbb{C}) \oplus \mathbb{C}) \) for \( U(2) \otimes U(1) \) in the order to include the strong interactions.

d) The modification and extension of the Dirac operator to have the general form given by Eq.(16). In the case of the standard model and to account for the strong interactions, quark masses and mixings the Dirac operator the following simplified expression:
Notice that, in order to insure that the gauge group $SU(3)$ remains unbroken, the mass matrix along the $M_3 (\mathbb{C})$ direction is taken to be zero, forcing the vanishing of the scalar field expectation value along the same direction. In the Chamssedine-Frohlich approach, the general form of the Dirac operator has the following general form:

$$D = \begin{pmatrix}
\partial \otimes 1 \otimes 1_N & 0 & \gamma_5 \otimes M_{13} \otimes K_{13} & 0 & 0 & 0 \\
0 & \partial \otimes 1 \otimes 1_N & 0 & \gamma_5 \otimes M_{24} \otimes K_{24} & 0 & 0 \\
\gamma_5 \otimes M_{31} \otimes K_{31} & 0 & \partial \otimes 1 \otimes 1_N & 0 & 0 & 0 \\
0 & \gamma_5 \otimes M_{24} \otimes K_{32} & 0 & \partial \otimes 1 \otimes 1_N & 0 & 0 \\
0 & 0 & 0 & 0 & \partial \otimes 1 \otimes 1_N & 0 \\
0 & 0 & 0 & 0 & 0 & \partial \otimes 1 \otimes 1_N 
\end{pmatrix}$$

(58)

and in the case of the standard model it becomes ($M_{13} = M_{23} = 0$):

$$D = \begin{pmatrix}
\partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{12} \otimes K_{12} & 0 \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{23} \otimes K_{23} \\
\gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \partial \otimes 1 \otimes 1_N 
\end{pmatrix}$$

(59)

and in the case of the standard model it becomes ($M_{13} = M_{23} = 0$):

$$D = \begin{pmatrix}
\partial \otimes 1 \otimes 1_N & \gamma_5 \otimes M_{12} \otimes K_{12} & 0 \\
\gamma_5 \otimes M_{21} \otimes K_{21} & \partial \otimes 1 \otimes 1_N & 0 \\
0 & 0 & \partial \otimes 1 \otimes 1_N 
\end{pmatrix}$$

(60)

Notice that, in our extended approach, the matrices $M_{13}$ and $M_{24}$ contain the quarks and leptons mass respectively. Similarly for the matrices $K_{13}$ and $K_{24}$ where they include the quarks and leptons mixings. However, for the Chamssedine-Frohlich approach, we have just the matrices $M_{12}$ and $K_{12}$ and therefore, we cannot include in a natural way the quarks and their mixings. Furthermore, even if we try to include in their approach the $SU(3)$ gauge group in a commutative way and decoupling it from the rest, we will face the quark mass problem.

In what follows, we will concentrate on the scalar fields sector. Starting from the definition of the bosonic action $\tilde{\mathcal{I}}_B$:

$$\tilde{\mathcal{I}}_B = \int d^4x \; Tr (tr C^* C)$$

(61)
and after diagonalization, by introducing a mixing angle $\vartheta$ such that:

$$\tan \vartheta = \frac{2 \text{tr} 3 \mu_{du}}{\text{tr} (3 (\mu_d - \mu_u) + \mu_u)} \approx \frac{2 m_t m_b}{m^2_t - m^2_b}$$

and spontaneous breaking of the gauge symmetry as:

$$U(2)_L \otimes U(1) \otimes U(3) \xrightarrow{\langle \varphi \rangle, \langle \kappa \rangle} U(1) \otimes U(1)' \otimes U(3)$$

where $\varphi$ and $\kappa$ are two complex doublets where:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \equiv \begin{pmatrix} \Phi^+_1 \\ \eta_1 + i \kappa_1 \end{pmatrix}$$

and

$$\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \equiv \begin{pmatrix} \Phi^+_2 \\ \eta_2 + i \kappa_2 \end{pmatrix}$$

The corresponding vacuum expectation values $\langle \varphi \rangle$ and $\langle \kappa \rangle$ are chosen as:

$$\langle \varphi \rangle = \begin{pmatrix} 0 \\ \langle \varphi_0 \rangle \end{pmatrix}, \langle \kappa \rangle = \begin{pmatrix} 0 \\ \langle \kappa_0 \rangle \end{pmatrix}$$

By redefining the scalar fields $\varphi$ and $\kappa$ as:

$$\varphi \rightarrow \alpha_1 \varphi$$

and

$$\kappa \rightarrow \alpha_2 \kappa$$

with

$$\alpha_{1,2} = \frac{\widetilde{L}}{m_t \cos \vartheta \pm m_b \sin \vartheta}$$

where
\[ \tilde{L} = \frac{1}{[3 (x + y) TrK^* K]^{1/2}} \] (68)

A lengthy calculation leads in the Higgs sector, to the following expression of the potential \( V(\varphi, \kappa) \):

\[ V(\varphi, \kappa) = \xi_1 (\varphi^* \varphi - \langle \varphi_0 \rangle^2)^2 + \xi_2 (\kappa^* \kappa - \langle \kappa_0 \rangle^2)^2 + \xi_3 (\varphi^* \varphi) + \xi_4 (\kappa^* \kappa) \]
\[ + \xi_5 (\varphi^* \varphi) (\kappa^* \kappa + \varphi^* \varphi) + \xi_6 (\varphi^* \varphi) (\varphi^* \varphi + \varphi^* \varphi) \]
\[ + \xi_7 (\kappa^* \kappa) (\varphi^* \varphi + \kappa^* \kappa) + \xi_8 (\varphi^* \varphi + \kappa^* \kappa)^2 \]
\[ + \xi_9 (\varphi^* + \kappa^* \varphi) \] (69)

where

\[ \xi_1 = 3y \Sigma \left( m_4^4 \cos^4 \vartheta + m_4^4 \sin^4 \vartheta + 2m_4^2 m_6^2 \cos^2 \vartheta \sin^2 \vartheta + 2m_4^2 m_b \sin 2\vartheta + 2m_4^2 m_t \sin^2 2\vartheta \right) (\alpha_1)^4 \]
\[ \xi_2 = 3y \Sigma \left( m_4^4 \sin^4 \vartheta + m_4^4 \cos^4 \vartheta + 2m_4^2 m_6^2 \cos^2 \vartheta \sin^2 \vartheta - 2m_4^2 m_b \sin 2\vartheta + 2m_4^2 m_t \sin^2 2\vartheta \right) (\alpha_2)^4 \]
\[ \xi_3 = 6y \Sigma \left( m_4^4 \cos^4 \vartheta + m_4^4 \sin^4 \vartheta + m_4^2 m_6^2 - 2m_4^2 m_b \sin 2\vartheta \right) (\alpha_1)^2 (\alpha_2)^2 \]
\[ \xi_4 = 6y \Sigma \left( m_4^4 \sin^4 \vartheta + m_4^4 \cos^4 \vartheta - m_4^2 m_6^2 - 2m_4^2 m_b \sin 2\vartheta \right) (\alpha_2)^2 \]
\[ \xi_5 = 6y \Sigma \left( (m_4^4 + m_b^4) \cos^2 \vartheta \sin^2 \vartheta + 2m_4^2 m_b^2 (\cos^4 \vartheta + \sin^4 \vartheta) - 4m_4^2 m_t \sin^2 2\vartheta \right) (\alpha_1)^2 (\alpha_2)^2 \]
\[ \xi_6 = 3y \Sigma \left( m_4^2 m_6^2 \cos 2\vartheta \sin 2\vartheta + 2m_4^2 m_b (\cos^2 \vartheta \cos 2\vartheta + \frac{1}{2} \sin^2 2\vartheta) + \right) (\alpha_1)^3 (\alpha_2) \]
\[ + 2m_4^2 m_t (\sin^2 \vartheta \cos 2\vartheta + \frac{1}{2} \sin^2 2\vartheta) \]
\[ \xi_7 = 3y \Sigma \left( \sin 2\vartheta \left( -m_4^4 \sin^2 \vartheta + m_b^4 \cos^2 \vartheta \right) - m_4^2 m_6^2 \cos 2\vartheta \sin 2\vartheta \right) (\alpha_1)^2 (\alpha_2)^2 \]
\[ + 2m_4^2 m_b (\cos 2\vartheta \sin^2 \vartheta + \frac{1}{2} \sin^2 2\vartheta) + \]
\[ + 2m_4^2 m_t (\cos 2\vartheta \cos^2 \vartheta + \frac{1}{2} \sin^2 2\vartheta) \]
\[ \xi_8 = 3y \Sigma \left( \frac{1}{4} (m_4^2 + m_b^2)^2 \sin^2 2\vartheta + 2m_4^2 m_t \cos^2 2\vartheta \right) (\alpha_1)^2 (\alpha_2)^2 \]

and

\[ \xi_9 = 3y \Sigma \left( (m_b^2 - m_4^2)^2 \sin 2\vartheta + 4m_4^2 m_b \cos 2\vartheta \right) (\alpha_1) (\alpha_2) \]

It is easy now, to obtain the following mass matrices for the real neutral, imaginary neutral and charged scalar fields \((\eta_1, \eta_2), (\kappa_1, \kappa_2)\) and \((\Phi_1^+, \Phi_2^+)\) respectively:
\[ M^2(\eta_1, \eta_2) = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \]

\[ M^2(\kappa_1, \kappa_2) = \begin{pmatrix} \tilde{\Delta}_{11} & \tilde{\Delta}_{12} \\ \tilde{\Delta}_{21} & \tilde{\Delta}_{22} \end{pmatrix} \]

\[ M^2(\Phi_1^\pm, \Phi_2^\pm) = \begin{pmatrix} \hat{\Delta}_{11} & \hat{\Delta}_{12} \\ \hat{\Delta}_{21} & \hat{\Delta}_{22} \end{pmatrix} \]

where

\[ \Delta_{11} = 6\xi_1 \langle \varphi_0 \rangle^2 + (\xi_5 + 4\xi_8) \langle \varkappa_0 \rangle^2 + 6\xi_6 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_3 \]

\[ \Delta_{12} = \Delta_{21} = 3\xi_6 \langle \varphi_0 \rangle^2 + 3\xi_7 \langle \varkappa_0 \rangle^2 + 2(\xi_5 + 2\xi_8) \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_9 \]

\[ \Delta_{22} = (\xi_5 + 4\xi_8) \langle \varphi_0 \rangle^2 + 6\xi_2 \langle \varkappa_0 \rangle^2 + 6\xi_7 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_4 \]

\[ \tilde{\Delta}_{11} = 2\xi_1 \langle \varphi_0 \rangle^2 + \xi_5 \langle \varkappa_0 \rangle^2 + 2\xi_6 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_3 \]

\[ \tilde{\Delta}_{12} = \tilde{\Delta}_{21} = \xi_6 \langle \varphi_0 \rangle^2 + \xi_7 \langle \varkappa_0 \rangle^2 + 2\xi_8 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_9 \]

\[ \tilde{\Delta}_{22} = 2\xi_1 \langle \varphi_0 \rangle^2 + \xi_5 \langle \varkappa_0 \rangle^2 + 2\xi_7 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_4 \]

\[ \hat{\Delta}_{11} = \xi_6/2 + \xi_1 \langle \varphi_0 \rangle^2 + \xi_5 \langle \varkappa_0 \rangle^2/2 + \xi_6 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle \]

\[ \hat{\Delta}_{12} = \hat{\Delta}_{21} = \xi_9/2 + +\xi_6 \langle \varphi_0 \rangle^2 + \xi_7 \langle \varkappa_0 \rangle^2 + 2\xi_8 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle \]

and

\[ \hat{\Delta}_{22} = \xi_5 \langle \varphi_0 \rangle^2/2 + \xi_2 \langle \varkappa_0 \rangle^2 + 2\xi_7 \langle \varphi_0 \rangle \langle \varkappa_0 \rangle + \xi_4/2 \]

After a diagonalisation of the above symmetric matrices (Eqs.71), and choosing \( \vartheta = \frac{\pi}{4} \), it is easy to show that:
\[ m_t \approx 2.4 m_b \] (72)

\[ g = (12N_f x)^{-1/2} \] (73)

\[ g' = \left( 12N_f \left( \frac{x}{4} + y \right) \right)^{-1/2} \] (74)

and

\[ g_3 = (12N_f z)^{-1/2} \]

where \( g, g' \) and \( g_3 \) are the coupling constants of the gauge groups \( U(2), U(1), U(3) \) respectively. Similarly,

\[ \xi_1 \approx \frac{0.17 \Sigma}{x (TrKK^*)^2}, \]

\[ \xi_2 \approx \frac{0.30 \Sigma}{x (TrKK^*)^2}, \]

\[ \xi_3 \approx \frac{0.30m_t^2 \Sigma}{TrKK^*}, \]

\[ \xi_4 \approx \frac{1.50m_t^2 \Sigma}{TrKK^*}, \]

\[ \xi_5 \approx \xi_1, \]

\[ \xi_6 = \xi_7 = 0, \]

\[ \xi_8 \approx \frac{0.20 \Sigma}{x (TrKK^*)^2}, \]

\[ \xi_9 \approx \frac{2m_t^2 \Sigma}{TrKK^*} \]

\[ \Sigma = Tr (KK^*)^2 - (3N)^{-1} (TrKK^*)^2 \] (76)

and

\[ \frac{\langle \varphi_0 \rangle}{\langle \chi_0 \rangle} \approx 70 \] (77)

which means that \( \langle \chi_0 \rangle \ll \langle \varphi_0 \rangle \). Straightforward calculations give
\[ \langle \varphi_0 \rangle^2 \approx 9x TrKK^*m_t^2 \]  

(78)

and if we take \( M_t \approx 2M_W \), we deduce that:

\[ \sqrt{\frac{TrKK^*}{x}} \approx 12 \]  

(79)

Regarding the Higgs masses, we obtain first from the eigenstates of Eq.(71); the CP-odd pseudoscalar \( A^0 \), CP-even scalars \( H^0 \) and \( h^0 \) and charged scalars \( H^\pm \) related to the state systems \((\eta_1, \eta_2), (\kappa_1, \kappa_2)\) and \((\Phi^+_1, \Phi^+_2)\) respectively, then we obtain in terms of the top quark mass the following approximate mass eigenvalues relations:

\[
\begin{align*}
M^2_{A^0} &= M^2_{H^\pm} \approx \frac{2.64\Sigma}{x^{\frac{1}{2}} (TrKK^*)^2} M_t^2, \\
M^2_{H^0} &\approx \frac{3.15\Sigma}{x^{\frac{1}{2}} (TrKK^*)^2} M_t^2, \\
M^2_{h^0} &\approx \frac{0.42\Sigma}{x^{\frac{1}{2}} (TrKK^*)^2} M_t^2,
\end{align*}
\]

(80)

with:

\[ M_t^2 = c^2 (m_t \langle \varphi_0 \rangle + m_b \langle \kappa_0 \rangle) \]

Now, using the Schwartz inequality, one can deduce that:

\[ \Sigma \leq \frac{26}{9} (TrK^*K)^2 \]

(81)

and therefore, we get for the CP-odd pseudoscalar \( A^0 \) and charged scalars \( H^\pm \) the constraint:

\[ M_{A^0} = M_{H^\pm} \lesssim 9.57M_t \]  

(82)

Similarly, for the CP-even scalars \( H^0 \) and \( h^0 \) one can show that:

\[ M_{H^0} \lesssim 10.45M_t \]  

(83)

and

\[ M_{h^0} \lesssim 3.81M_t \]  

(84)
Notice that for the $M_{H^\pm}$ and $M_{H^0}$, our results are compatible with the actual experimental bounds which allow for a heavy Higgs scenario.

Regarding the tree level NCG unitarity bound, we have to consider the scalar elastic and inelastic processes of the form $B_1 + B_2 \rightarrow B_3 + B_4$ by using the partial wave decomposition technique to the corresponding amplitude $M$:

$$M(s, t, u) = 16\pi \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \theta) A_l(s)$$  \hspace{1cm} (85)

To get upper bound limits to the scalar potential parameters of Eq. (75), one has to require that the tree level unitarity has to be preserved in all possible scattering processes. This is equivalent to the fact that the $s$ partial wave amplitude $A_0$ for the scalar-scalar, gauge boson- gauge boson and gauge boson-scalar has to satisfy $|A_0| \leq 1/2$ in the high energy limit. We remind the reader that at a very high energy, the equivalence theorem states that the amplitude of a scattering process involving longitudinal gauge bosons $W^\pm \mu$ and $Z^\mu$ may be approximated by the scalar amplitude in which gauge bosons are replaced by their corresponding Goldstone bosons and the unitarity constraints can be implemented by only considering the pure scalar scatterings. In what follows, we limit ourselves to pure scalar scattering processes dominated by quartic interactions and follow the technique introduced in ref.[25]. The $S$ matrix expressed in terms of the physical fields can be transformed into an $S$ matrix for the non physical fields $w_j^\pm, h_j$ and $z_j \ (i = 1, 2)$ such that:

$$\varphi = \begin{pmatrix} w_1^+ \\ \langle \varphi_0 \rangle + \frac{1}{2} (h_1 + z_1) \end{pmatrix},$$

$$\kappa = \begin{pmatrix} w_2^+ \\ \langle \kappa_0 \rangle + \frac{1}{2} (h_2 + z_2) \end{pmatrix}$$ \hspace{1cm} (86)

and

$$w_j^\pm = \alpha_j^{-1} \Phi_j^\pm; \quad h_j = 2 \alpha_j^{-1} \eta_j; \quad z_i = 2 \alpha_i^{-1} \zeta_i; \quad \langle \varphi_0 \rangle = \alpha_1^{-1} \langle \varphi_0 \rangle; \quad \langle \kappa_0 \rangle = \alpha_2^{-1} \langle \kappa_0 \rangle$$ \hspace{1cm} (87)

by making a unitary transformation. Thus, the full set of the scalar scattering processes can be expressed as an $S$ matrix $22x22$ composed of 4 submatrices $M_{6x6}^1, M_{6x6}^2, M_{2x2}^3$ and $M_{8x8}^4$ which do not couple with each other and where the entries are the quartic.
couplings mediating the scattering processes. In our case, straightforward simplifications lead to:

(a) Basis \((w_1^+w_2^-, w_2^+w_1^-, h_1z_2, h_2z_1, z_1z_2, h_1h_2)\):

\[
M_{6 \times 6}^1 = \begin{bmatrix}
4\xi_1 & \xi_5 + \frac{1}{2}\xi_8 & \sqrt{2}\xi_1 & \frac{1}{\sqrt{2}}\xi_5 & \sqrt{2}\xi_1 & \frac{1}{\sqrt{2}}\xi_5 \\
\xi_5 + \frac{1}{2}\xi_8 & 4\xi_2 & \frac{1}{\sqrt{2}}\xi_5 & \sqrt{2}\xi_2 & \frac{1}{\sqrt{2}}\xi_5 & \sqrt{2}\xi_2 \\
\sqrt{2}\xi_1 & \frac{1}{\sqrt{2}}\xi_5 & 3\xi_1 & \xi_5 + \frac{1}{2}\xi_8 & \xi_1 & \frac{1}{2}\xi_5 \\
\frac{1}{\sqrt{2}}\xi_5 & \sqrt{2}\xi_2 & \xi_5 + \frac{1}{2}\xi_8 & 3\xi_2 & \frac{1}{2}\xi_5 & \xi_2 \\
\sqrt{2}\xi_1 & \frac{1}{\sqrt{2}}\xi_5 & \xi_1 & \frac{1}{2}\xi_5 & 3\xi_1 & \xi_5 + \frac{1}{2}\xi_8 \\
\frac{1}{\sqrt{2}}\xi_5 & \sqrt{2}\xi_2 & \frac{1}{2}\xi_5 & \xi_2 & \xi_5 + \frac{1}{2}\xi_8 & 3\xi_2
\end{bmatrix}
\]  

(b) Basis \((w_1^+w_1^-, w_2^+w_2^-, \frac{z_1z_1}{\sqrt{2}}, \frac{z_2z_2}{\sqrt{2}}, \frac{h_1h_1}{\sqrt{2}}, \frac{h_2h_2}{\sqrt{2}})\):

\[
M_{6 \times 6}^2 = \begin{bmatrix}
\xi_5 - \frac{1}{4}\xi_8 & \xi_8 & 0 & 0 & \xi_8 & \xi_8 \\
\xi_8 & \xi_5 + \frac{1}{4}\xi_8 & 0 & 0 & \xi_8 & \xi_8 \\
0 & 0 & \xi_5 & \xi_8 & 0 & 0 \\
0 & 0 & \frac{\xi_8}{2} & \xi_5 & 0 & 0 \\
\frac{\xi_8}{2} & \frac{\xi_8}{2} & 0 & 0 & \xi_5 + \xi_8 & \frac{\xi_8}{2} \\
\frac{\xi_8}{2} & \frac{\xi_8}{2} & 0 & 0 & \frac{\xi_8}{2} & \xi_5 + \xi_8
\end{bmatrix}
\]  

(c) Basis \((h_1z_1, h_2z_2)\):

\[
M_{2 \times 2}^3 = \begin{bmatrix}
2\xi_1 & \frac{1}{2}\xi_8 \\
\frac{1}{2}\xi_8 & 2\xi_2
\end{bmatrix}
\]  

(d) Basis \((h_1w_1^+, h_2w_1^+, z_1w_1^+, z_2w_1^+, h_1w_2^+, h_2w_2^+, z_1w_2^+, z_2w_2^+)\):
Now, using Mathematica one can diagonalize the $M_i$ ($i = 1, 4$) matrices to get the following eigensvalues:

\[ \Omega_1^1 = \Omega_2^1 = \xi_5 - \frac{1}{2} \xi_8, \quad \Omega_3^1 = \xi_5 + \frac{5}{2} \xi_8, \quad \Omega_4^1 = \Omega_5^1 = \xi_5 + \frac{1}{2} \xi_8 \]  

(92)

\[ \Omega_{1,2}^2 = 3(\xi_1 + \xi_2) \pm \sqrt{(\xi_1 - \xi_2)^2 + (2\xi_5 + \xi_8)^2}, \quad \Omega_{3,4}^2 = \Omega_{5,6}^2 = \xi_1 + \xi_2 \pm \sqrt{(\xi_1 - \xi_2)^2 + \frac{1}{4} \xi_8^2} \]  

(93)

and

\[ \Omega_{1,2}^3 = \Omega_{3,4}^3 \]  

(94)

\[ \Omega_1^4 = \Omega_4^4, \quad \Omega_2^4 = \Omega_2^1, \quad \Omega_3^4 = \Omega_5^1, \quad \Omega_4^4 = \Omega_5^1, \quad \Omega_5^4 = \Omega_{5,6}^2, \quad \Omega_6^4 = \Omega_{4,6}^2, \quad \Omega_1^4 = \Omega_1^1 \]  

(95)

where $\Omega_i^j$ stands for the $i^{th}$ eigenvalue of the matrices $M^j$. Now, imposing the unitarity condition

\[ |\Omega_i^j| \leq 8\pi \]  

(96)

and replacing the $\xi_i$’s by their expressions (Eqs.75), we end up with the following solution:

\[ \frac{\Sigma}{(Tr K K^*)^2} \lesssim \frac{8\pi x}{14.6} \]  

(97)
which can be considered as a new unitarity constraints between the various NCG parameters. Notice that in order that Eq.(97) is compatible with the Schwartz inequality (Eq.(81)), one has to have $x \gtrsim 1.7$.

Regarding the quantization, in the framework of NCG, there is no satisfactory procedure which has been developed yet treating the gauge and higgs bosons in an equal footing. Therefore, we expect that the quantum fluctuations may badly violate the tree level NCG constraints and relations. In principle, the change in the quantization rules is needed around certain energy scale and we have to assume that just below such a scale, the standard quantization method makes a good approximation. In this case, we can start from the classical lagrangian and use the conventional quantization.

In what follow, we apply this approach to our NCG approach of the two doublets Higgs model through simple examples and test whether it is possible to preserve the tree level mass, coupling and unitarity relations. Let us take for example the ratio $\frac{g}{\sin \theta_w}$. It is easy to see from the relations Eqs.(73)-(74) and Eq.(75) that:

$$\frac{\xi_8}{g} \approx \frac{1.2 \Sigma}{x^{1/2} (TrKK^*)^2}$$  \hspace{1cm} (98)

where

$$g = \frac{g_1}{\sin \theta_w}$$

Now, if eq.(98) will hold at any scale $\mu$ for given independent values of the NCG parameters $x, \Sigma$ and $TrKK^*$, the corresponding $\beta$-functions $\beta_{\xi_8}$ and $\beta_{g}$ have to verify the same relation of Eq.(98). However, the one loop $\beta$-functions with complicated expressions in terms of the various couplings (see Eq.(99)) do not seem to satisfy this relation for any values of $x, \Sigma$ and $TrKK^*$. Therefore, at the one loop order, the constraint Eq.(98) is not preserved under the renormalization flow. To see the complexity of the $\beta$-functions expressions, we have derived (based on refs.[26], [27] and [28]) some of the coupling constants renormalization group equations (R.G.) for the two doublets Higgs model:

\begin{align*}
16\pi^2 \frac{d\xi_8}{dt} &= b_8 g_1^3; \quad (b_1,b_2,b_3)=(1,3,5) \\
16\pi^2 \frac{d\xi_8}{dt} &= (\Lambda_1+\Lambda_2 g_1^2)g_1; \quad \Lambda_1=-\frac{17}{20}g_1 \xi_6^2-\frac{9}{4}g_1^2-8g_1^3; \quad \Lambda_2=-\frac{2}{3} \\
16\pi^2 \frac{d\xi_8}{dt} &= (\Lambda_3+\Lambda_4 g_1^2)g_1; \quad \Lambda_3=-\frac{1}{2}g_1^5-\frac{9}{4}g_1^3-8g_1^3; \quad \Lambda_4=1 \\
16\pi^2 \frac{d\xi_8}{dt} &= 24\xi_8^2+24\xi_8^2\xi_8+24\xi_8^2-(9g_1^2+\frac{9}{4}g_1^5)\xi_1+(\frac{7}{2}g_1^4+\frac{17}{20}g_1^2)+\frac{9}{4}g_1^3+12g_1^3 \xi_1-6g_1^4 \\
16\pi^2 \frac{d\xi_8}{dt} &= 24\xi_8^2+24\xi_8^2\xi_8+24\xi_8^2-(9g_1^2+\frac{9}{4}g_1^5)\xi_2+(\frac{7}{2}g_1^4+\frac{17}{20}g_1^2)+\frac{9}{4}g_1^3+12g_1^3 \xi_2-6g_1^4 \\
16\pi^2 \frac{d\xi_8}{dt} &= 12\xi_8^2\xi_1+24\xi_8\xi_1+24\xi_8^2\xi_2+24\xi_8\xi_2-\frac{1}{2}(9g_1^2+\frac{9}{4}g_1^5)\xi_3+(\frac{7}{2}g_1^4+\frac{17}{20}g_1^2)-\frac{9}{4}g_1^3+3g_1^3 \xi_3-6g_1^4 \xi_3^2
\end{align*}
and
\[ 16\pi^2 \frac{d\xi_8}{dt} = 32\xi_8^2 + 4\xi_5\xi_8 + 2\xi_5^2 - (9g_2^2 + \frac{9}{5}g_1^2)\xi_8 + 6(g_t^2 + g_b^2)\xi_8 - 6g_b^2g_t^2 \]

Here \( g_1, g_2, g_3, g_t \) and \( g_b \) denote the electromagnetic, weak, strong, top Yukawa and bottom Yukawa couplings respectively. Moreover, an \( SU(5) \) normalization is used for which the Weinberg \( \theta_w \) angle is defined through the relation
\[ g_2^2 \sin^2 \theta_w = \frac{3}{5}g_1^2 \cos^2 \theta_w \]

and the evolution parameter \( t \) is defined as:
\[ t = \ln \left( \frac{\mu}{\mu_0} \right) \]

(\( \mu_0 \) is some reference scale e.g. \( \mu_0 = M_Z \)). We have also neglected all fermionic Yukawa couplings except those of the top and bottom quarks.

We notice that if the NCG parameters \( x, \Sigma \) and \( TrKK^* \) are scale dependent, the NCG relations, Schwartz and unitarity bounds (Eqs.(75),(81) and (97)) cannot be simultaneously verified. To be more explicit, let us assume that the relation in Eq.(73) holds for any scale \( \mu \). Plugging the latter into the NCG relations Eq.(75), using the fact that:
\[ TrKK^* = \frac{4}{9xg_t^2} \]

and the solutions of the renormalization group equations of \( g_1, g_2, g_t \) and \( \xi_1 \) as are given by Eqs.(99), one can fix exactly the \( \mu \)-dependence of the \( \Sigma \) parameter. However, the latter will not be compatible with the unitarity constraint(Eq.(97)), Schwartz inequality and the other NCG relations (Eqs.(81) and Eqs.(75)) under the renormalization flow of the other couplings \( \xi_2, \xi_3 \) etc. Thus, the NCG constraints are not renormalization group invariant and we cannot have one scale \( \mu \) where all the relations are satisfied simultaneously.

**Conclusion**

As a conclusion, we cannot satisfy simultaneously all the tree level NCG mass, couplings and unitarity relations at the same energy scale and therefore we will face a sort of incompatibility with the renormalization group approach if the NCG constraints holds. Thus, the two doublet higgs model can be nicely constructed classically within this extended Chamesseddine-Frohlich approach to NCG but we have to look for an appropriate quantization procedure in the context of Noncommutative geometry.
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